Kinematics and Lorentz Transformations

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Exotic Hadron Topical Collaboration

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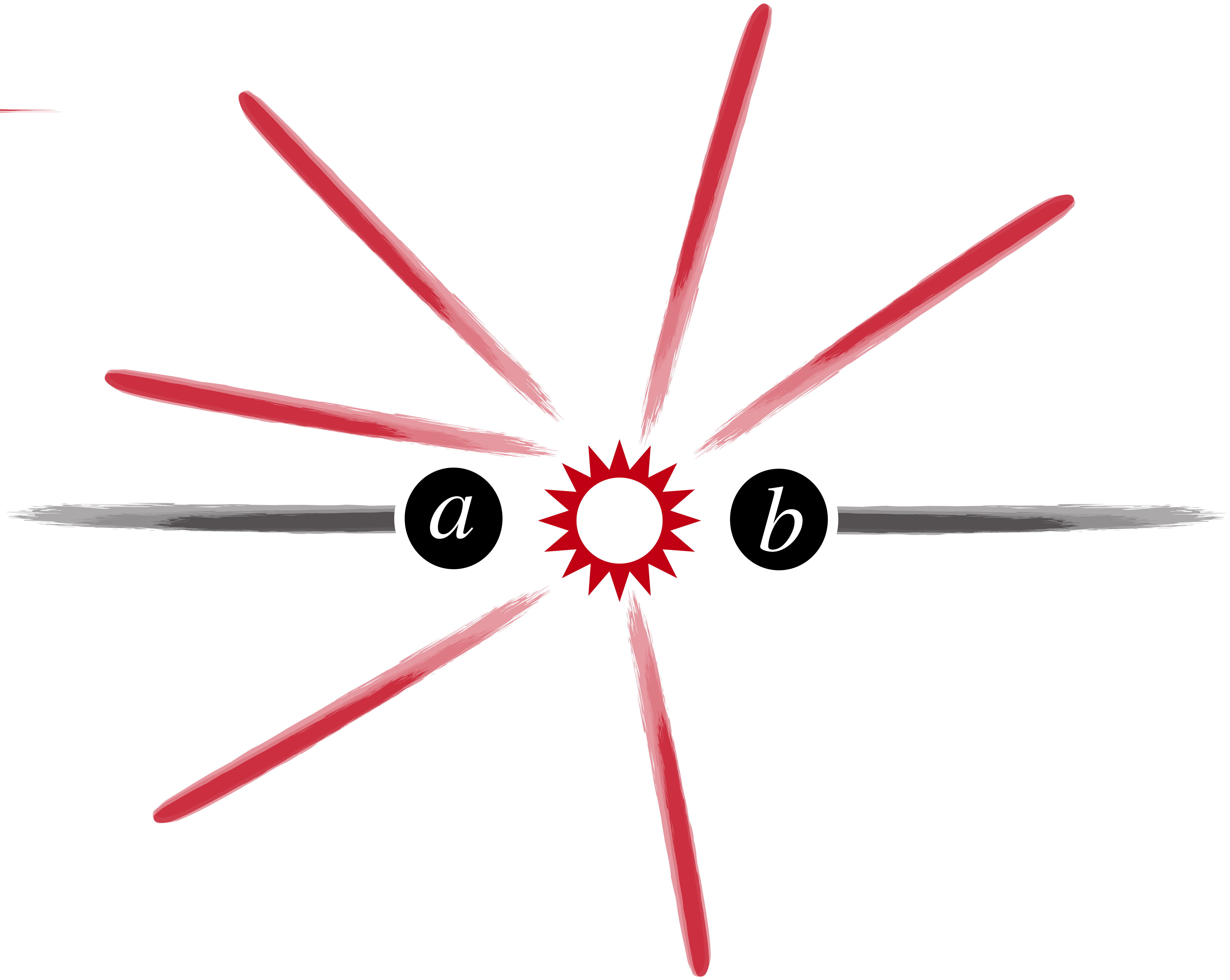
Outline

1. Kinematics and Lorentz transformations
2. Cross sections
3. Helicity formalism

References

1. Martin & Spearman,
   Elementary Particle Physics
2. Perl,
   High Energy Hadron Physics
3. Weinberg
   The Quantum Theory of Fields, vol I
4. Byckling & Kajantie,
   Particle Kinematics
5. Chung
   Spin Formalisms

Experiments



A “beam” and a “target” collide

Other particles are created

Each particle has a 4-vectors

$$p^\mu = (E, p_x, p_y, p_z) = (E, \vec{p})$$

They depend on the frame

On their mass shell:

Notations: $p_a + p_b \to p_1 + p_2 + \ldots + p_n$

$$p_i^2 = E_i^2 - \vec{p}_i^{\,2} = m_i^2$$

Frames

Need to specify which particle or group of particles is at rest and two axes.

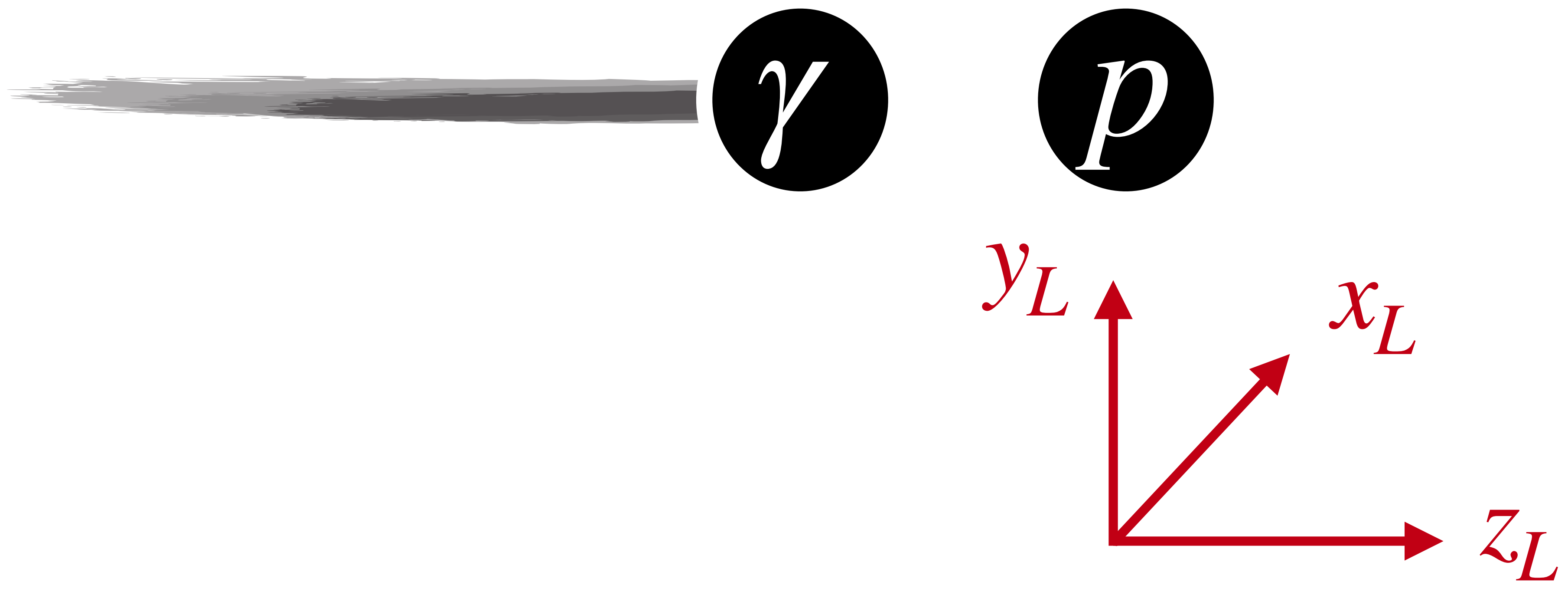
The third one is given by $\vec{x} = \vec{y} \times \vec{z}$

The laboratory frame (or Lab frame)

$$\vec{p}_b = \vec{0}$$

$\vec{z}$ is parallel to the beam $\vec{p}_a$

$\vec{y}$ is pointing “upward”

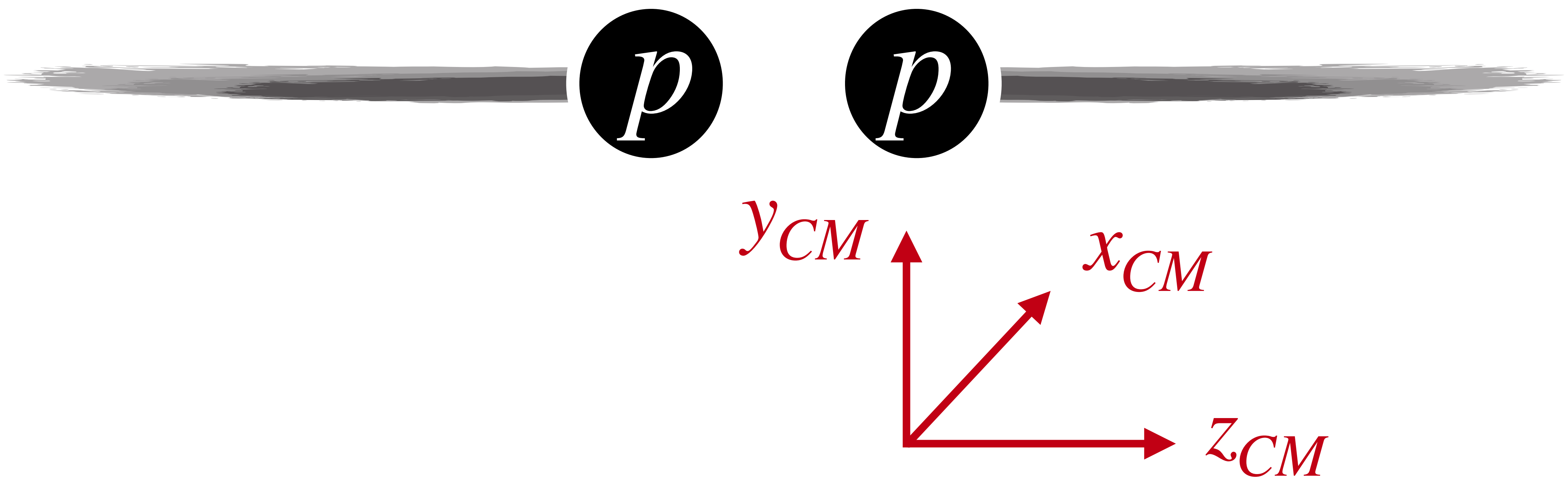


The center-of-mass frame (or CoM frame)

$$\vec{p}_a + \vec{p}_b = \vec{0}$$

$\vec{z}$ is parallel to the beam $\vec{p}_a$

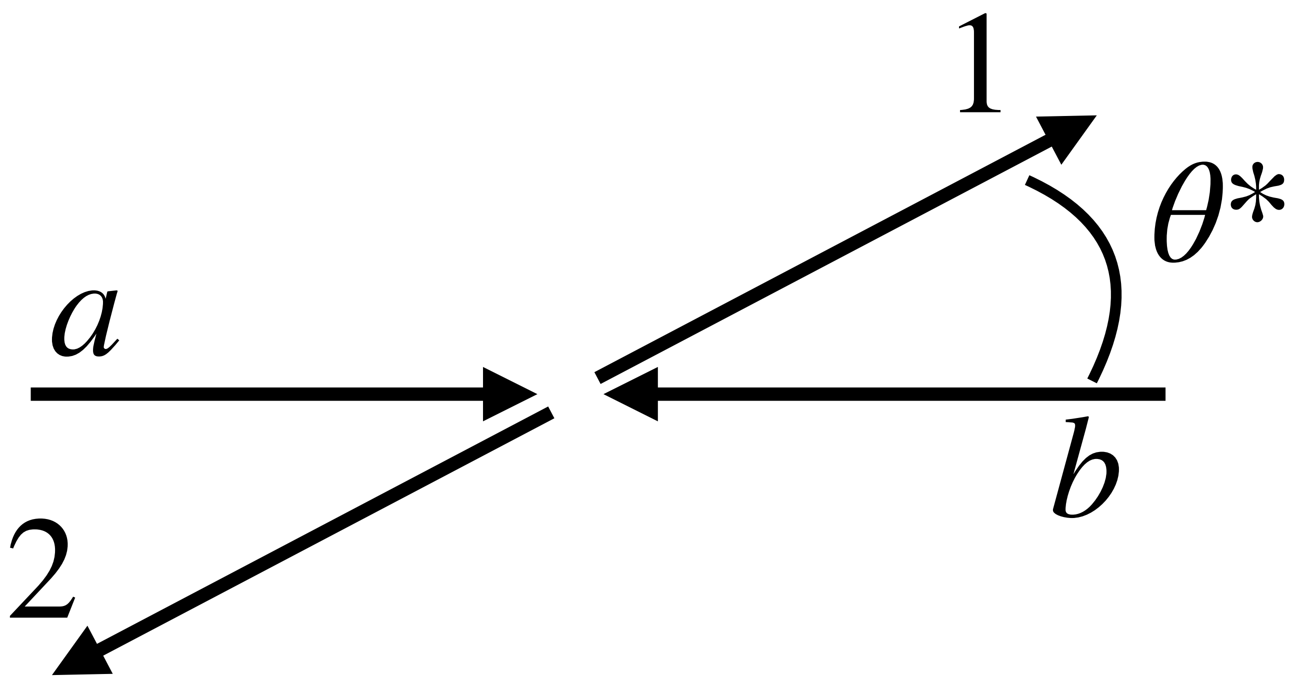
$\vec{y}$ is pointing “upward”



Dimensionality

2-final particles

4×3 d.o.f



Conservation of energy-momentum $\vec{p}_a + \vec{p}_b = \vec{p}_1 + \vec{p}_2$ -4

Choose a (rest) frame $= \vec{0}$ -3

Choose an orientation (if possible) -3

2 degrees of freedom

- Total energy in CoM
- Scattering angle

n-final particles

$$\vec{p}_a + \vec{p}_b = \vec{p}_1 + \ldots + \vec{p}_n = \vec{0}$$

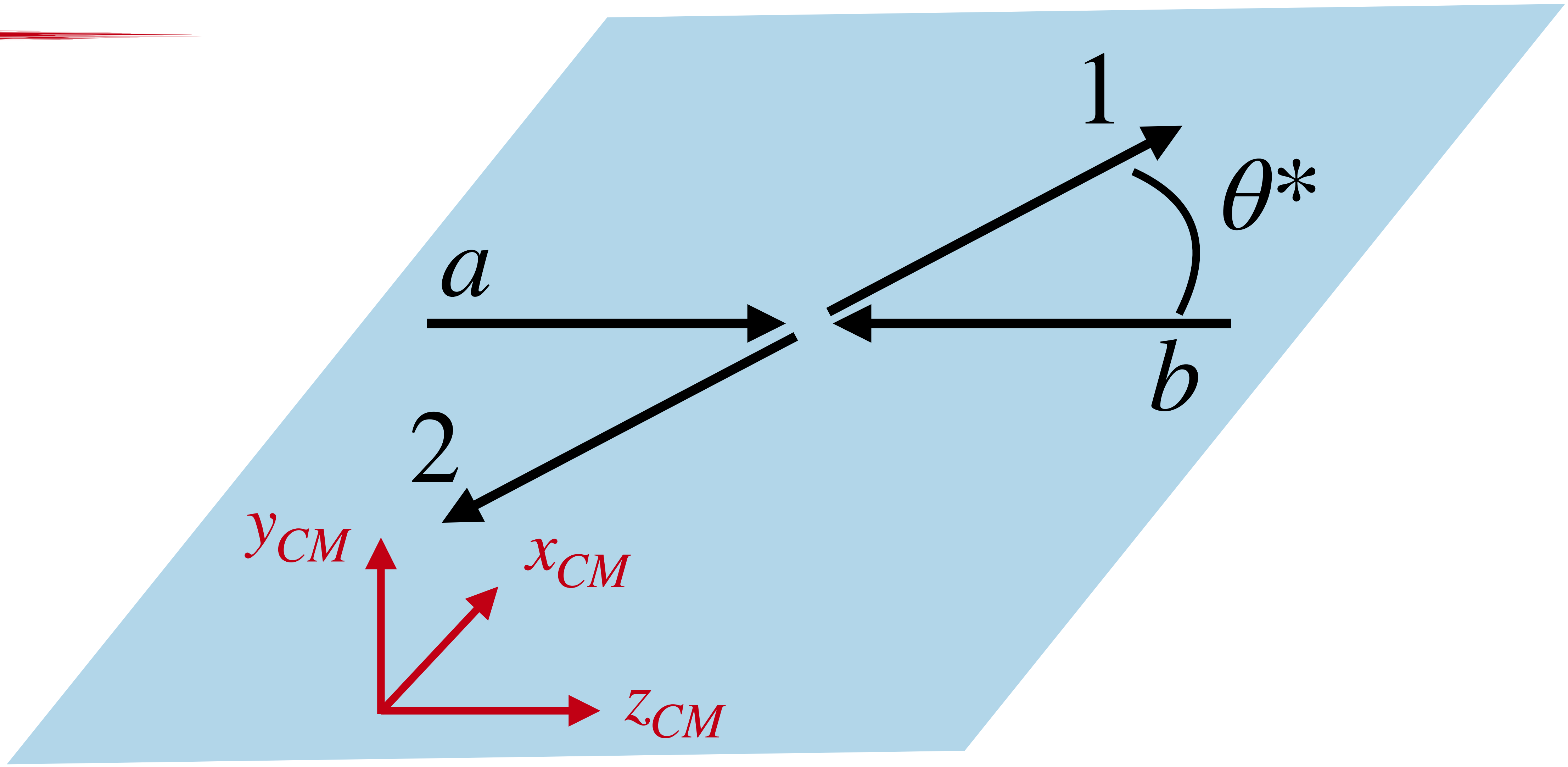
$$3(n+2) - 10 = 3n - 4$$

For n-particles in the final states:
$3n - 4$ degrees of freedom

Two-particles Final States

Standard frames:

CoM frame and Lab frame

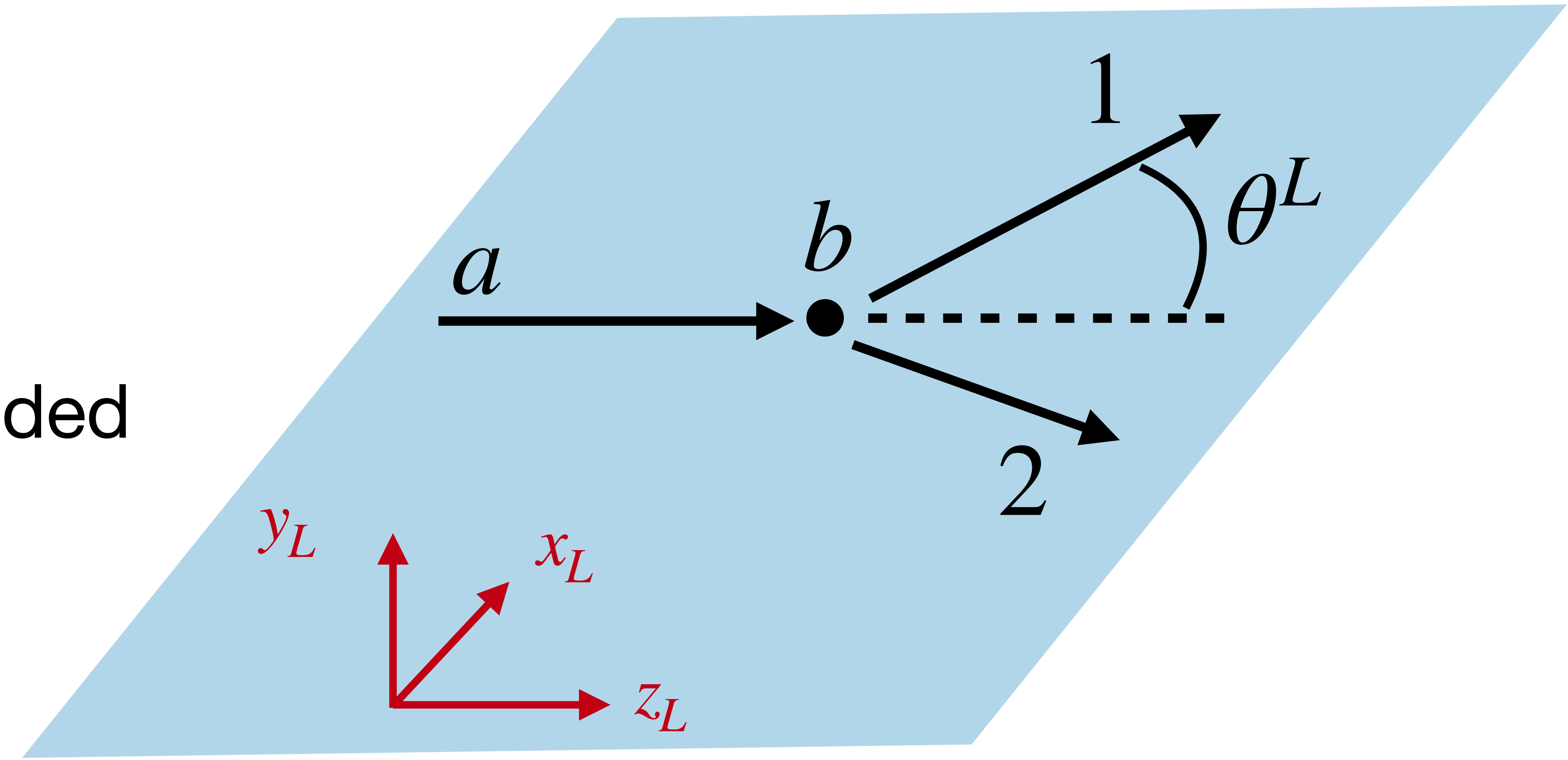


In both frames,
the scattering happens in the x-z plane

$\vec{z}$ is parallel to the beam $\vec{p}_a$

$\vec{y}$ is parallel to $\vec{p}_a \times \vec{p}_1$ ➔ $\theta^{*,L} \in [0,\pi]$

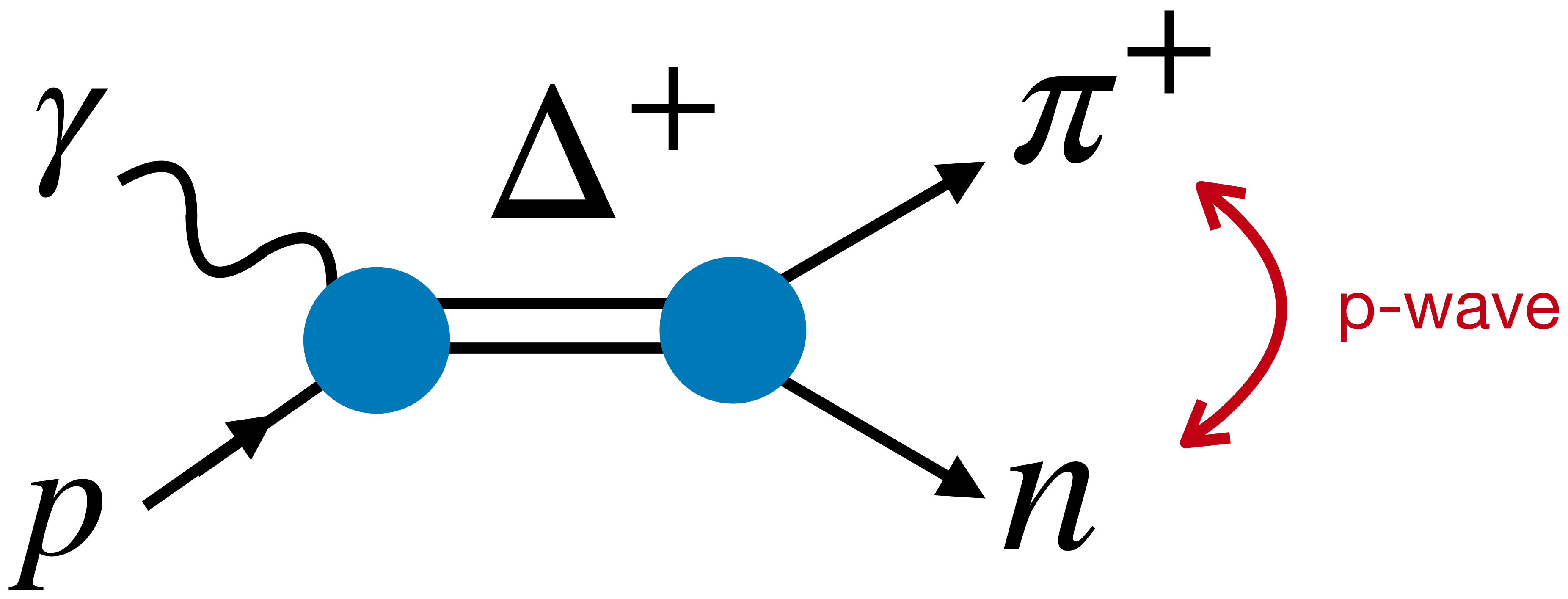
➔ Only $\cos\theta^{*,L}$ needed

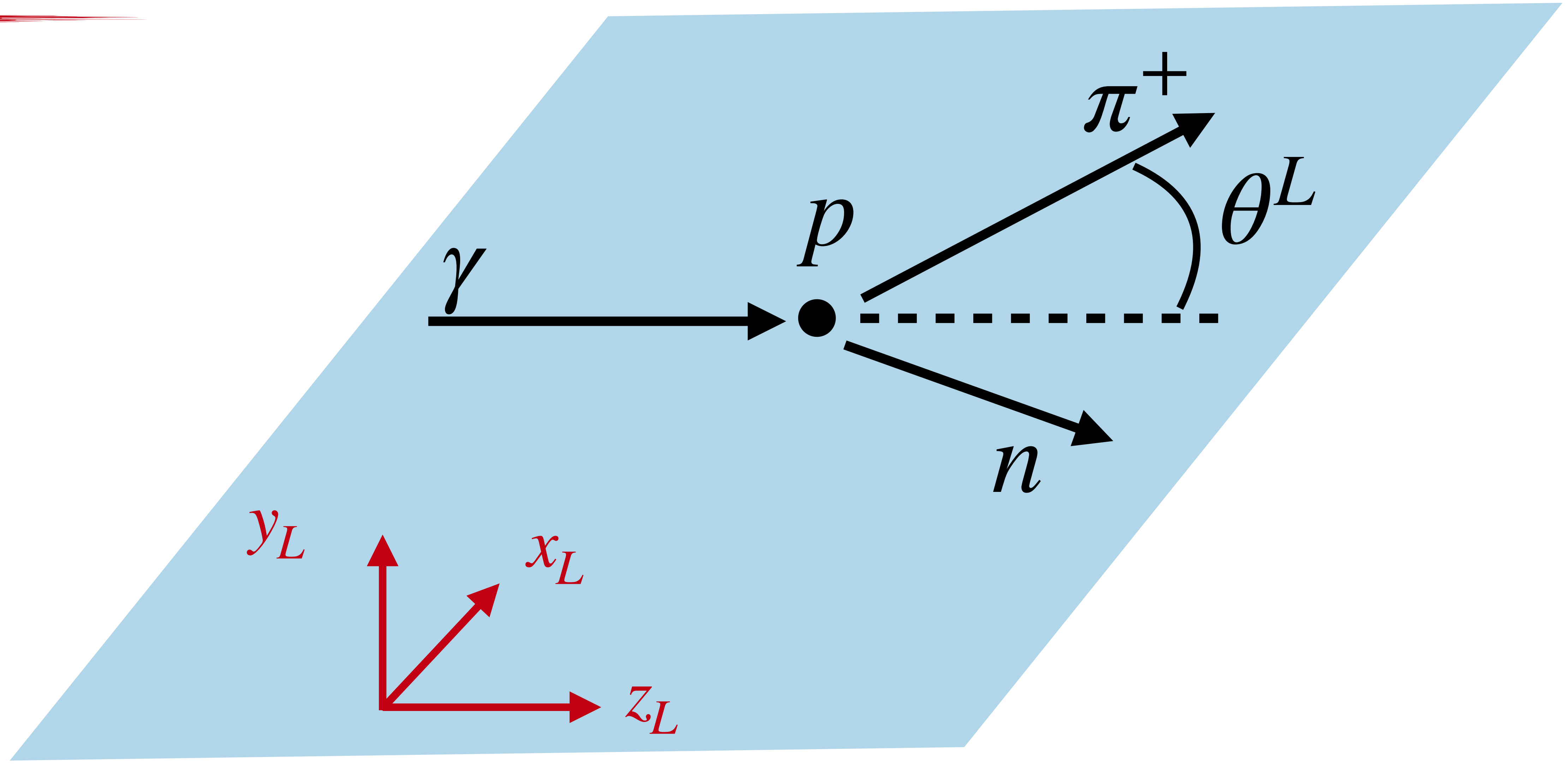


The two frames are related by
a boost along the z axis

Two-particles Final States

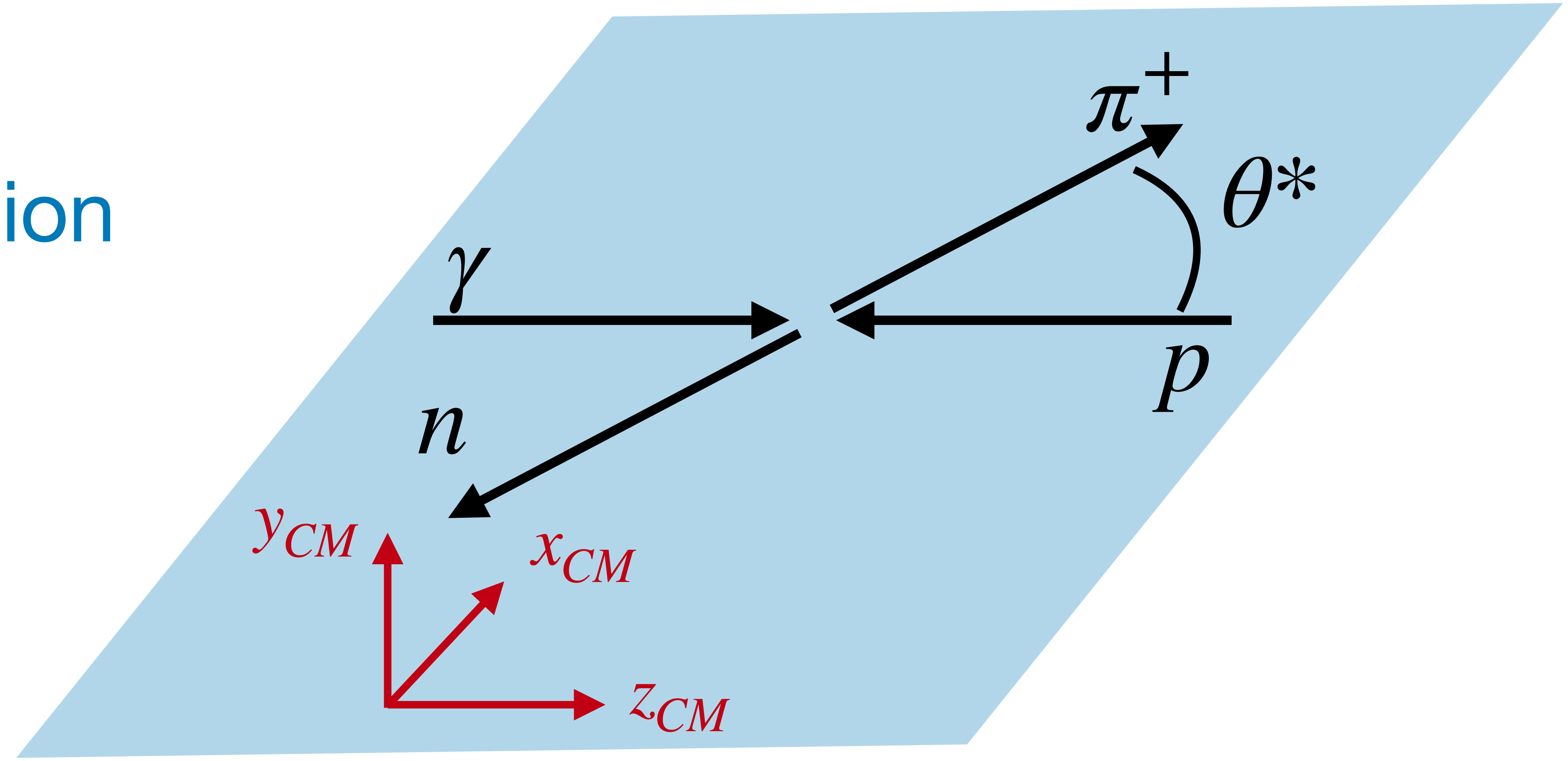
Data in Lab frame but physics in CoM





Resonance spin determine CoM angular distribution

$$I \propto \left| Y^1_M(\theta^*, \phi^*) \right|^2 \neq \left| Y^1_M(\theta^L, \phi^L) \right|^2$$



The true intensity involves $D^{\frac{3}{2}}(\phi^*, \theta^*, 0)$ but we'll see that later…

Boost between Lab and CoM

Only the energy and z component change

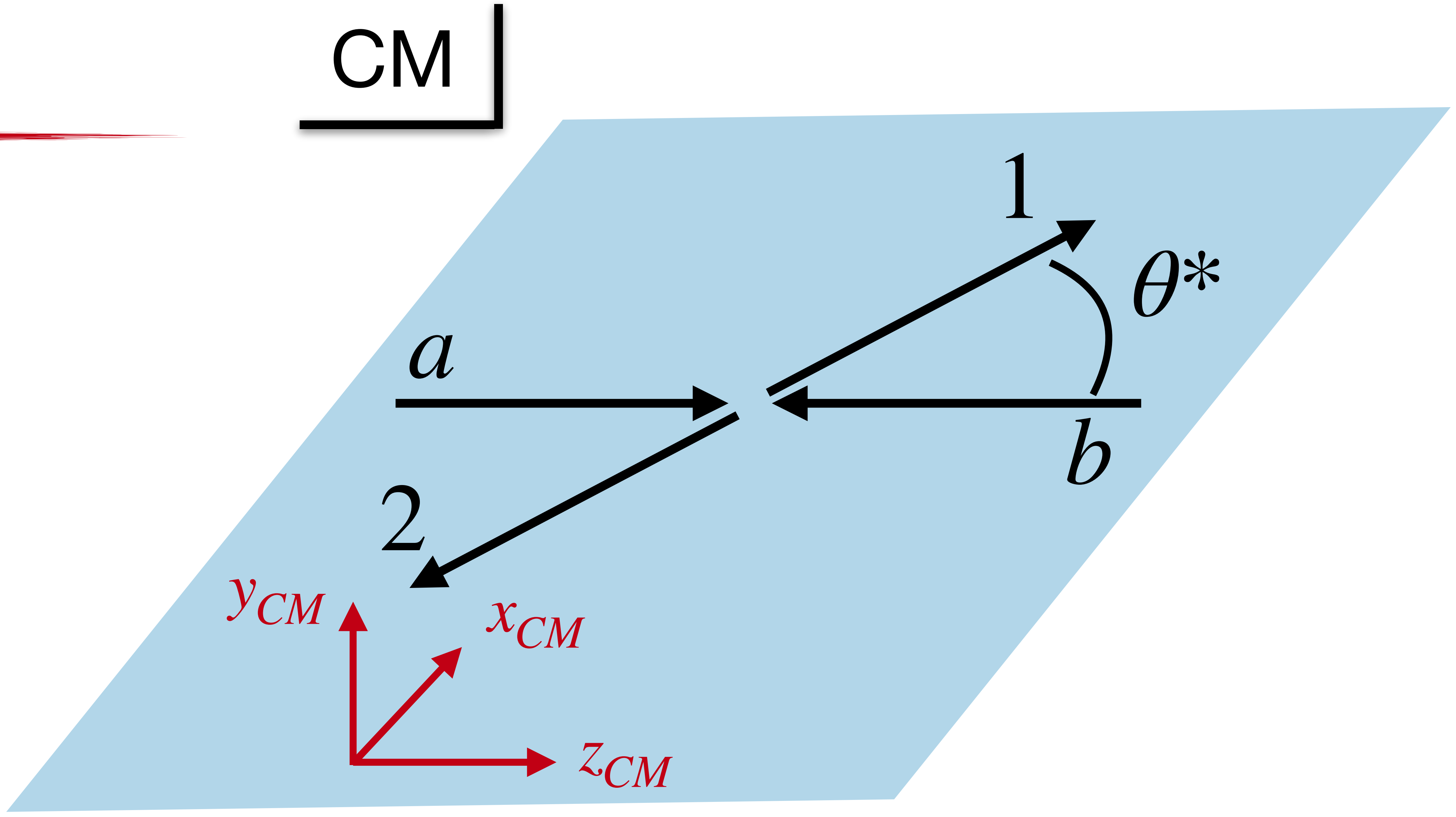
$$\begin{pmatrix} E^* \\ p_z^* \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E^L \\ p_z^L \end{pmatrix}$$

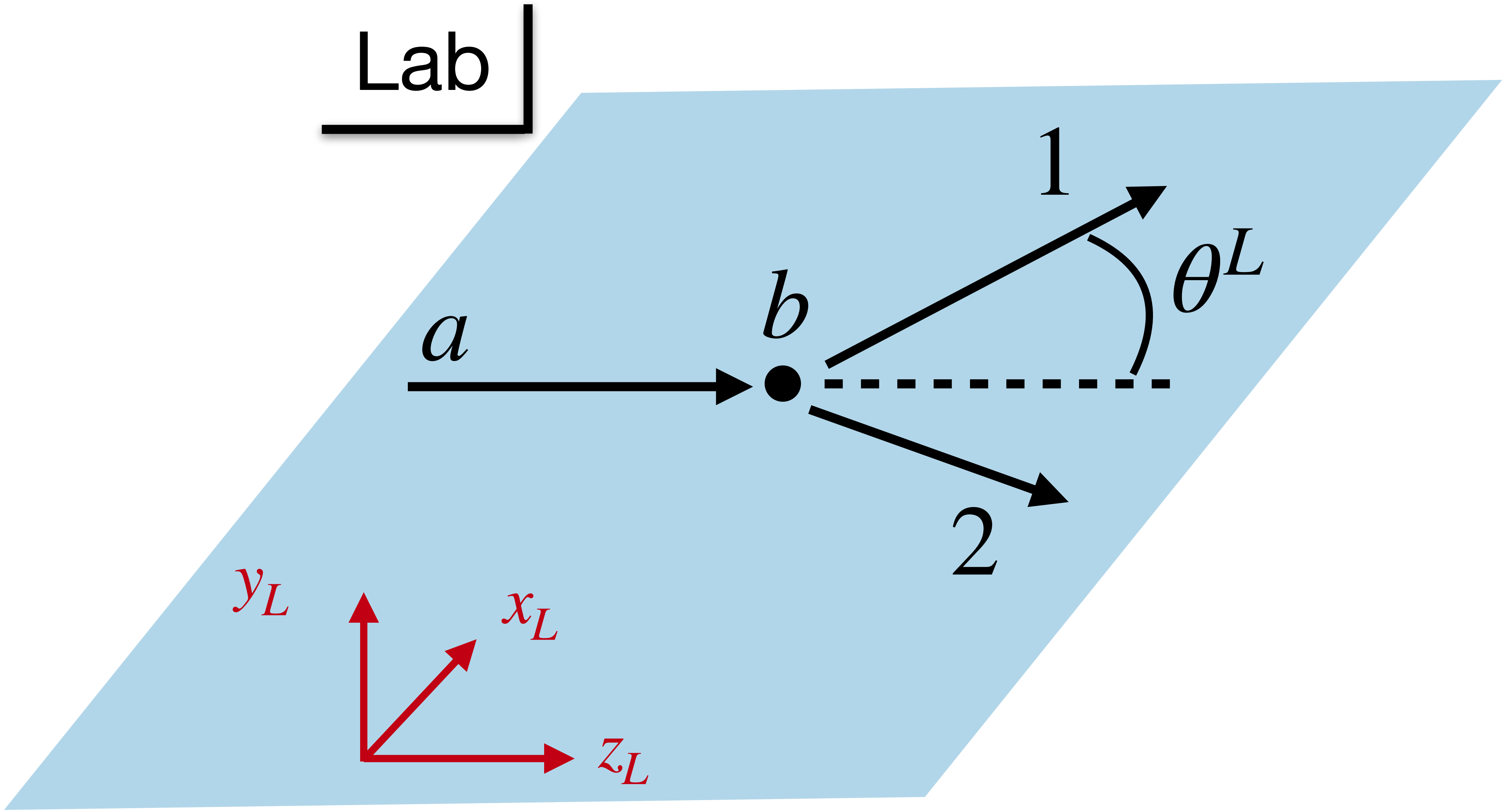
Inverse relation

$$\begin{pmatrix} E^L \\ p_z^L \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E^* \\ p_z^* \end{pmatrix}$$

With $\beta = \frac{p_b^*}{m_b}$ $\gamma\beta = \frac{E_b^*}{m_b}$

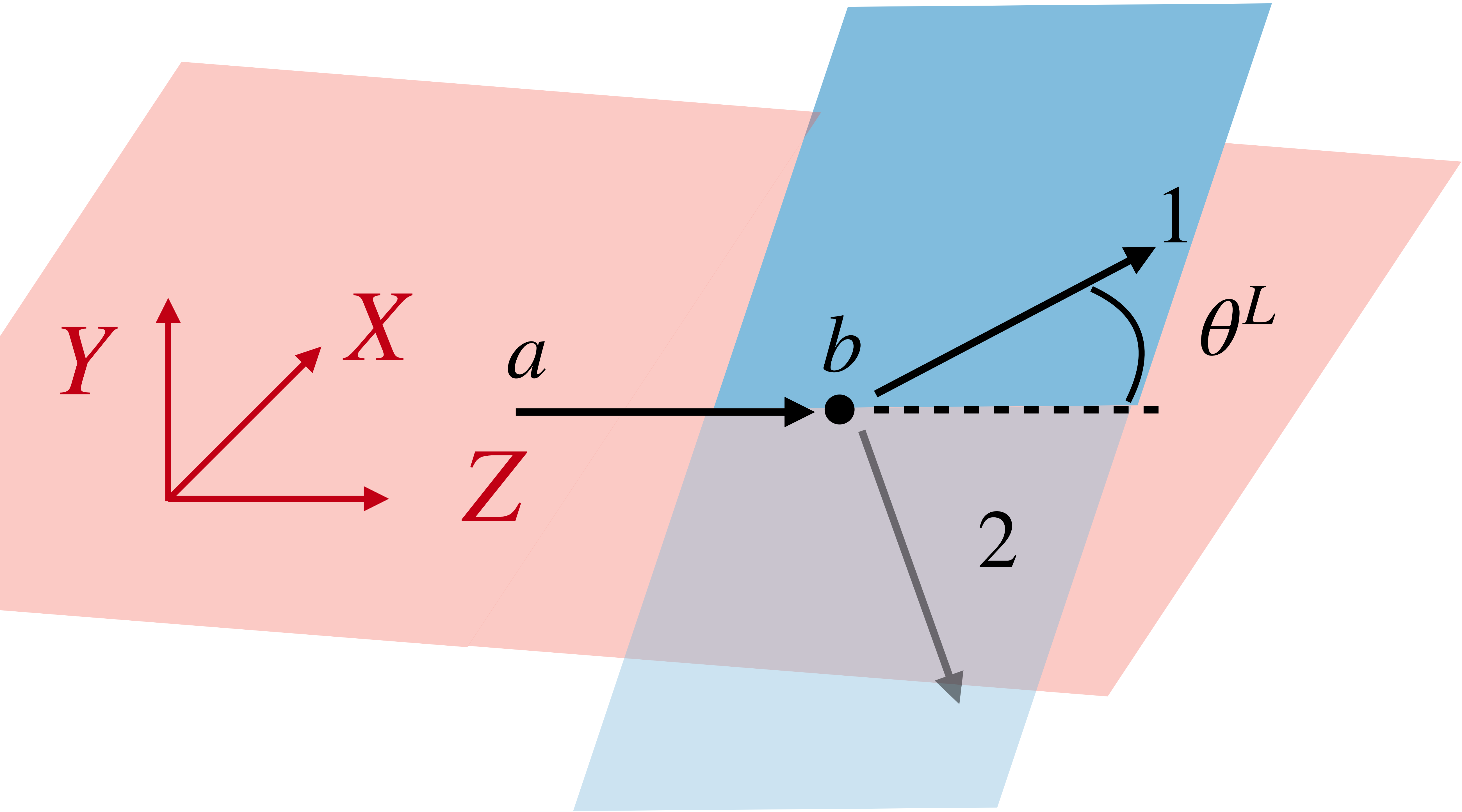
Exercise: check that it brings the target at rest

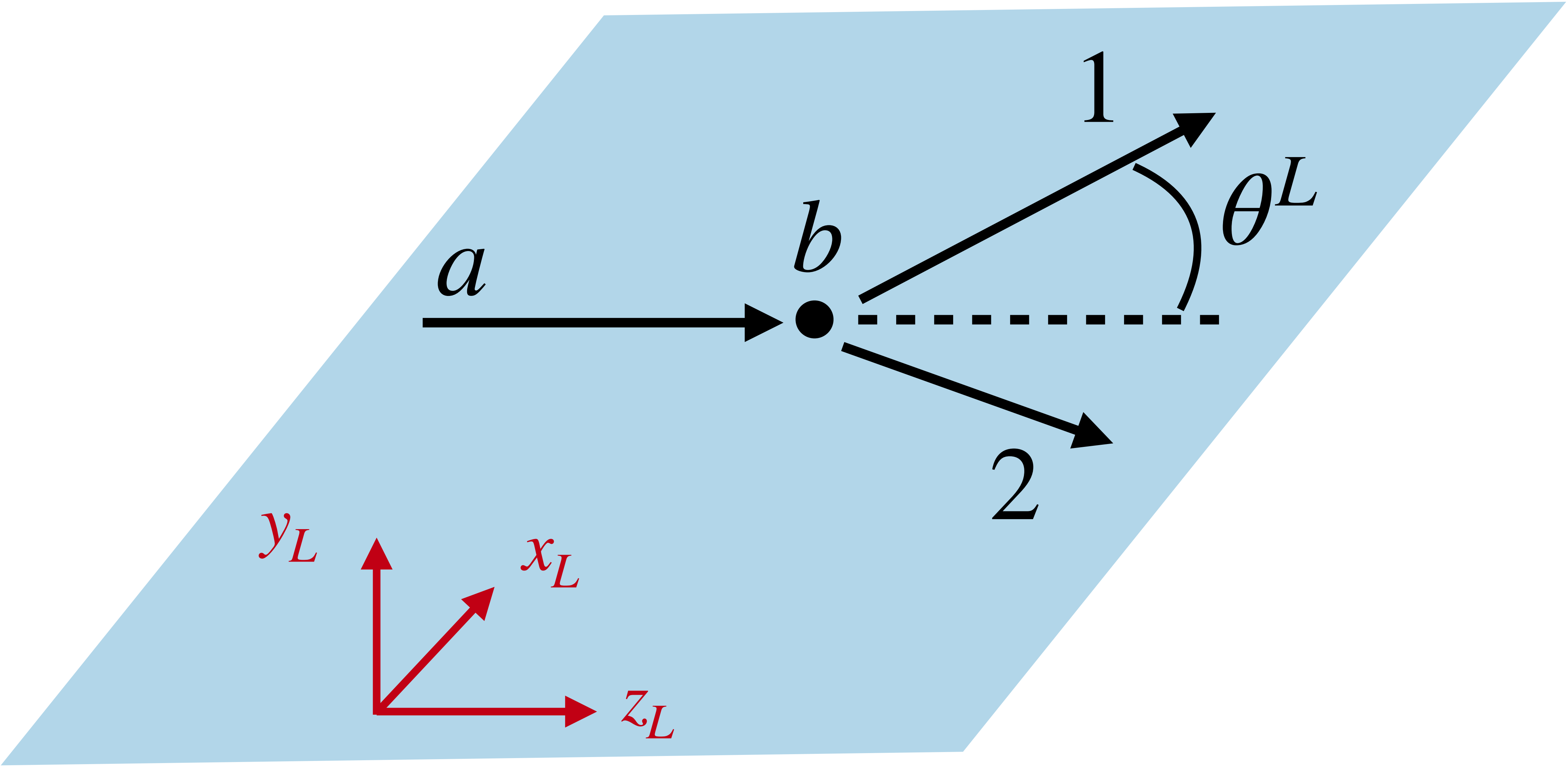




Events

Events are collected in a detector-fixed frame XYZ





Every event will lie in a different plane in the fixed XYZ frame

The orientation of the blue plane in the fixed frame only matters if either beam or target is polarized

Rotations

Under a rotation, the energy is conserved, only the spacial components change

$$(E, \vec{p}) = (E, p_x, p_y, p_z) \quad \rightarrow \quad R\left[(E, \vec{p})\right] = (E, \vec{p}\,') = (E, p'_x, p'_y, p'_z)$$

Any rotation can be decomposed into rotations around the z and y axes

$$R_z(\omega) = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_y(\omega) = \begin{pmatrix} \cos\omega & 0 & \sin\omega \\ 0 & 1 & 0 \\ -\sin\omega & 0 & \cos\omega \end{pmatrix}$$

For completeness:

$$R_x(\omega) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{pmatrix}$$

Active rotations

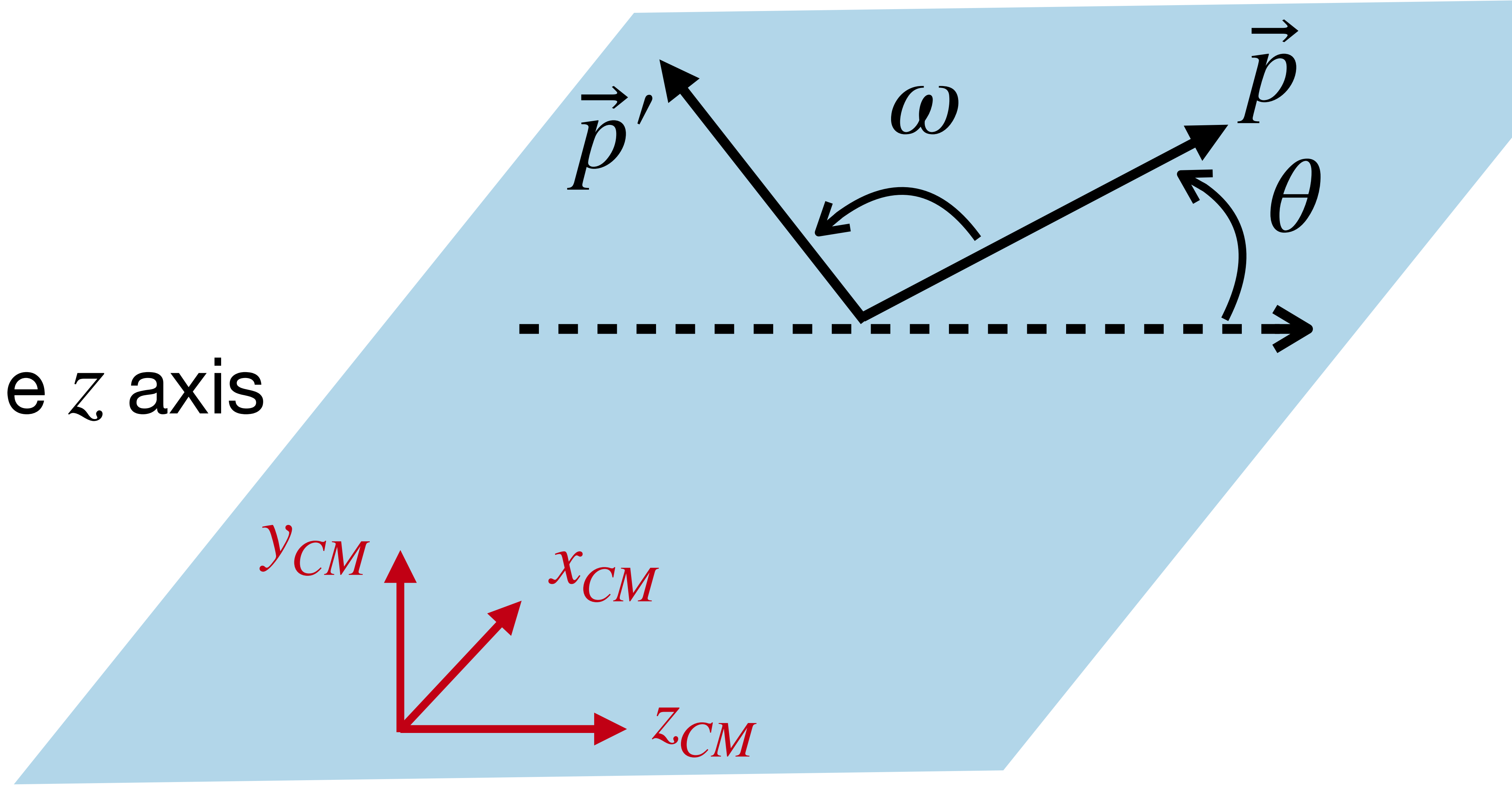
Under an active rotation, the momentum is changed and the axes are fixed

Example:
a momentum of unit length forming an angle θ with the z axis

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$$

After a rotation of ω around y, it forms an angle $\theta + \omega$ with the z axis

$$\begin{pmatrix} p'_x \\ p'_y \\ p'_z \end{pmatrix} = R_y(\omega) \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \sin(\theta+\omega) \\ 0 \\ \sin(\theta+\omega) \end{pmatrix}$$



Exercice: check the result of the rotation

Example 1

File: Two-Particles-1.dat

Format:

$$E_a,\ p_{a,x},\ p_{a,y},\ p_{a,z}$$

The data are in the lab frame

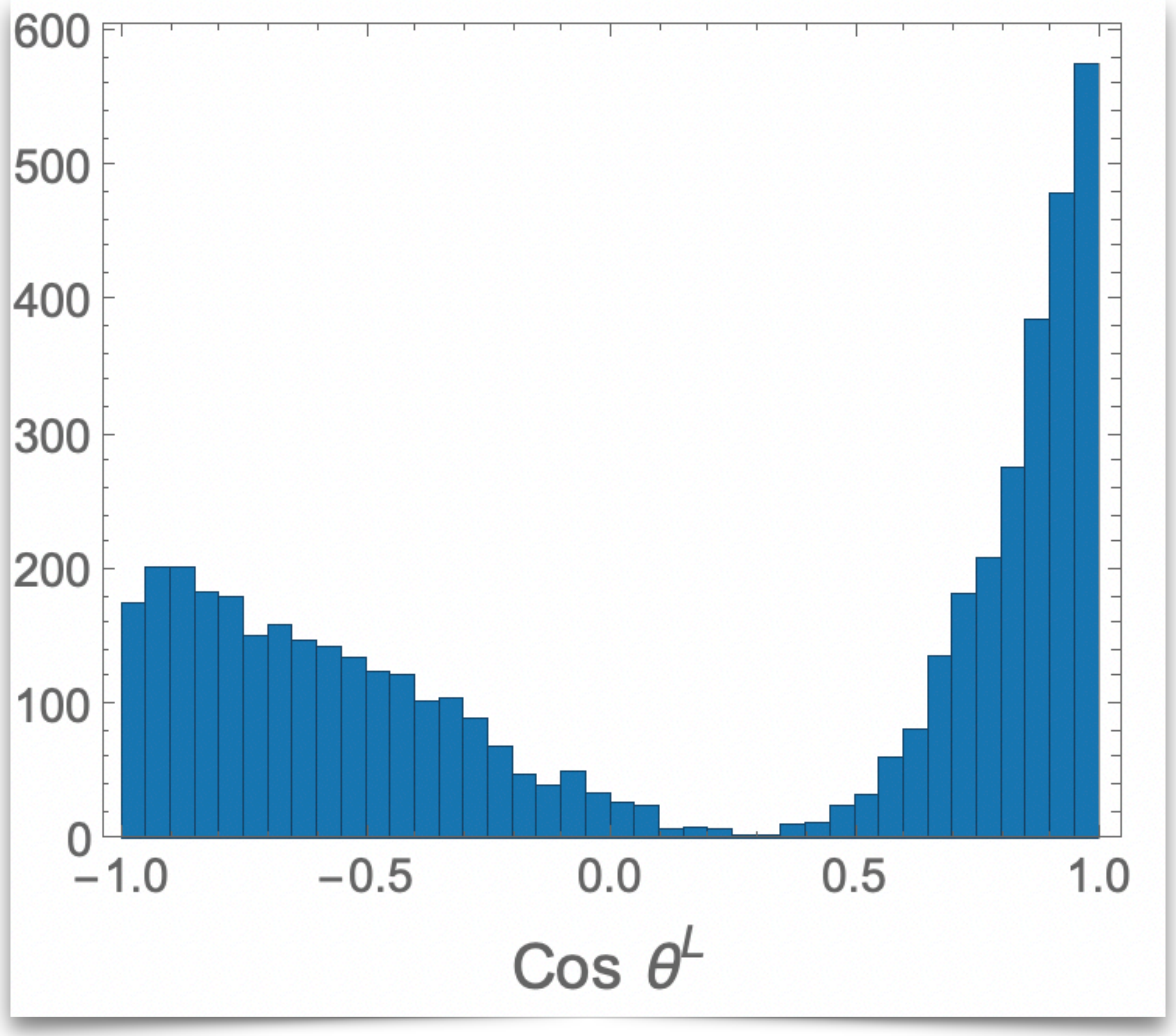
$$E_1,\ p_{1,x},\ p_{1,y},\ p_{1,z}$$

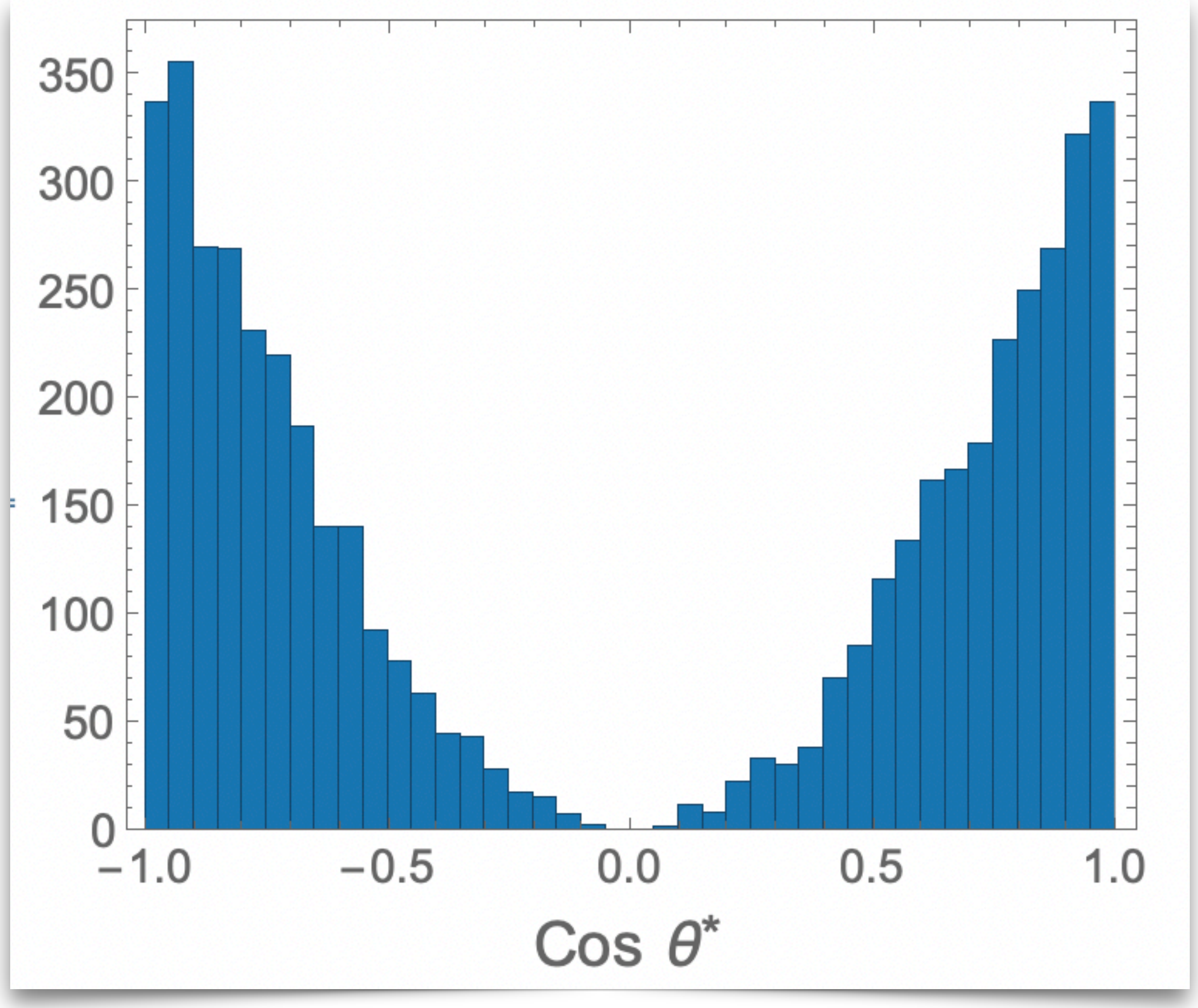
$$E_2,\ p_{2,x},\ p_{2,y},\ p_{2,z}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

Boost to the CoM and Compute and plot the $\cos\theta^*$ distribution

Compute and plot the $\cos\theta^L$ distribution





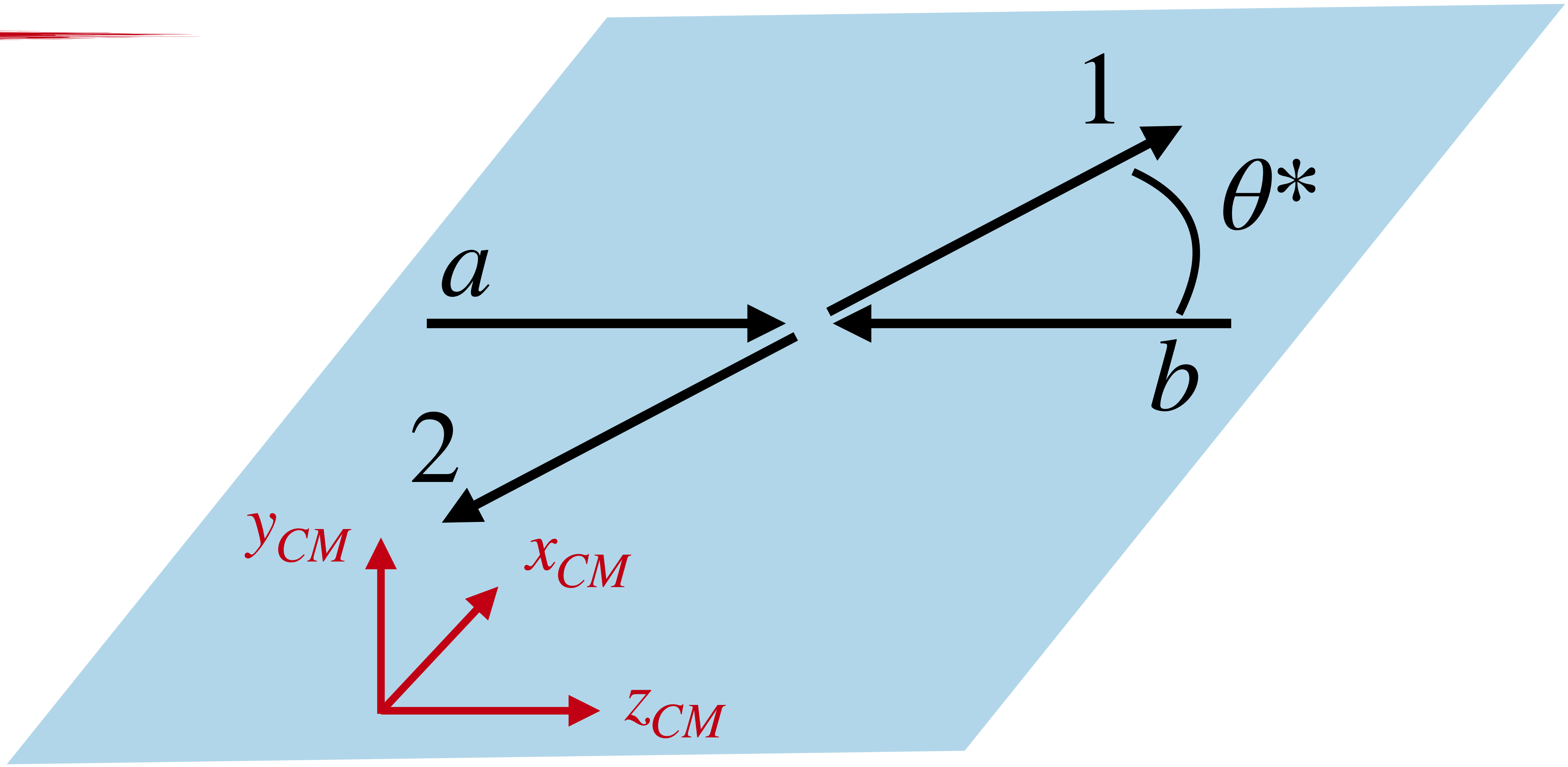
Computing angles

First method:

Perform appropriate Lorentz transformations

Extract angles from scalar and vectorial products

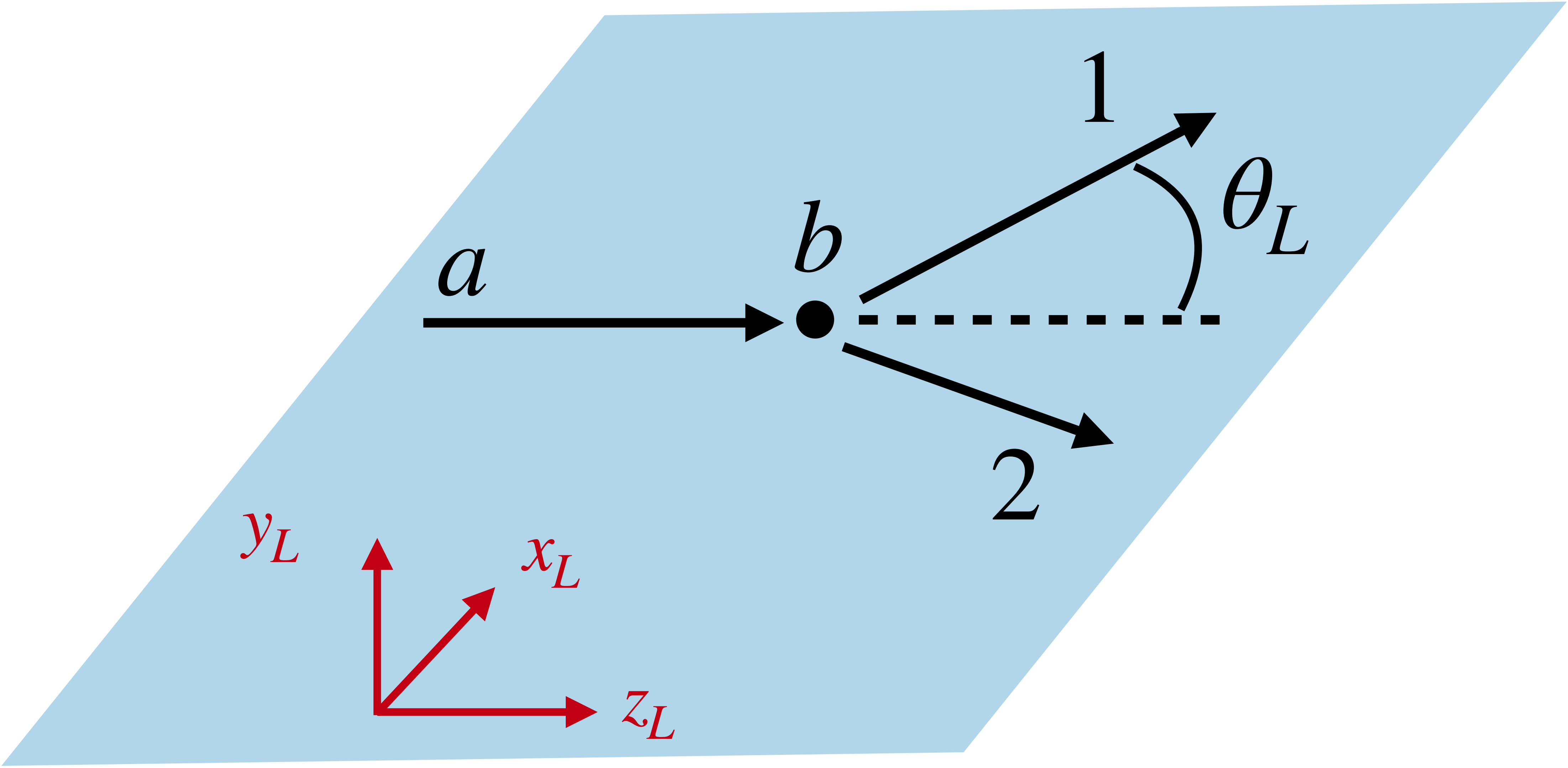
Polar angles $\in [0,\pi]$ require only cos
Azimuthal angles $\in [0,2\pi[$ require cos and sin



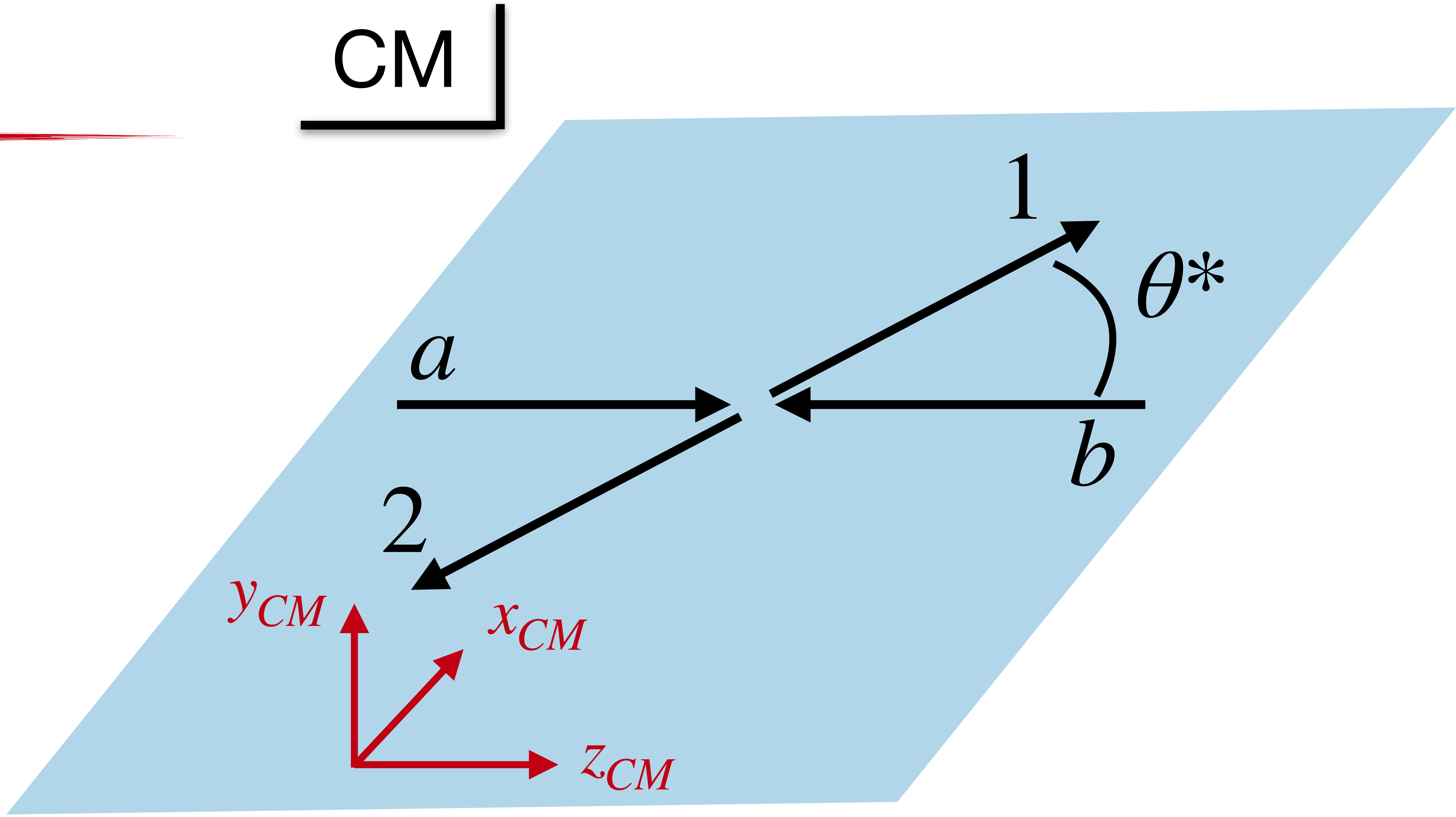
Alternative method:

Extract Lorentz invariants

Compute angles from Lorentz invariants



Mandelstam variables



Two variables: Total energy E_{CM}

Scattering angle θ^*, θ^L

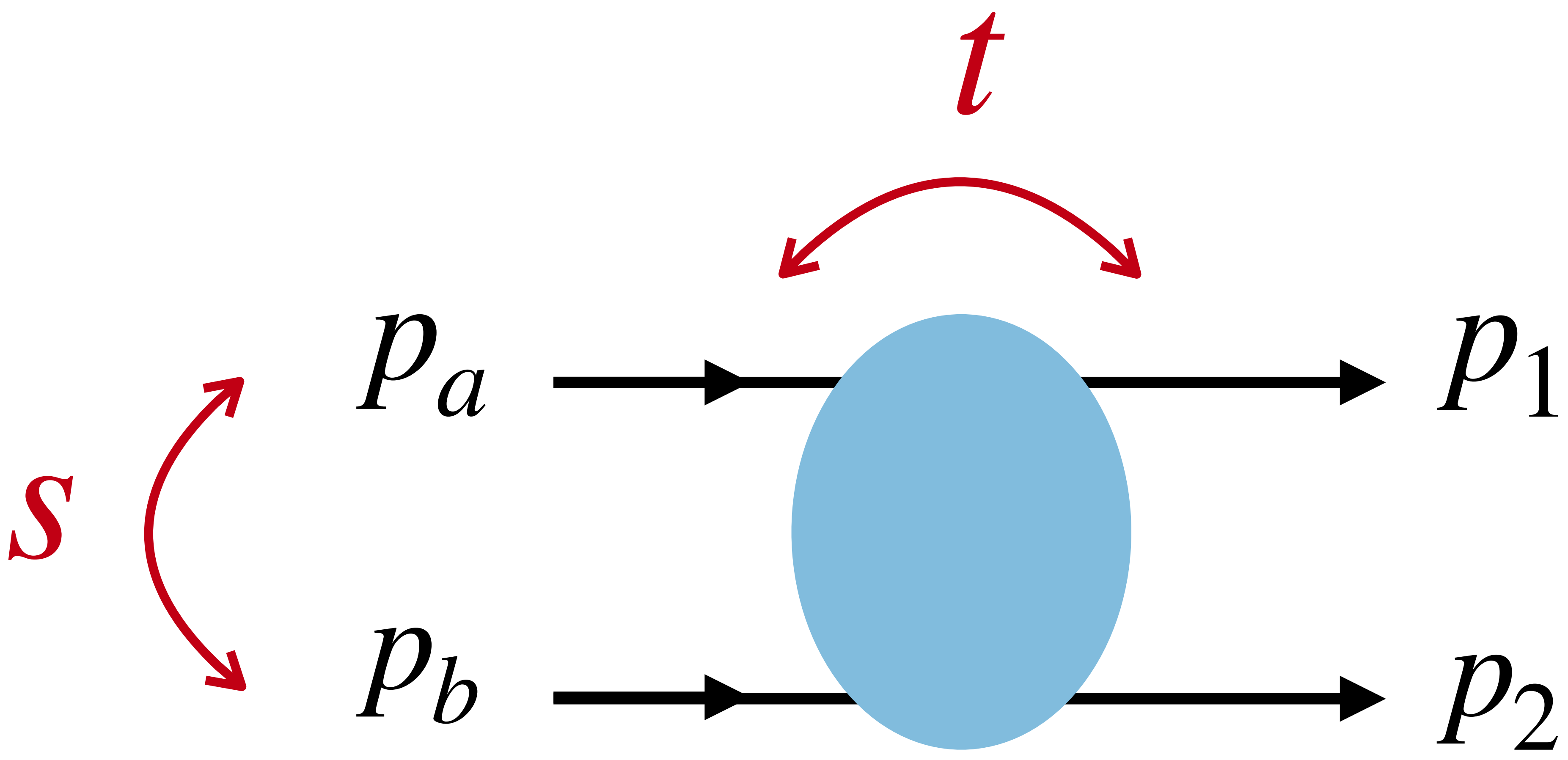
The mass shell $p_i^2 = m_i^2$

Lorentz invariants:

$$s = (p_a + p_b)^2 = (p_1 + p_2)^2$$

$$t = (p_a - p_1)^2 = (p_b - p_2)^2$$

$$u = (p_a - p_2)^2 = (p_b - p_1)^2$$



Only two independent

Check that $s + t + u = m_a^2 + m_b^2 + m_1^2 + m_2^2$, using $p_a + p_b = p_1 + p_2$

Example 1

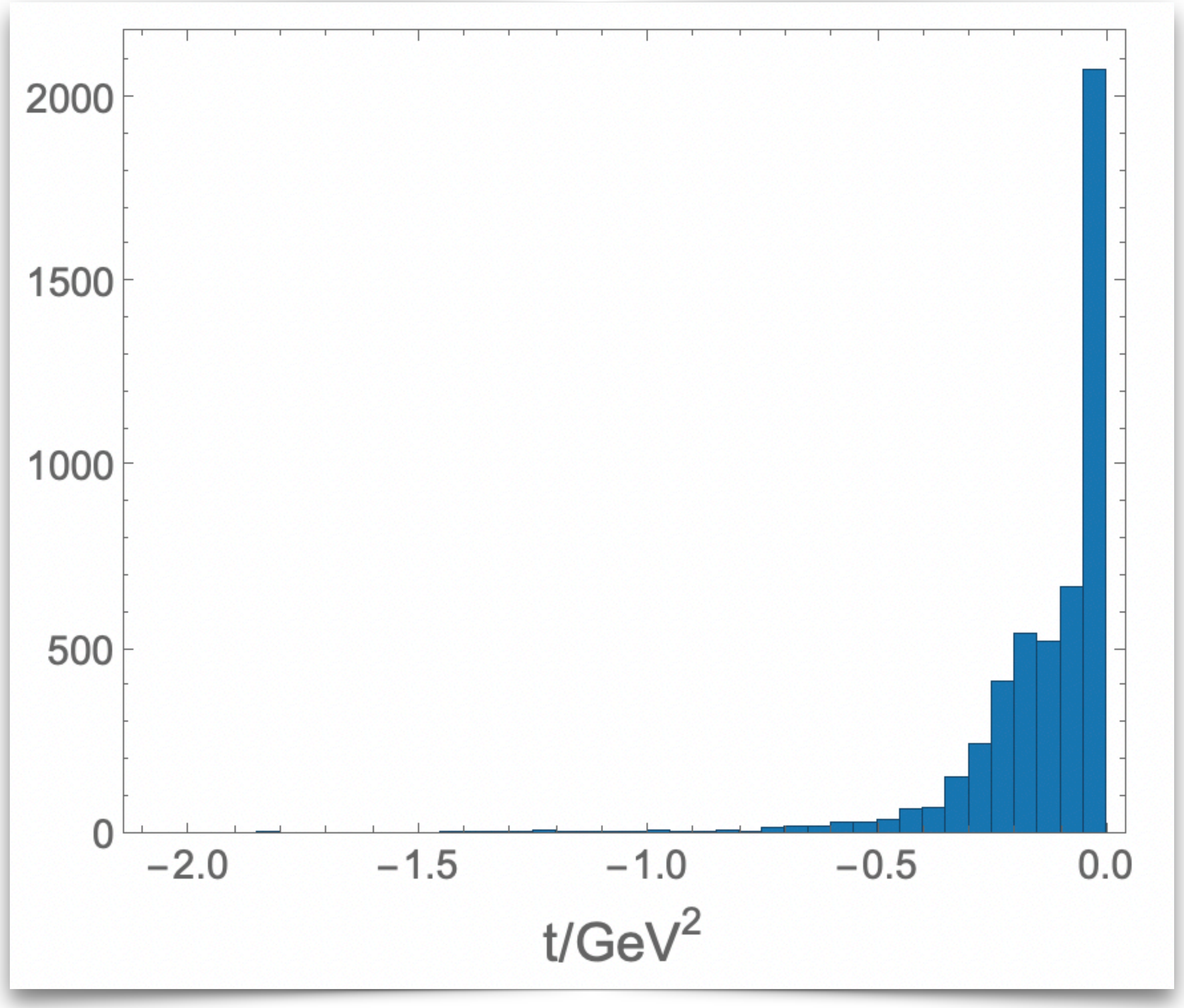
File: Two-Particles-1.dat

Format:

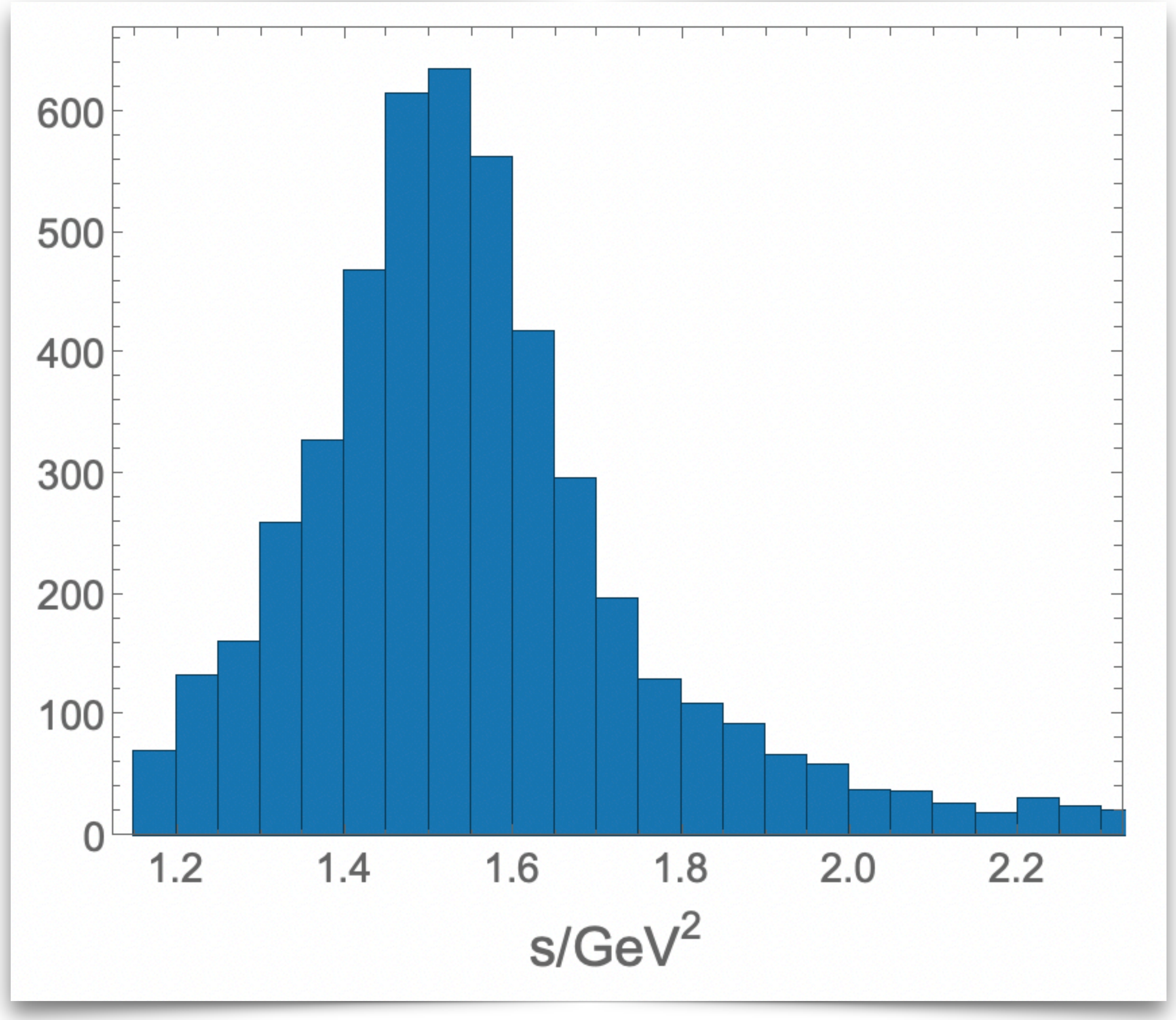
$$\begin{array}{cccc} E_a, & p_{a,x}, & p_{a,y}, & p_{a,z} \\ E_1, & p_{1,x}, & p_{1,y}, & p_{1,z} \\ E_2, & p_{2,x}, & p_{2,y}, & p_{2,z} \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

The data are in the lab frame

t distribution

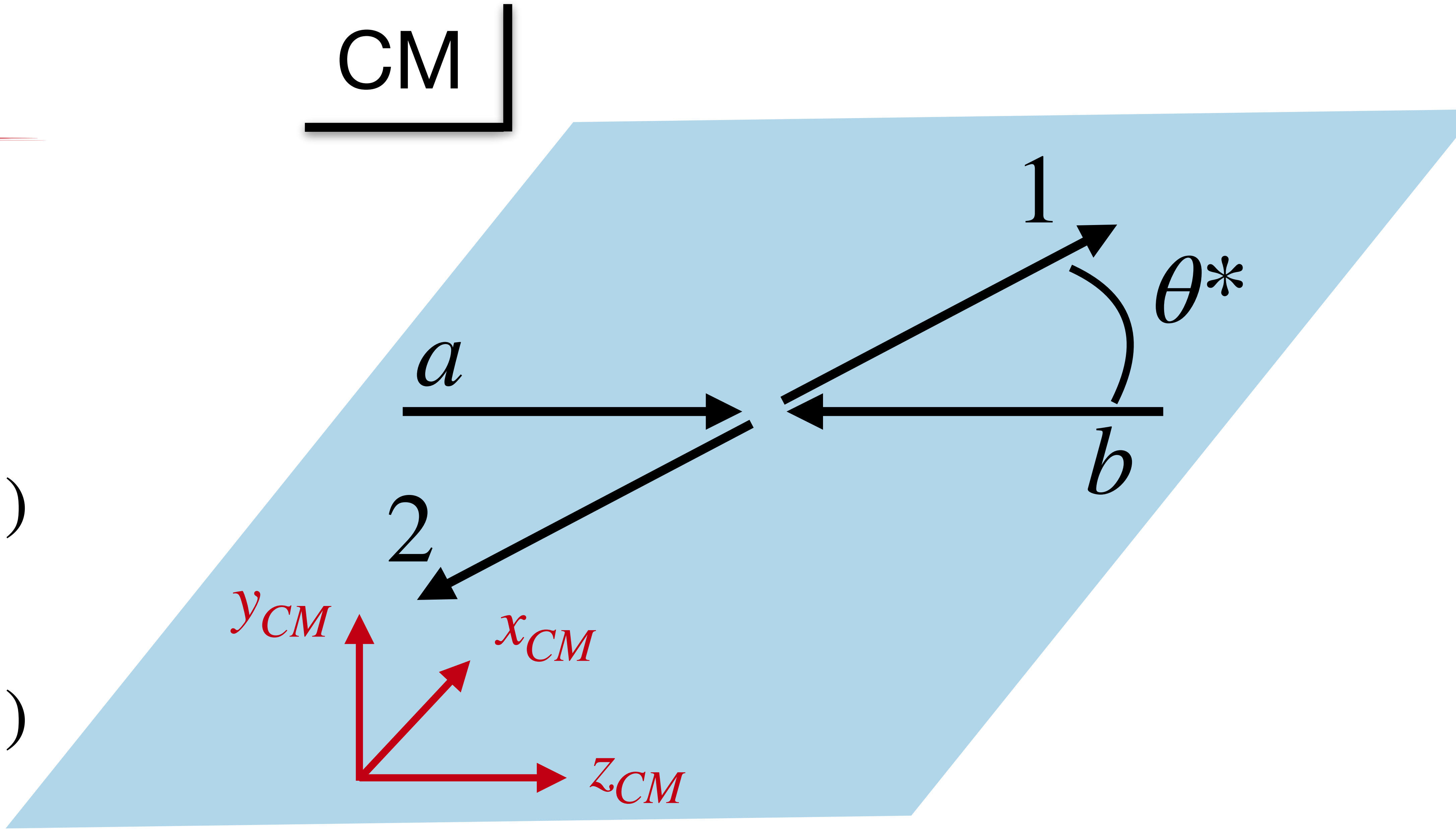


$s = (E^*)^2$ distribution



CM Kinematics

Every frame dependent quantities
is expressed with Mandelstam variables



$$p_a = (E_a^*, 0, 0, \quad |\vec{p}_a^*|) \qquad p_1 = (E_1^*, \quad |\vec{p}_1^*| \sin\theta^*, 0, \quad |\vec{p}_1^*| \cos\theta^*)$$

$$p_b = (E_b^*, 0, 0, -|\vec{p}_a^*|) \qquad p_2 = (E_2^*, -|\vec{p}_1^*| \sin\theta^*, 0, -|\vec{p}_1^*| \cos\theta^*)$$

Invariant: $s = (E_a^* + E_b^*)^2 = E_{CM}^2$ $\qquad t = (p_a - p_1)^2 = m_a^2 + m_1^2 - 2E_a^* E_1^* + 2|\vec{p}_a^*||\vec{p}_1^*| \cos\theta^*$

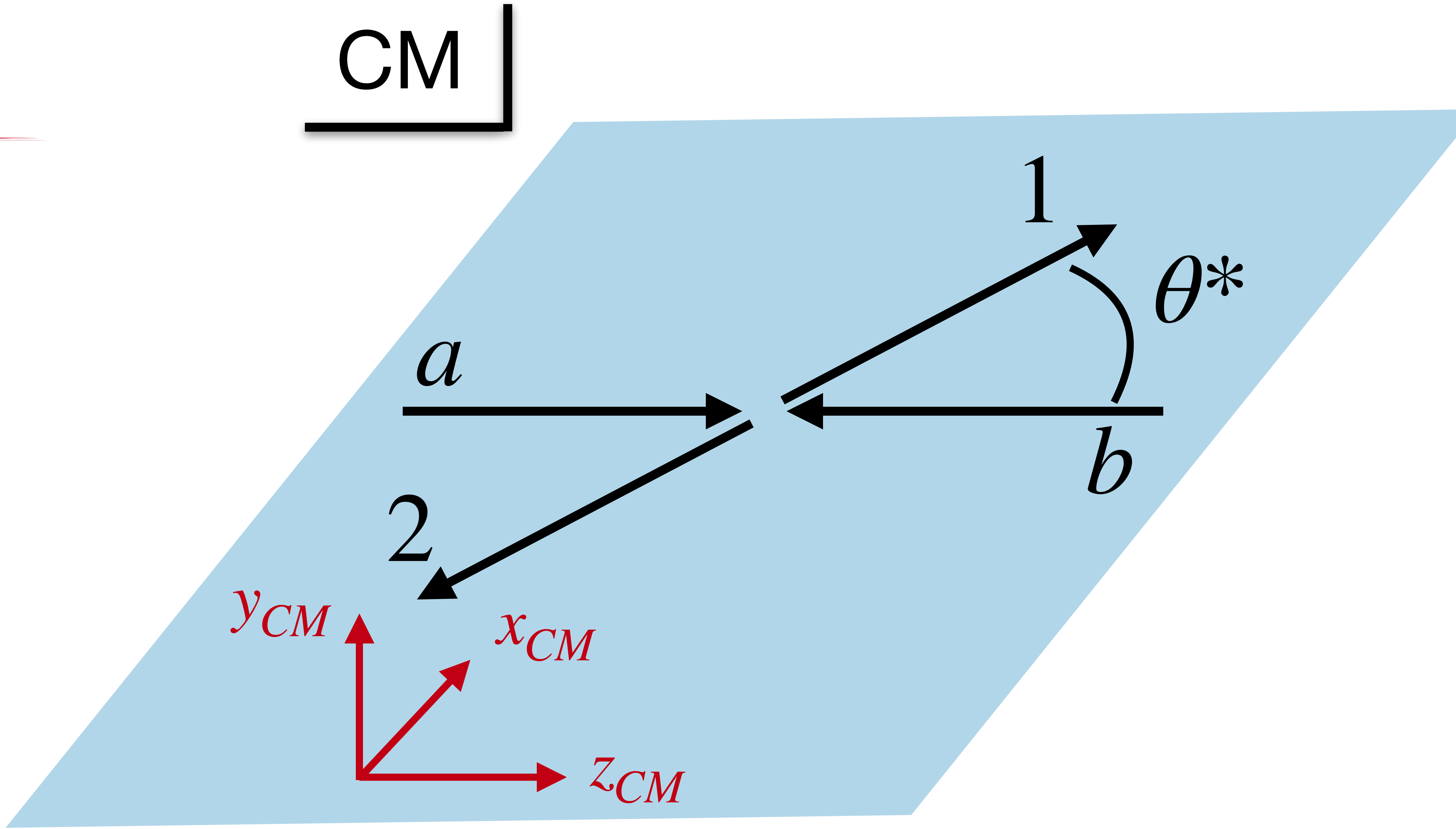
From $p_1^2 = ([p_a + p_b] - p_2)^2$ we obtain $E_2^* = \dfrac{s + m_2^2 - m_1^2}{2\sqrt{s}}$

And $|\vec{p}_1^*| = \sqrt{(E_1^*)^2 - m_1^2} = \dfrac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}}$ With $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ca)$

CoM Kinematics

$$t = m_a^2 + m_1^2 - 2E_a^* E_1^* + 2\,|\vec{p}_a^{\,*}|\,|\vec{p}_1^{\,*}|\cos\theta^*$$

Angles are physical: $-1 \leq \cos\theta^* \leq 1$



$$t_{min,max} = m_a^2 + m_1^2 - 2E_a^* E_1^* \pm 2\,|\vec{p}_a^{\,*}|\,|\vec{p}_1^{\,*}|$$

$$= \frac{1}{2s}\left[(m_1 - m_a)^2 + (m_2 - m_b)^2\right]^2 + \left(|\vec{p}_a^{\,*}| \pm |\vec{p}_1^{\,*}|\right)^2$$

Note that $t_{min} = 0$, $t_{max} = -4|p^*|^2$
if $m_a = m_1$ and $m_b = m_2$
(elastic scattering)

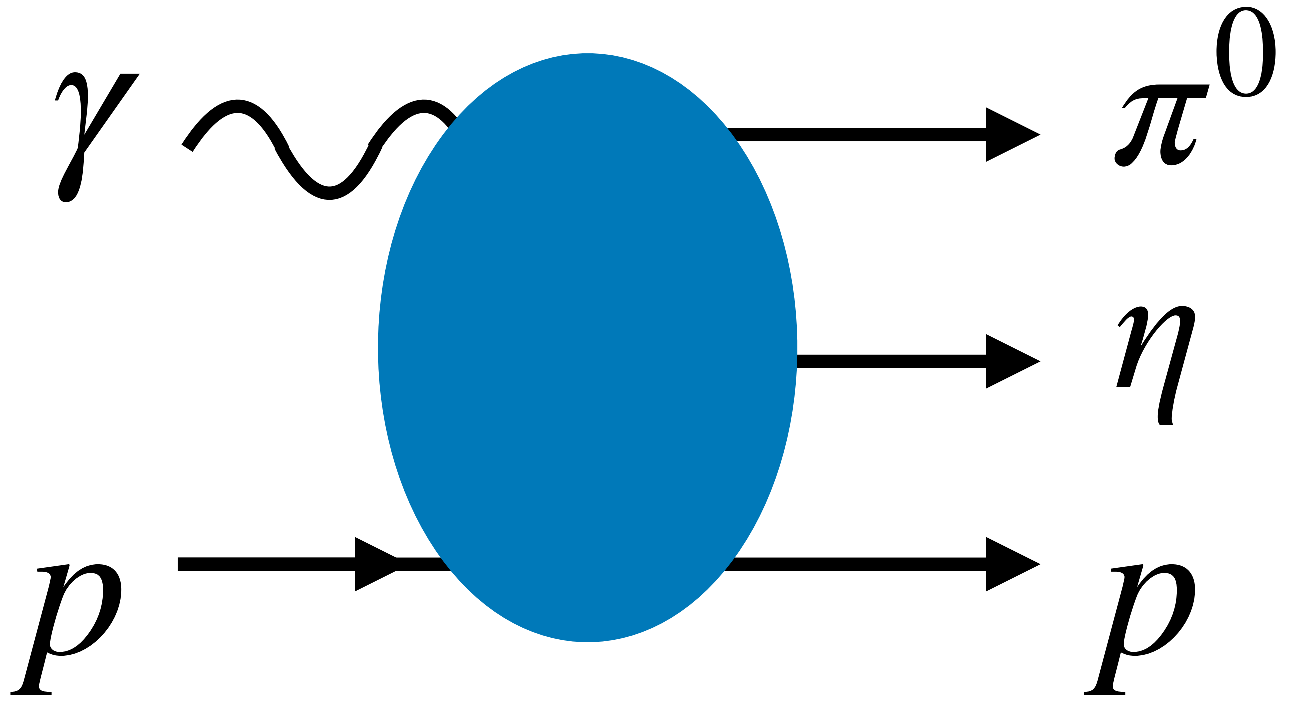
Equivalent expression for the scattering angle in the CoM

$$\cos\theta^* = 1 + \frac{t - t_{min}}{2\,|\vec{p}_a^{\,*}|\,|\vec{p}_1^{\,*}|} = \frac{s(t-u) + (m_a^2 - m_b^2)(m_1^2 - m_2^2)}{\lambda^{1/2}(s, m_a^2, m_b^2)\lambda^{1/2}(s, m_1^2, m_2^2)}$$

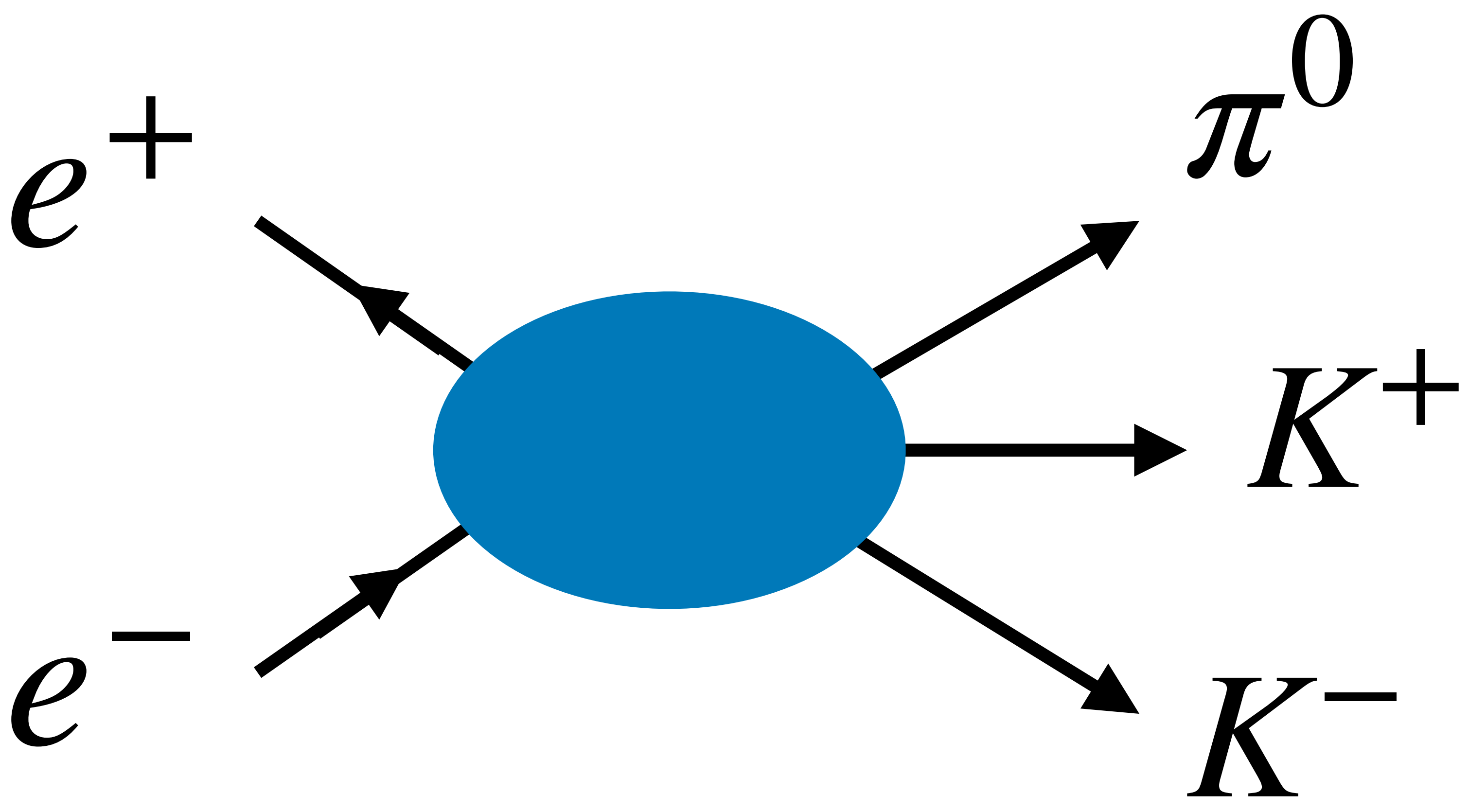
Exercices:
- Check all relations
- Compute $\cos\theta^L$ as a function of s, t, u

Three-particles Final States

Peripheral production $\gamma p \to \eta \pi^0 p$

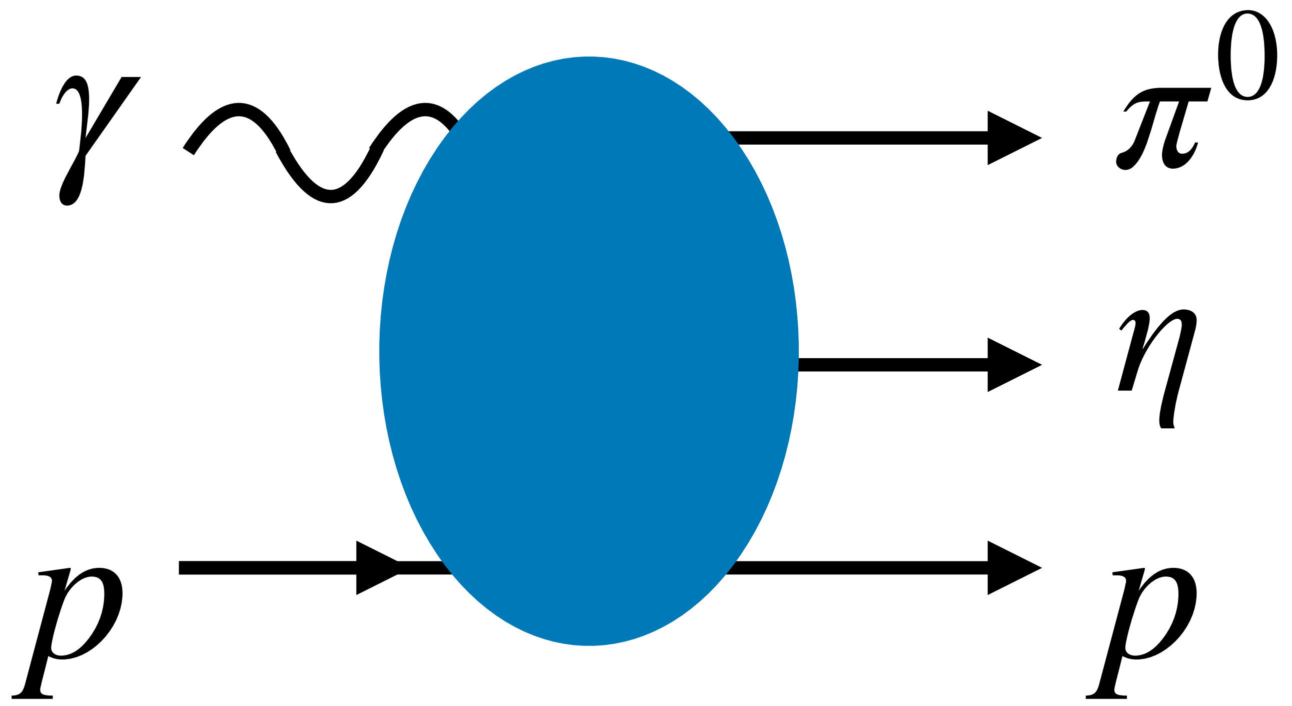


Annihilation $e^+e^- \to K^+K^-\pi^0$



Three-particles Final States

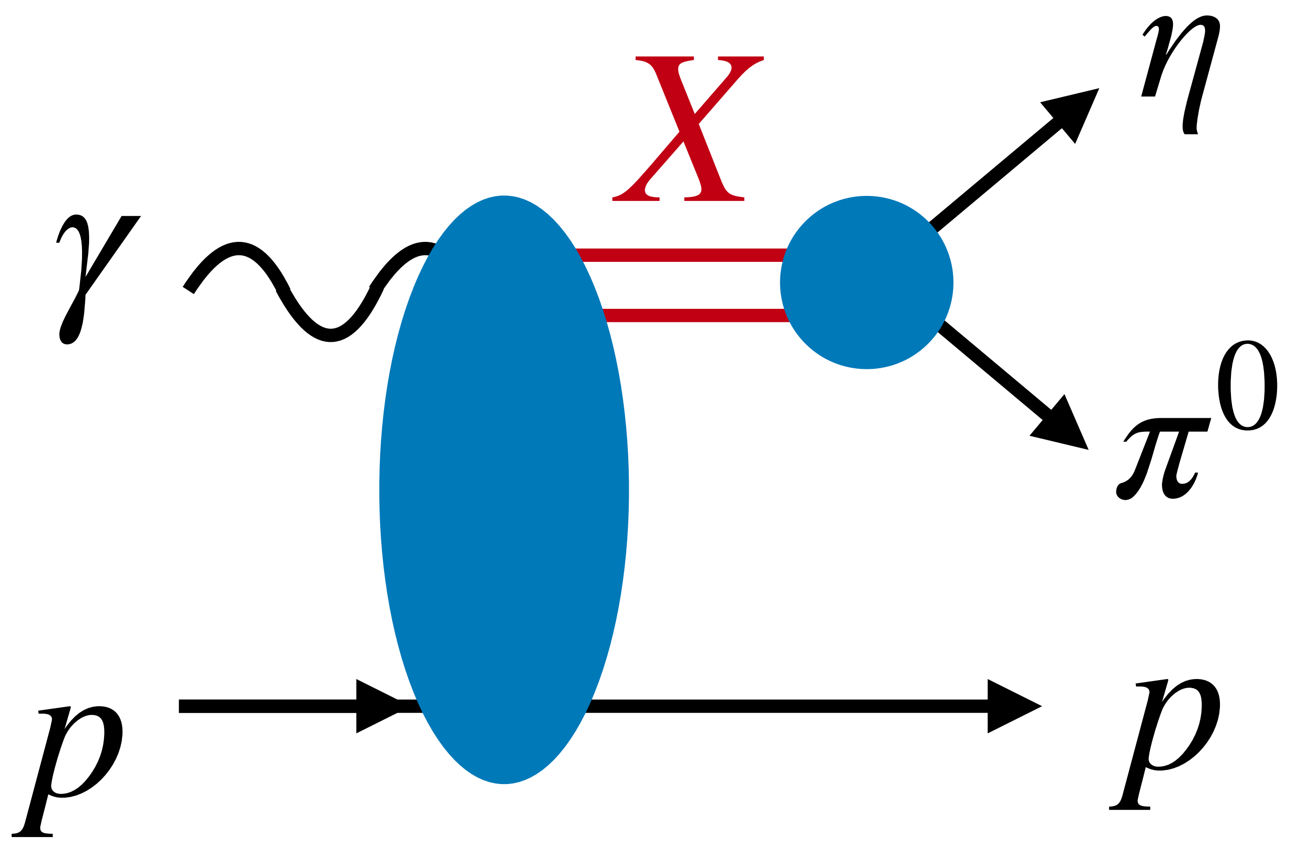
Peripheral production $\gamma p \to \eta\pi^0 p$



Mesonic resonances produced on a nucleon target

$$X = a_0, \pi_1, a_2, \ldots \to \eta\pi$$

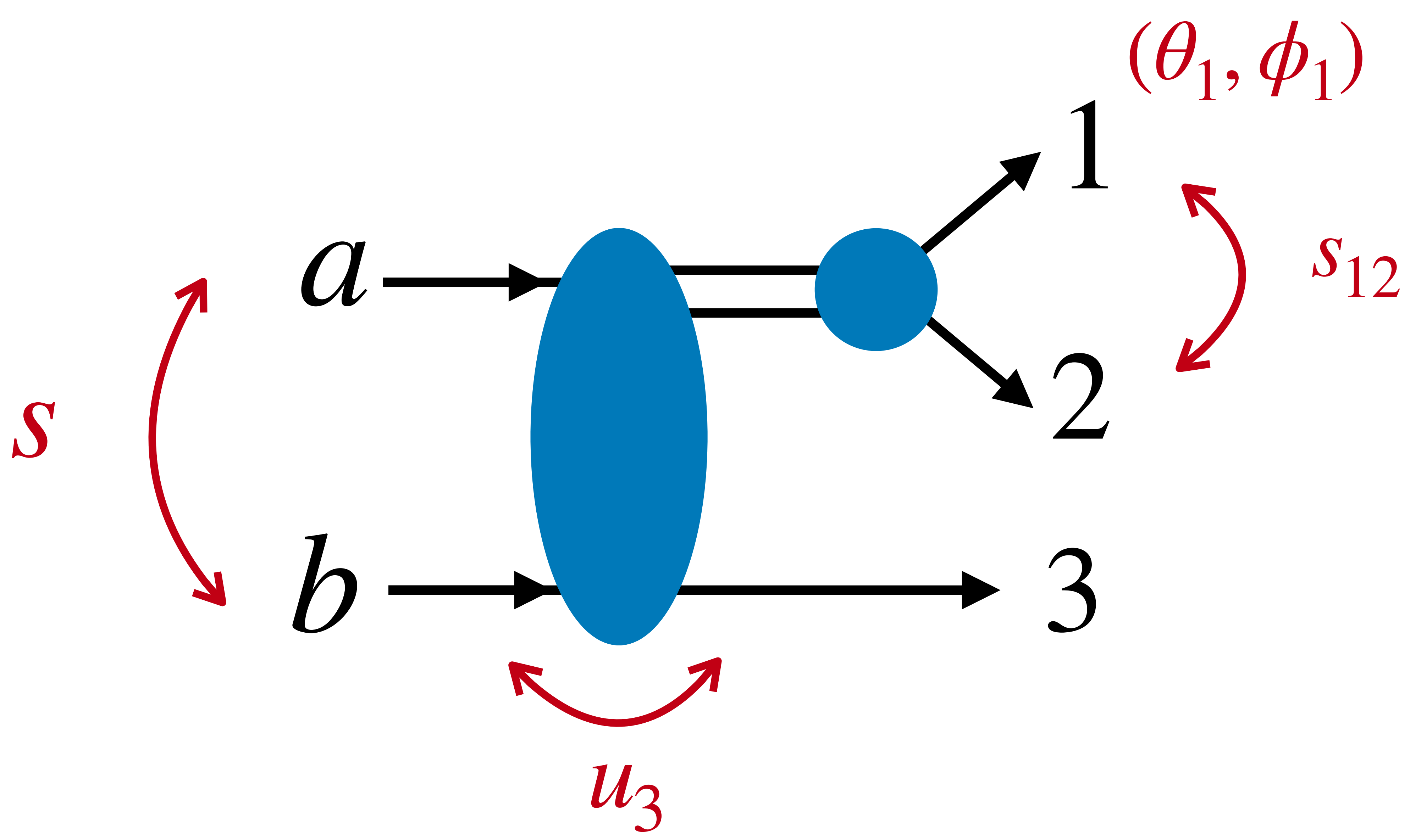
Corresponds to S,P,D,… waves



Need to study the angular distribution in the $\eta\pi$ rest frame

Relevant variables

There are 5 independent variables

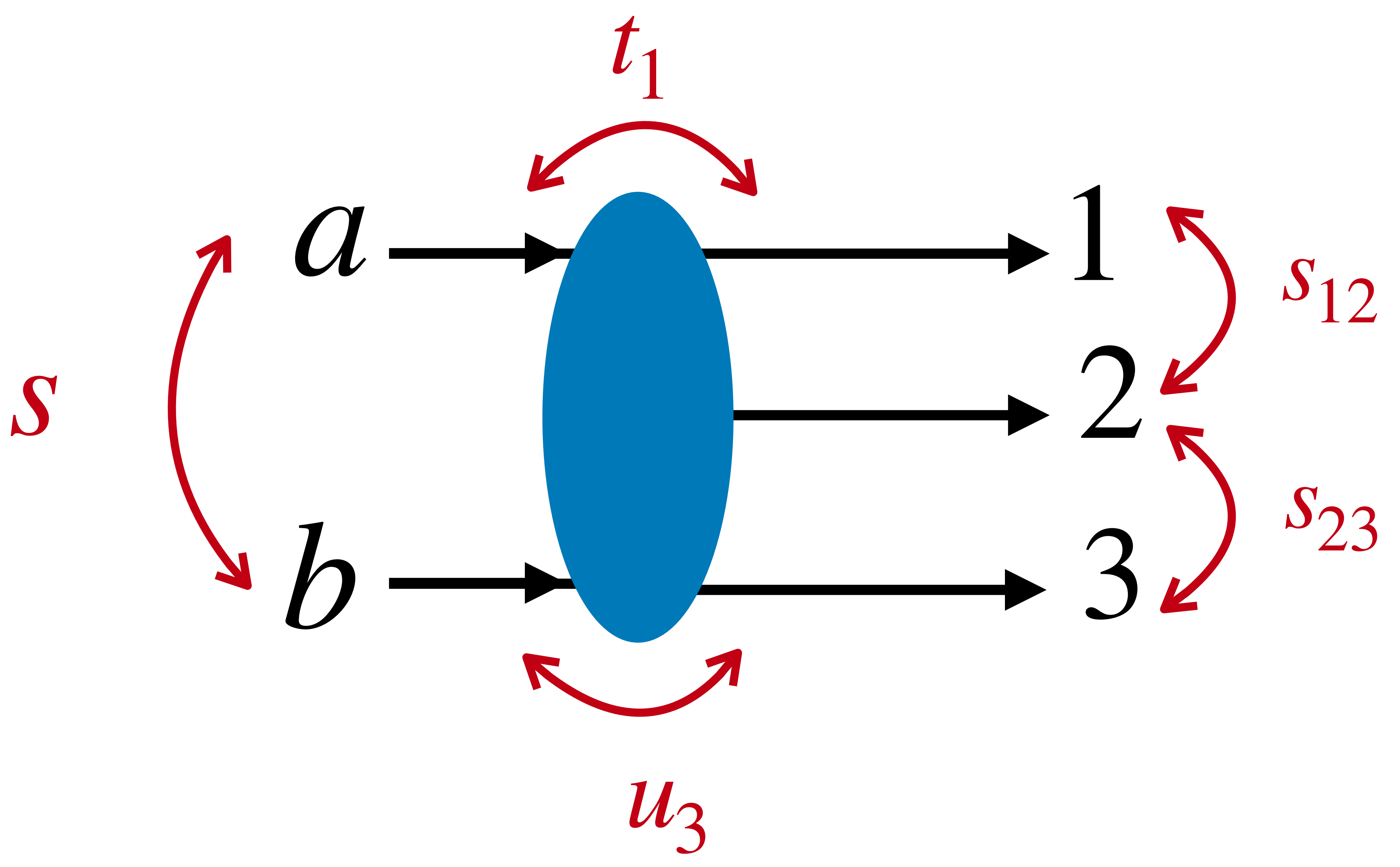


(θ_1, ϕ_1): Angles of 1 in (12) RF

$s_{12} = (p_1 + p_2)^2$ $\quad s = (p_a + p_b)^2$

$u_3 = (p_b - p_3)^2$ $\quad (\theta_1, \phi_1) \leftrightarrow (t_1, s_{23})$

5 Mandelstam variables: $s, t_1, u_3, s_{12}, s_{23}$



$$s_{ij} = (p_i + p_j)^2$$

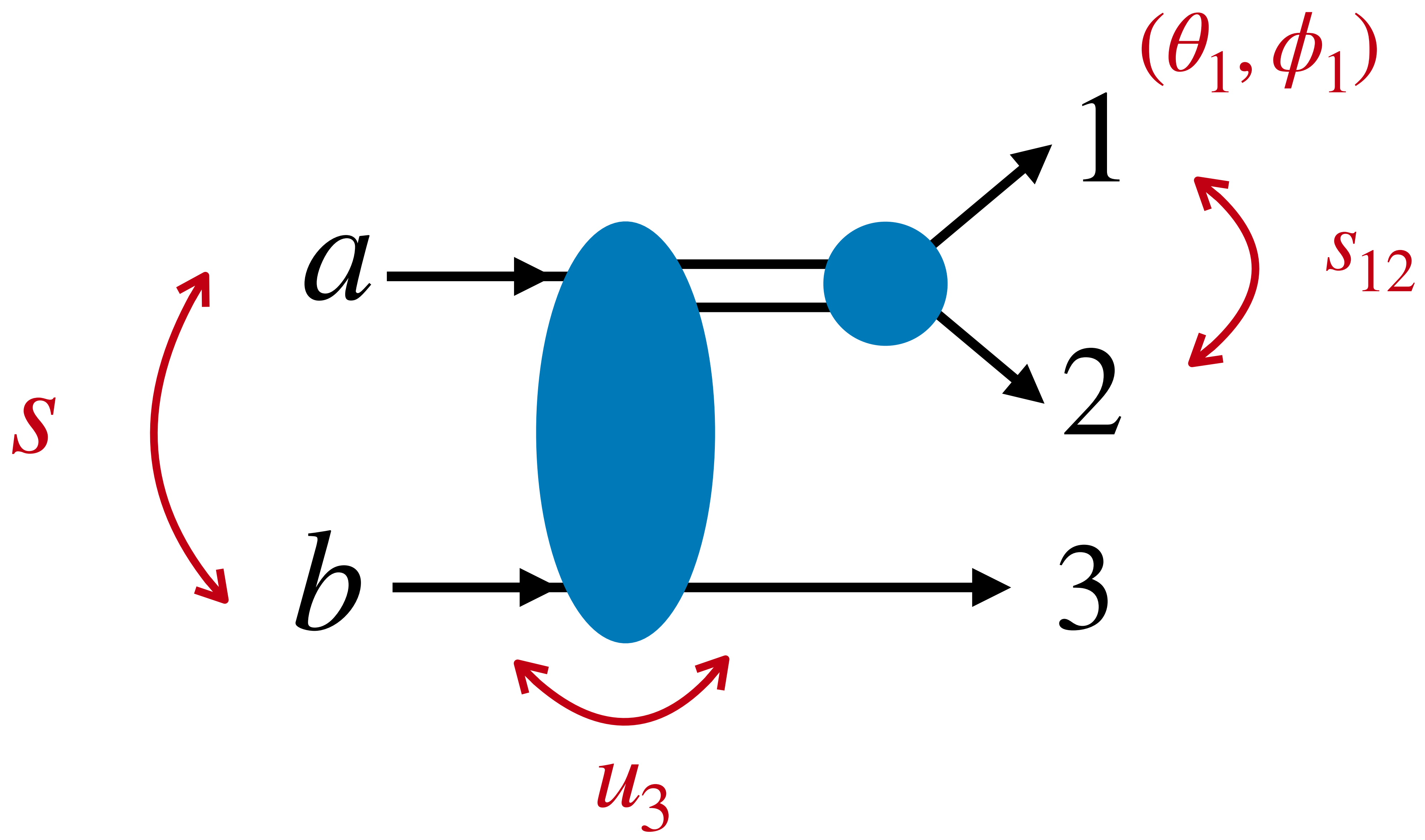
$$t_i = (p_a - p_i)^2$$

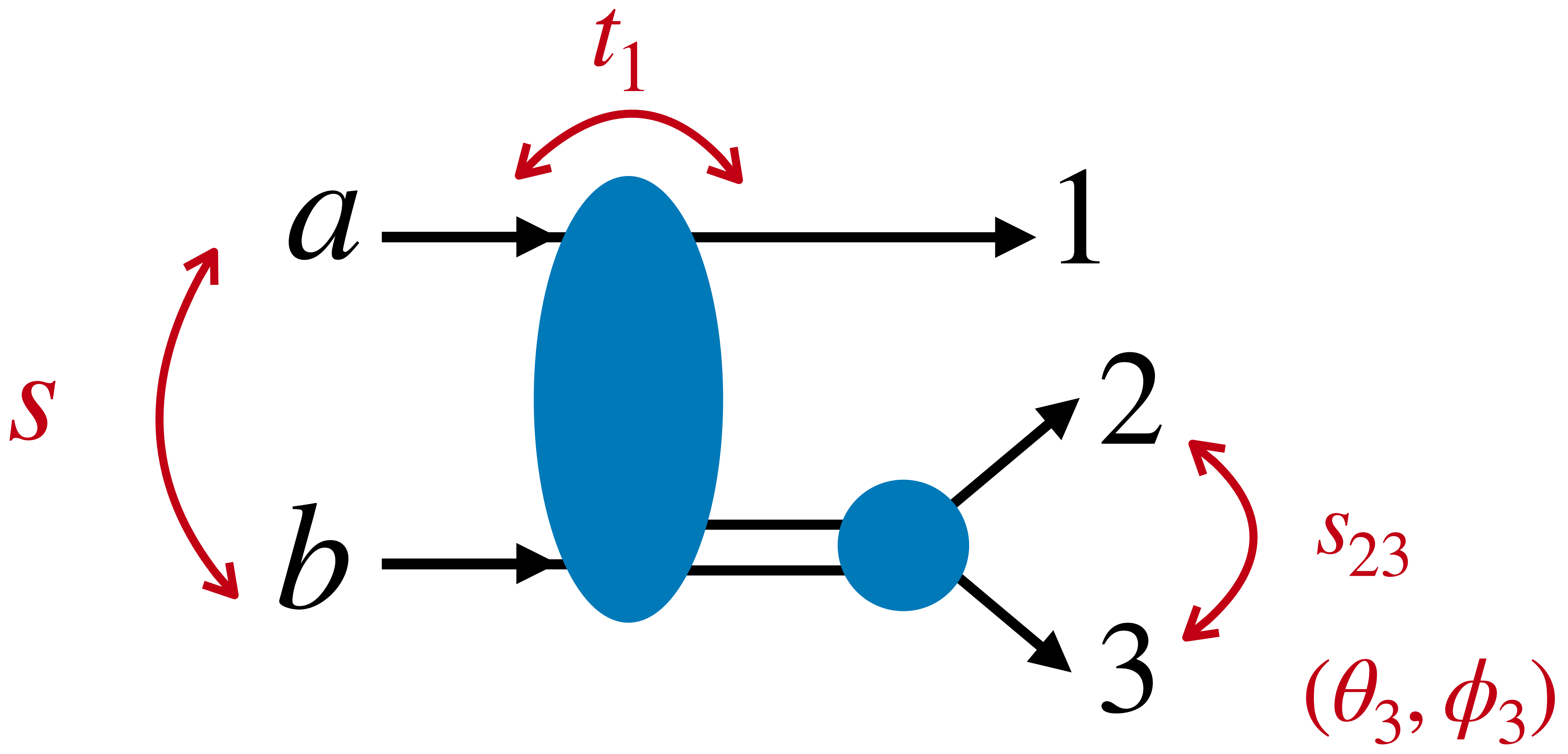
$$u_i = (p_b - p_i)^2$$

Relevant variables

There are 5 independent variables

5 Mandelstam variables: $s, t_1, u_3, s_{12}, s_{23}$





(θ_1, ϕ_1): Angles of 1 in (12) RF

$$s_{12} = (p_1 + p_2)^2 \qquad s = (p_a + p_b)^2$$

$$u_3 = (p_b - p_3)^2 \qquad \boxed{(\theta_1, \phi_1) \leftrightarrow (t_1, s_{23})}$$

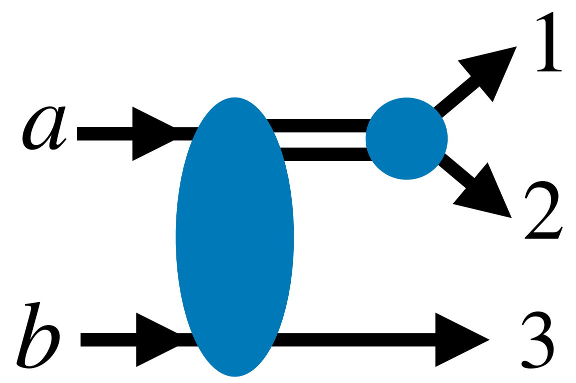
(θ_3, ϕ_3): Angles of 3 in (23) RF

$$s_{23} = (p_2 + p_3)^2 \qquad s = (p_a + p_b)^2$$

$$t_1 = (p_a - p_1)^2 \qquad \boxed{(\theta_3, \phi_3) \leftrightarrow (u_3, s_{12})}$$

How to get angles from Mandelstam variables?

Gottfried-Jackson Frame



(12) Rest Frame (12RF)

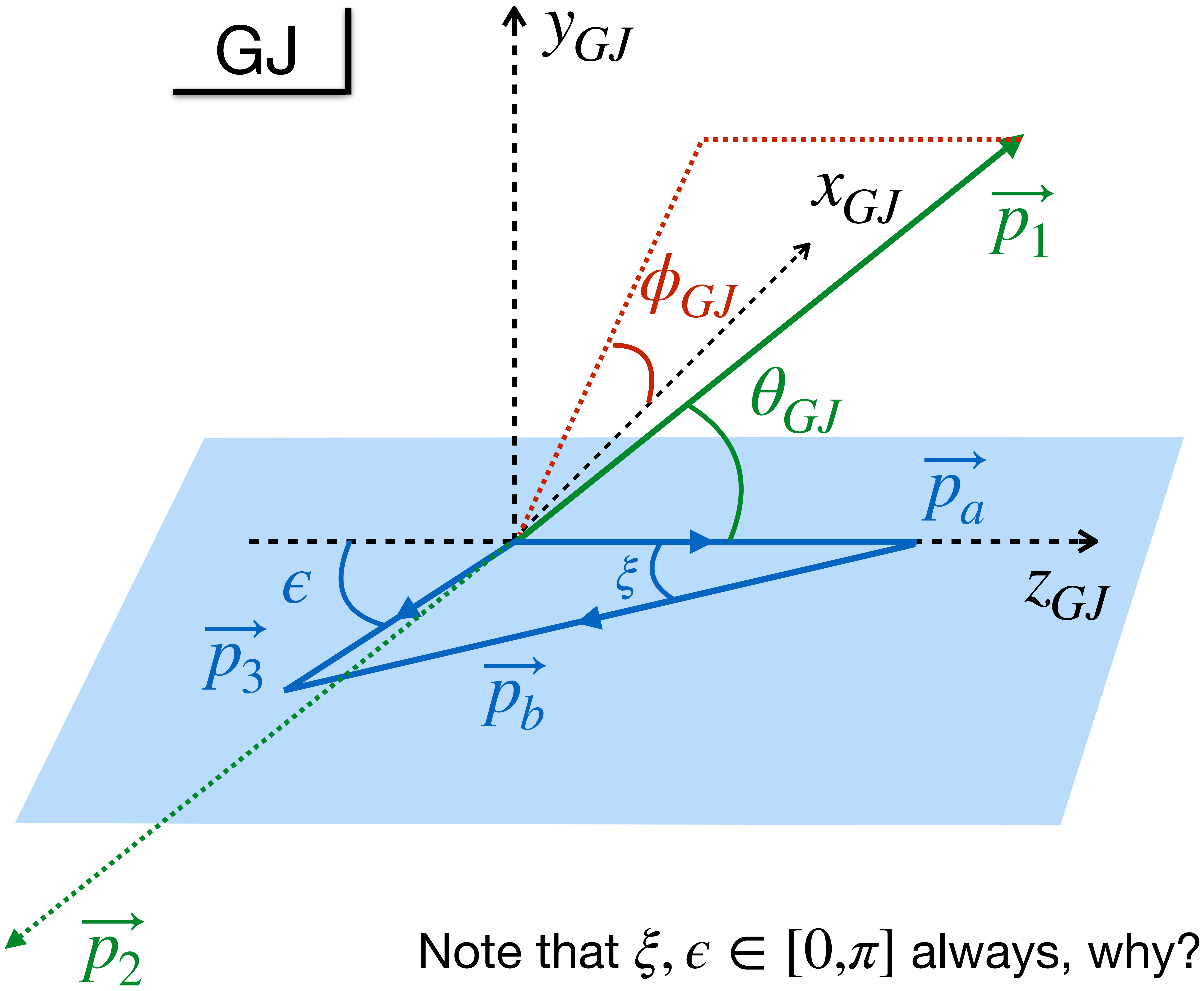
$$\vec{p}_a + \vec{p}_b = \vec{p}_3 + \underbrace{(\vec{p}_1 + \vec{p}_2)}_{\vec{0}}$$

The reaction plane is $x - z$ with

$$\vec{y} \propto \vec{p}_b \times \vec{p}_a \Big|_{12RF}$$

z axis along the beam (a)

$$\vec{z} \propto \vec{p}_a \Big|_{12RF}$$



Note that $\xi, \epsilon \in [0,\pi]$ always, why?

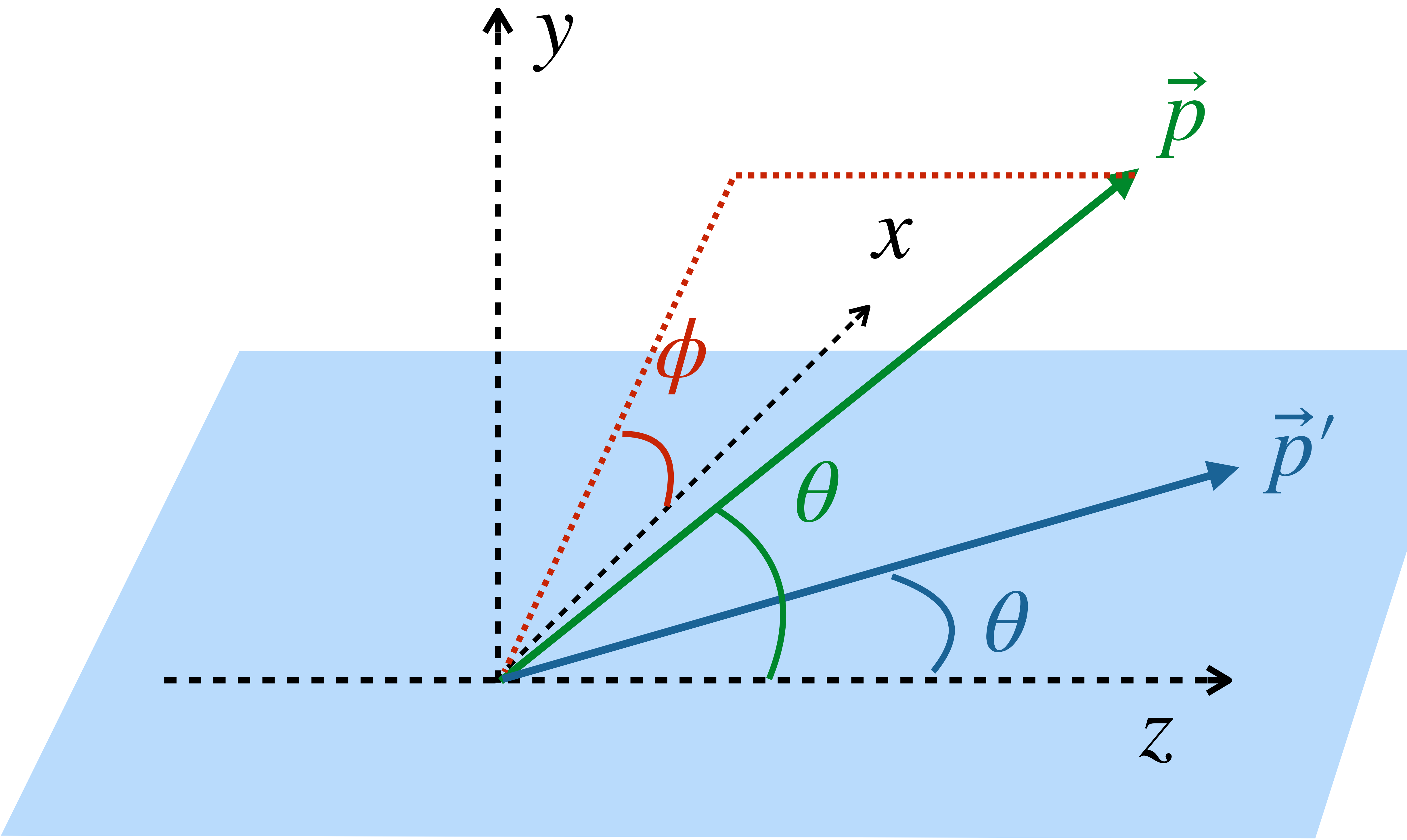
Rotation of momenta

$$\vec{p} = |\vec{p}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{p} = |\vec{p}|R_z(\phi)\cdot R_y(\theta)\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\vec{p} = |\vec{p}|R_z(\phi)\cdot R_y(\theta)\cdot \underset{\substack{\downarrow\\ \text{Irrelevant rotation}}}{R_z(\gamma)}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\vec{p} = |\vec{p}|R(\phi,\theta,\gamma)\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$



Conventions: $\boxed{\gamma = 0}$ or $\gamma = -\phi$

Gottfried-Jackson Frame

Momenta in the (12)-GJ frame

$$\vec{p}_1 = |\vec{p}_1|(\sin\theta_{GJ}\cos\phi_{GJ}, \sin\theta_{GJ}\sin\phi_{GJ}, \cos\theta_{GJ})$$

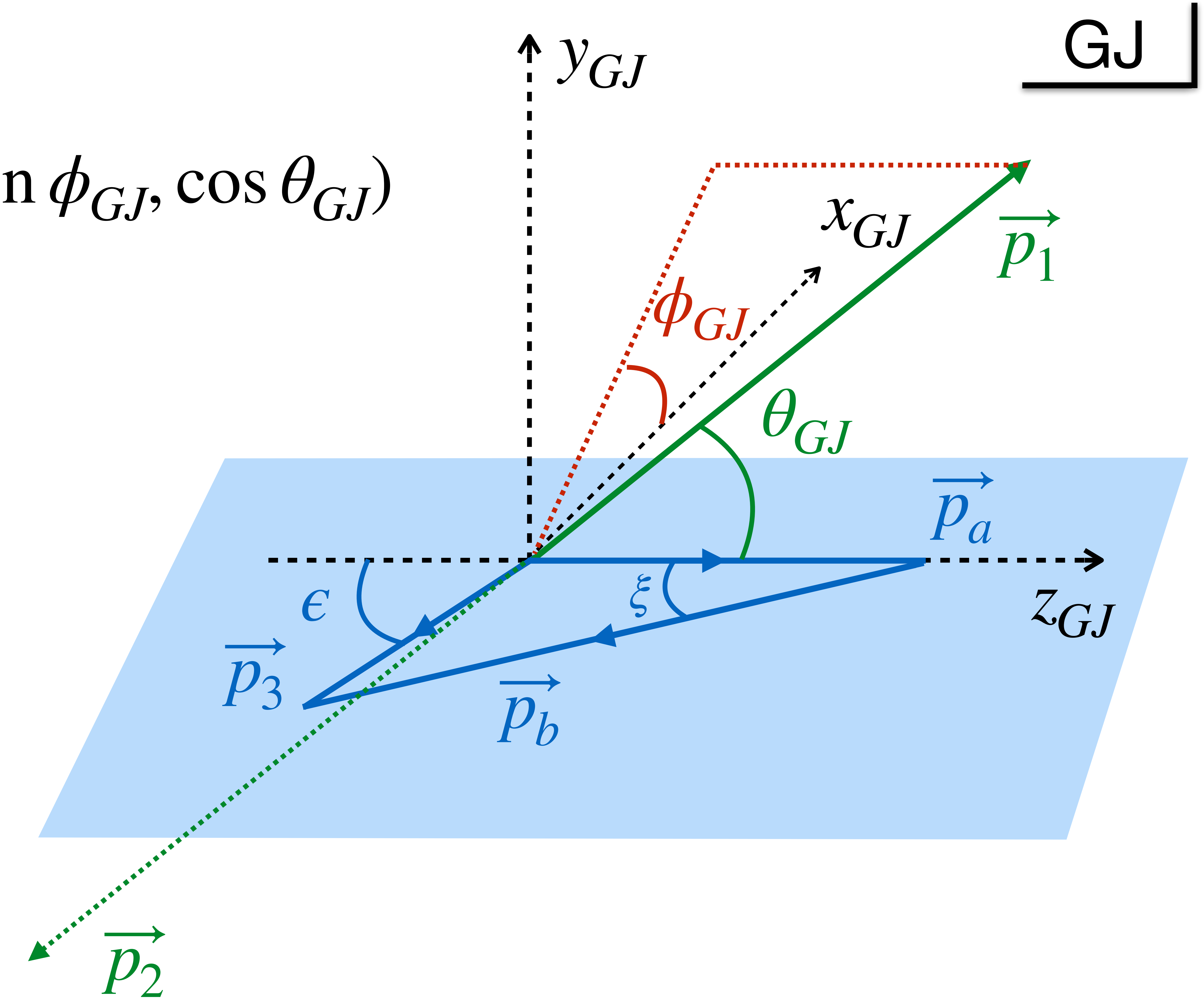
$$\vec{p}_2 = -\vec{p}_1$$

$$\vec{p}_a = |\vec{p}_a|(0,0,1)$$

$$\vec{p}_b = |\vec{p}_b|(-\sin\xi, 0, -\cos\xi)$$

$$\vec{p}_3 = |\vec{p}_3|(-\sin\epsilon, 0, -\cos\epsilon)$$

Momentum $|\vec{p}_i| = \sqrt{E_i^2 - m_i^2}$



Gottfried-Jackson Frame

Calculating energies using $\vec{p}_1 + \vec{p}_2 = \vec{0}$

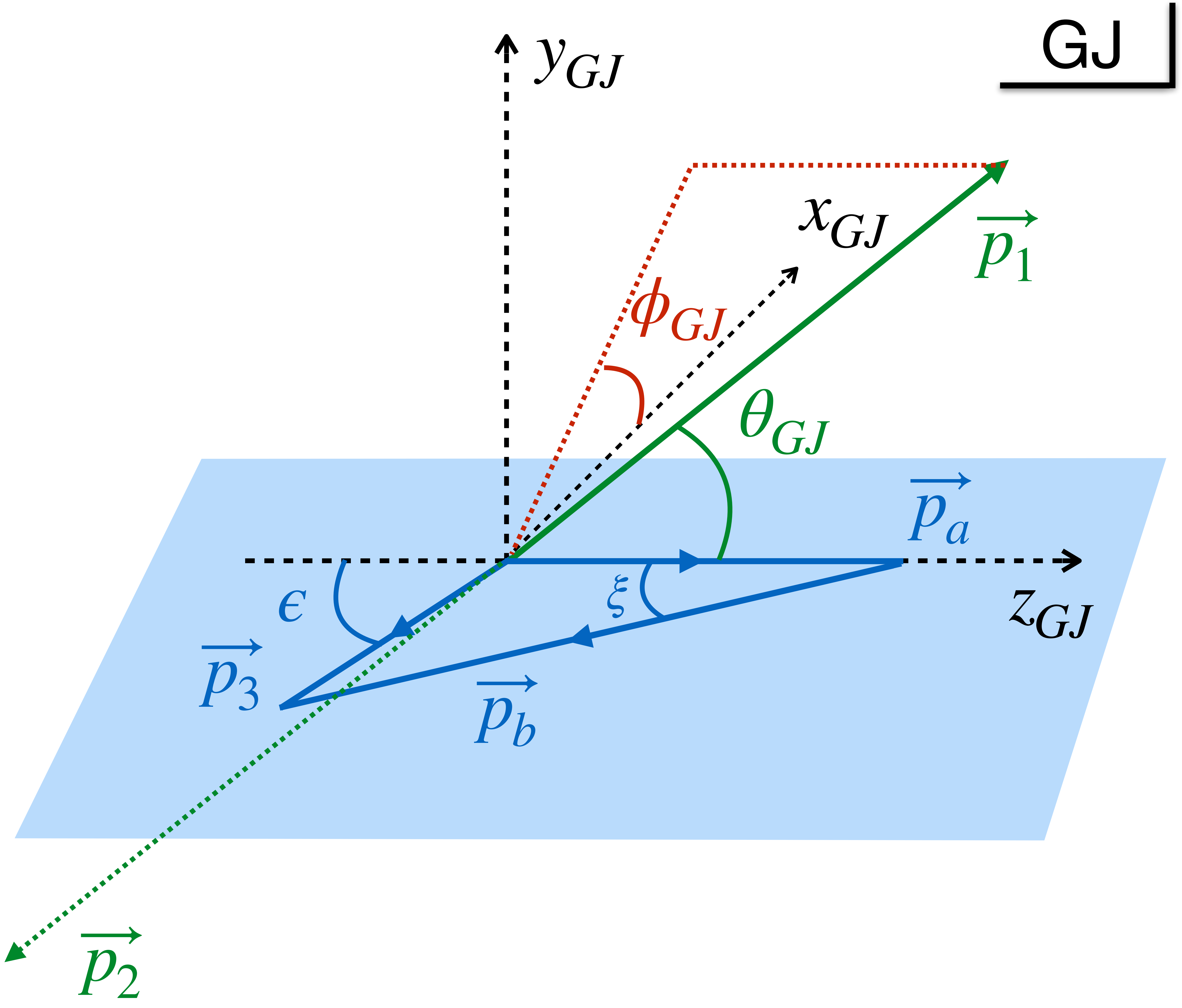
$$u_3 = (p_b - p_3)^2 = ([p_1 + p_2] - p_a)^2$$

$$= s_{12} + m_a^2 - 2E_a\sqrt{s_{12}}$$

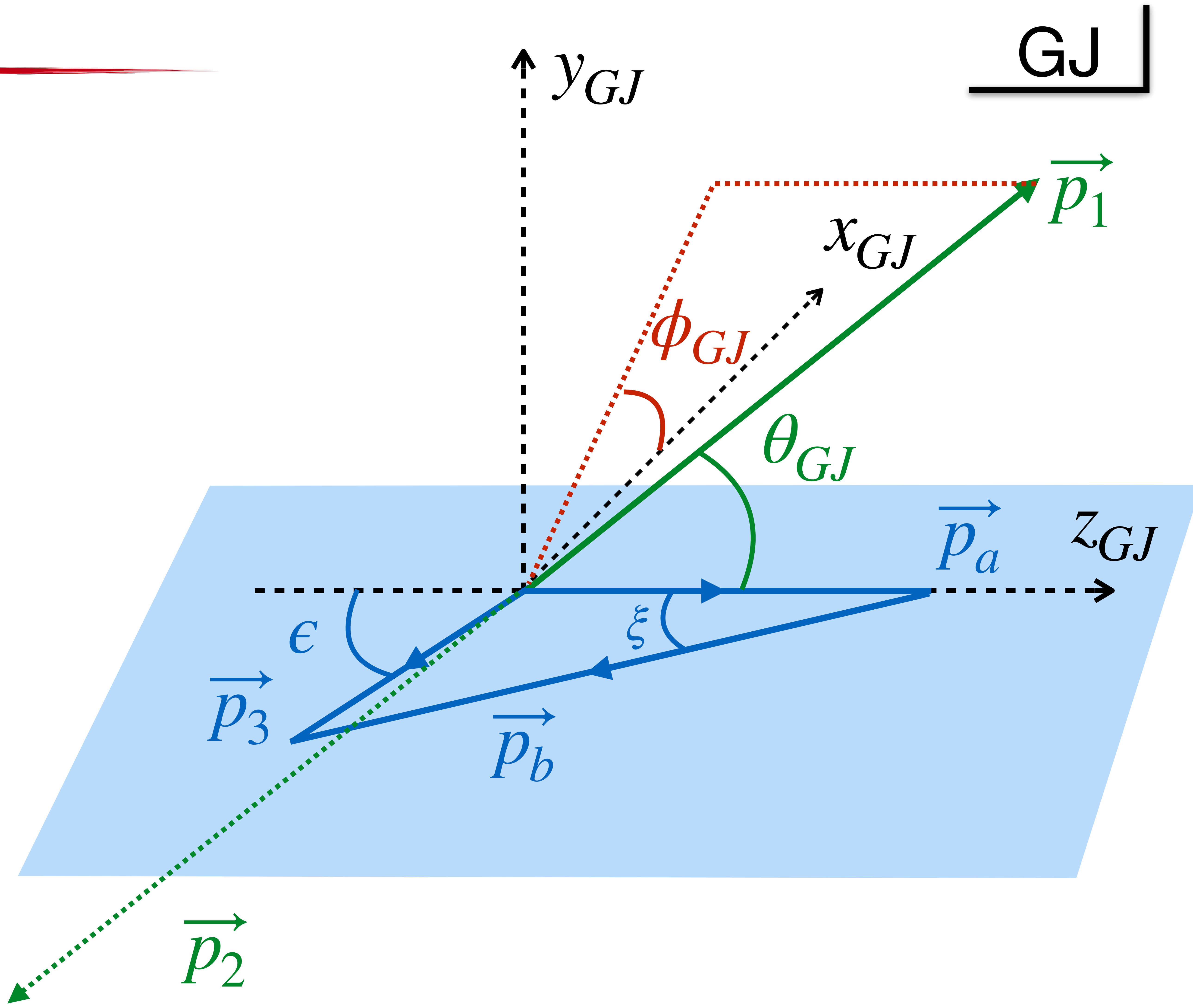
$$\boxed{E_a = \frac{s_{12} + m_a^2 - u_3}{2\sqrt{s_1 2}}}$$

All 5 energies depend only on s, s_{12}, u_3

Exercice: Compute the other energies E_b, E_1, E_2, E_3



Gottfried-Jackson Frame



Polar angle from $\vec{p}_a \cdot \vec{p}_1$

$$t_1 = (p_a - p_1)^2$$

$$= m_a^2 + m_1^2 - 2E_aE_1 + 2|\vec{p}_1||\vec{p}_a|\cos\theta_{GJ}$$

Azimuthal angle from $\vec{p}_2 \cdot \vec{p}_3$

$$s_{23} = (p_2 + p_3)^2$$

$$= m_2^2 + m_3^2 + 2E_2E_3$$

$$-2|\vec{p}_1||\vec{p}_3|\left[\sin\epsilon\sin\theta_{GJ}\cos\phi_{GJ} + \cos\epsilon\cos\theta_{GJ}\right]$$

Exercice: Compute the other angles $\cos\xi$ and $\cos\epsilon$

Helicity Frame

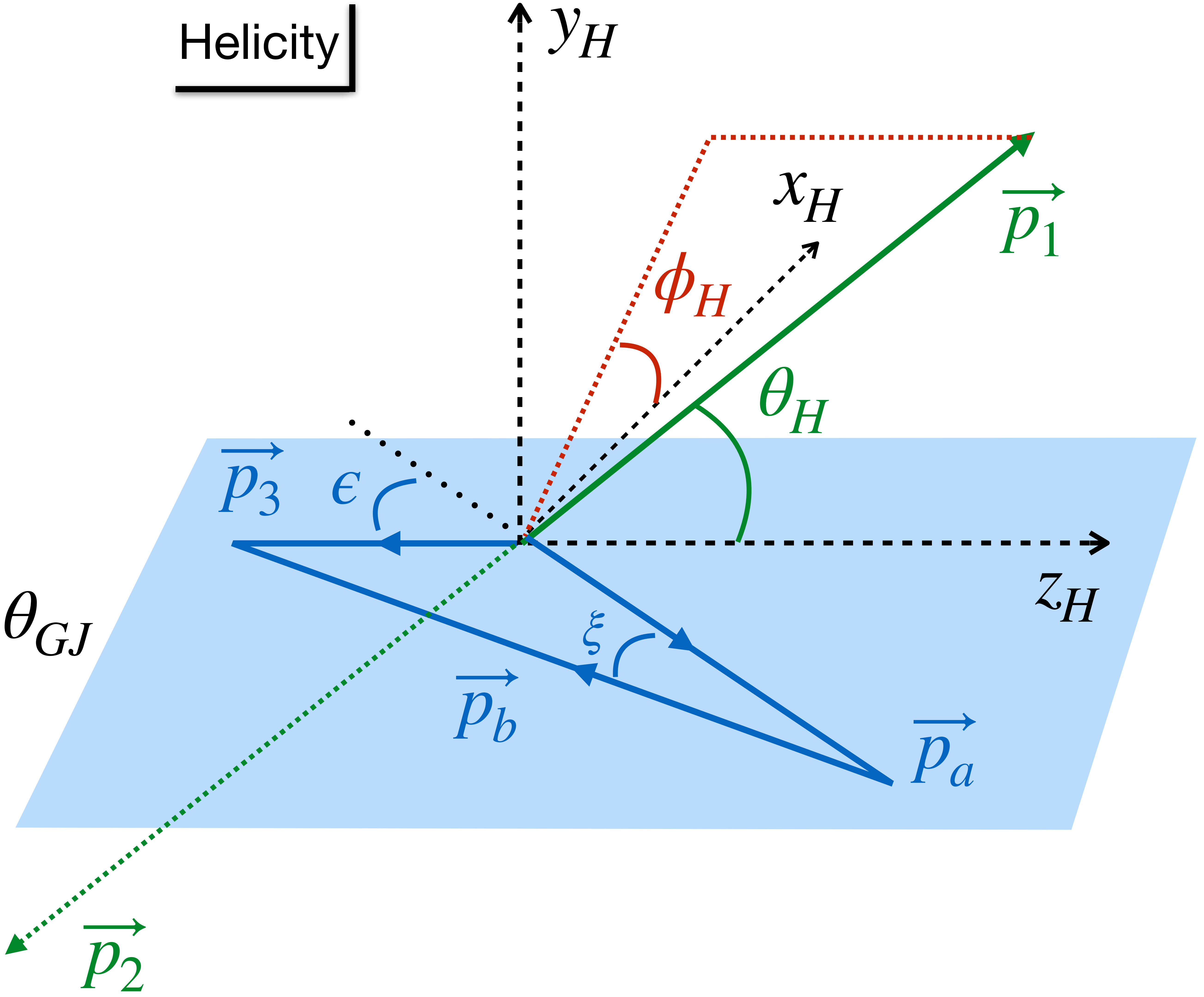
z is opposite to the recoil, $\vec{z} \propto -\vec{p}_3$

Gottfried-Jackson and helicity frames are related by a rotation of angle ω around y

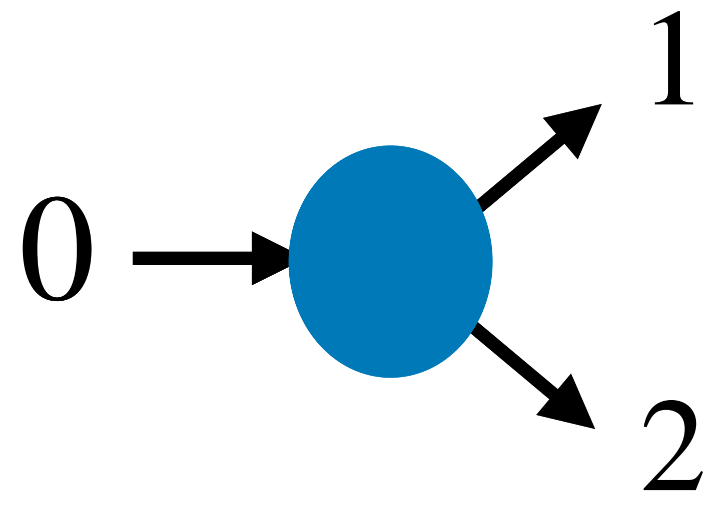
$$\vec{p}\,|_{GJ} = R_y(\epsilon)\ \vec{p}\,|_H$$

$$\cos\theta_H = \cos\epsilon\cos\theta_{GJ} + \sin\epsilon\cos\phi_{GJ}\sin\theta_{GJ}$$

$$\cot\phi_H = \cos\epsilon\cot\phi_{GJ} - \sin\epsilon\frac{\cot\theta_{GJ}}{\sin\phi_{GJ}}$$



Decays into Two Particles



Momentum fixed

$$|\vec{p}_1| = |\vec{p}_2| = \frac{1}{2M}\lambda^{1/2}(M^2, m_1^2, m_2^2)$$

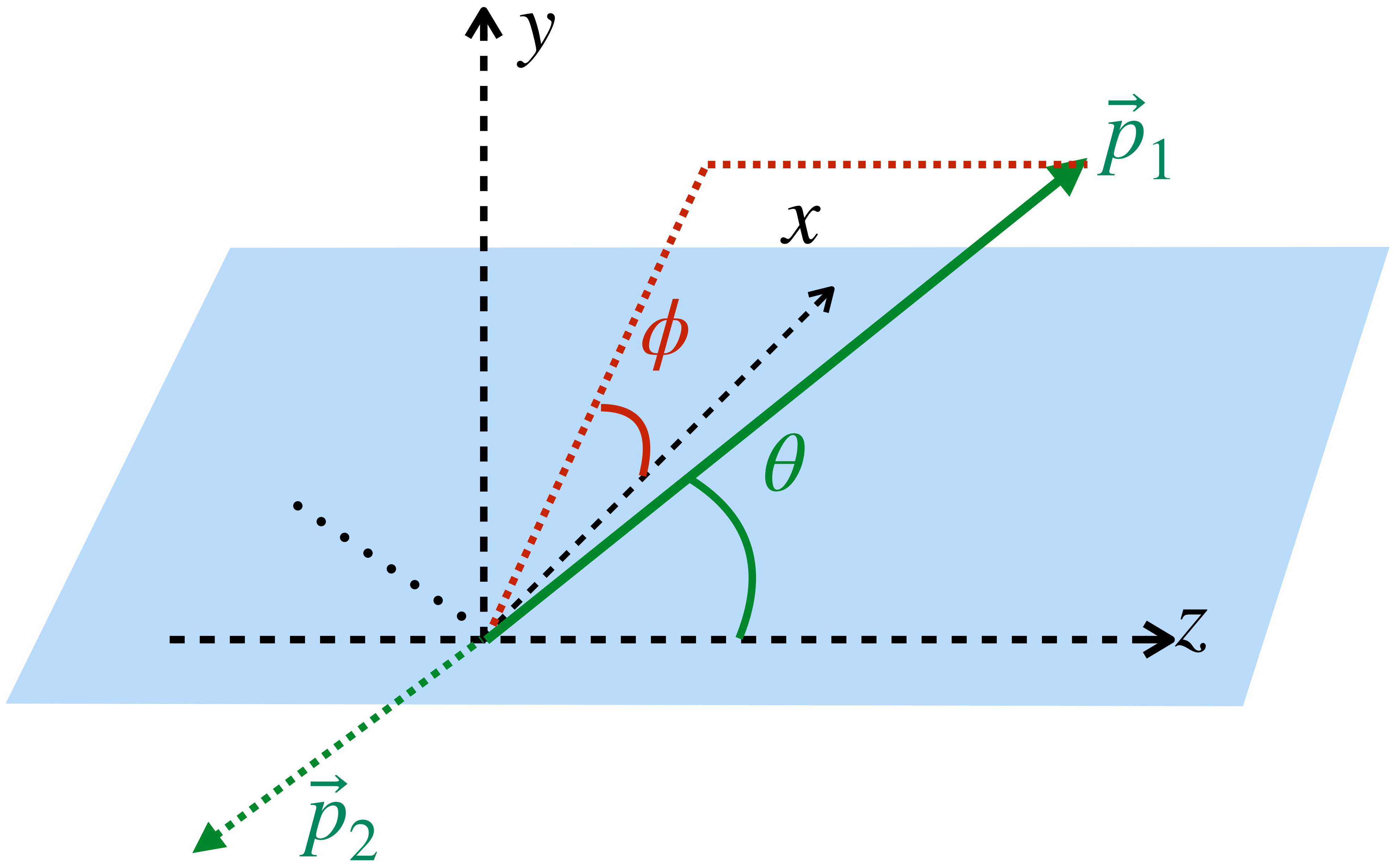
Particle 0 has mass M

Angular dependence fixed by the spin of 0

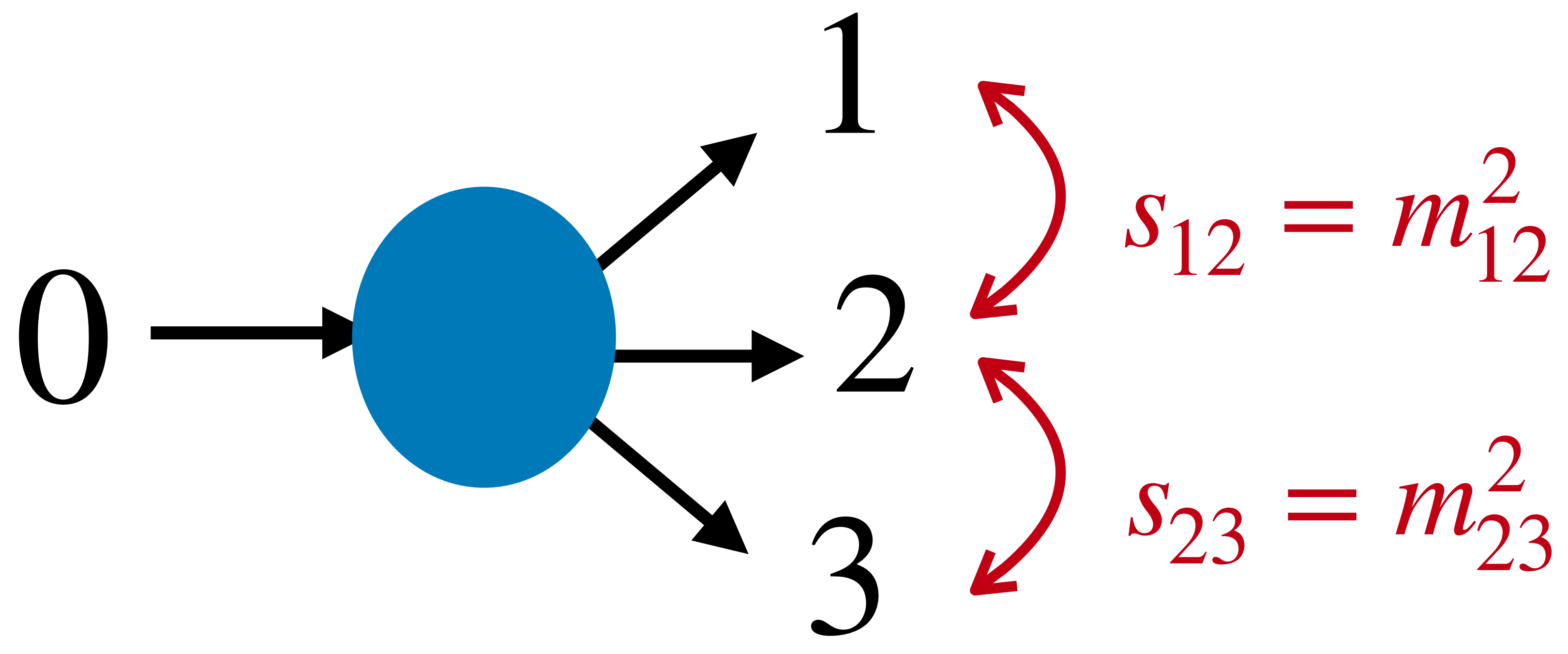
Only z axis matters if not polarized

Intensity depends on θ only

Example $$\frac{d\Gamma}{d\Omega} \propto |Y_m^\ell(\theta,\phi)|^2 \propto |P_\ell^{(m)}(\cos\theta)|^2$$



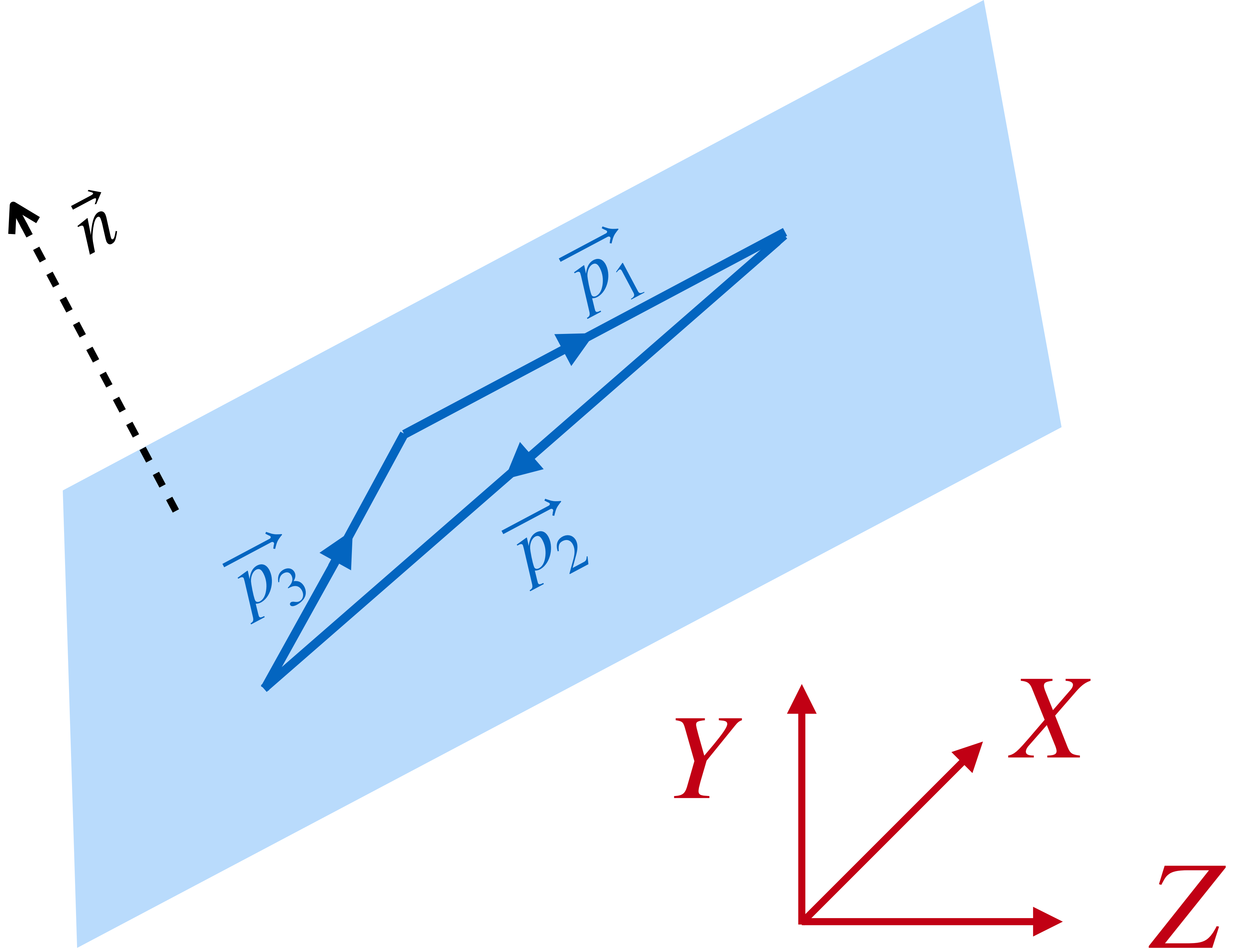
Decays into Three Particles



The three momenta determine a plane

The orientation of the plane is determined by three Euler angles α, β, γ

The orientation of the plane does not matters if not polarized



The decay is described by two variables s_{12}, s_{23}

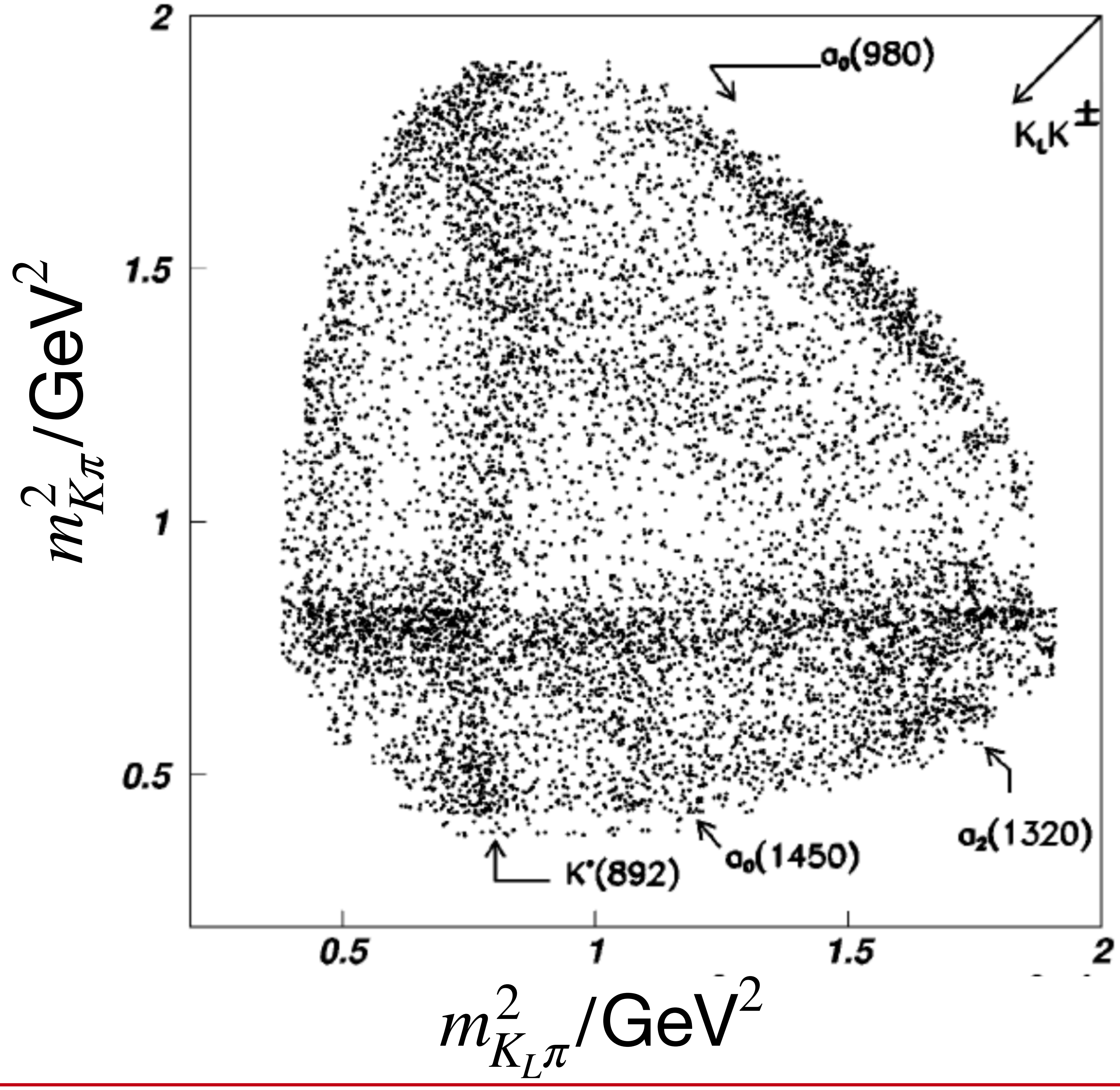
Representation in a Dalitz plot

$$\frac{d\Gamma}{ds_{12}ds_{23}} \propto |A|^2$$

Dalitz Plot

$$p\bar{p} \to K_L^0 K^\pm \pi^\mp$$

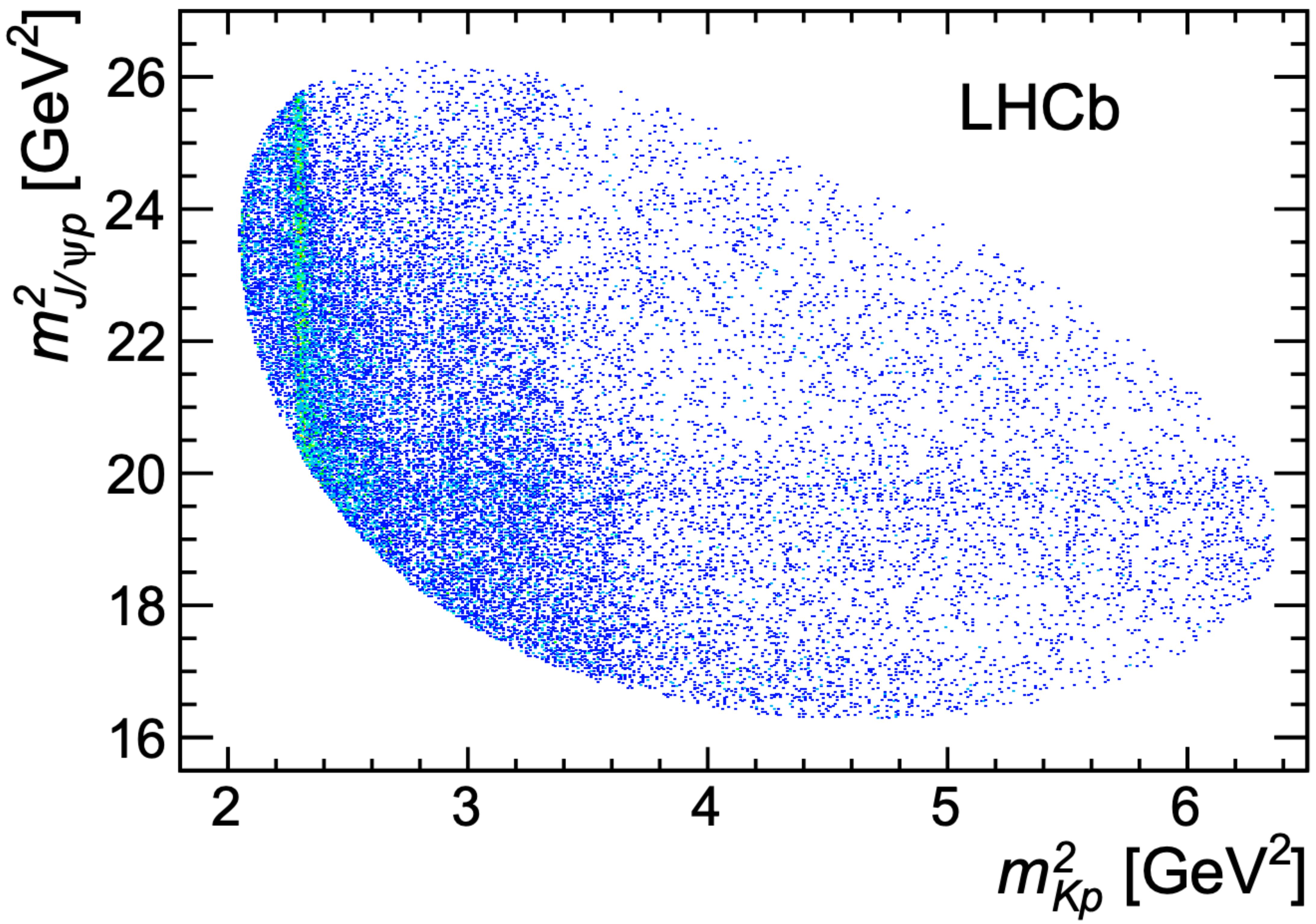
Crystal Barrel, Phys. Rev. D57 (1998) 3860



Every event is a point (scatter plot)

$$\Lambda_b^0 \to J/\psi\, K^- p$$

LHCb, Phys. Rev. Lett. 115 (2015) 072001

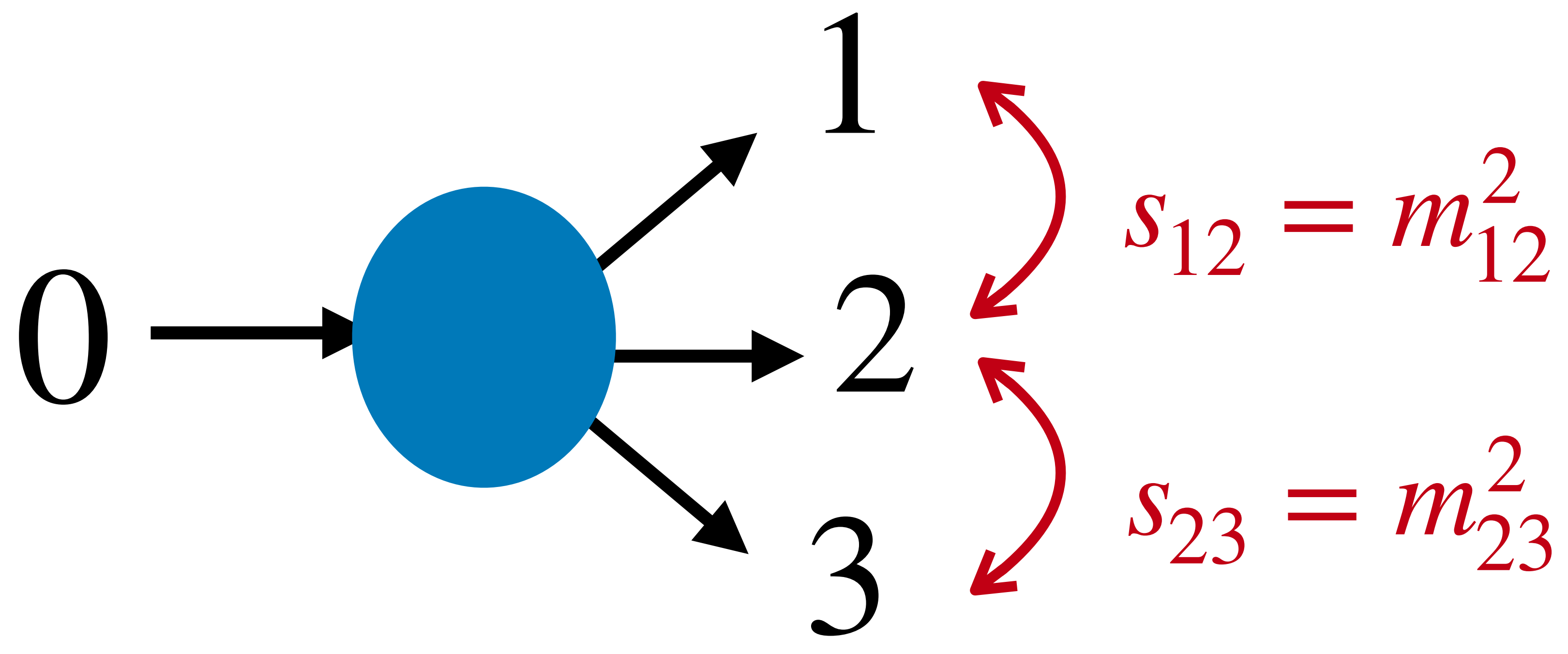


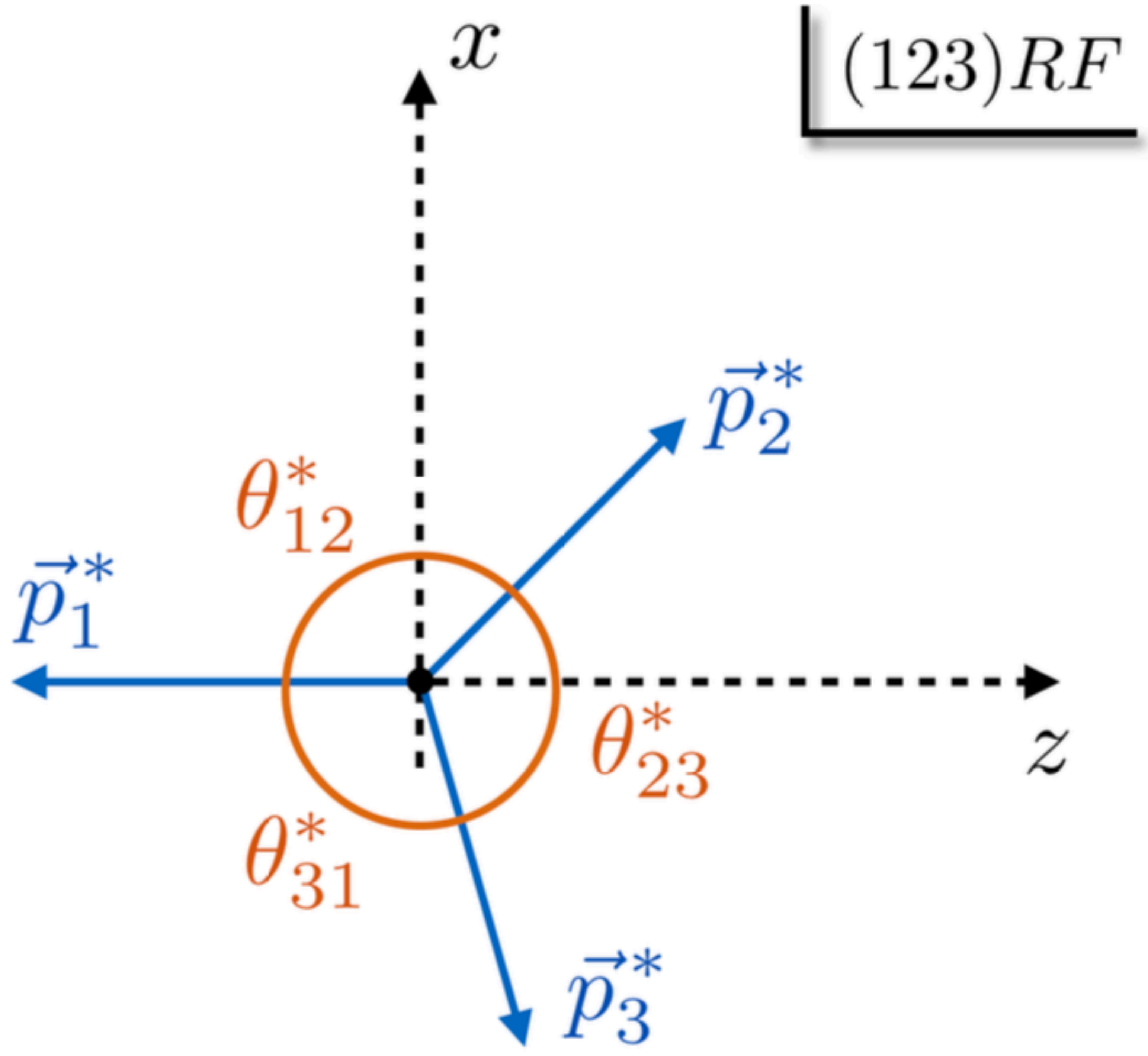
Dalitz Plot

Each CoM breakup momentum depends on only one invariant

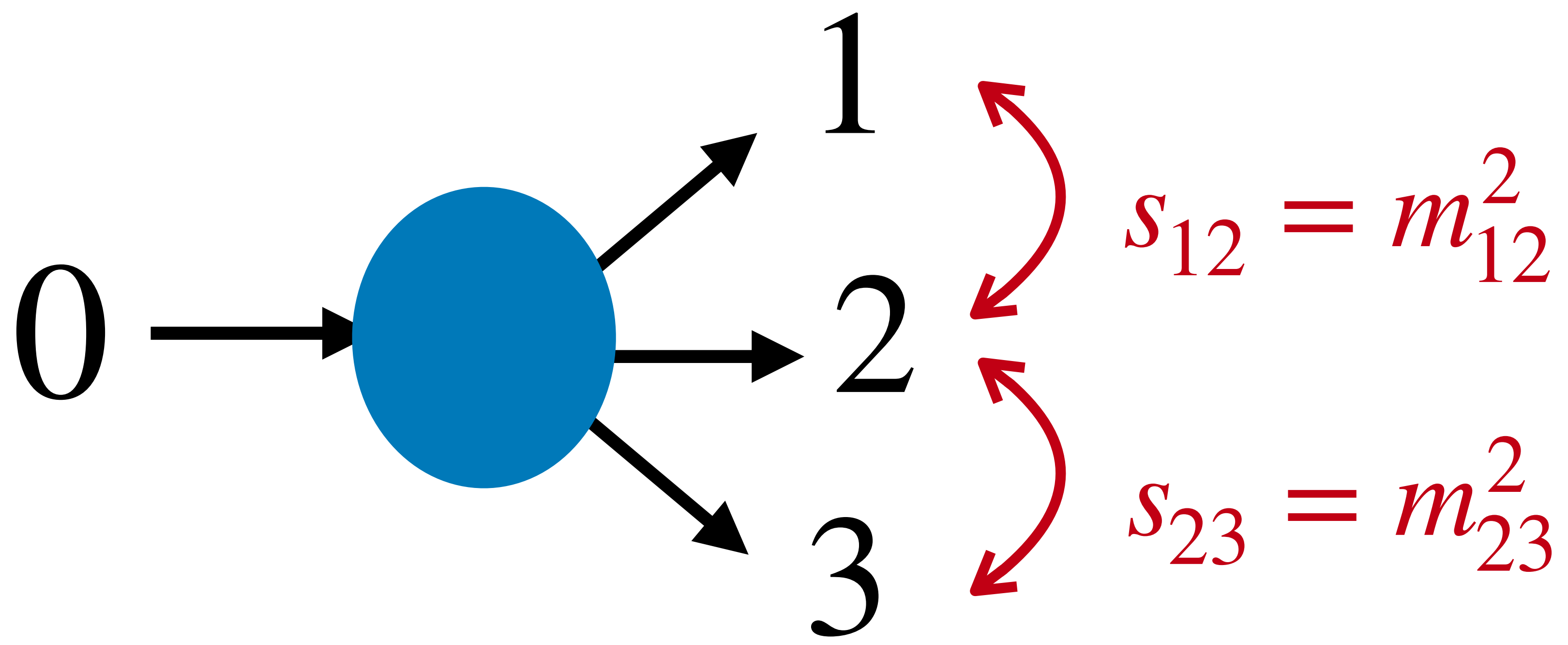
$$s_{12} = \underbrace{(p_1 + p_2 + p_3}_{(M,\vec{0})} - p_3)^2 \quad = M^2 + m_3^2 - 2ME_3^*$$

$$\boxed{\begin{aligned} E_i^* &= \frac{M^2 + m_i^2 - s_{jk}}{2M} \\ |\vec{p}_i^{\,*}| &= \frac{\lambda^{1/2}(M^2, m_i^2, s_{jk})}{2M} \end{aligned}}$$





Decays into Three Particles



Angle between particles i and j from s_{ij}

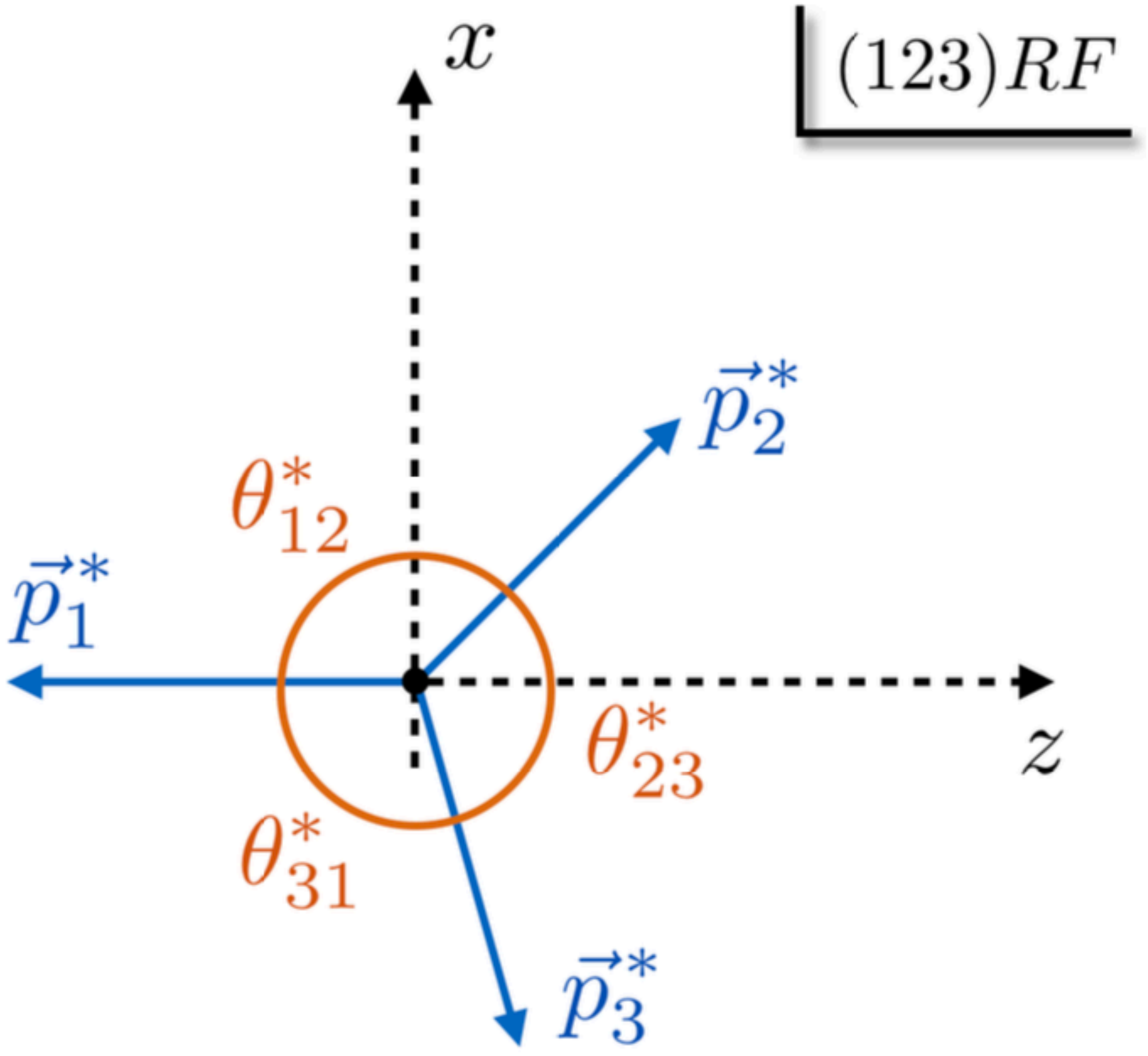
$$\cos\theta_{ij}^* = \frac{(M^2 + m_i^2 - s_{jk})(M^2 + m_j^2 - s_{ki}) + 2M^2(m_i^2 + m_j^2 - s_{ij})}{\lambda^{1/2}(M^2, m_i^2, s_{jk})\ \lambda^{1/2}(M^2, m_j^2, s_{ki})}$$

Exercice: check $\cos\theta_{12}^*$

There are 2 independent angles, corresponding to 2 independent invariants

$$\theta_{12}^* + \theta_{23}^* + \theta_{31}^* = 2\pi$$

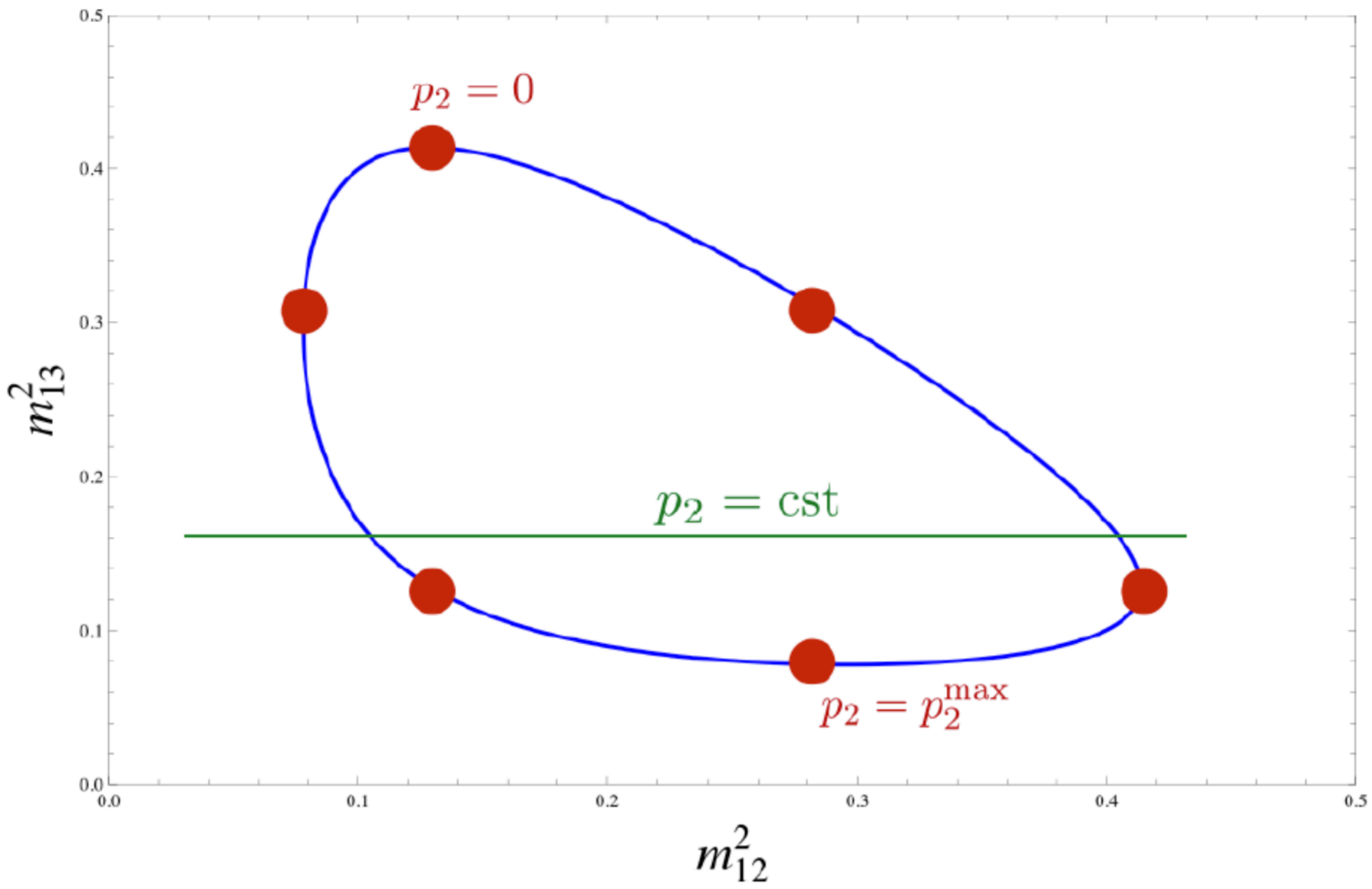
Exercice: check that relation using $s_{12} + s_{23} + s_{31} = M^2 + m_1^2 + m_2^2 + m_3^2$



Dalitz Plot: kinematics

Each CoM breakup momentum depends on only one invariant

$$|\vec{p}_2^*| = \frac{\lambda^{1/2}(M^2, m_2^2, s_{31})}{2M}$$



The Mandelstam invariants obey $s_{12} + s_{23} + s_{31} = M^2 + m_1^2 + m_2^2 + m_3^2$

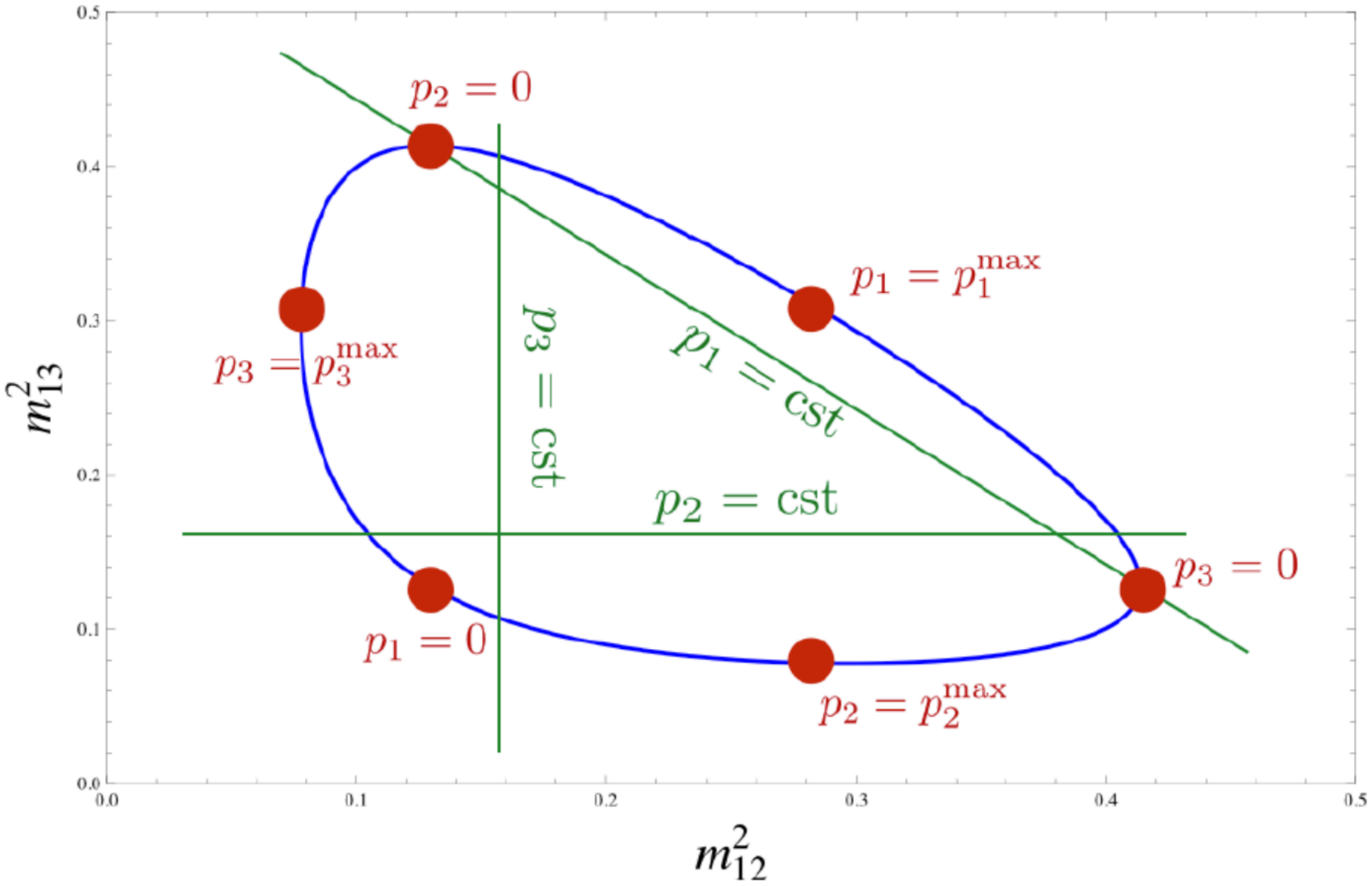
Dalitz Plot: kinematics

Each CoM breakup momentum depends on only one invariant

$$|\vec{p}_1^*| = \frac{\lambda^{1/2}(M^2, m_1^2, s_{12})}{2M}$$

$$|\vec{p}_2^*| = \frac{\lambda^{1/2}(M^2, m_2^2, s_{31})}{2M}$$

$$|\vec{p}_3^*| = \frac{\lambda^{1/2}(M^2, m_3^2, s_{12})}{2M}$$



The Mandelstam invariants obey $$s_{12} + s_{23} + s_{31} = M^2 + m_1^2 + m_2^2 + m_3^2$$

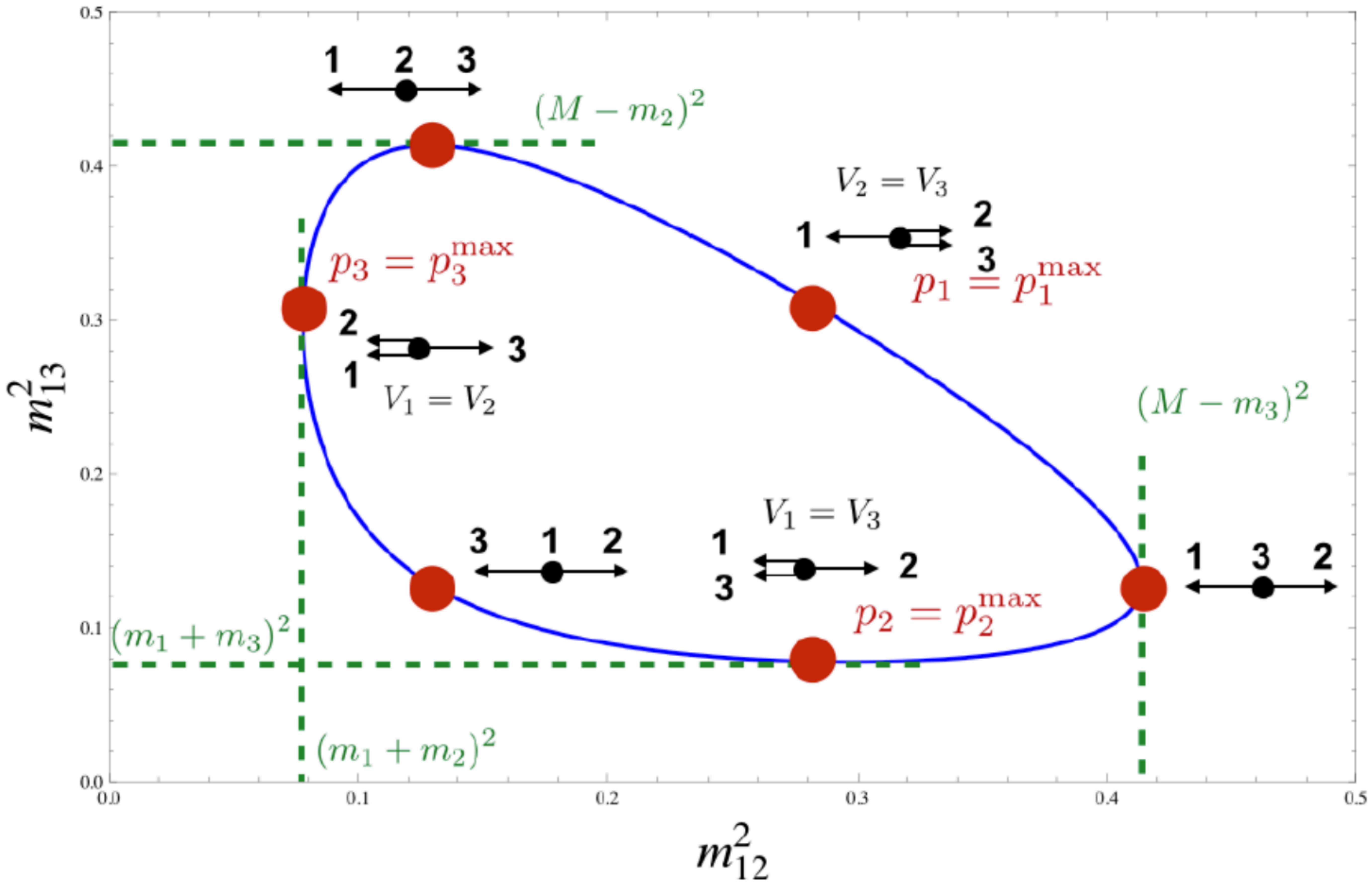
Dalitz Plot: kinematics

Each CoM breakup momentum depends on only one invariant

$$|\vec{p}_1^*| = \frac{\lambda^{1/2}(M^2, m_1^2, s_{12})}{2M}$$

$$|\vec{p}_2^*| = \frac{\lambda^{1/2}(M^2, m_2^2, s_{31})}{2M}$$

$$|\vec{p}_3^*| = \frac{\lambda^{1/2}(M^2, m_3^2, s_{12})}{2M}$$

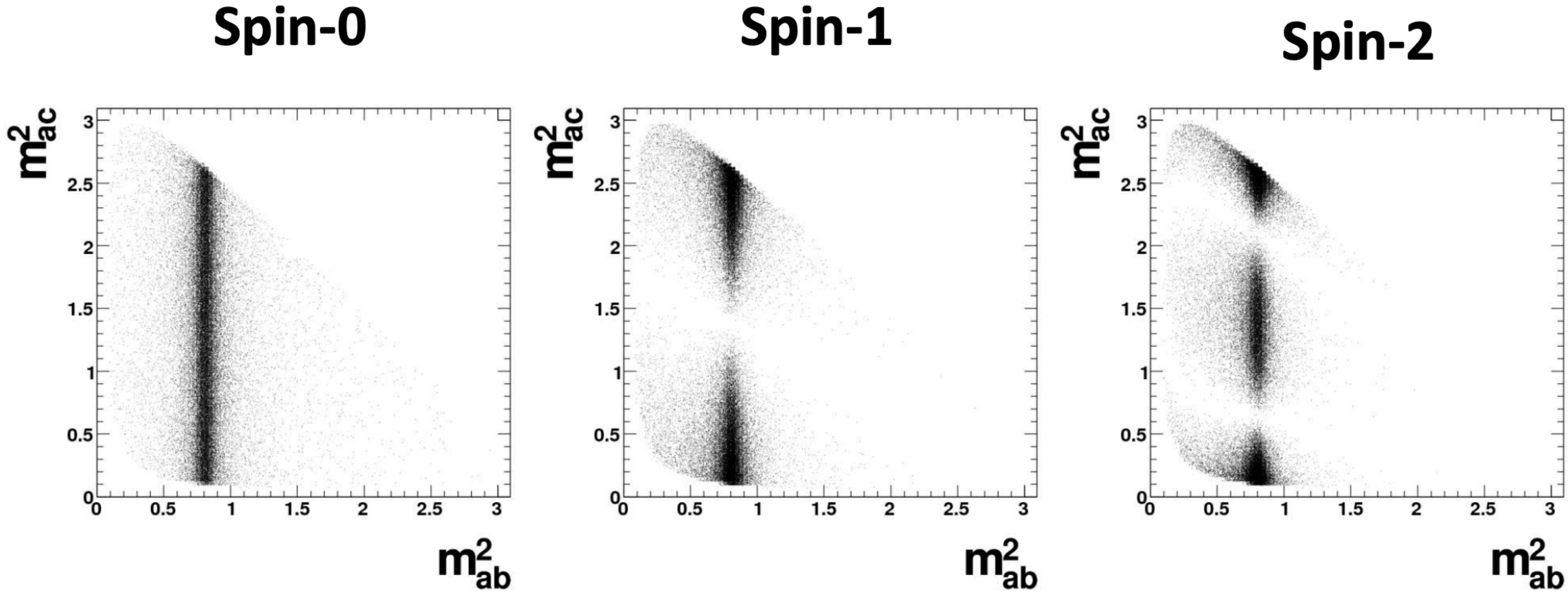


The Mandelstam invariants obey $s_{12} + s_{23} + s_{31} = M^2 + m_1^2 + m_2^2 + m_3^2$

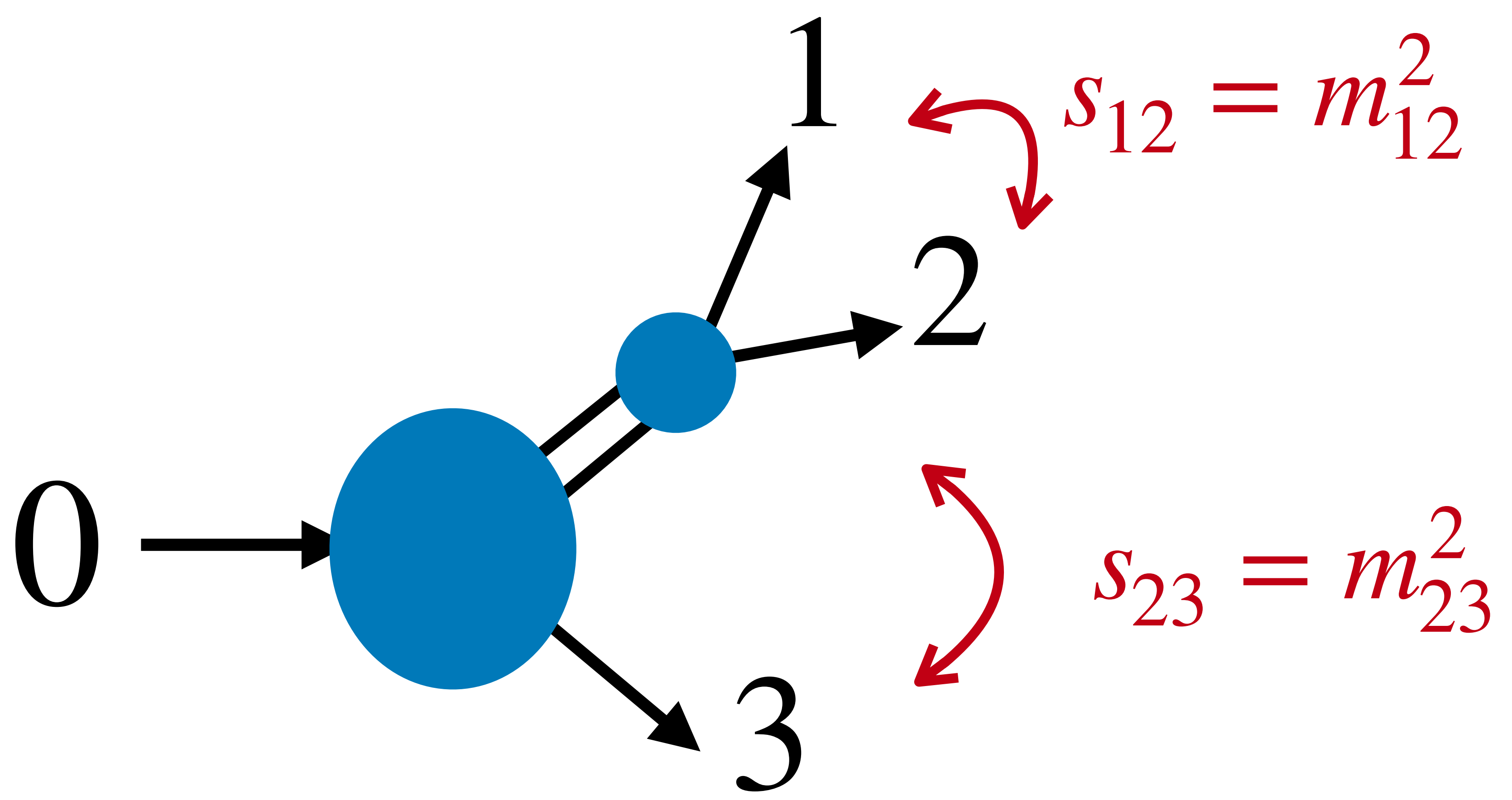
Dalitz Plot

Spin provides angular modulations

$$\frac{d\Gamma}{d\Omega} \propto |\, Y^{\ell}_{m}(\theta,\phi)\,|^2 \propto |\, P^{(m)}_{\ell}(\cos\theta)\,|^2$$



Decays into Three Particles



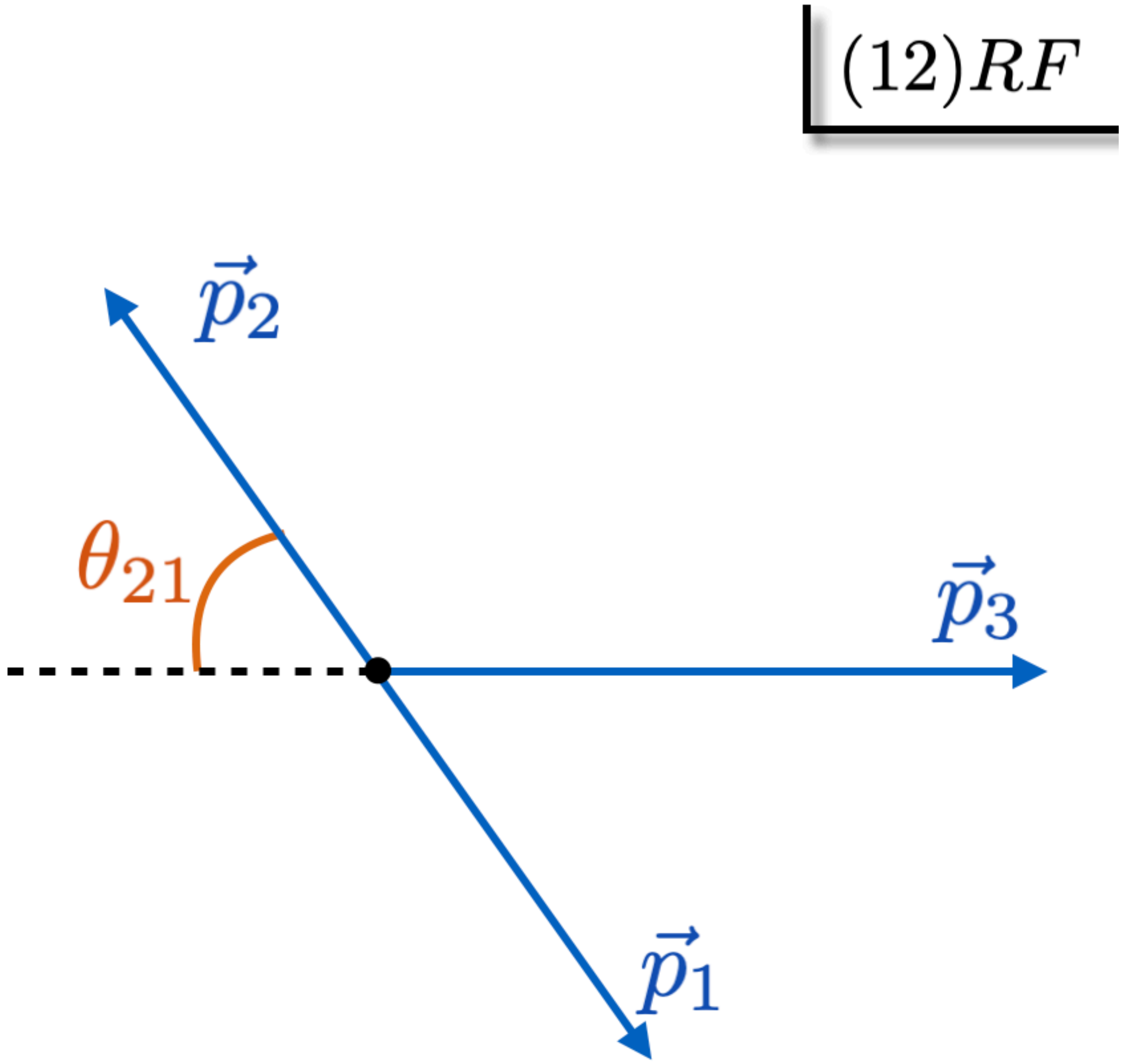
Kinematics in the (12) rest frame

same as 2-to-2 kinematics with $p_{\bar{3}} = -p_3 = (-E_3, -\vec{p}_3)$

Crossing: $0 \to 1 + 2 + 3 \quad \Longleftrightarrow \quad 0 + \bar{3} \to 1 + 2$

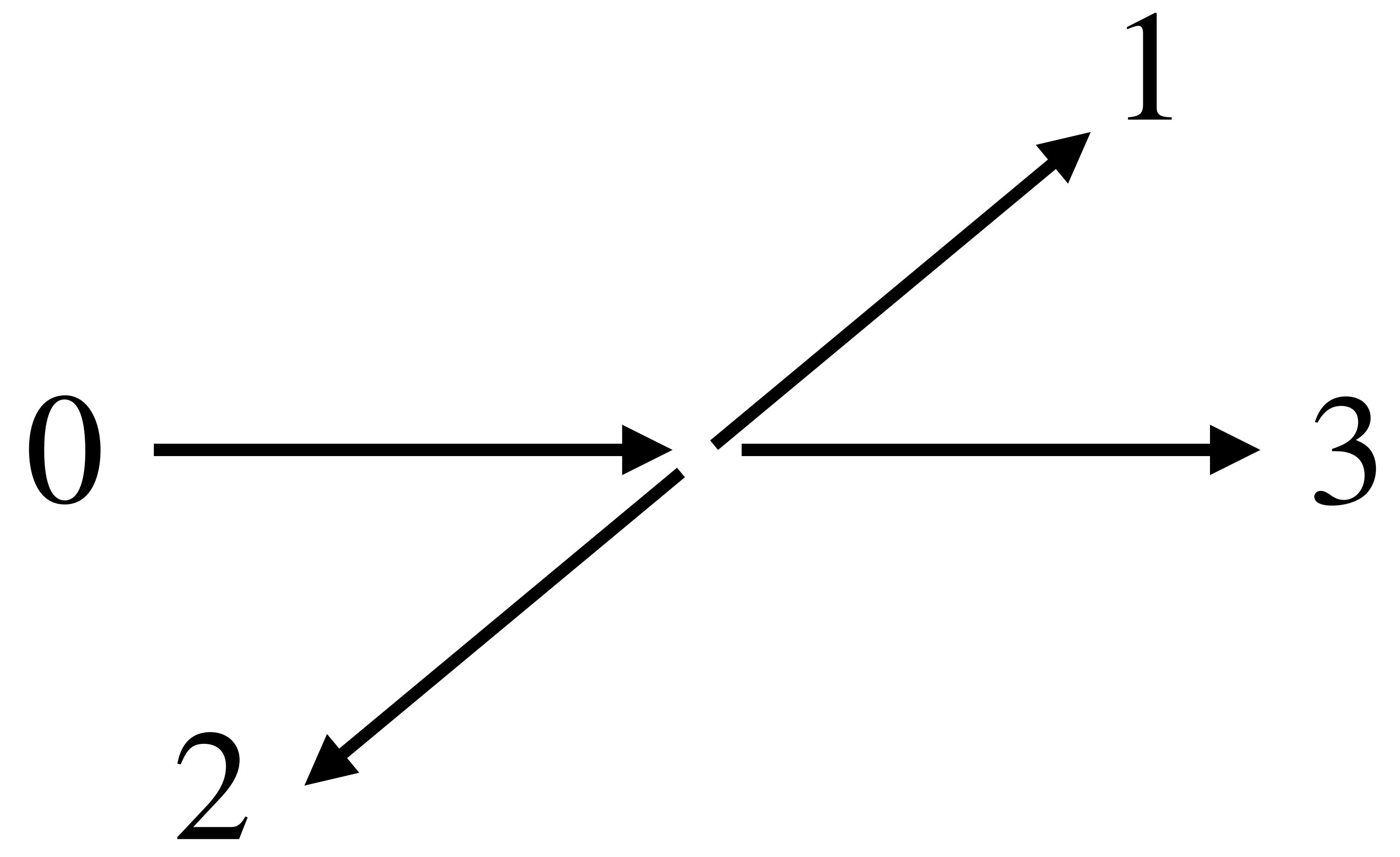
$$E_1 = \frac{s_{12} + m_1^2 - m_2^2}{2\sqrt{s_{12}}} \qquad E_3 = -\frac{s_{12} + m_3^2 - M^2}{2\sqrt{s_{12}}}$$

$$E_2 = \frac{s_{12} - m_1^2 + m_2^2}{2\sqrt{s_{12}}} \qquad = \frac{M^2 - s_{12} - m_3^2}{2\sqrt{s_{12}}}$$



In the (ij) rest frame, the angle θ_{ij} is between $\vec{p}_i$ and $-\vec{p}_k$

Dalitz Plot Boundaries



Boundaries of $s_{23} = (p_2 + p_3)^2$

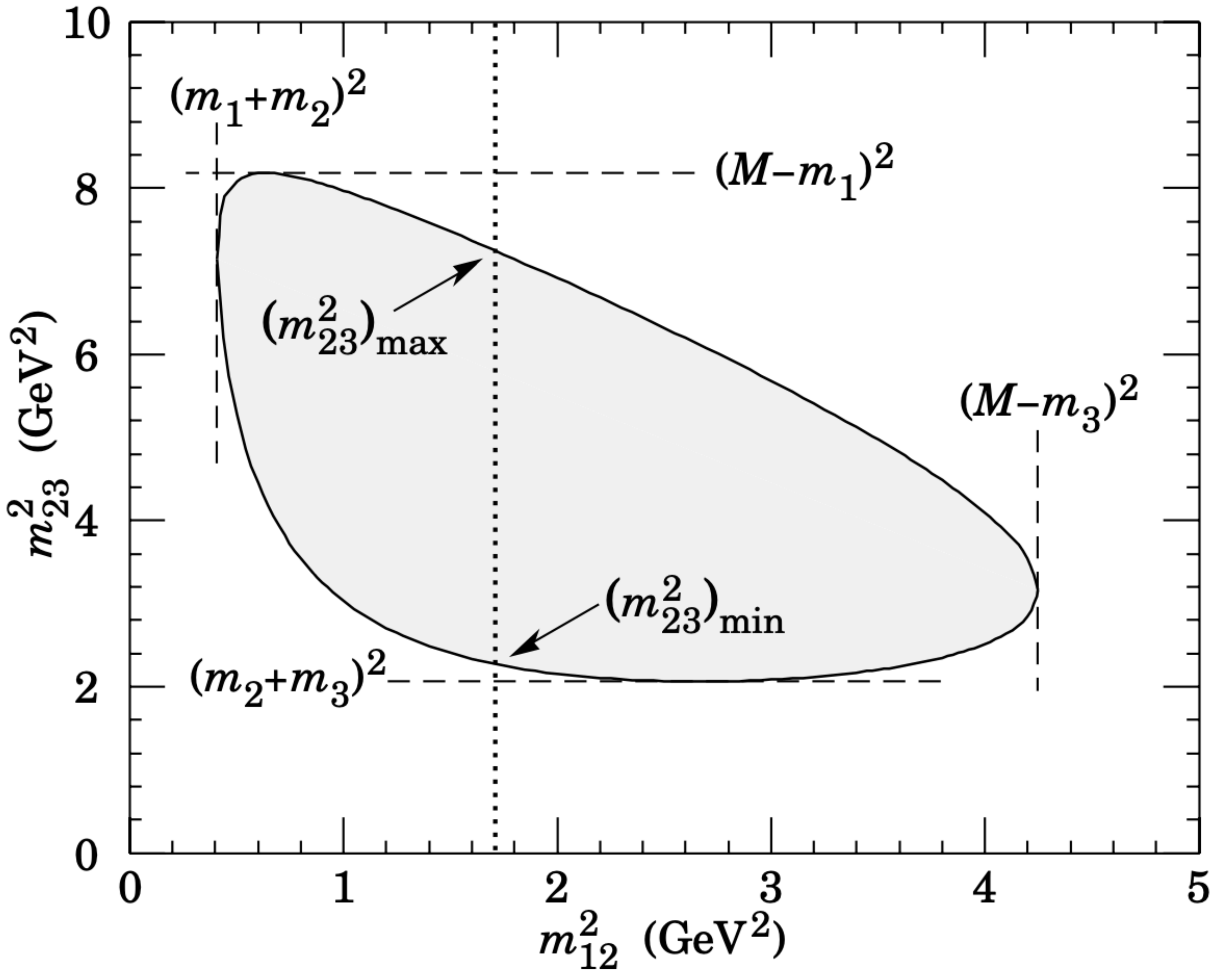
from $p_2 + p_3 = (E_2 + E_3, \vec{p}_2 + \vec{p}_3)$

$$s_{23,min} = (E_2 + E_3)^2 - \left(\sqrt{E_2^2 - m_2^2} + \sqrt{E_3^2 - m_3^2}\right)$$

$$s_{23,max} = (E_2 + E_3)^2 - \left(\sqrt{E_2^2 - m_2^2} - \sqrt{E_3^2 - m_3^2}\right)$$

$$E_2 = \frac{s_{12} - m_1^2 + m_2^2}{2\sqrt{s_{12}}} \qquad E_3 = \frac{M^2 - s_{12} - m_3^2}{2\sqrt{s_{12}}}$$

RPP, section 47 Kinematics



Exercice 2

File: Three-Particles-1.dat

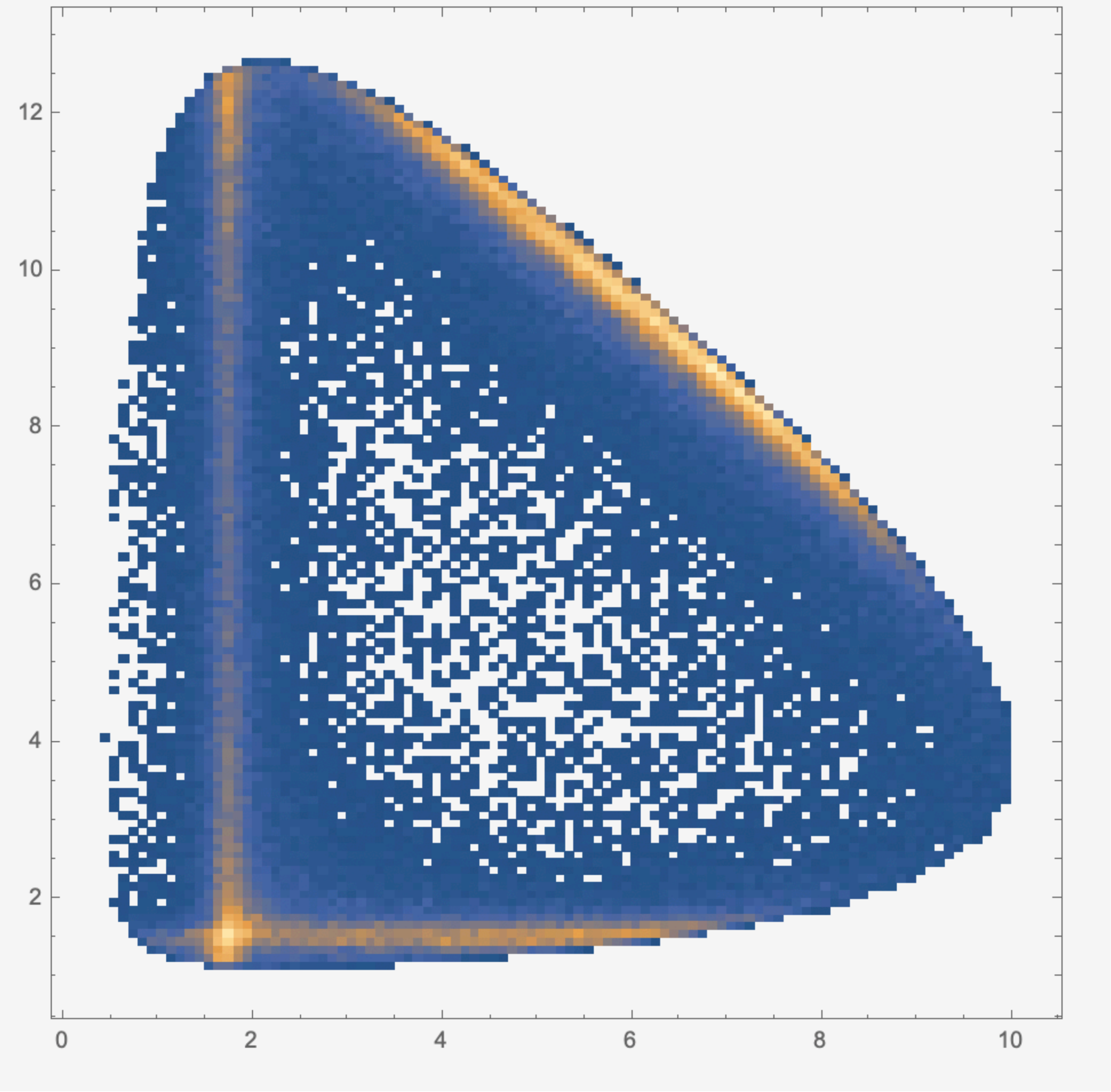
Format:

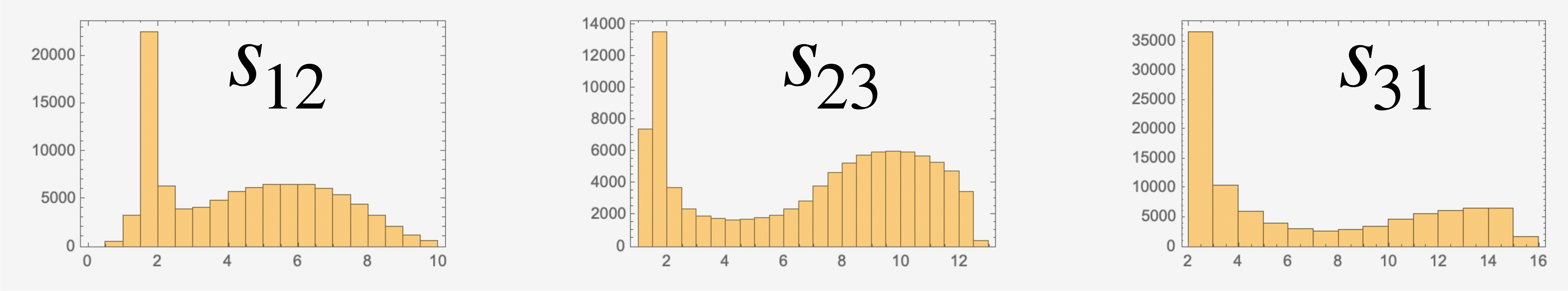
$$E_a,\ p_{a,x},\ p_{a,y},\ p_{a,z}$$
$$E_1,\ p_{1,x},\ p_{1,y},\ p_{1,z}$$
$$E_2,\ p_{2,x},\ p_{2,y},\ p_{2,z}$$
$$E_3,\ p_{3,x},\ p_{3,y},\ p_{3,z}$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

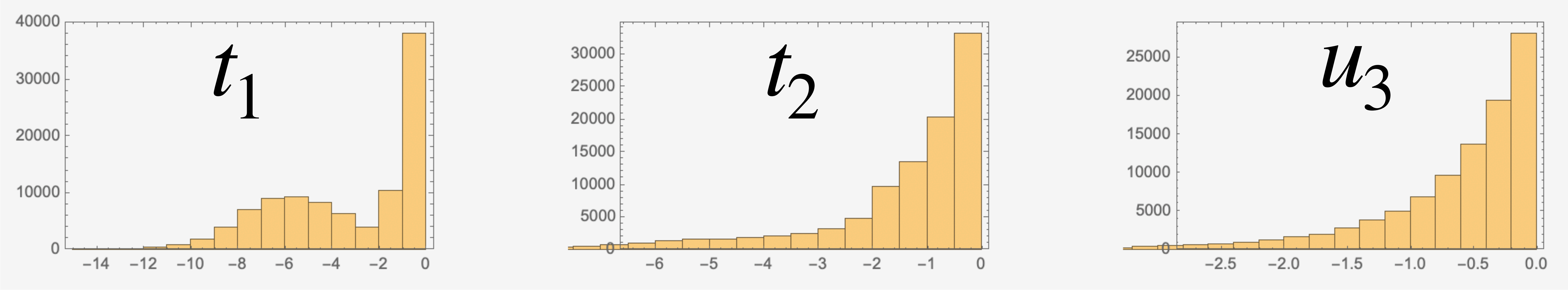
The data are in the lab frame

The file Three-Particles-flat.dat
has no dynamics, only phase

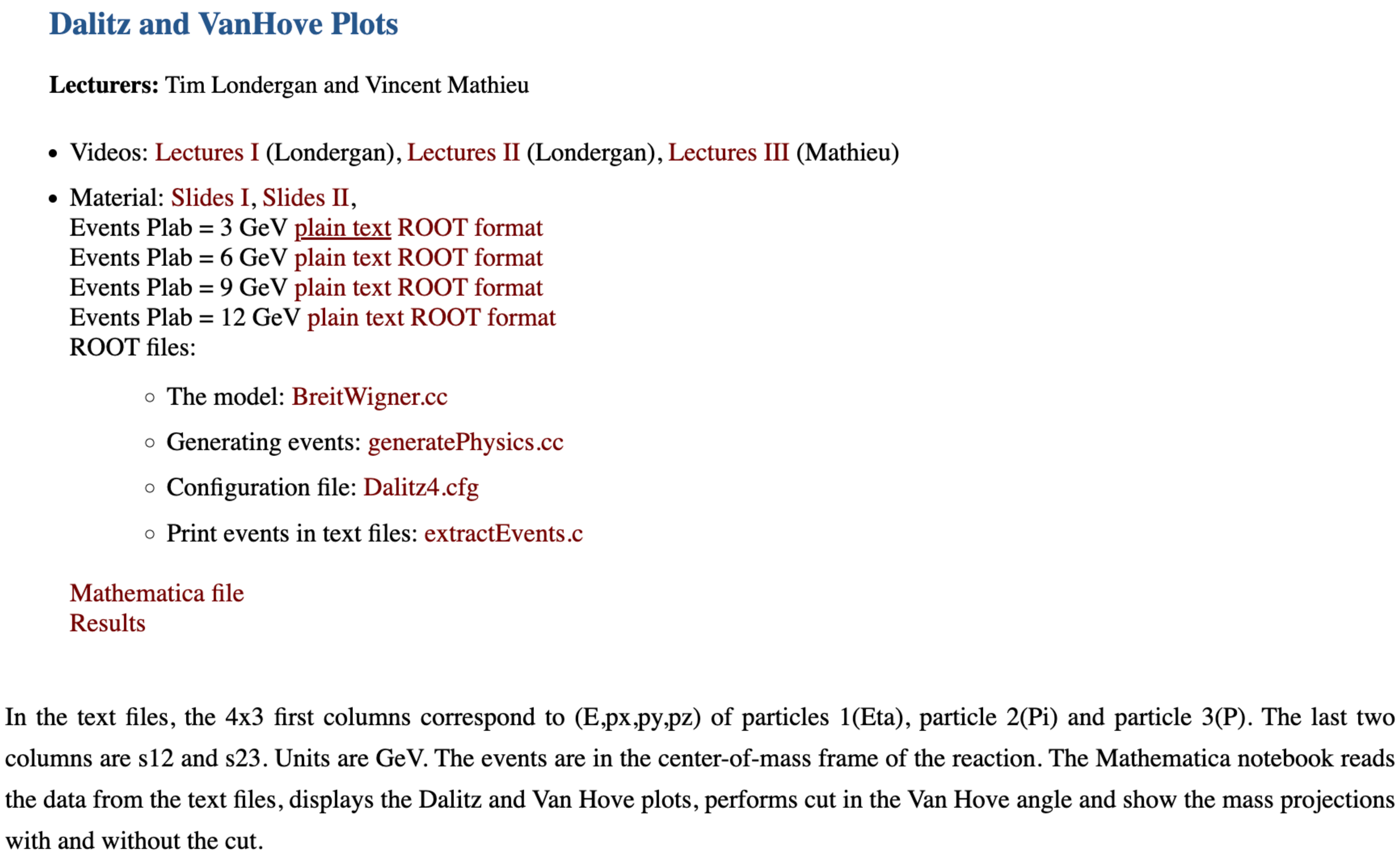
Which reaction is it?
What resonances are included?
What is the spin of the resonances?







http://jpac.nucleares.unam.mx/schools.html#tim



Dalitz and VanHove Plots

Lecturers: Tim Londergan and Vincent Mathieu

- Videos: Lectures I (Londergan), Lectures II (Londergan), Lectures III (Mathieu)
- Material: Slides I, Slides II,
  Events Plab = 3 GeV plain text ROOT format
  Events Plab = 6 GeV plain text ROOT format
  Events Plab = 9 GeV plain text ROOT format
  Events Plab = 12 GeV plain text ROOT format
  ROOT files:
    - The model: BreitWigner.cc
    - Generating events: generatePhysics.cc
    - Configuration file: Dalitz4.cfg
    - Print events in text files: extractEvents.c

  Mathematica file
  Results

In the text files, the 4x3 first columns correspond to (E,px,py,pz) of particles 1(Eta), particle 2(Pi) and particle 3(P). The last two columns are s12 and s23. Units are GeV. The events are in the center-of-mass frame of the reaction. The Mathematica notebook reads the data from the text files, displays the Dalitz and Van Hove plots, performs cut in the Van Hove angle and show the mass projections with and without the cut.