Cross Sections and Observables

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Joint Physics Analysis Center
Exotic Hadron Topical Collaboration

Horizon2020 Summer School

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EXOHAD
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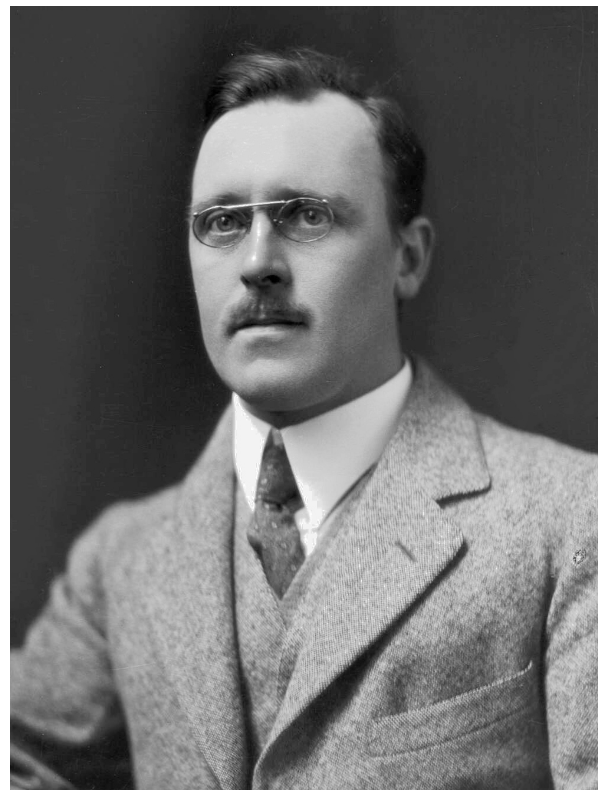
Outline

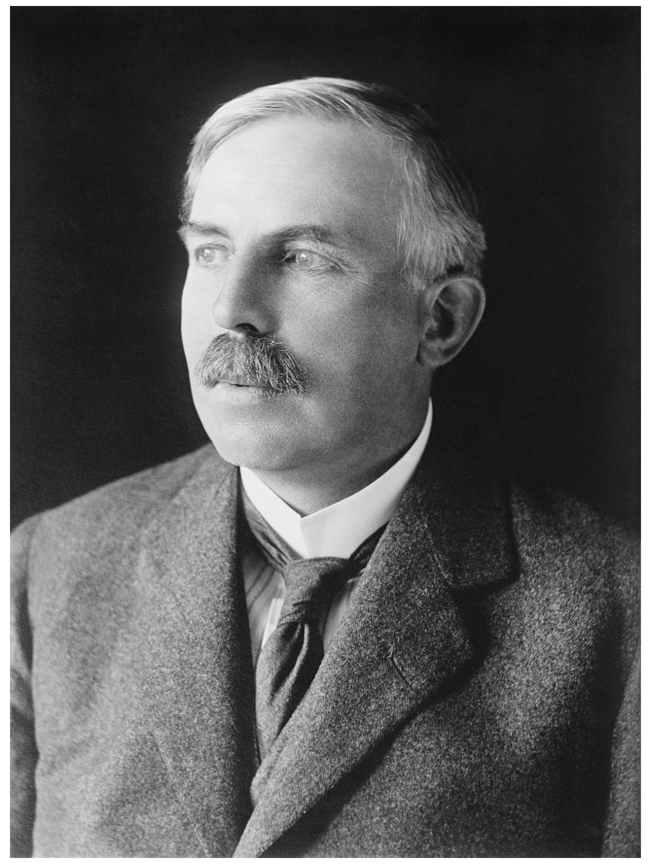
The Geiger-Marsden experiment (1908)

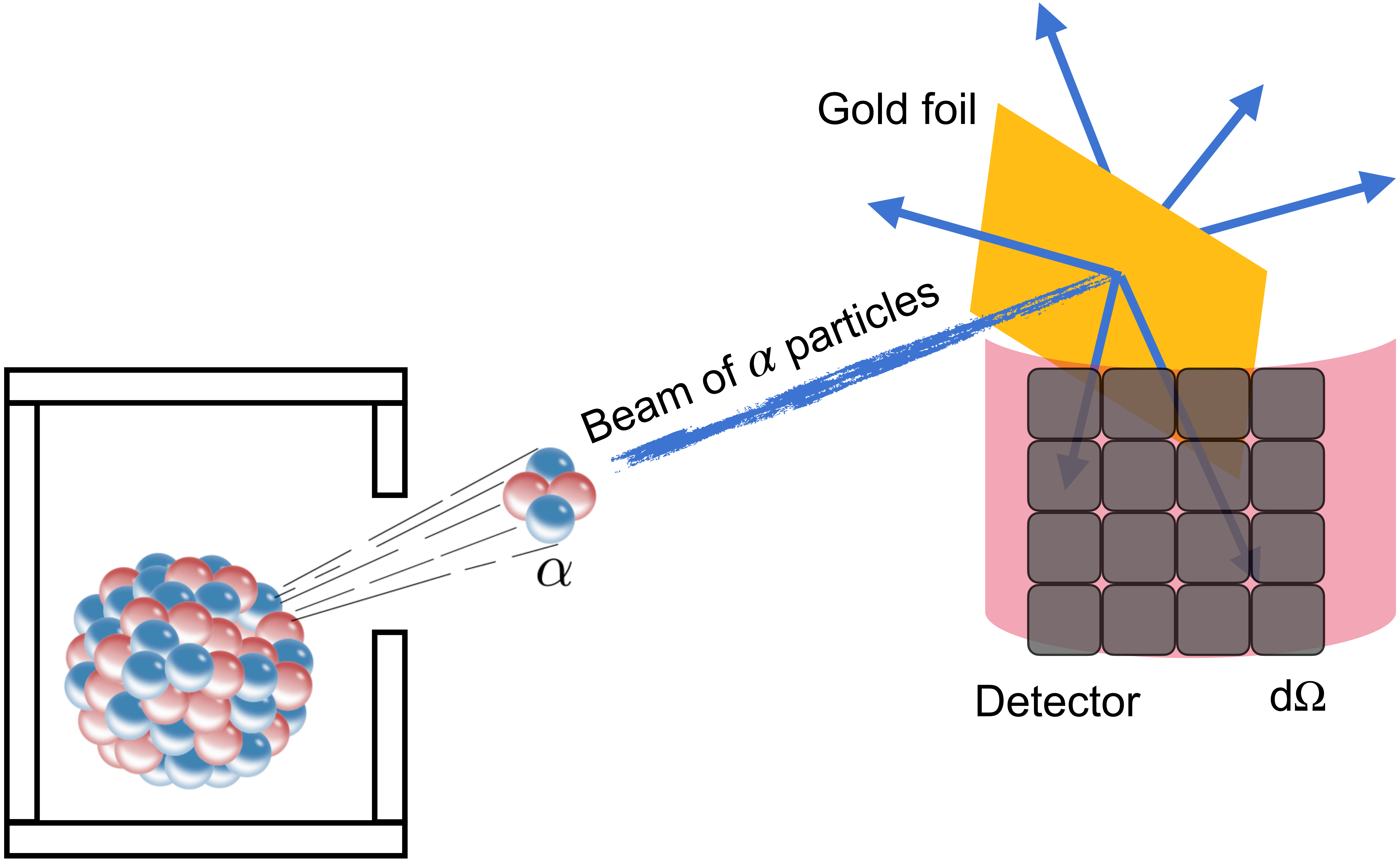
H. Geiger
1882 - 1945

E. Marsden
1889 - 1970



E. Rutherford
1871 - 1937

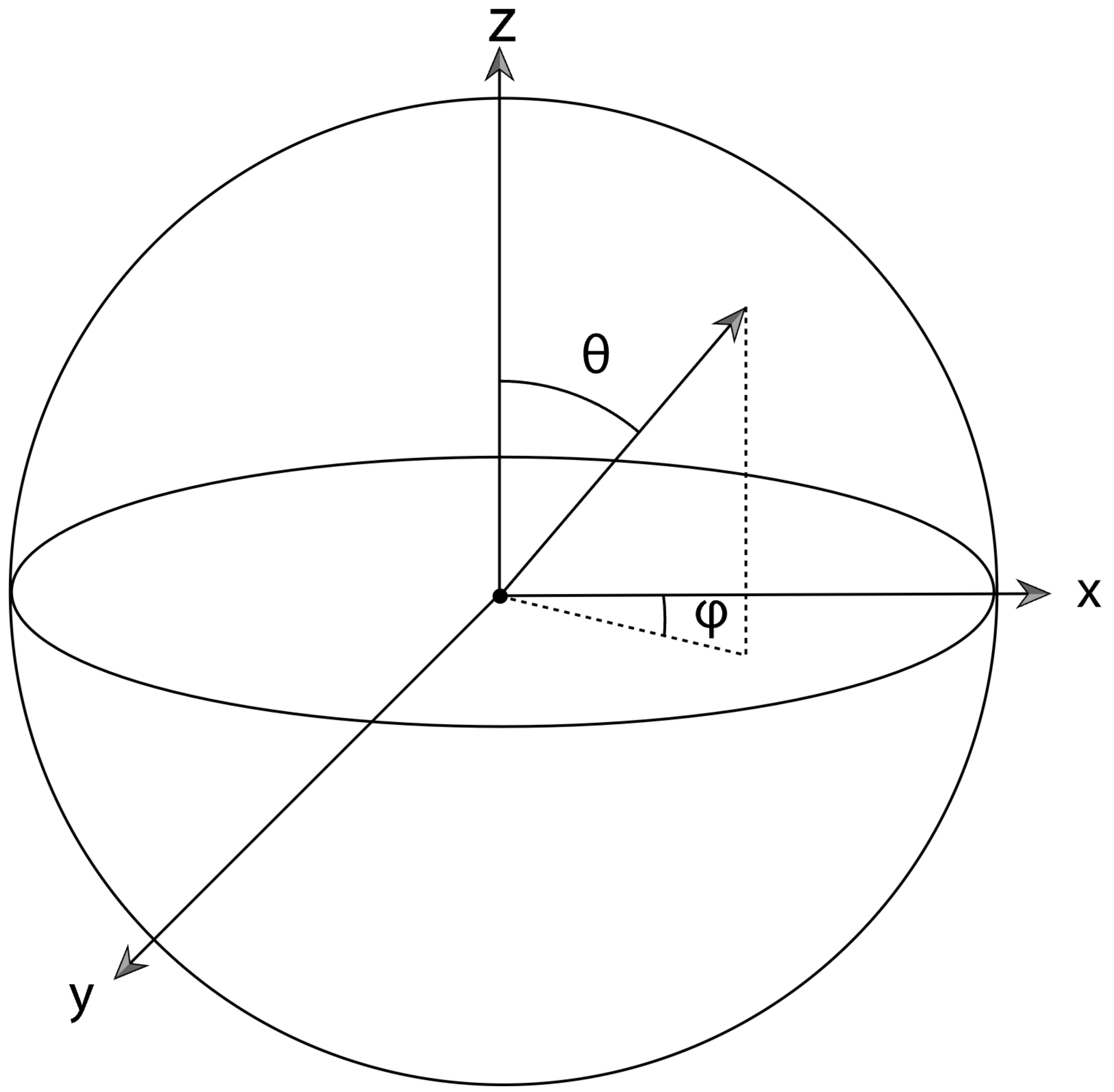


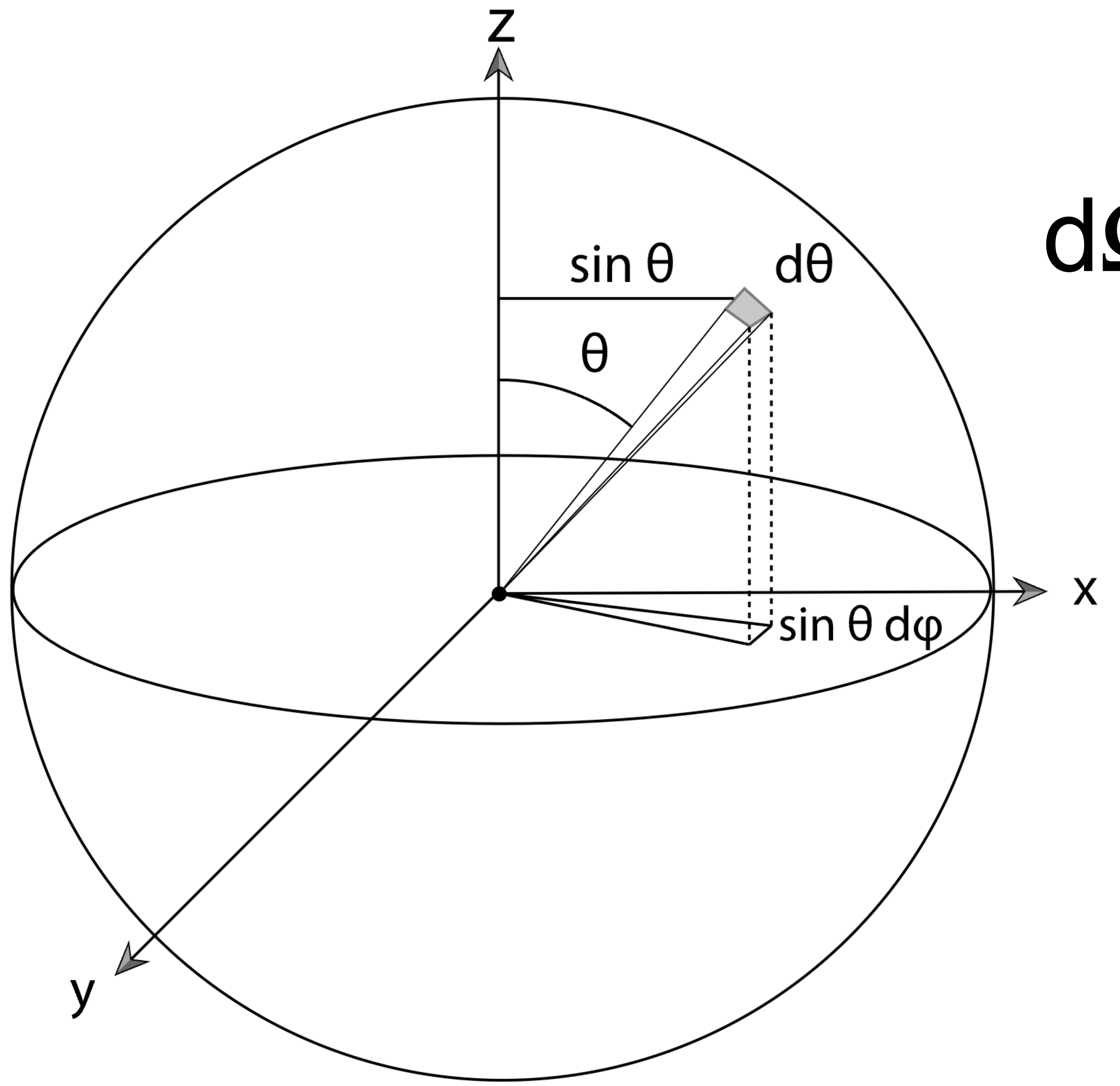


Spherical coordinates

Points on the (unit radius) sphere are identified by two angles

$d\Omega$ is the area when angles are in $[\theta, \theta + d\theta]$ and $[\phi, \phi + d\phi]$



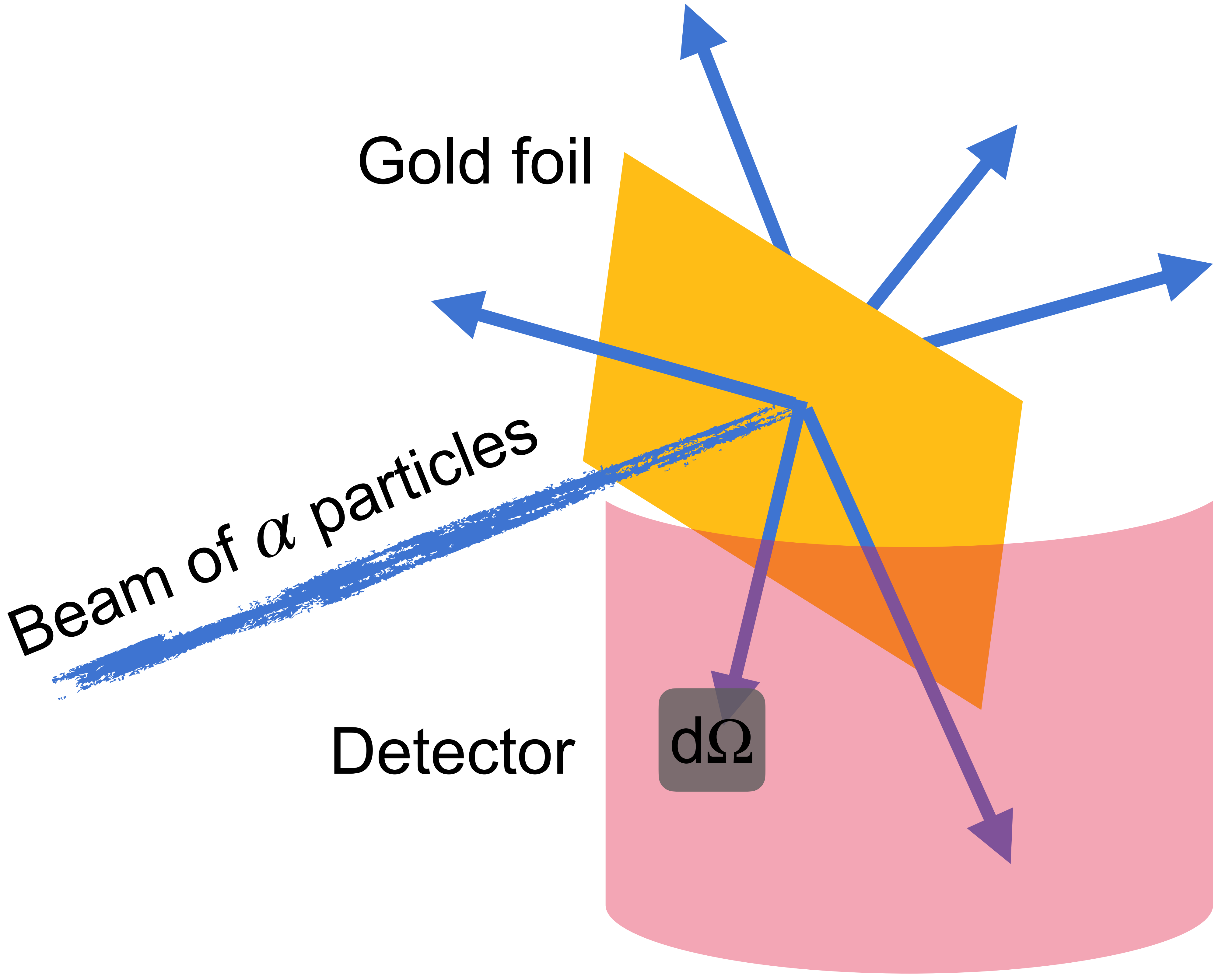


$$d\Omega = d\theta \times \sin\theta d\phi$$

Differential cross-section (experimentalist point of view)

Differential cross-section:

$$\frac{d\sigma}{d\Omega} = \frac{\text{“\#detected particles”}}{\text{unit of solid angle}}$$



The number of particles depends on the time and on the initial flux!

$$\frac{d\sigma}{d\Omega} = \frac{1}{\mathscr{L}} \frac{dN}{d\Omega}$$

Luminosity $\mathscr{L} = \frac{\text{\#incident particles}}{\text{time x area}}$

Modern order of magnitude:

$$\mathscr{L}_{LHC} \sim 10^{34}/(\text{cm}^2\,.\,\text{s})$$

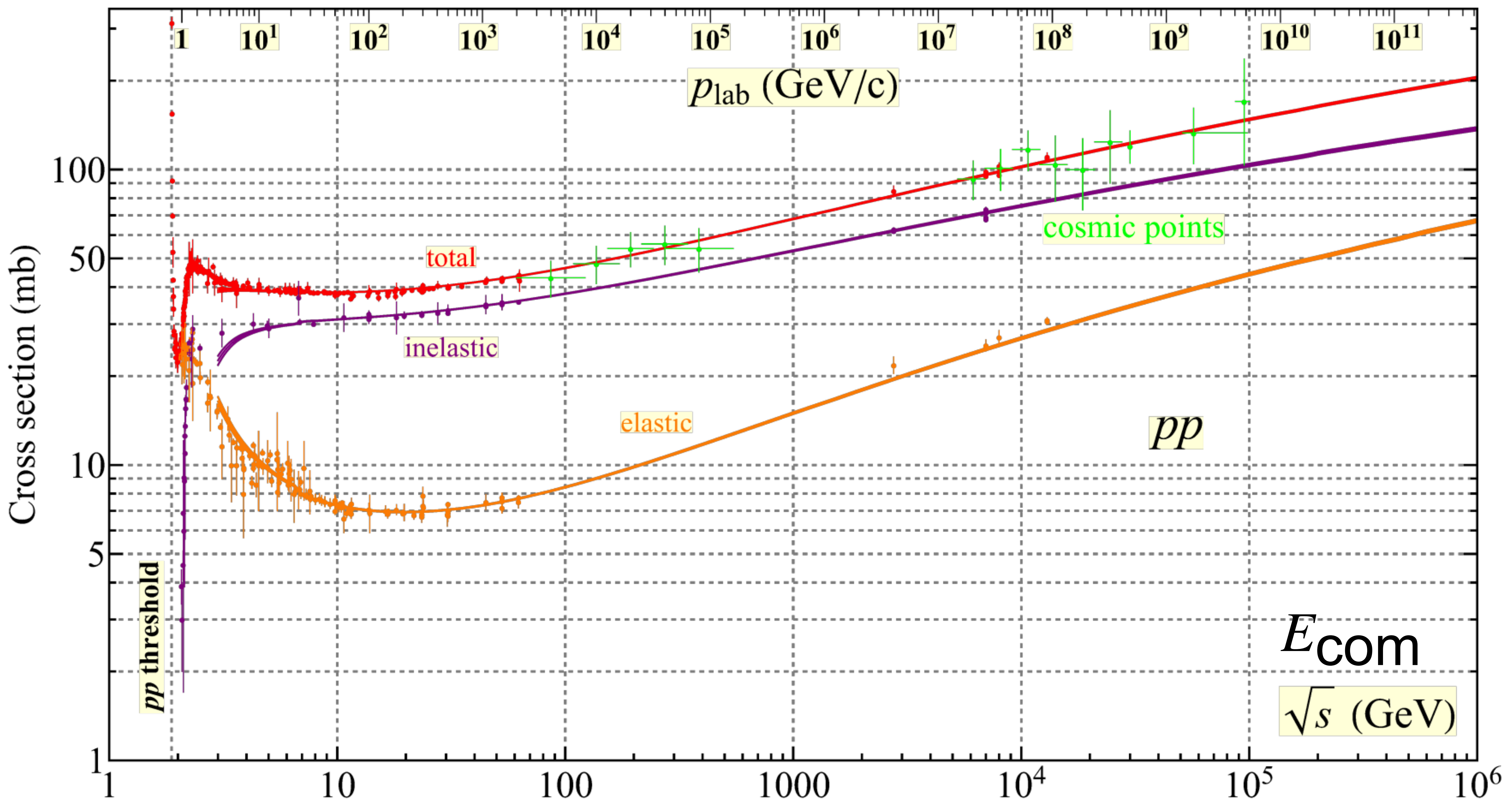
cross-section (experimentalist point of view)

Differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{\mathscr{L}}\frac{dN}{d\Omega}$$

Cross section:

$$\sigma = \int_{\text{sphere}} \frac{d\sigma}{d\Omega} d\Omega = \frac{N}{\mathscr{L}}$$



Source: Particle Data Group (https://pdg.lbl.gov/)

Modern order of magnitude:

$\mathscr{L}_{LHC} \sim 10^{34}/(\text{cm}^2\,.\,\text{s})$

$\sigma_{pp}(13\ \text{TeV}) \sim 0.1\ \text{b} = 10^{-25}\ \text{cm}^2$

Number of collisions at LHC: $N = \mathscr{L}\sigma \sim 10^9$ /s

Let's talk about units!

$\sigma_{pp}(13\ \text{TeV}) \sim 0.1\ \text{b} = 10^{-25}\ \text{cm}^2$

Electron-Volt (eV):

Energy acquired by an electron
accelerated by a potential of 1 Volt

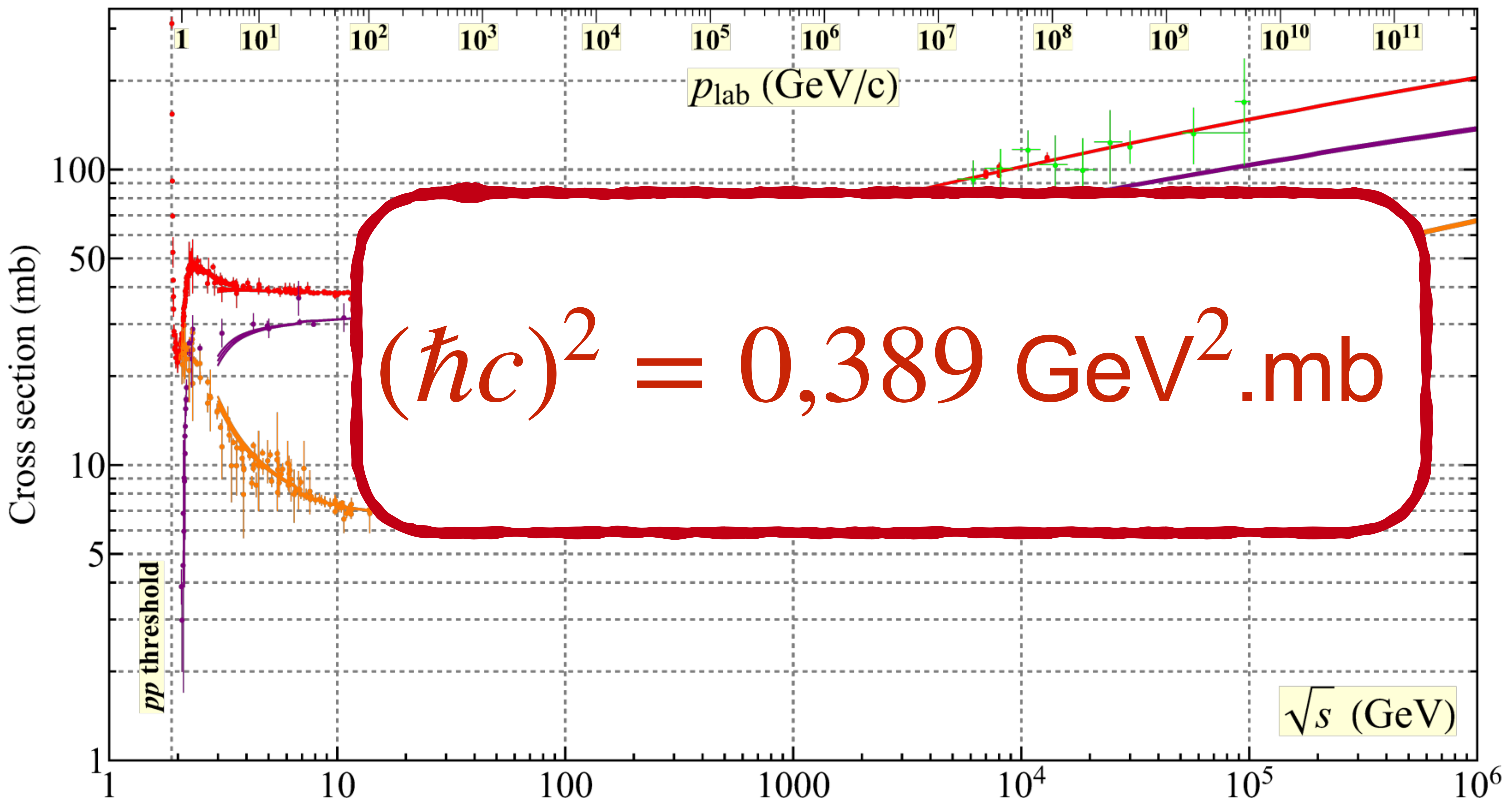
$1\ \text{eV} \simeq 1.6\ 10^{-19}\ \text{J}$ (J = kg.m^2/s^2)

Einstein relation: $E = mc^2$

Masses are expressed in eV/c^2

Nuclear physicists take c = 1
and express masses in eV

Proton mass $m_p \simeq 1\ \text{GeV} = 10^6\ \text{eV}$



$(\hbar c)^2 = 0{,}389\ \text{GeV}^2.\text{mb}$

Source: Particle Data Group (https://pdg.lbl.gov/)

Barn (b): ~ transverse area of uranium nucleus

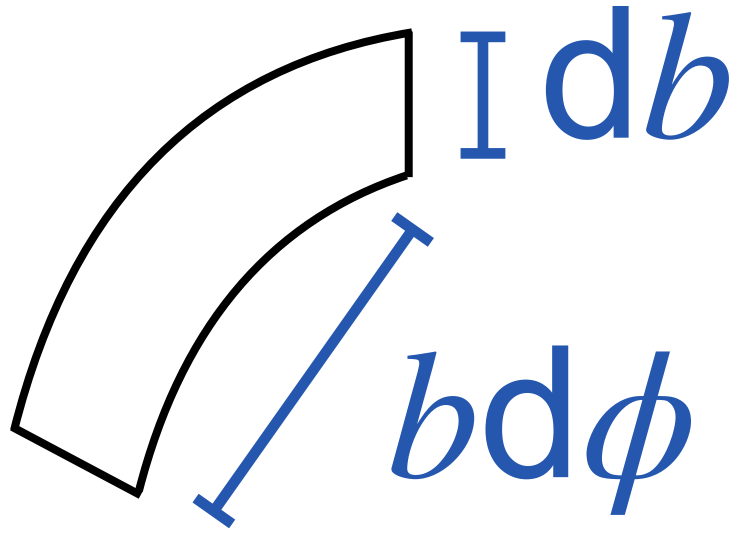
$1\text{b} = 10^{-28}\ \text{m}^2 = 100\ \text{fm}^2$

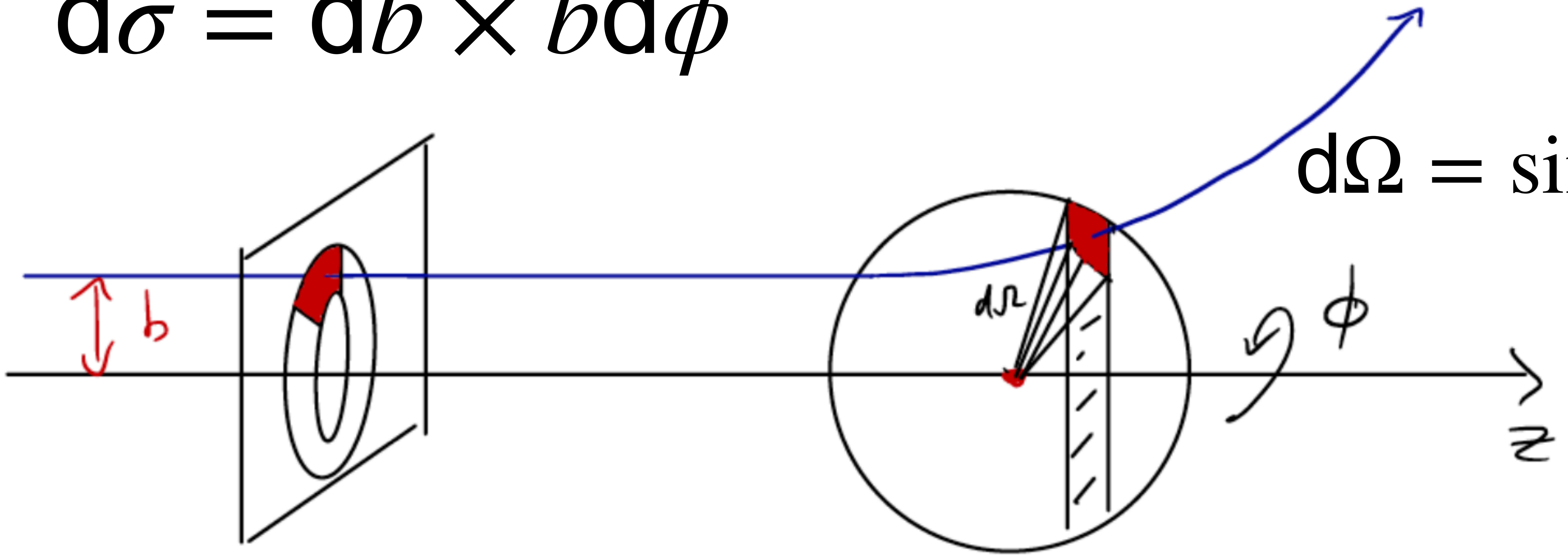
$1\ \text{fm} = 10^{-15}\ \text{m}$

Differential cross-section (theorist point of view)

cross-section: “transverse area where a collision happens”

$$\mathrm{d}\sigma = \mathrm{d}b \times b\mathrm{d}\phi$$





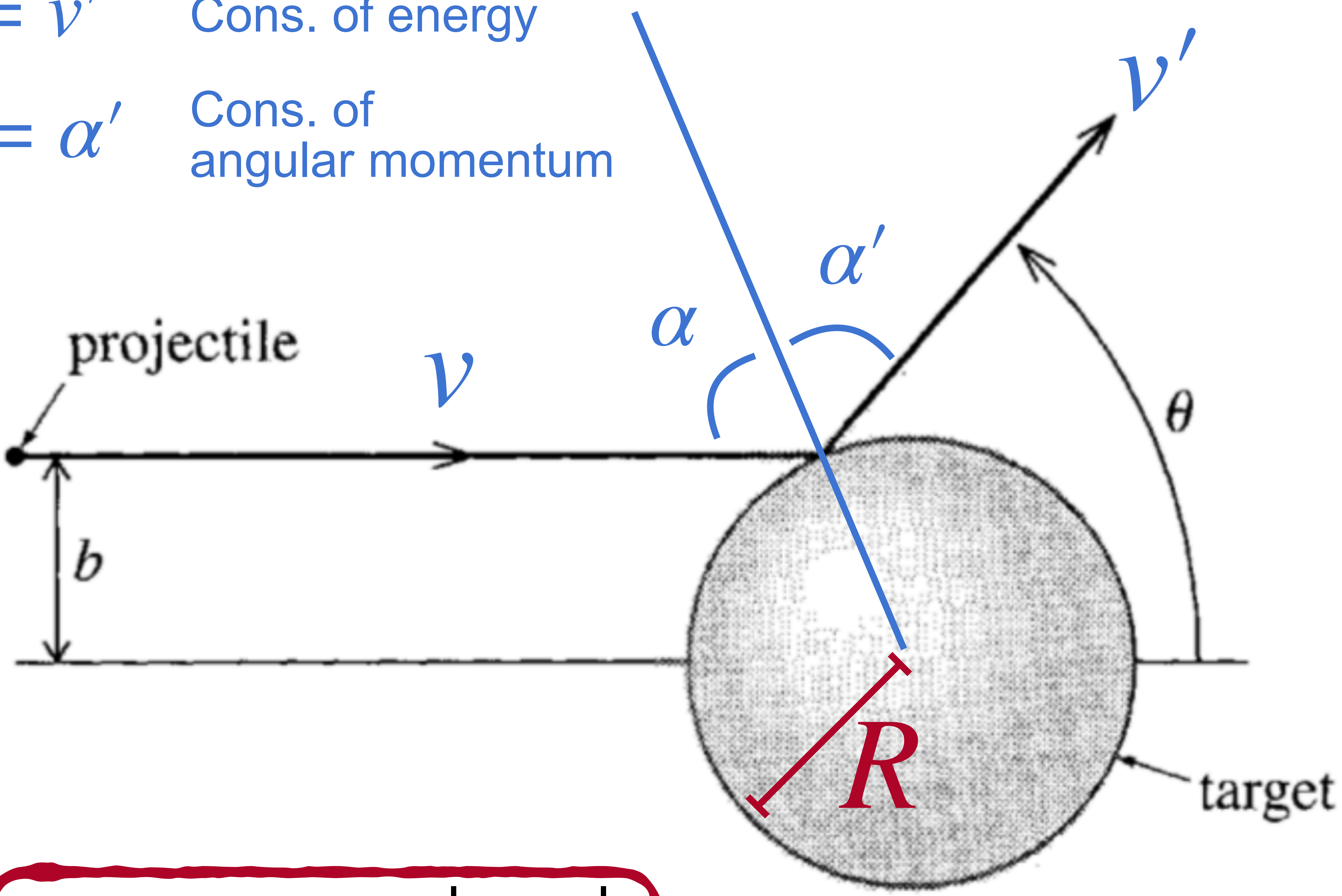
$$\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

Hard sphere scattering

$v = v'$ Cons. of energy

$\alpha = \alpha'$ Cons. of angular momentum



$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{R^2}{4}$$

$$2\alpha + \theta = \pi$$

$$b = R\sin\alpha$$
$$= R\cos\frac{\theta}{2}$$

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{R}{2}\sin\frac{\theta}{2}$$

$$\sigma = \int_{\text{sph.}} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega$$
$$= \frac{R^2}{4}\int_{4\pi}\mathrm{d}\Omega = \pi R^2$$

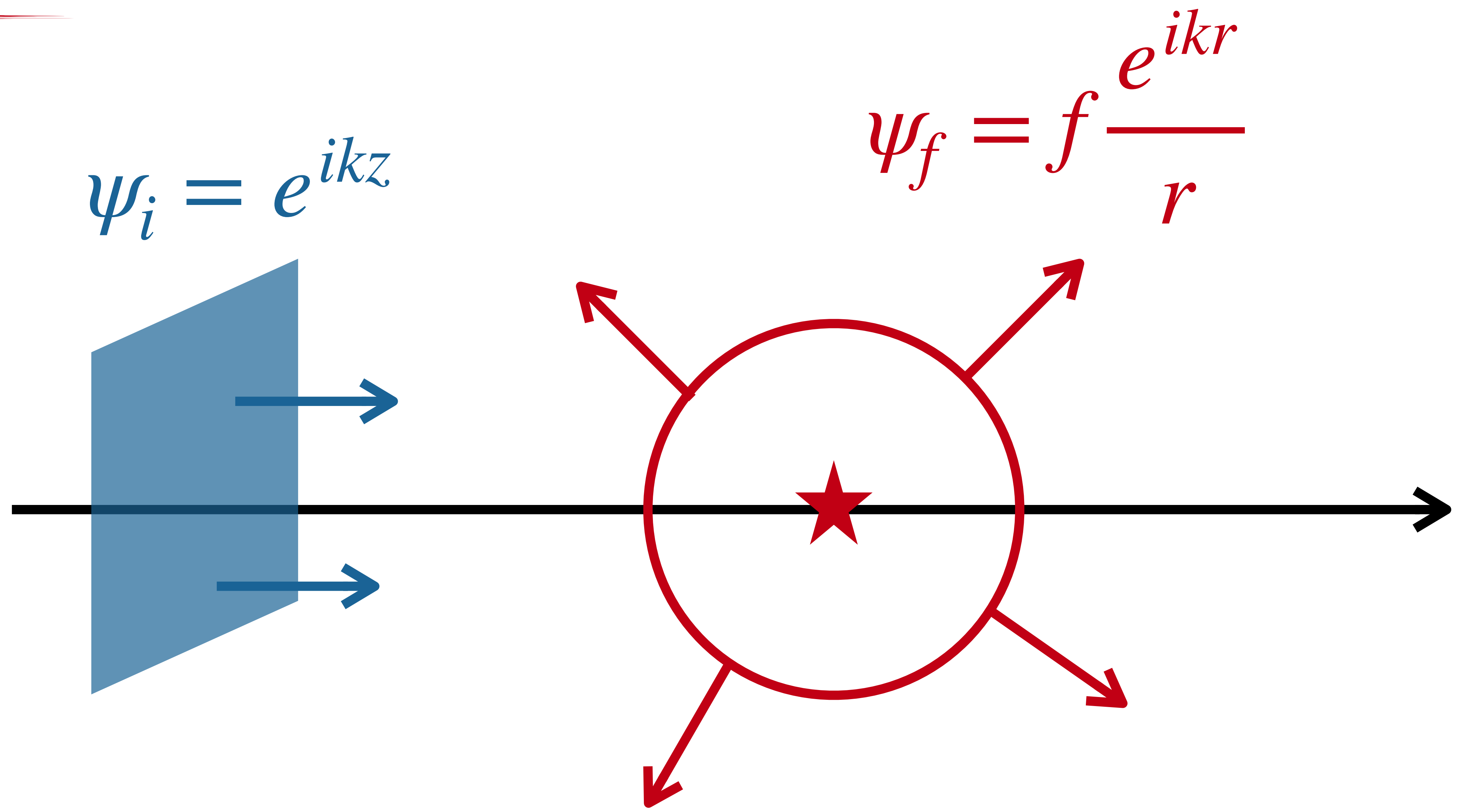
The projectile sees a disk a radius R

The transverse area of a sphere is indeed πR^2

Non-relativistic cross section

A plane wave enters, a spherical wave comes out

$$\psi(r,E,\theta) = A\left[e^{ikz} + f(E,\theta)\frac{e^{ikr}}{r}\right]$$



Dimensions of ψ, $[\psi] = ?$

$[r] = L$

$[f] = L$

$$\int d^3\vec{r}\,|\psi|^2 = 1$$

$[\psi] = L^{-3/2}$

So $|\psi|^2 \times L^3$ is dimensionless

Conservation of probability

$$dP = |\psi_i|^2 dV = |A|^2 d\sigma\ vdt$$

$$dP = |\psi_f|^2 dV' = |A|^2 \frac{|f|^2}{r^2} r^2 d\Omega\ vdt$$

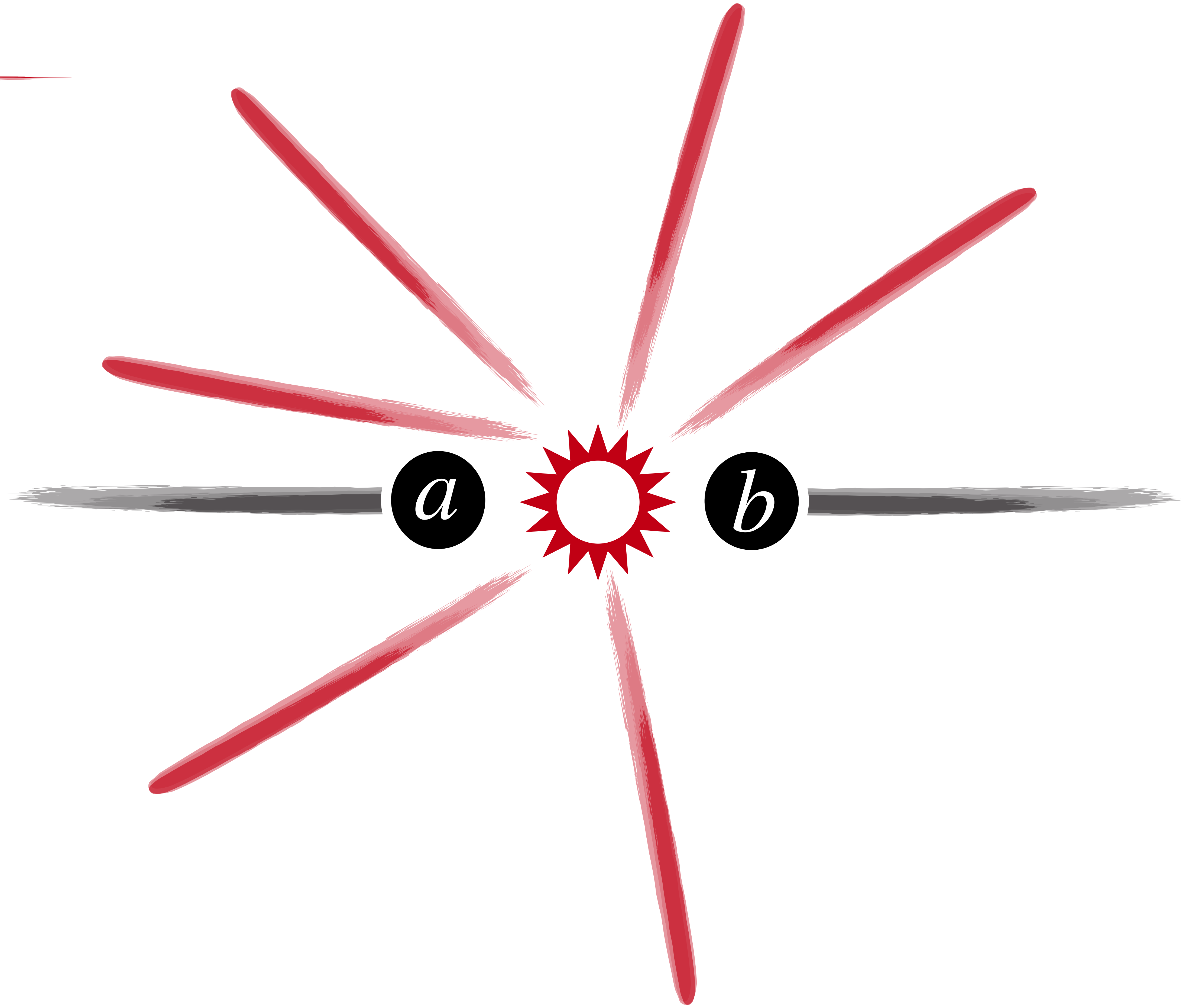
$$\frac{d\sigma}{d\Omega} = |f(E,\theta)|^2$$

Phase space

Every final state particle contributes to

$$d^4p \;\times\; \delta(p^2 - m^2) \;\times\; \theta(p^0)$$

$dp^0 d^3\vec{p}$ mass shell Positive energy



Use $\delta\left[f(x)\right] = \dfrac{\delta(x - x_0)}{|f'(x_0)|}$ with $f(x_0) = 0$

$$\delta\left[(p^0)^2 - \vec{p}^2 - m^2\right] = \frac{\delta(p^0 - E_p)}{2E_p} \qquad E_p = \sqrt{\vec{p}^2 + m^2}$$

Only one root since $\theta(p^0)$, thus

$$d^4p\,\delta(p^2 - m^2)\,\theta(p^0) = \frac{d^3p}{2E_p}$$

Producing n final state particles

$$R_n(s) = \delta^4(P - P')\prod_{i=1}^{n}\int\frac{d^3\vec{p}_i}{2E_i}$$

Relativistic cross section

Add the flux and conventional $(2\pi)^3$ factors

$$d\sigma = \frac{1}{F}\frac{(2\pi)^4}{(2\pi)^{3n}} \times \delta^4(P - P')\prod_{i=1}^{n}\frac{d^3\vec{p}_i}{2E_i} \times \overline{\sum} |A|^2$$

$$F = 2\lambda^{1/2}(s, m_a^2, m_b^2) = 4m_b p_a^L = 4\sqrt{s}p_a^*$$

sum final/average initial helicities

$$\overline{\sum} = \frac{1}{2s_a+1}\frac{1}{2s_b+1}\sum_{\lambda_a=-s_a}^{s_a}\sum_{\lambda_b=-s_b}^{s_b}\sum_{\lambda_1=-s_1}^{s_1}\cdots\sum_{\lambda_n=-s_n}^{s_n}$$

We will introduce helicities tomorrow…

States are conventionally normalized as

$$\langle \vec{p}\lambda | \vec{p}'\lambda' \rangle = (2\pi)^3 \; 2E \; \delta^3(\vec{p} - \vec{p}') \; \delta_{\lambda,\lambda}$$

For a decay, the flux is different

$$d\Gamma = \frac{1}{2M}\frac{(2\pi)^4}{(2\pi)^{3n}} \times \delta^4(P - P')\prod_{i=1}^{n}\frac{d^3\vec{p}_i}{2E_i} \times \overline{\sum} |A|^2$$

2-body cross section

2-body phase space in the CoM, $P = (\sqrt{s}, \vec{0})$

$$R_2(s) = \int \frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}\, \delta^4(P - p_1 - p_2) \qquad = \int \frac{d^3\vec{p}_1}{2E_1}\, d^4p_2\, \delta^4(P - p_1 - p_2)\, \delta(p_2^2 - m_2^2)\theta(p_2^0)$$

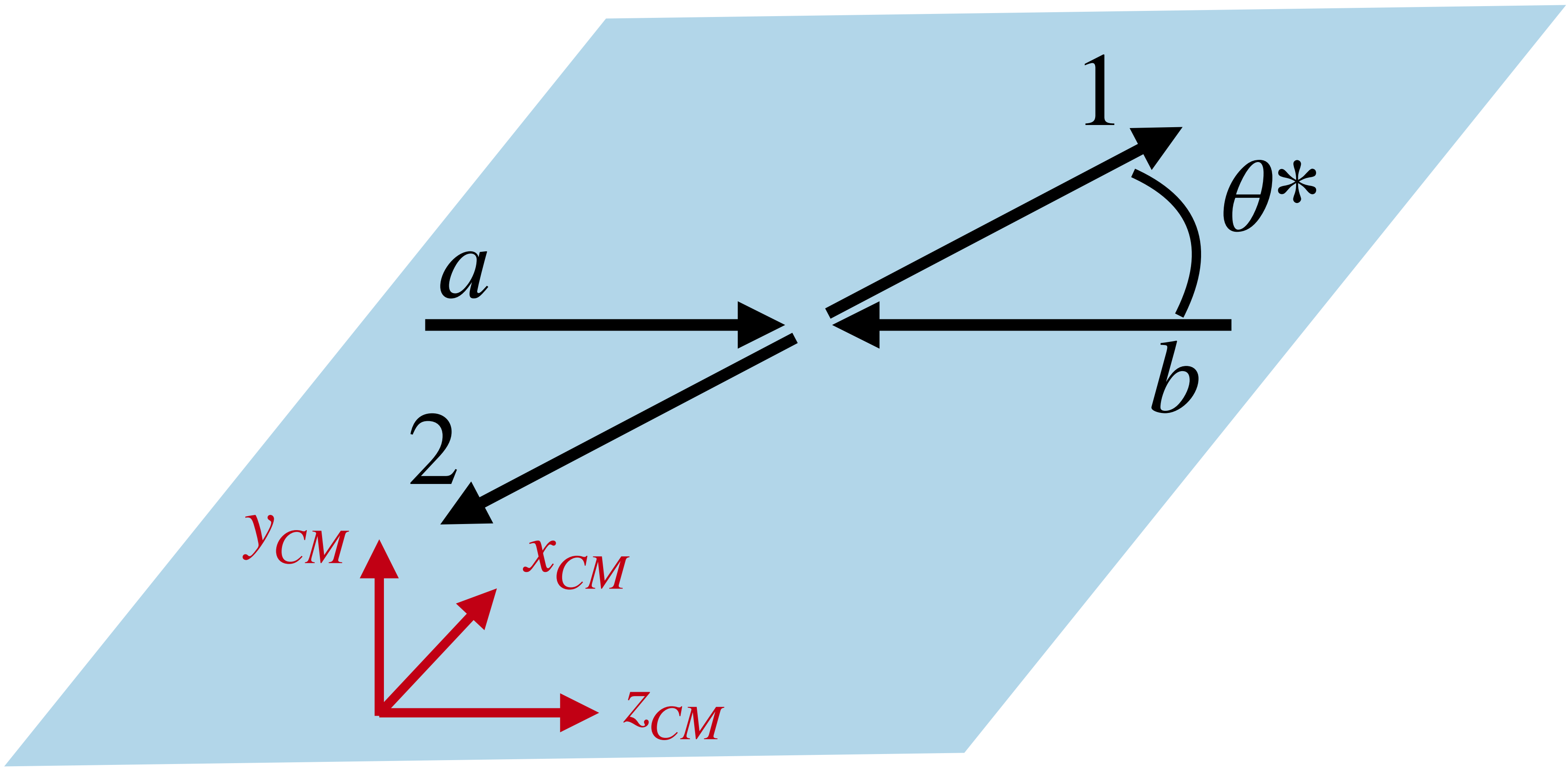
$$= \int \frac{d^3\vec{p}_1}{2E_1}\, \delta\left[(P - p_1)^2 - m_2^2\right] \qquad = \int \frac{p^2 dp d\Omega^*}{2E_1}\, \delta(s - 2\sqrt{s}\sqrt{p^2 + m_1^2} + m_1^2 - m_2^2)$$

With the solid angle elements $d\Omega^* = d\cos\theta^*\ d\phi^*$

$$R_2(s) = \int \frac{p^2 dp d\Omega^*}{2E_1} \frac{\delta(p - p^*)}{\frac{2\sqrt{s}}{2E_1} 2p^*}$$

$$R_2(s) = \frac{p^*}{4\sqrt{s}} \int d\Omega^*$$

$$p^* = \frac{1}{2\sqrt{s}} \lambda^{1/2}(s, m_1^2, m_2^2)$$



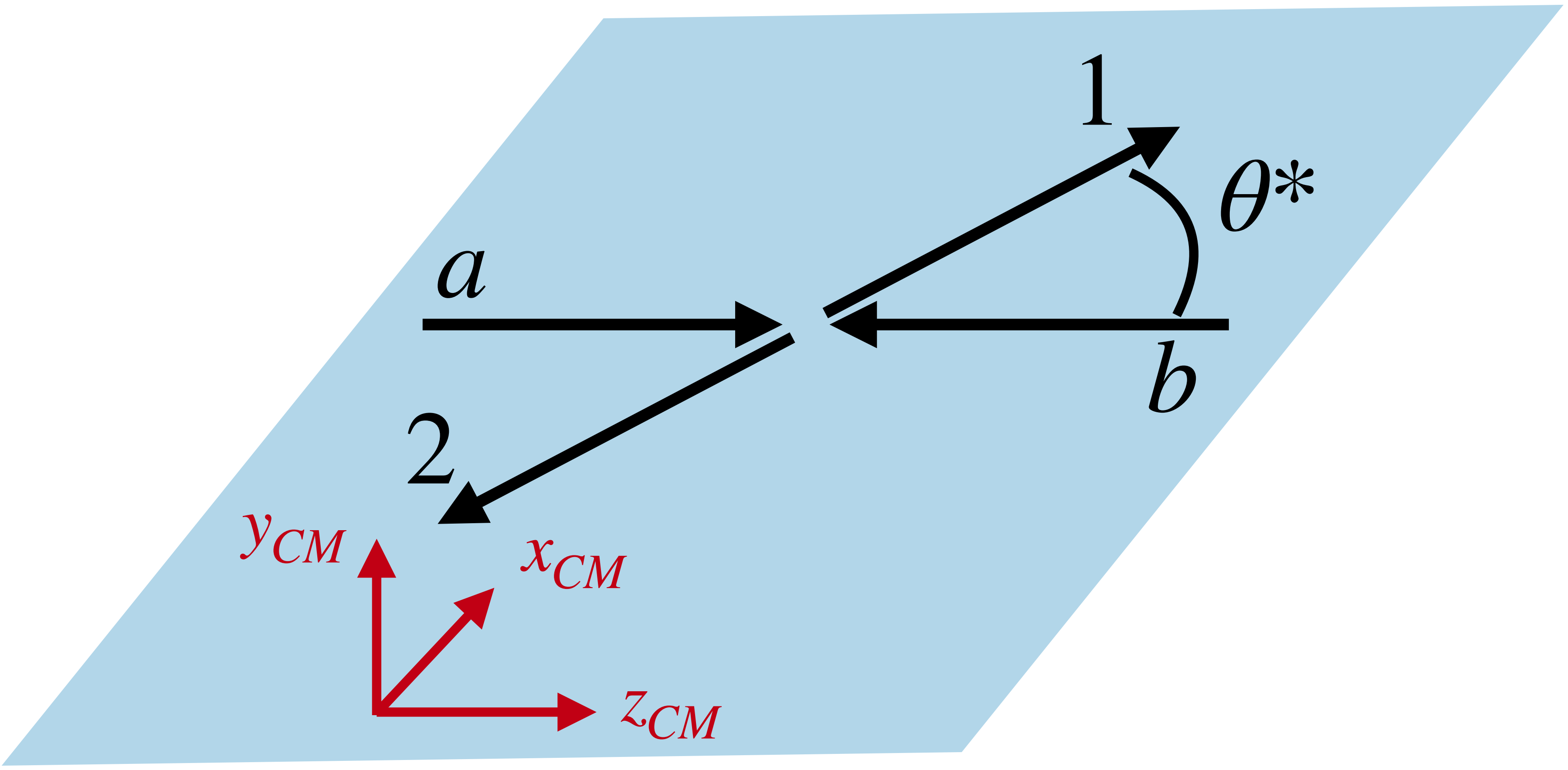
2-body cross section

Production of 2 particles

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s}\frac{p_1^*}{p_a^*}\overline{\sum}|A|^2$$

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s (p_a^*)^2}\overline{\sum}|A|^2$$

We used $dt = 2p_a^* p_1^* d\cos\theta^*$



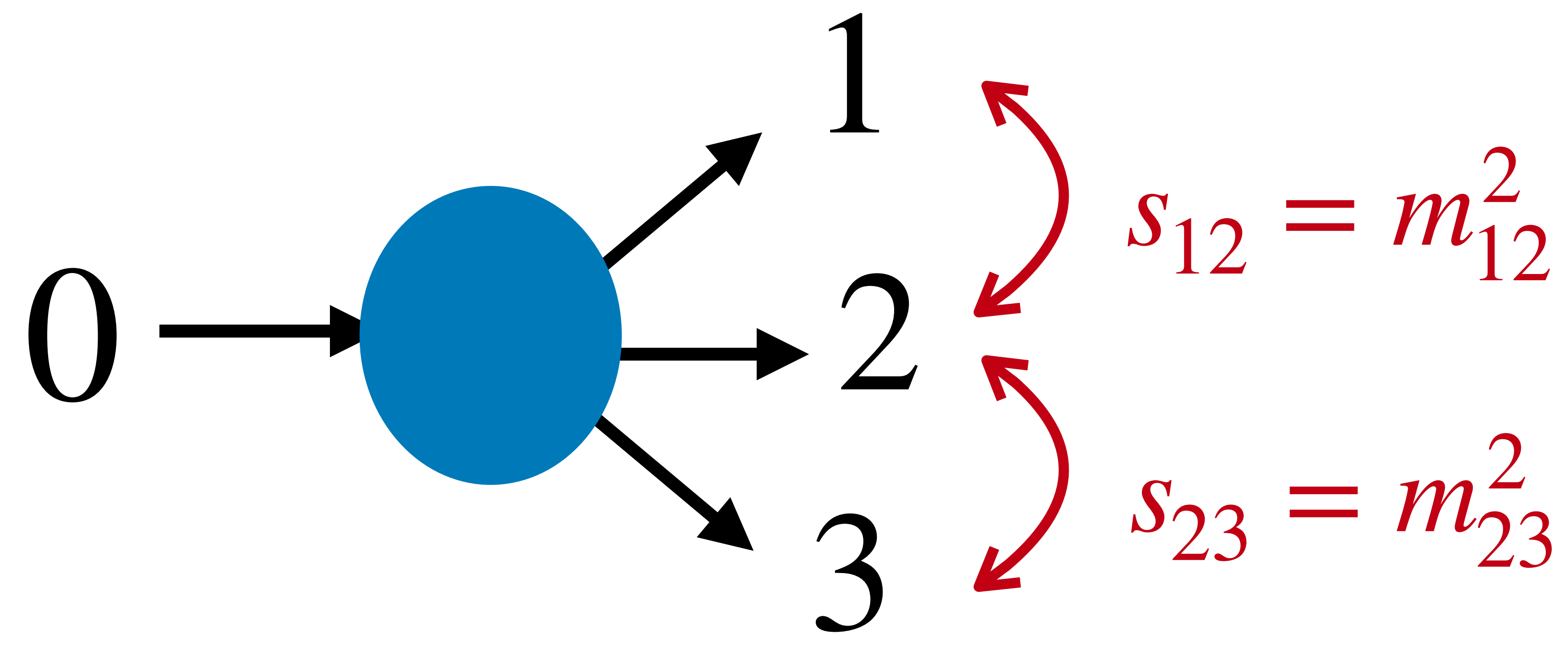
$$p_1^* = \frac{1}{2\sqrt{s}}\lambda^{1/2}(s, m_1^2, m_2^2)$$

$$p_a^* = \frac{1}{2\sqrt{s}}\lambda^{1/2}(s, m_a^2, m_b^2)$$

Decay into 2 particles

$$\Gamma = \frac{1}{2J+1}\frac{1}{8\pi}\frac{p_1^*}{M^2}\sum|A|^2 \times \frac{d\Omega^*}{4\pi}$$

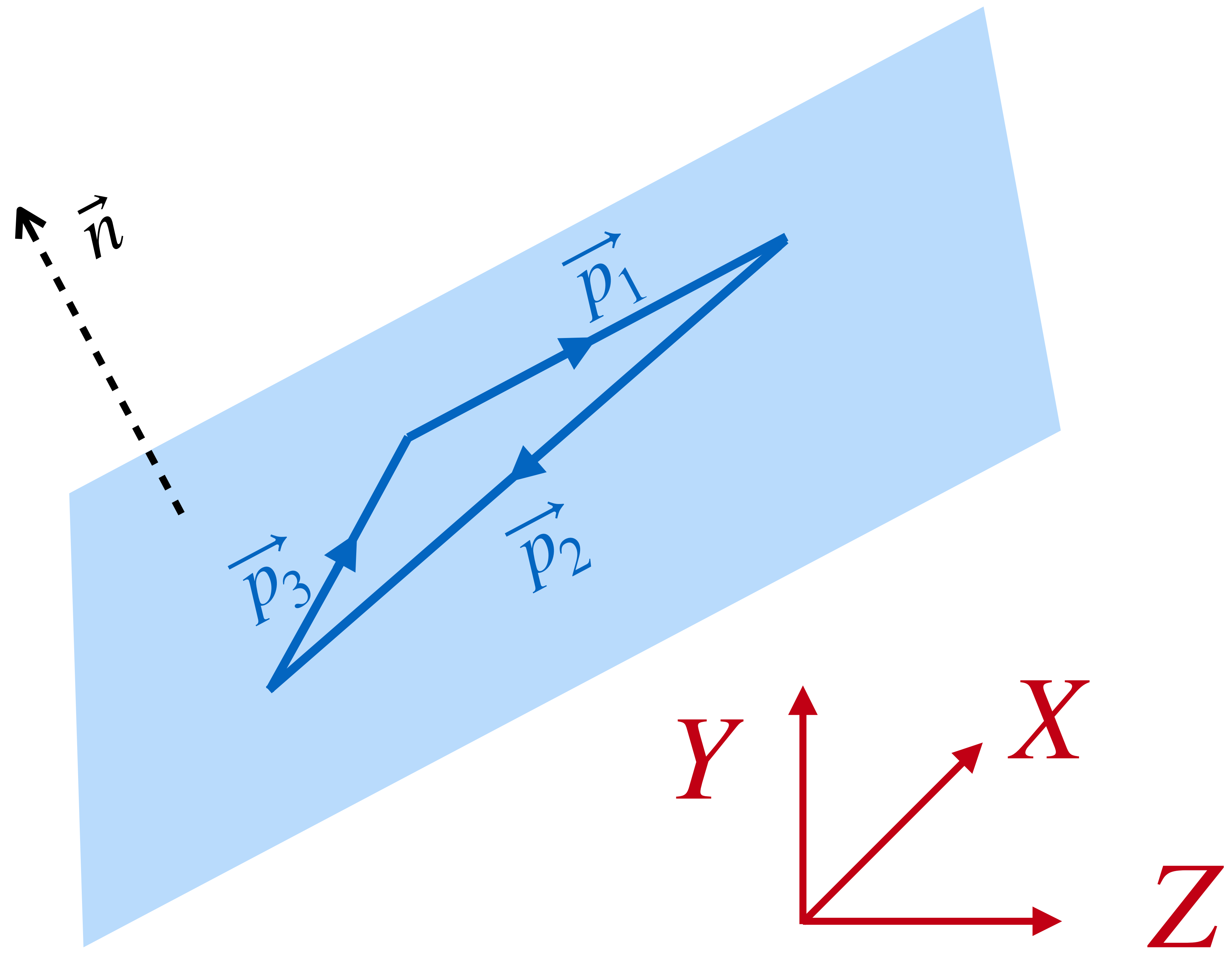
Decays into Three Particles



The three momenta determine a plane

The orientation of the plane is determined by three Euler angles α, β, γ

The orientation of the plane does not matters if not polarized



The decay is described by two variables s_{12}, s_{23}

Representation in a Dalitz plot

$$\frac{d\Gamma}{ds_{12}ds_{23}} \propto |A|^2$$

Dalitz plot

Three-body decay width

$$d\Gamma = \frac{1}{2M}\frac{1}{(2\pi)^5}\overline{\sum}|A|^2\, dR_3$$

The phase space is

$$R_3(s) = \delta^4(P-P')\int \frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}\frac{d^3\vec{p}_3}{2E_3}$$

$$= \frac{1}{8}\int \frac{d^3\vec{p}_1 d^3\vec{p}_3}{E_1E_2E_3}\delta(\sqrt{s}-E_1-E_2-E_3)$$

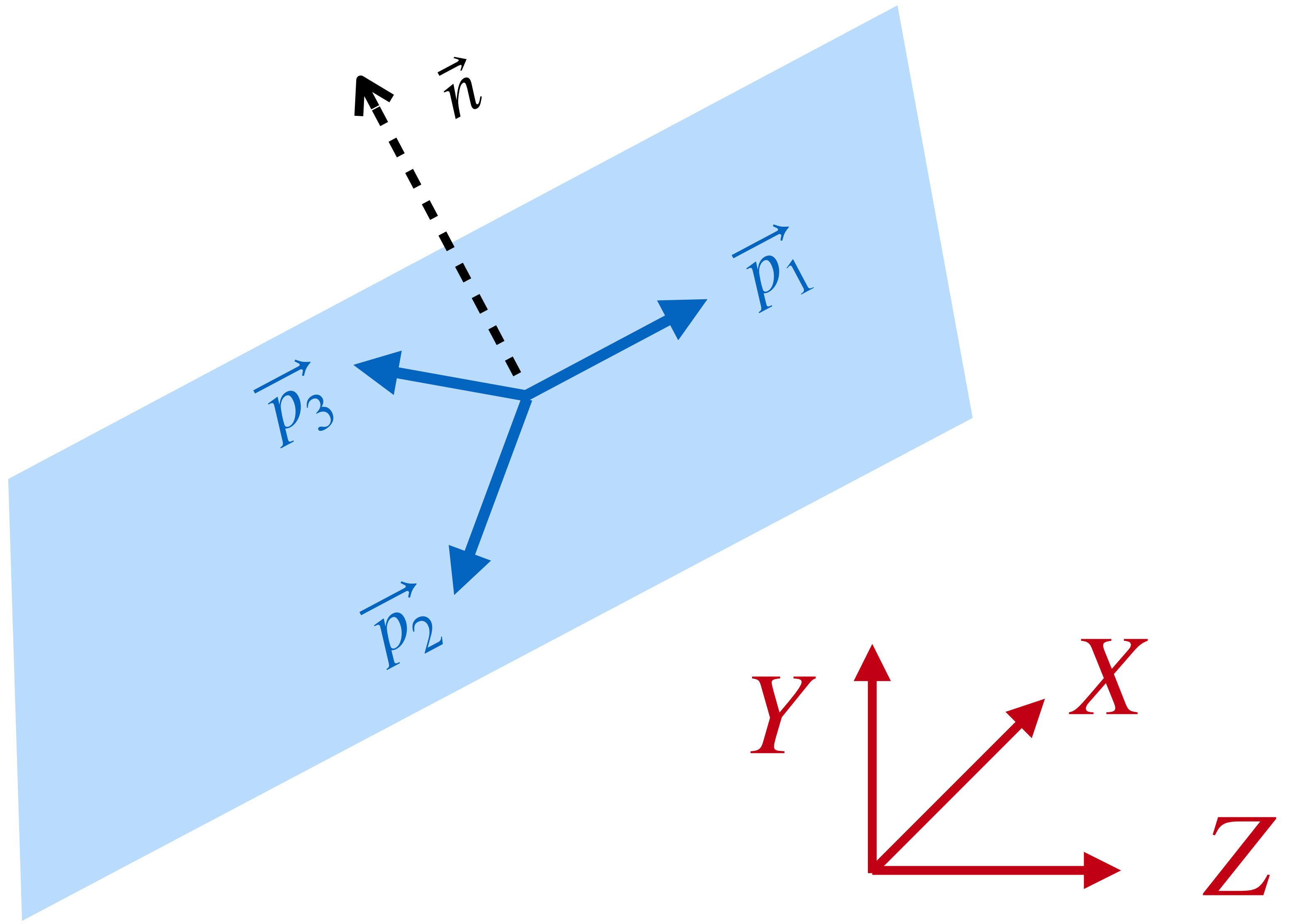
$$= \frac{1}{8}\int \frac{p_1p_3}{E_2} dE_1 dE_3 d\Omega_1 d\Omega_{31}\delta(\sqrt{s}-E_1-E_2-E_3)$$

Let's use the notation $|\vec{p}_1| \equiv p_1, \ldots$

$$d^3\vec{p}_1 d^3\vec{p}_3 = p_1^2 dp_1 d\Omega_1 \times p_3^2 dp_3 d\Omega_{31}$$

$$= p_1E_1dE_1d\Omega_1 \times p_3E_3dE_3d\Omega_{31}$$

We used $dp^2 = d(p^2+m^2) = dE^2 = 2EdE$



Dalitz plot

Three-body decay width

$$d\Gamma = \frac{1}{2M}\frac{1}{(2\pi)^5}\overline{\sum}|A|^2\, dR_3$$

The phase space is

$$d\Omega_{31} = d\cos\theta_{31}\, d\phi_3$$

$$R_3(s) = \frac{1}{8}\int \frac{p_1 p_3}{E_2} dE_1 dE_3 d\Omega_1 d\Omega_{31} \delta(\sqrt{s} - E_1 - E_2 - E_3)$$

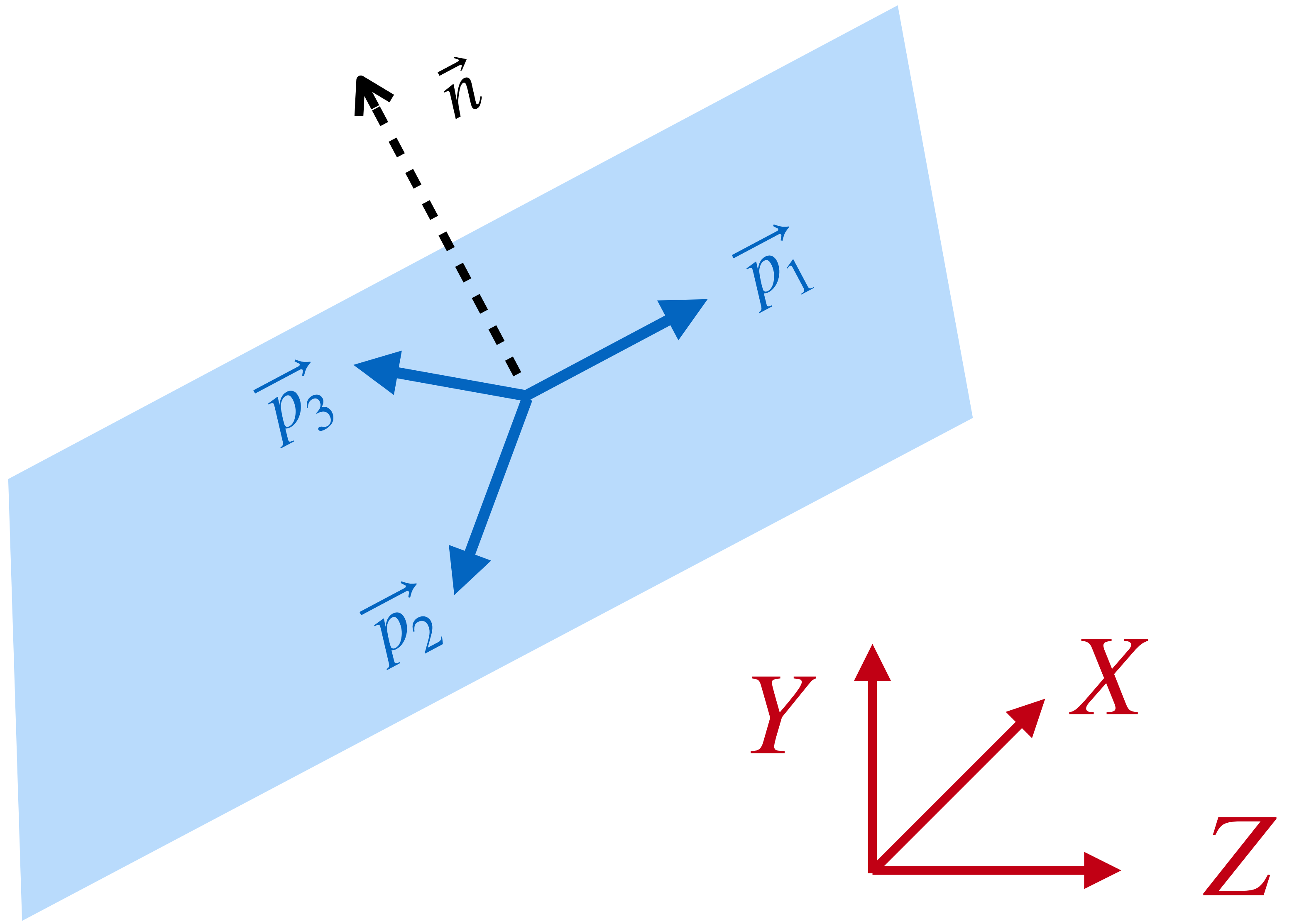
$$= \frac{1}{8}\int dE_1 dE_3 d\Omega_1 d\phi_3 dE_2 \delta(\sqrt{s} - E_1 - E_2 - E_3)$$

$$= \frac{1}{8}\int dE_1 dE_3 d\Omega_1 d\phi_3 \qquad = \frac{1}{32M^2}\int ds_{12} ds_{23} d\Omega_1 d\phi_3$$

We will need

$$dE_2^2 = 2p_1 p_3\, d\cos\theta_3 \qquad dE_2 = \frac{p_1 p_3}{E_2}\, d\cos\theta_3$$

$$dE_1 dE_3 = \frac{1}{4M^2} ds_{12} ds_{23}$$



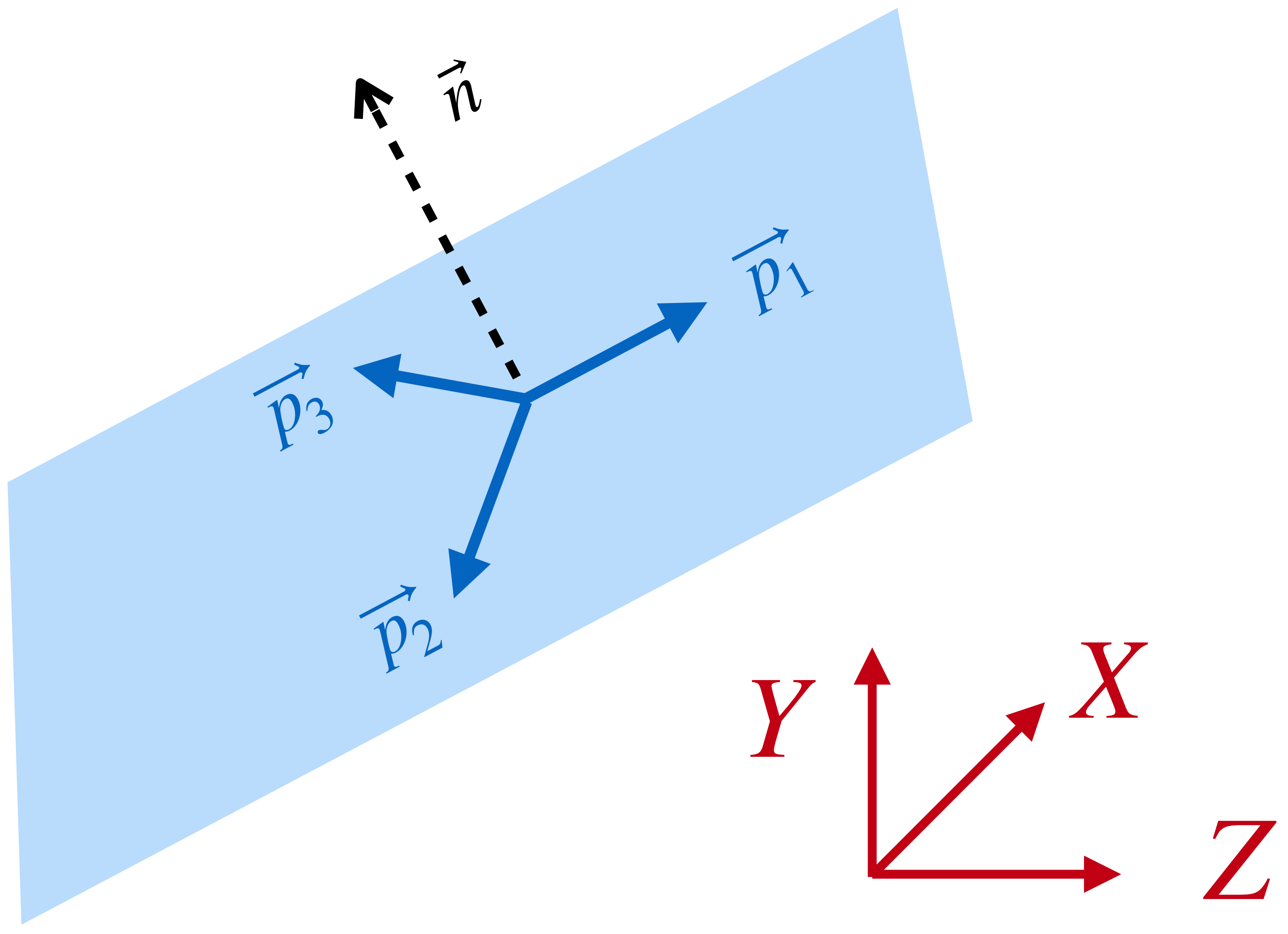
Dalitz plot

Combining everything we obtain

$$d\Gamma = \frac{1}{64M^3}\frac{1}{(2\pi)^5}\overline{\sum}|A|^2\, ds_{12}ds_{23} \times\ d\alpha\, d\cos\beta\, d\gamma$$

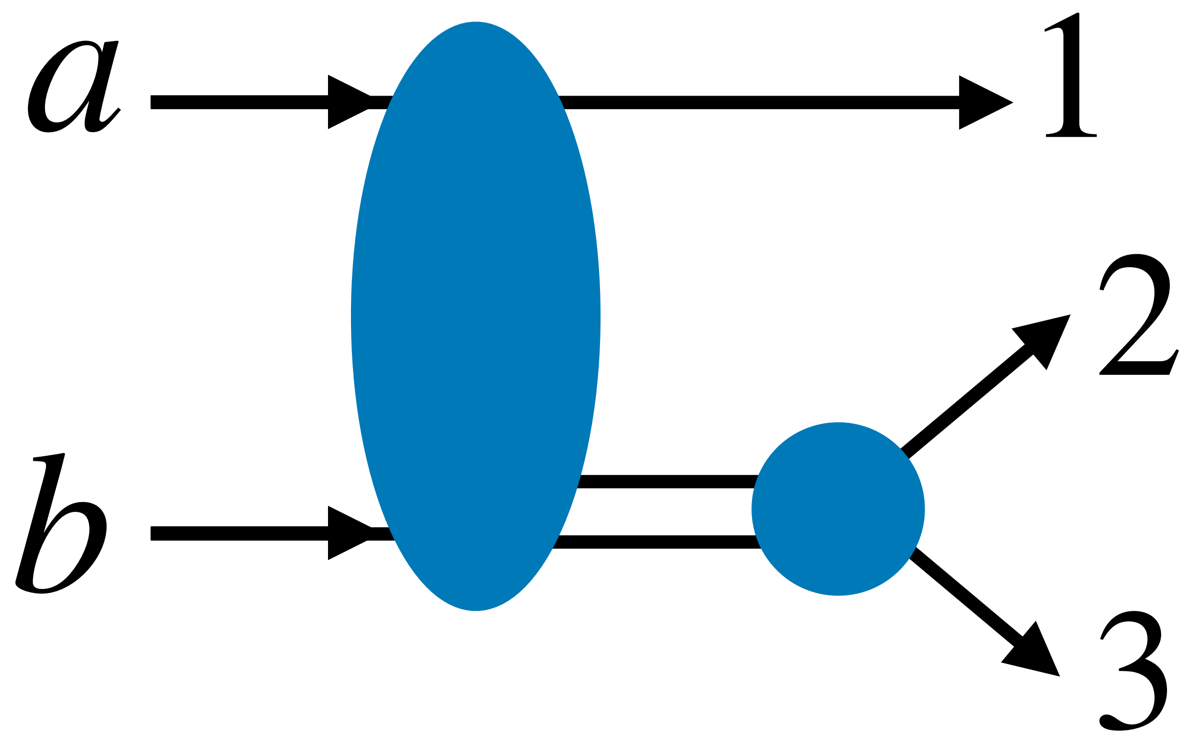
If the decaying particle is not polarized

$$\frac{d\Gamma}{ds_{12}ds_{23}} = \frac{1}{32M^3}\frac{1}{(2\pi)^3}\overline{\sum}|A|^2$$



Three Particles Production

$$R_3(s) = \int \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\frac{d^3p_3}{2E_3}\delta^4(p_a + p_b - p_1 - p_2 - p_3)$$



Insert completeness relation $1 = \int ds_{23} \int \frac{d^3p_{23}}{2E_{23}} \delta^4(p_{23} - p_2 - p_3)$

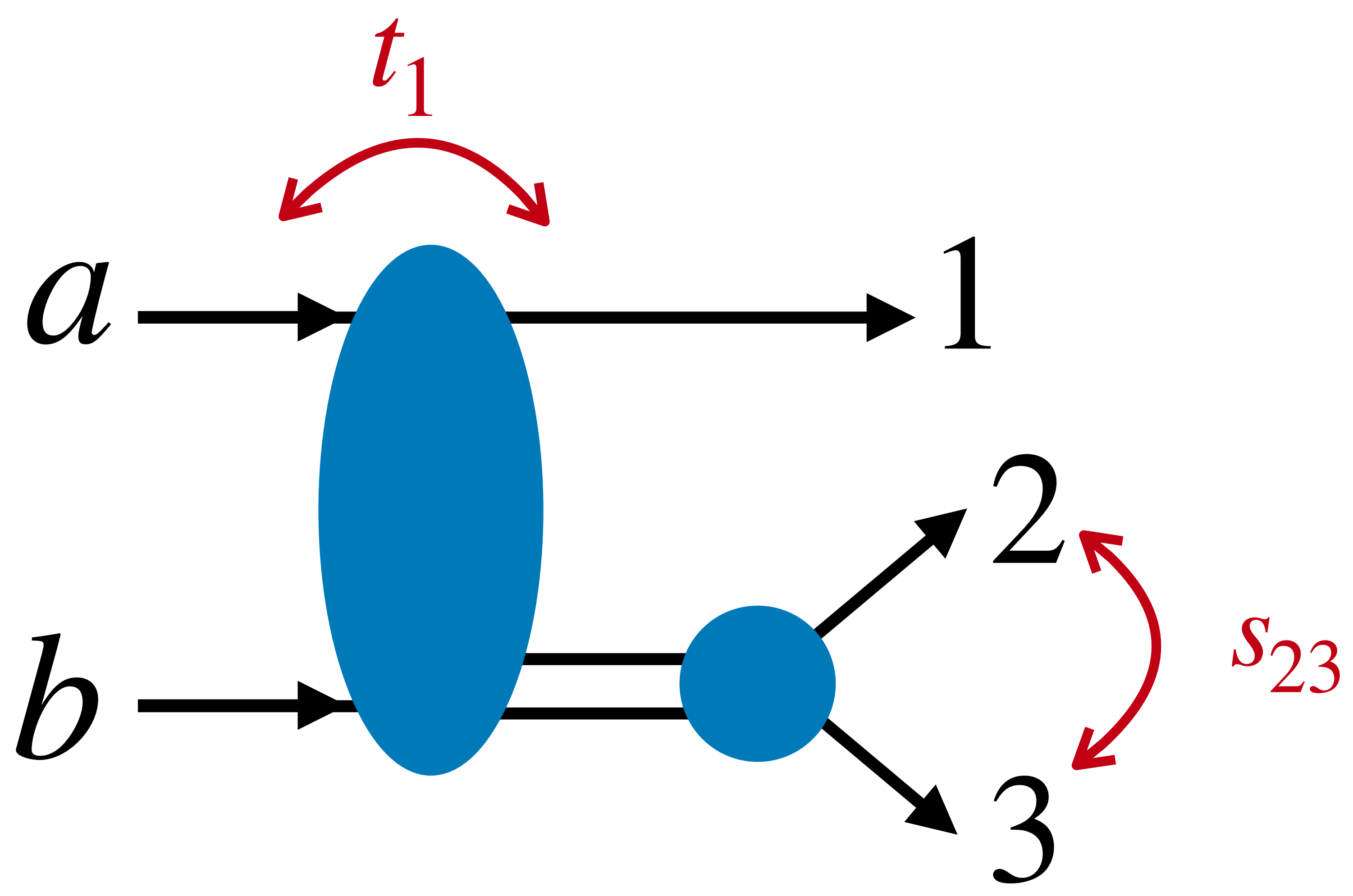
$$R_3(s) = \int ds_{23} \left\{ \int \frac{d^3p_1}{2E_1}\frac{d^3p_{23}}{2E_{23}}\delta^4(p_a + p_b - p_1 - p_{23}) \right\}$$

$$\times \left\{ \int \frac{d^3p_2}{2E_2}\frac{d^3p_3}{2E_3}\delta^4(p_{23} - p_2 - p_2) \right\}$$

Phase space as a convolution

$$R_3(s) = \int ds_{23}\ R_2(s, m_1^2, s_{23})\ R_2(s_{23}, m_2^2, m_3^2)$$

Three Particles Production



$$R_3(s) = \int ds_{23}\, R_2(s, m_1^2, s_{23})\, R_2(s_{23}, m_2^2, m_3^2)$$

The cross section reads

$$d\sigma = \frac{1}{(2\pi)^3} \frac{1}{(2m_b p_a^L)^2} \frac{1}{16} \int \frac{d\Phi}{2\pi} \int_{t_1^-}^{t_1^+} \int_{s_{23}^-}^{s_{23}^+} \frac{p_3}{\sqrt{s_{23}}} dt_1 ds_{23} \int \frac{d\Omega_3}{4\pi} \overline{\sum} |A|^2$$

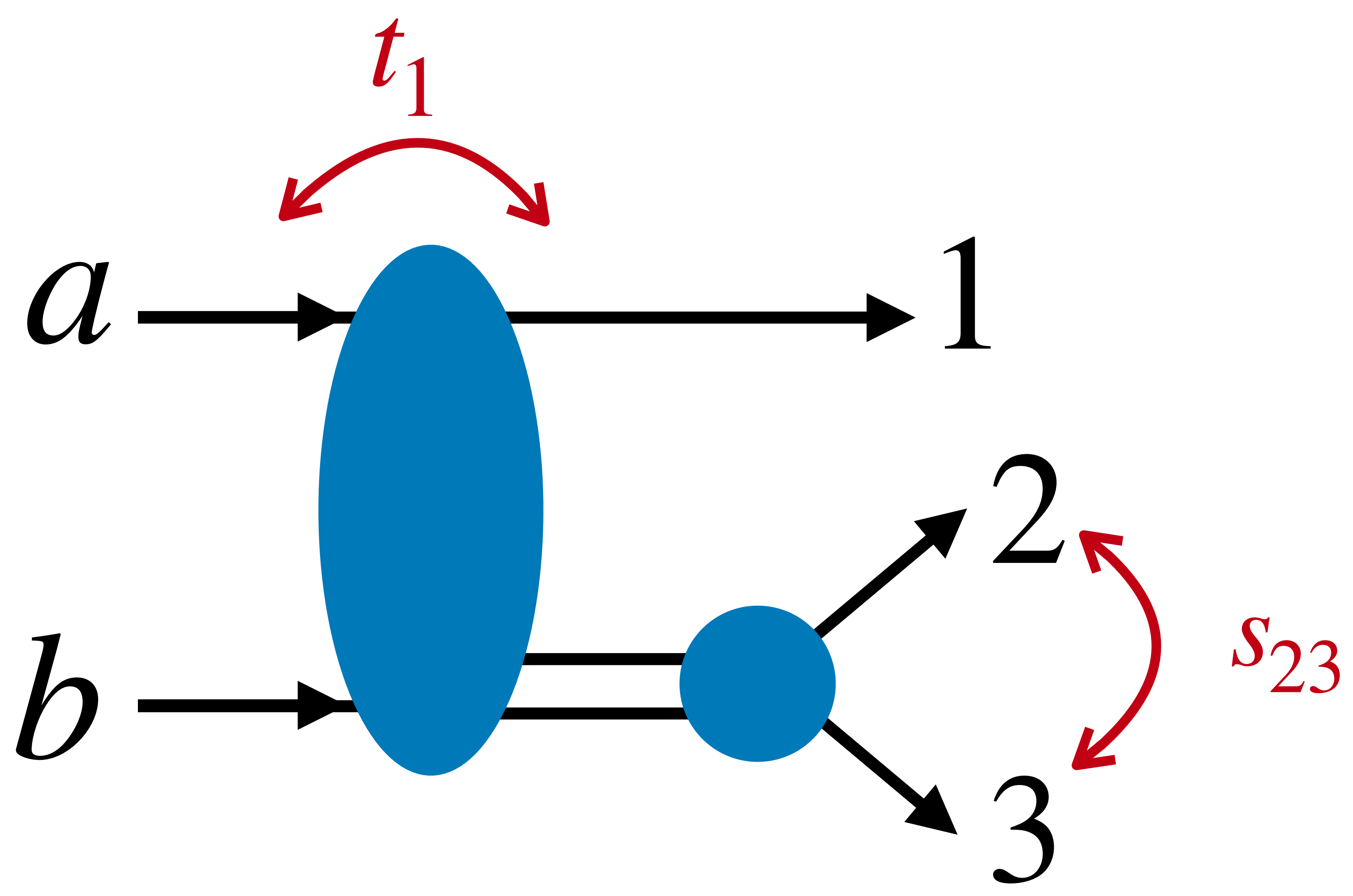
Ω_3 are the angle of particle 3 in (23) frame

Φ is the azimutal angle of the production plane

p_3 is the breakup momentum in the (23) frame

$$p_3 = \frac{\lambda^{1/2}(s_{23}, m_2^2, m_3^2)}{2\sqrt{s_{23}}}$$

Three Particles Production



$$R_3(s) = \int ds_{23}\ R_2(s, m_1^2, s_{23})\ R_2(s_{23}, m_2^2, m_3^2)$$

The cross section reads

$$d\sigma = \frac{1}{(2\pi)^3} \frac{1}{(2m_b p_a^L)^2} \frac{1}{16} \int \frac{d\Phi}{2\pi} \int_{t_1^-}^{t_1^+} \int_{s_{23}^-}^{s_{23}^+} \frac{p_3}{\sqrt{s_{23}}} dt_1 ds_{23} \int \frac{d\Omega_3}{4\pi} \overline{\sum} |A|^2$$

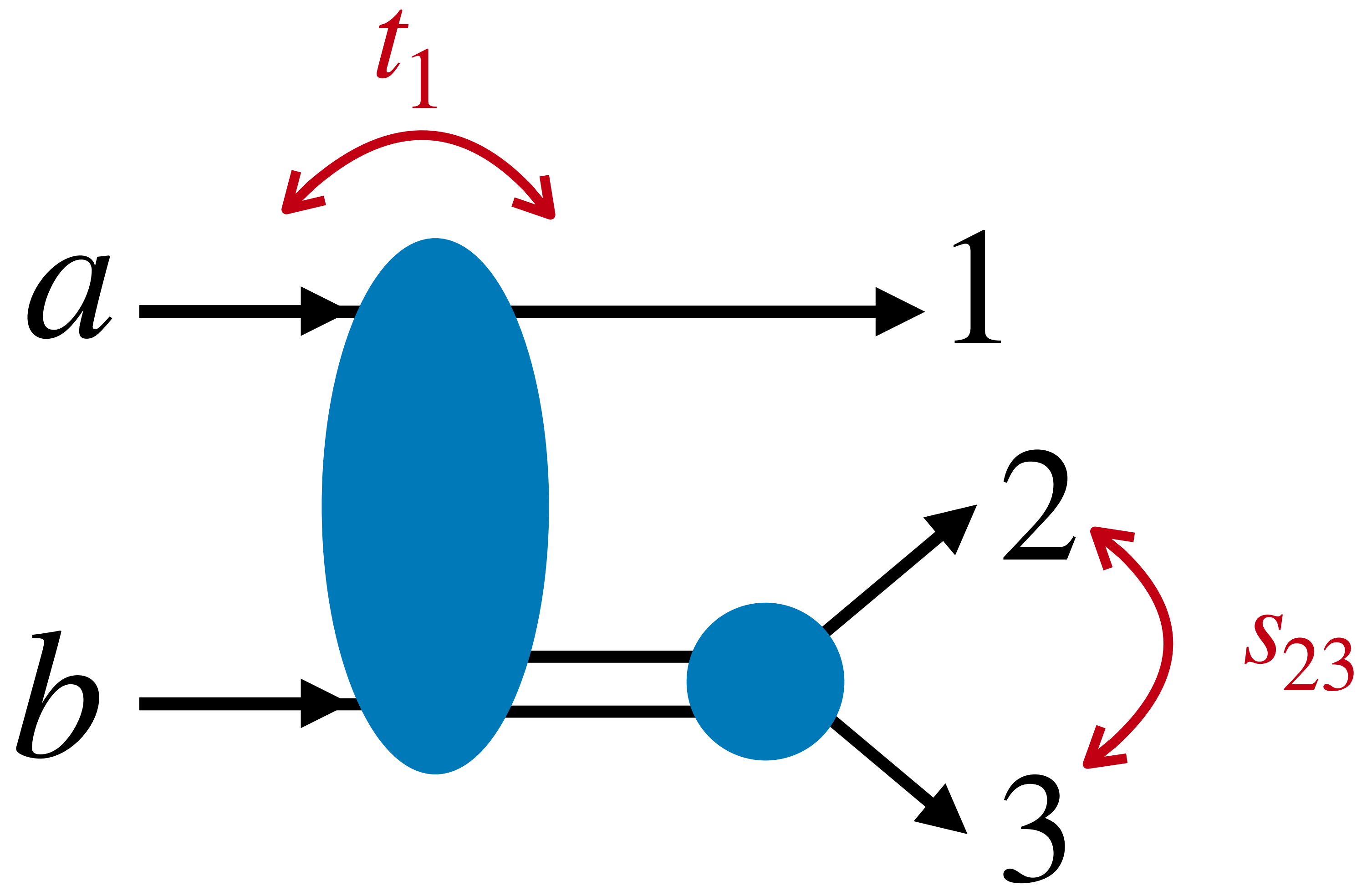
The boundaries of t_1 are given by $t_{min/max}$ for the reaction $a + b \to 1 + (23)$

$$t_1^{\pm} = m_a^2 + m_1^2 - 2E_a^* E_1^* \pm 2\,|\vec{p}_a^{\,*}|\,|\vec{p}_1^{\,*}|$$

$$t_1^{\pm} = m_a^2 + m_1^2 - \frac{1}{2s}\left[(s + m_a^2 - m_b^2)(s + m_1^2 - s_{23}) \mp \lambda^{1/2}(s, m_a^2, m_b^2)\lambda^{1/2}(s, m_1^2, s_{23})\right]$$

Three Particles Production

$$R_3(s) = \int ds_{23}\; R_2(s, m_1^2, s_{23})\; R_2(s_{23}, m_2^2, m_3^2)$$



The cross section reads

$$d\sigma = \frac{1}{(2\pi)^3}\frac{1}{(2m_b p_a^L)^2}\frac{1}{16}\int \frac{d\Phi}{2\pi}\int_{t_1^-}^{t_1^+}\int_{s_{23}^-}^{s_{23}^+}\frac{p_3}{\sqrt{s_{23}}}dt_1 ds_{23}\int\frac{d\Omega_3}{4\pi}\overline{\sum}|A|^2$$

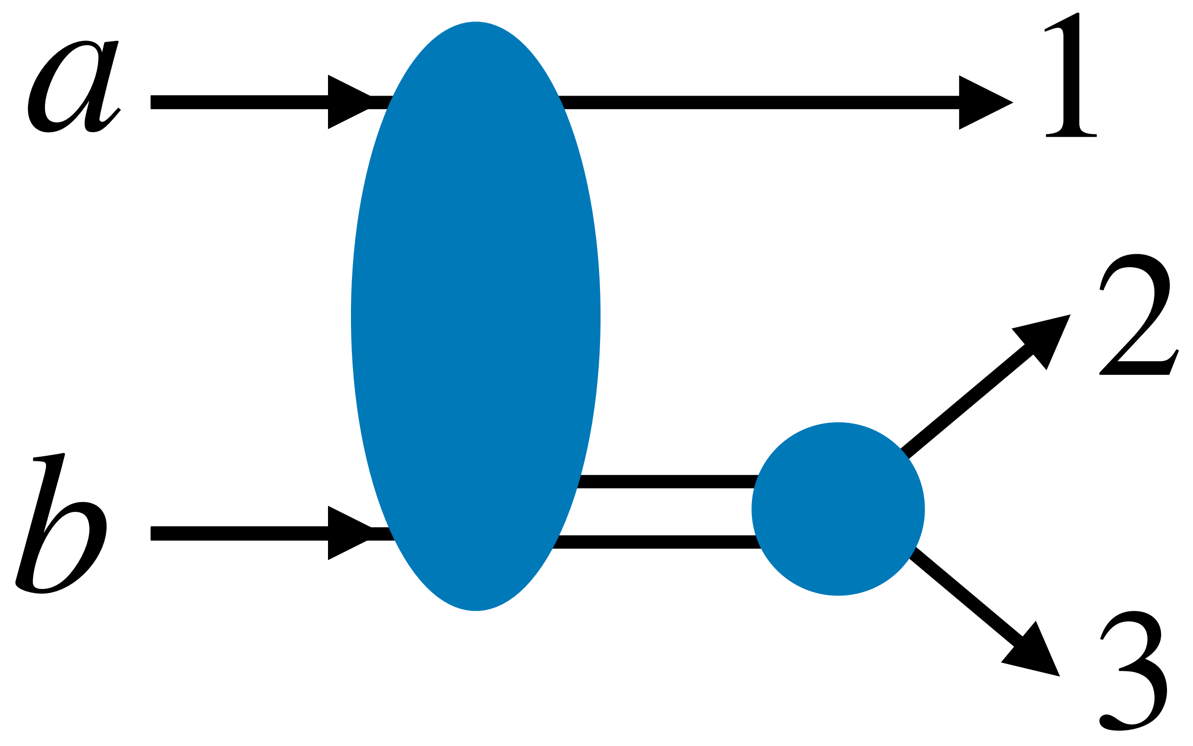
The boundaries of s_{23} are given by evaluating $s_{23} = ([p_a + p_b] - p_1)^2$ in the a rest frame

$$s_{23}^{\pm} = s + m_1^2 - \frac{1}{2m_a^2}\left[(s + m_a^2 - m_b^2)(m_a^2 + m_1^2 - t_1) \mp \lambda^{1/2}(s, m_a^2, m_b^2)\lambda^{1/2}(t_1, m_a^2, m_1^2)\right]$$

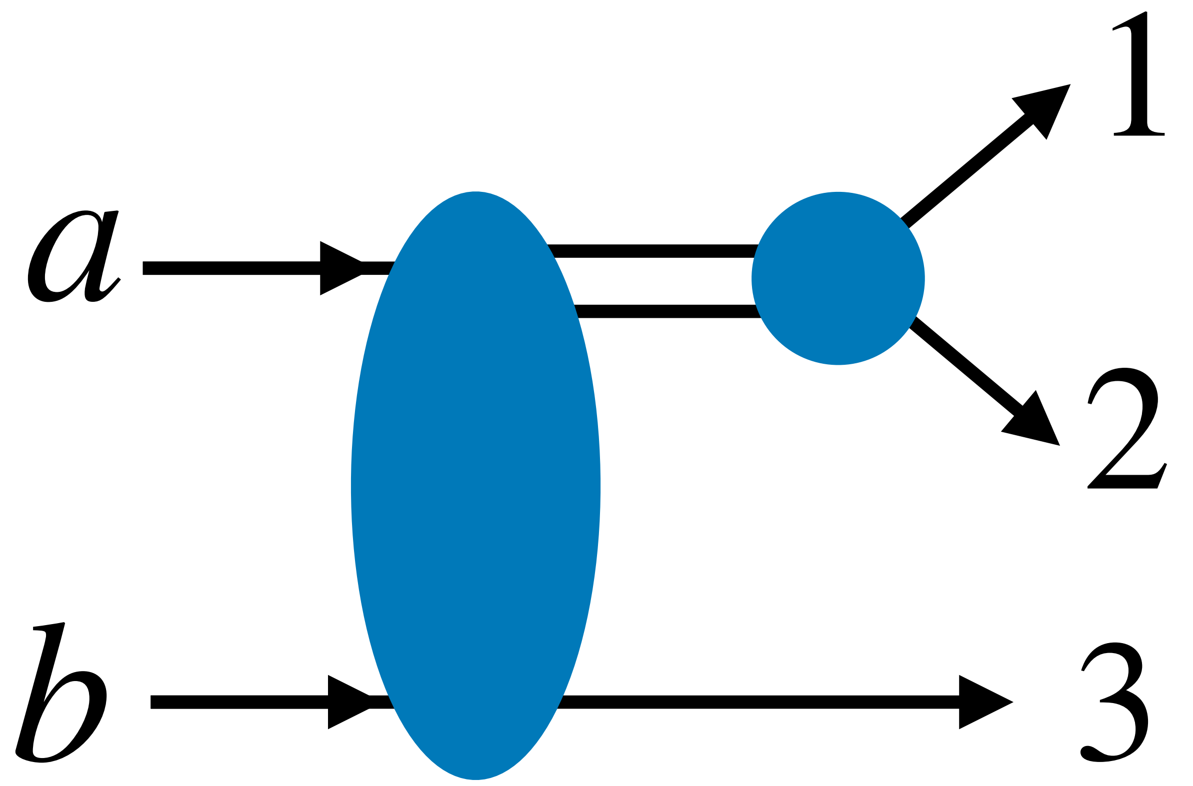
Note the boundaries of t_1 depend on s_{23} and vice-versa

Three Particles Production

$$R_3(s) = \int ds_{23}\ R_2(s, m_1^2, s_{23})\ R_2(s_{23}, m_2^2, m_3^2)$$

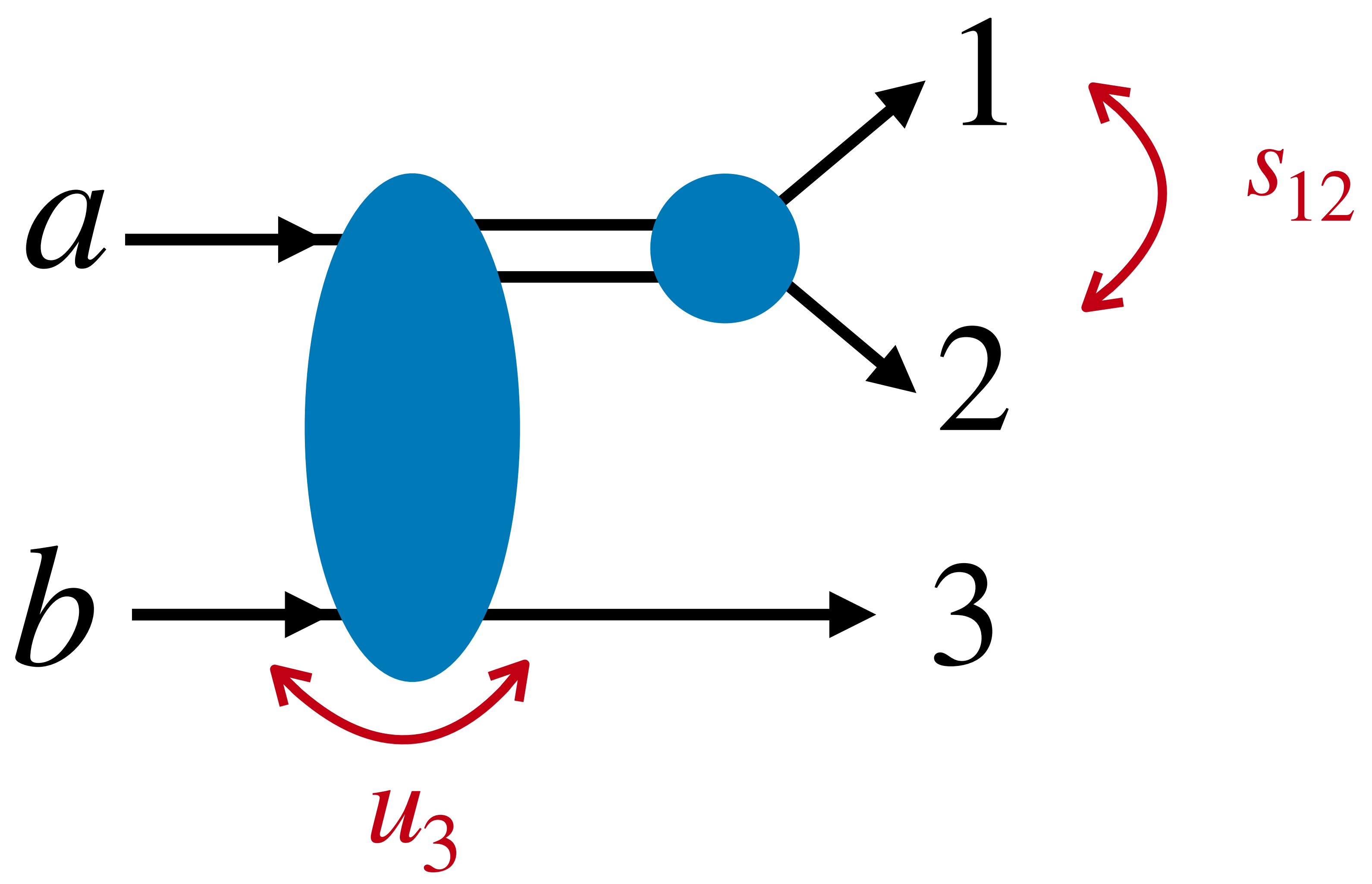


$$R_3(s) = \int ds_{12}\ R_2(s, m_3^2, s_{12})\ R_2(s_{12}, m_1^2, m_2^2)$$



Three Particles Production

$$R_3(s) = \int ds_{12}\ R_2(s, m_3^2, s_{12})\ R_2(s_{12}, m_1^2, m_2^2)$$



The cross section reads

$$d\sigma = \frac{1}{(2\pi)^3} \frac{1}{(2m_b p_a^L)^2} \frac{1}{16} \int \frac{d\Phi}{2\pi} \int_{u_3^-}^{u_3^+} \int_{s_{12}^-}^{s_{12}^+} \frac{p_1}{\sqrt{s_{12}}} du_2 ds_{12} \int \frac{d\Omega_1}{4\pi} \overline{\sum} |A|^2$$

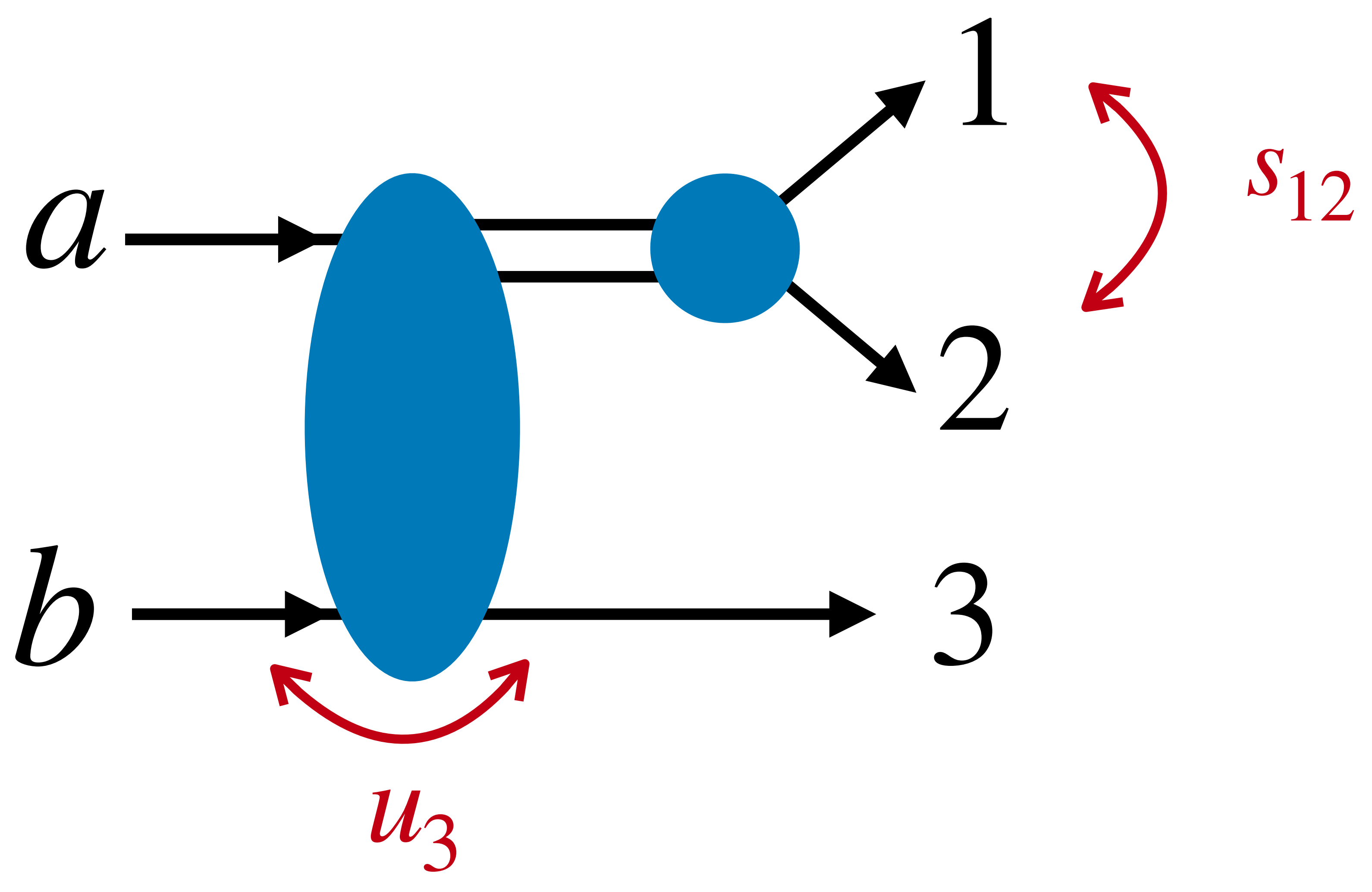
Ω_1 are the angle of particle 1 in (12) frame

Φ is the azimutal angle of the production plane

p_1 is the breakup momentum in the (12) frame $\qquad p_1 = \dfrac{\lambda^{1/2}(s_{12}, m_1^2, m_2^2)}{2\sqrt{s_{12}}}$

Three Particles Production

$$R_3(s) = \int ds_{12}\ R_2(s, m_3^2, s_{12})\ R_2(s_{12}, m_1^2, m_2^2)$$



The cross section reads

$$d\sigma = \frac{1}{(2\pi)^3}\frac{1}{(2m_b p_a^L)^2}\frac{1}{16}\int\frac{d\Phi}{2\pi}\int_{u_3^-}^{u_3^+}\int_{s_{12}^-}^{s_{12}^+}\frac{p_1}{\sqrt{s_{12}}}du_2 ds_{12}\int\frac{d\Omega_1}{4\pi}\overline{\sum}|A|^2$$

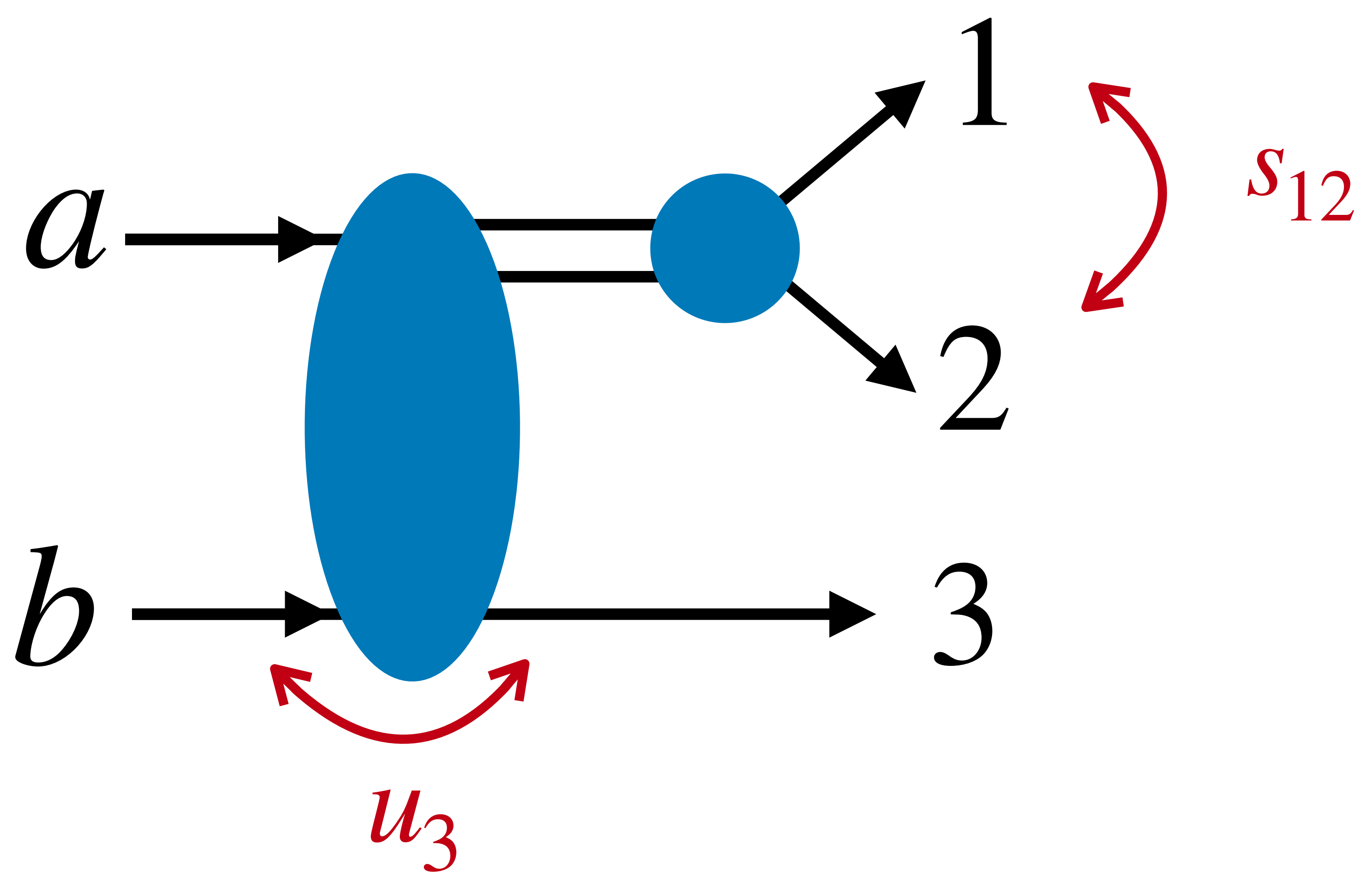
The boundaries of u_3 are given by $t_{min/max}$ for the reaction $a + b \to (12) + 3$

$$u_3^\pm = m_b^2 + m_3^2 - 2E_b^* E_3^* \pm 2|\vec{p}_b^*||\vec{p}_3^*|$$

$$u_3^\pm = m_b^2 + m_3^2 - \frac{1}{2s}\left[(s + m_b^2 - m_a^2)(s + m_3^2 - s_{12}) \mp \lambda^{1/2}(s, m_a^2, m_b^2)\lambda^{1/2}(s, m_3^2, s_{12})\right]$$

Three Particles Production

$$R_3(s) = \int ds_{12}\ R_2(s, m_3^2, s_{12})\ R_2(s_{12}, m_1^2, m_2^2)$$



The cross section reads

$$d\sigma = \frac{1}{(2\pi)^3}\frac{1}{(2m_b p_a^L)^2}\frac{1}{16}\int\frac{d\Phi}{2\pi}\int_{u_3^-}^{u_3^+}\int_{s_{12}^-}^{s_{12}^+}\frac{p_1}{\sqrt{s_{12}}}du_2 ds_{12}\int\frac{d\Omega_1}{4\pi}\overline{\sum}|A|^2$$

The boundaries of s_{12} are given by evaluating $s_{12} = ([p_a + p_b] - p_3)^2$ in the b rest frame

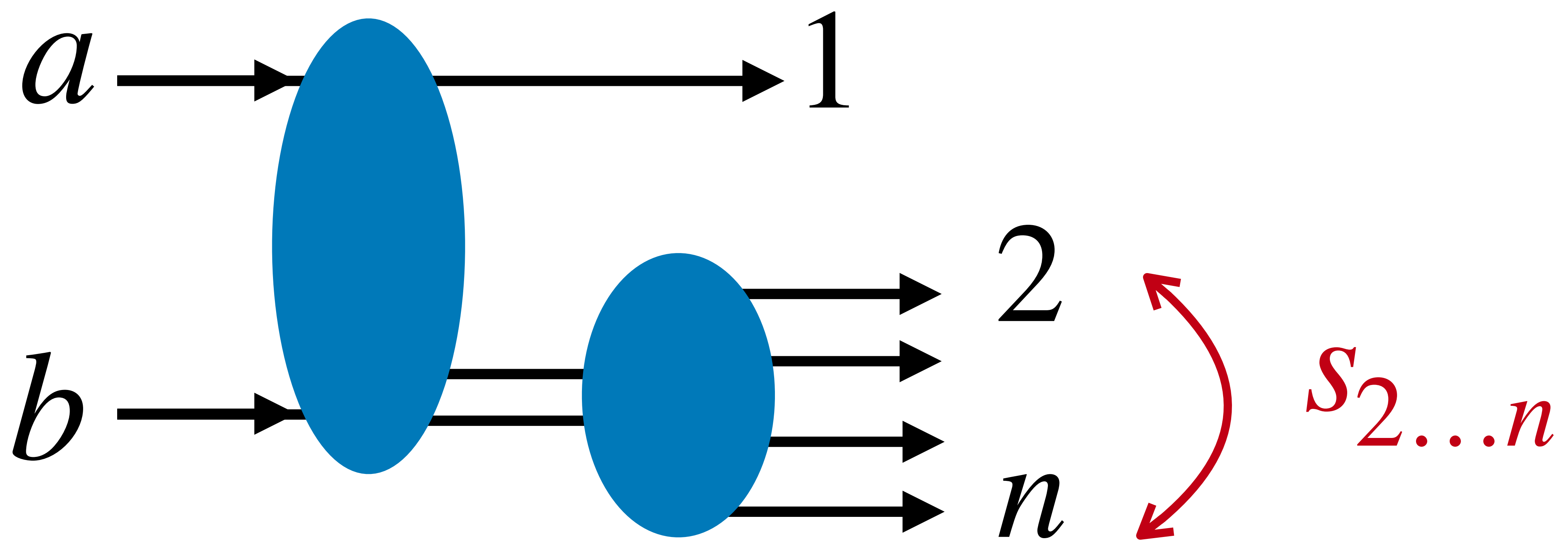
$$s_{12}^{\pm} = s + m_3^2 - \frac{1}{m_b^2}\left[(s + m_a^2 - m_b^2)(m_b^2 + m_3^2 - u_3) \mp \lambda^{1/2}(s, m_a^2, m_b^2)\lambda^{1/2}(u_3, m_b^2, m_3^2)\right]$$

Note the boundaries of u_3 depend on s_{12} and vice-versa

N Particle Production

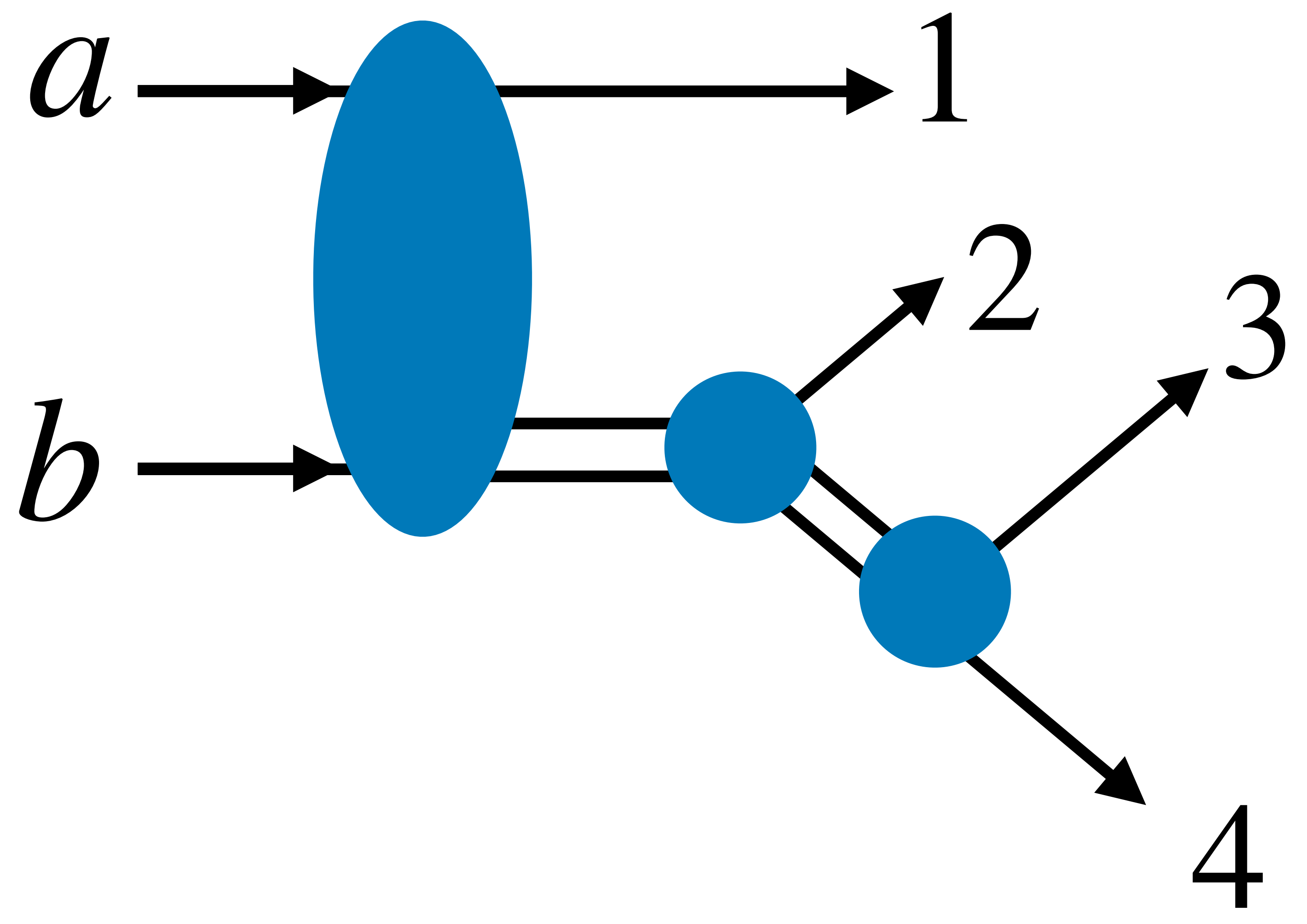
N-particle phase space as a convolution

$$R_n(s) = \int ds_{2\ldots n}\, R_2(s, m_1^2, s_{2\ldots n})\, R_{n-1}(s_{2\ldots n}, m_2^2, \ldots, m_n^2)$$



Iterative procedure

$$R_2(s) = \frac{p^*}{4\sqrt{s}} \int d\Omega^*$$



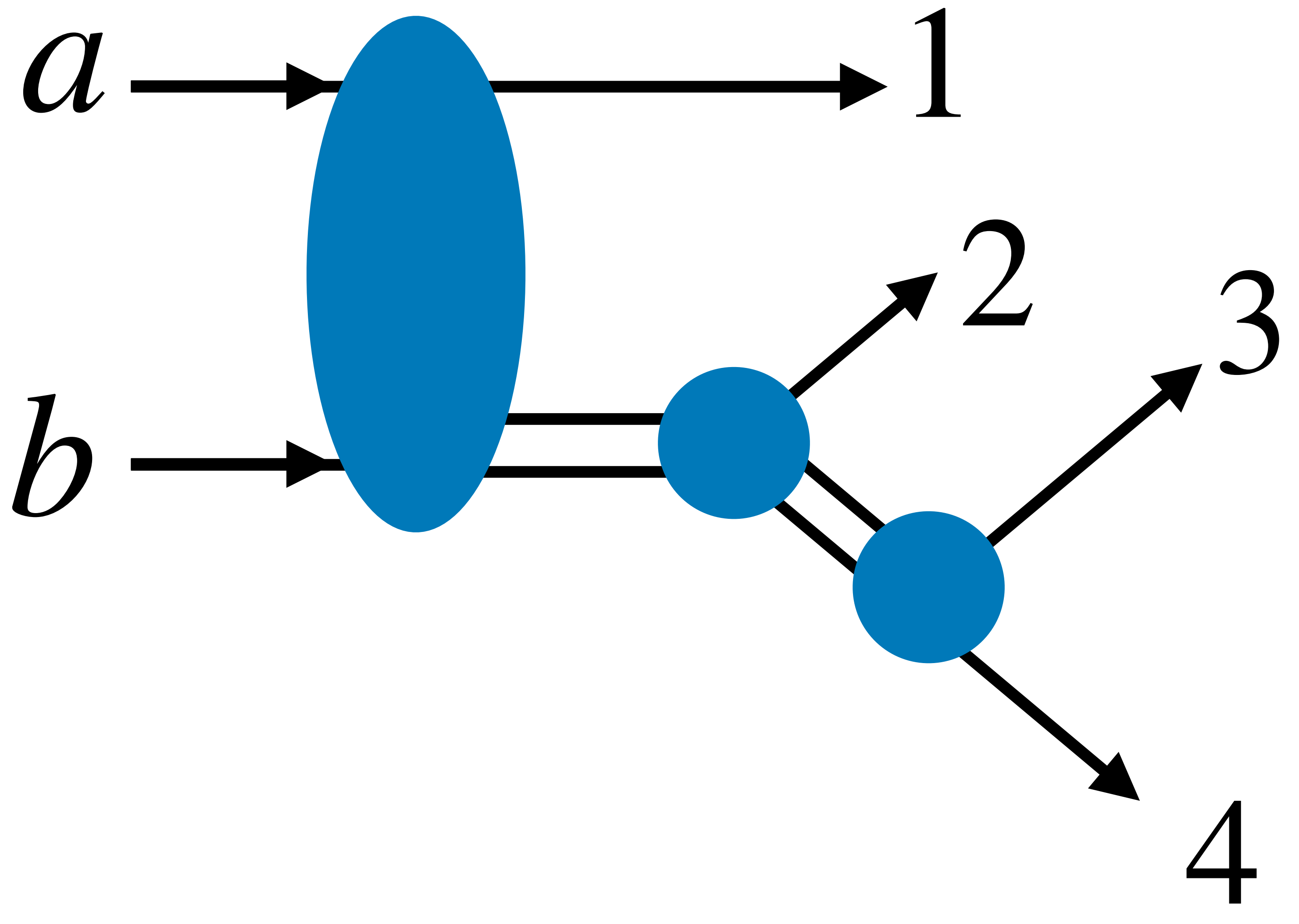
One can also simplify using

$$\int ds R_2(s, m^2, s') = \int ds \frac{p^*}{4\sqrt{s}} \int d\Omega^* \quad = \frac{1}{2} \int p^* d\sqrt{s} \int d\Omega^*$$

N=4 Particle Production

4-particle phase space as a 2 convolutions

$$R_4(s) = \int ds_{234}\ R_2(s, m_1^2, s_{234}) \ \int ds_{34} R_2(s_{234}, m_2^2, s_{34}) R_2(s_{34}, m_2^2, m_4^2)$$



More explicitly

$$R_4(s) = \int ds_{234}\ \frac{p_1}{4\sqrt{s}} \int d\Omega_1 \times \int ds_{34} \frac{p_2}{4\sqrt{s_{234}}} \int d\Omega_2 \times \frac{p_3}{4\sqrt{s_{34}}} \int d\Omega_3$$

$$= \frac{1}{2^3} \frac{1}{\sqrt{s}} \int_{m_2+m_3+m_4}^{\sqrt{s}-m_1} p_1 dm_{234} \ \int d\Omega_1 \times \int_{m_3+m_4}^{m_{234}-m_2} p_2 dm_{34} \int d\Omega_2 \times p_3 \int d\Omega_3$$

Monte-Carlo Techniques

From RPP section 40. Monte-Carlo techniques

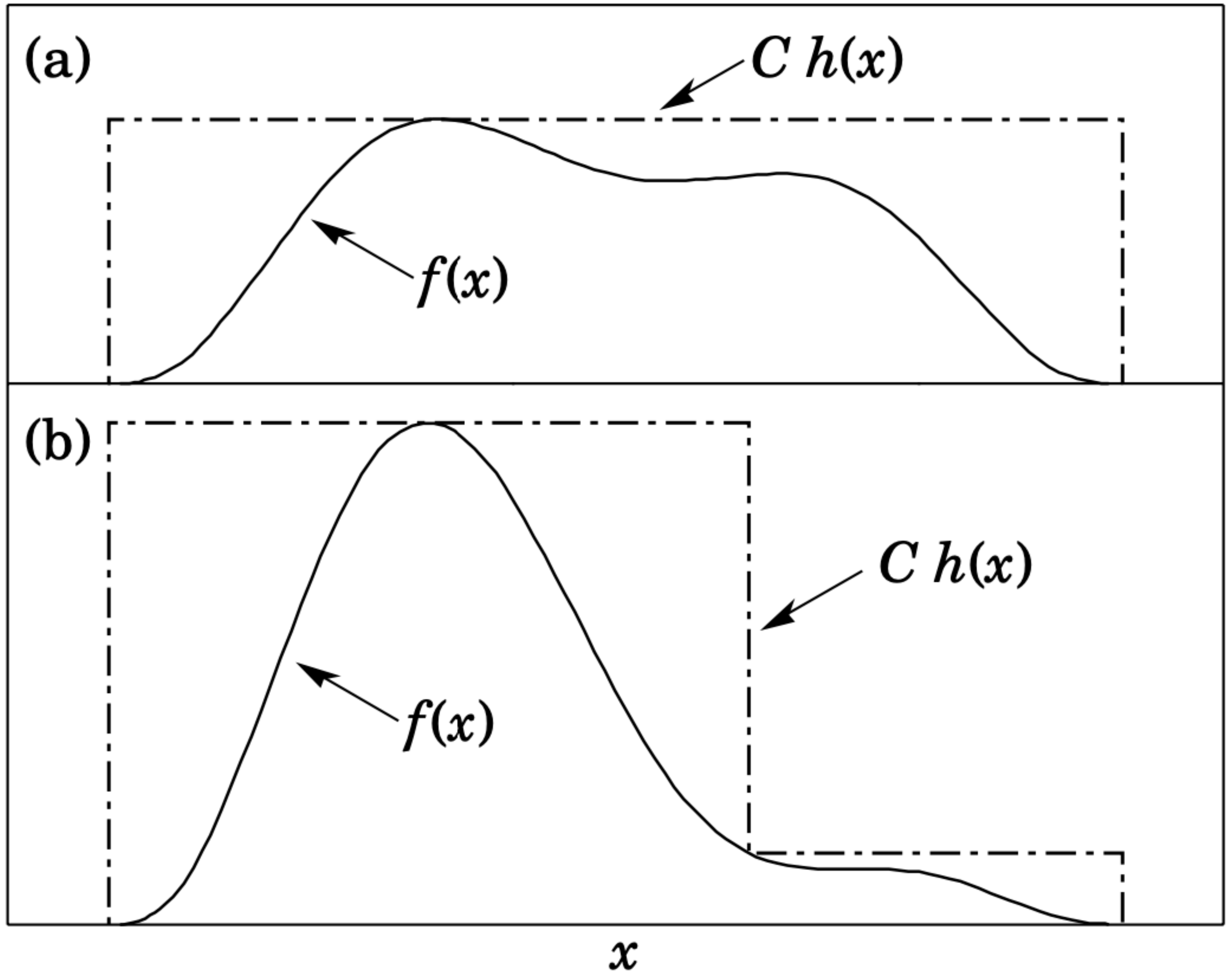


Figure 40.2: Illustration of the acceptance-rejection method. Random points are chosen inside the upper bounding figure, and rejected if the ordinate exceeds $f(x)$. The lower figure illustrates a method to increase the efficiency (see text).