Dispersive techniques for hadron physics

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Barcelona, February 22, 2024

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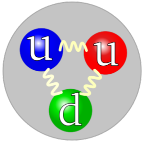
The Standard Model of Particle Physics

Particles	Generations		
Quarks	$\begin{pmatrix} u \\ d \end{pmatrix}$	$\begin{pmatrix} c \\ s \end{pmatrix}$	$\begin{pmatrix} t \\ b \end{pmatrix}$
Leptons	$\begin{pmatrix} e \\ \nu_e \end{pmatrix}$	$\begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}$	$\begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}$

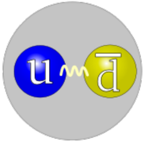
Exchange particles	Interaction
gluons (g)	strong
photons (γ)	electromagnetic
$W^\pm, Z^0$	weak

- Hadron physics $\subset$ particle physics: physics of strongly interacting particles ("hadrons")
- Built from quarks and gluons:

Baryons (three quarks)
ex: proton (uud)

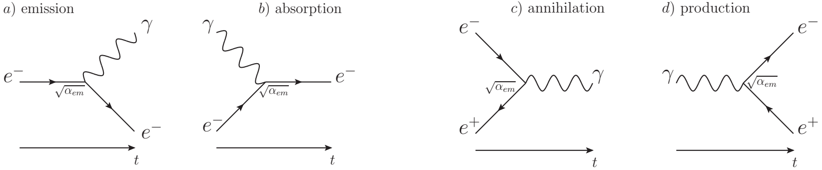


Mesons (quark anti-quark)
ex: pion ($u\bar{d}$)



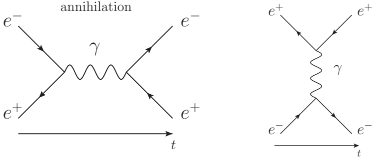
The paradigm of QED: perturbation theory

- Fundamental **vertices** of Feynman diagrams (visual representations of particle interactions) $\propto e$ ($\alpha_{em} = e^2/4\pi = 1/137$)

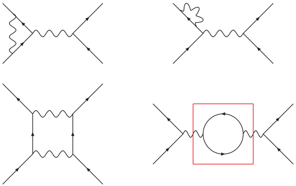


- **Perturbation theory**: expand amplitudes in powers of α_{em}!
- Leading order: α_{em}^1

  e.g. $e^+e^- \to e^+e^-$ (Bhabha scattering, 1935)



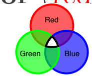
- Corrections of order α_{em}^2: "loops"

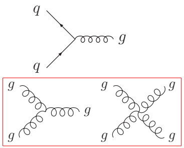
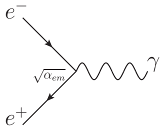


vacuum polarisation

Quantum chromodynamics (QCD)

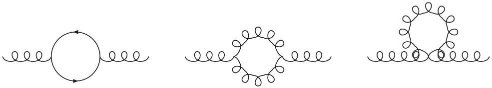
- QCD is a quantum field theory with similarity and differences with respect to QED describing the electromagnetic interaction
- Quarks have a new type of charge called **color** (red,blue,green)
- Quarks combine to form colorless **hadrons**



	Strong	**Electromagnetic**
Current theory	QCD	QED
Charge types	3 color charges	electric charge (e)
Acts on	color charged objects: quarks and gluons	electrically charged particles
Mediators	gluon (g)	photon (γ)
Relative strength	10^2	1
Range (in m)	10^{-15}	∞
Fundamental vertices:		

Vacuum polarisation in QCD (I)

- Gauge-field theories are typified by the feature that nothing is constant
- Couplings and masses are renormalized via processes involving virtual particles
- Such effects make these quantities depend on the energy scale at which one observe them



- running coupling constant in QCD:

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{12\pi}(11N_c - 2N_f)\log\frac{Q^2}{\mu^2}}$$

$N_c = 3$: number of colours; $N_f = 6$: number of "flavours"

nobel prize in physics 2004 for Gross, Politzer, Wilczek

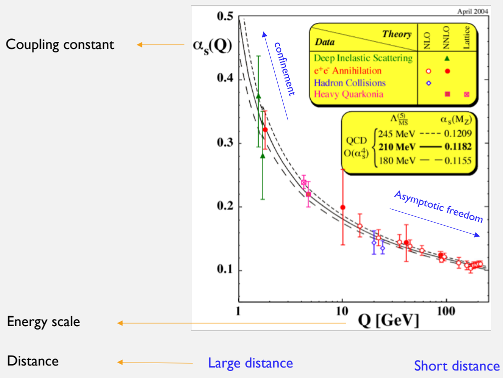
- compare QED:

$$(11N_c - 2N_f) \rightarrow -4$$

$\longrightarrow$ opposite sign

Vacuum polarisation in QCD (II)

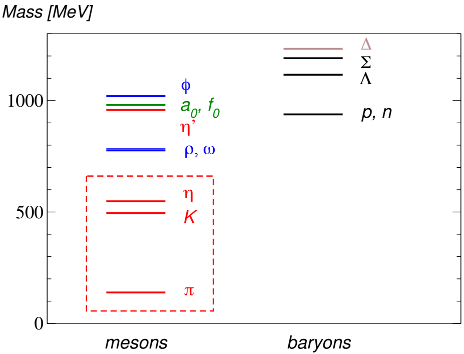
- **Asymptotic freedom** at high-energies:
  - "like QED", but only at high energies (perturbative QCD regime $Q \gg 1$ GeV)
  - In such regime quarks and gluons appear to be "quasi-free"
- **Confinement** at low-energies ($Q < 1$ GeV):
  - The gluons bind the quarks together to form the **hadrons** (the particles we observe in nature)
  - **Perturbation theory** in α_s at low-energies: **impossible**!



QCD: open questions

- **Particle spectrum**:
  why these states?
- **Interactions**:
  scattering, decays of hadrons?

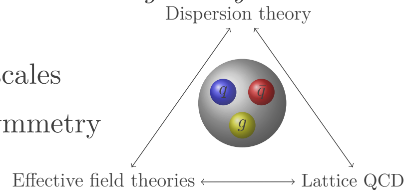
⟶ (perturbative) QCD does
not answer any of these questions!



- Quark models give gross structure of hadron spectrum
- Need non-pertubative **approaches** to describe these hadrons *rigorously*:

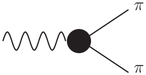
Remedies:

- Effective Field Theories: symmetries, separation of scales
- Dispersion Theory: unitarity, analyticity, crossing symmetry
- Lattice QCD: approach to solving a discretized version of QCD on a computer



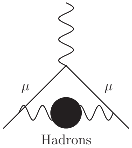
Case of study: the pion vector form factor

- photon conversion into two pions



$$\longrightarrow \quad \langle \pi^+(p)\pi^-(p')|J_\mu(0)|0\rangle = i(p-p')_\mu F_\pi(s)\,, \tag{1}$$

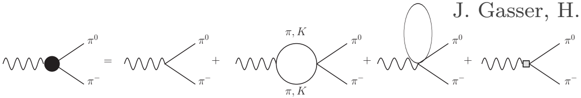
- $J_\mu(0) = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s$ is the electromagnetic current
- $F_\pi(s)$ contain the s dependent response of the $\pi\pi$ to $J_\mu(0)$
- Key object in many hadronic reactions, *e.g.* muon $(g-2)$



- Good pedagogic advent towards the challenges in the description of low-energy QCD

Chiral perturbation theory calculation

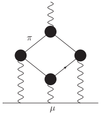
J. Gasser, H. Leutwyler, Nucl.Phys.B 250 (1985)

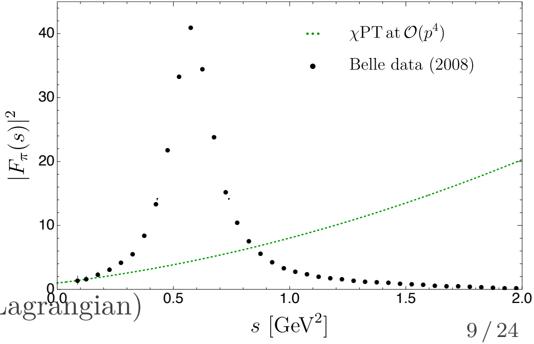


$$F_\pi(s)\big|_{\chi\mathrm{PT}}^{\mathcal{O}(p^4)} = 1 + \frac{2L_9^r(\mu)}{F_\pi^2}s - \frac{s}{96\pi^2 F_\pi^2}\left(A_\pi(s,\mu^2) + \frac{1}{2}A_K(s,\mu^2)\right),$$

$$A_P(s,\mu^2) = \log\frac{m_P^2}{\mu^2} + \frac{8m_P^2}{s} - \frac{5}{3} + \sigma_P^3(s)\log\left(\frac{\sigma_P(s)+1}{\sigma_P(s)-1}\right), \quad \sigma_P(s) = \sqrt{1-4\frac{m_P^2}{s}},$$

- **Drawbacks**:
  - limited energy range
  - $|F_V^\pi(s)|_{\chi\mathrm{PT}}^{\mathcal{O}(p^4)} \propto s$ vs $|F_V^\pi(s)|_{\mathrm{QCD}} \propto 1/s$
  - Non-predictive: divergent calculations

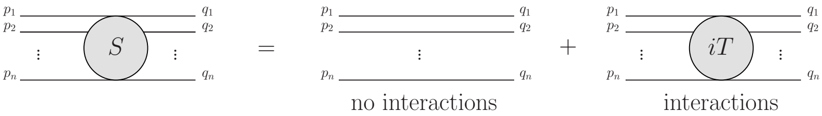




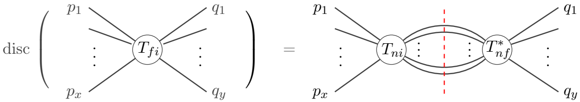
- **Remedy**: Dispersive approach (no need of a Lagrangian)

Dispersive framework

- Model independent and non-perturbative resummation of final-state interactions (FSI), based on:
  - **Unitarity** ($\sim$ probability conservation) gives rise to the *optical theorem*:



$$\sum_f \mathrm{Prob}_{i\to f} = \sum_f |\langle f|S|i\rangle|^2 = 1 \Rightarrow SS^\dagger = S^\dagger S = \mathbb{1} \quad (S = \mathbb{1} + iT)\,,$$

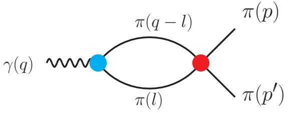


$$\mathrm{disc}T_{fi} = T_{fi} - T^*_{if} = i\sum_n \int d\Pi_n \delta^4\left(P_i - K_n\right) T^*_{nf} T_{ni}\,,$$

  - **Analyticity**: Dispersion relations reconstruct the whole amplitude with knowledge about discontinuity

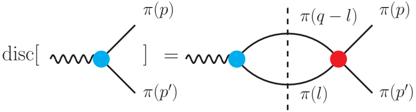
Pion vector form factor, dispersive approach

- **Full** loop calculation



$$\mathcal{M} = \frac{i}{2}\int \frac{d^4 l}{(2\pi)^4} \frac{T^{I*}_{\pi\pi}(s, z_l)(q-2l)_\mu F_\pi(s)}{[l^2 - m_\pi^2 + i\varepsilon][(q-l)^2 - m_\pi^2 + i\varepsilon]}\,,$$

- **Cutkosky** rule: propagators yield Im parts on-shell $\frac{1}{p^2 - M^2 + i\varepsilon} \longrightarrow -2\pi i \delta(p^2 - M^2)$
- Calculation of the **discontinuity** (unitarity)



$$(p-p')_\mu \mathrm{disc} F_\pi(s) = \frac{i}{2}\int \frac{d^4 l}{(2\pi)^4}(2\pi)\delta(l^2 - m_\pi^2)(2\pi)\delta((q-l)^2 - m_\pi^2) T^{I*}_{\pi\pi}(s, z_l)(q-2l)_\mu F_\pi(s)\,,$$

Calculation of the discontinuity

- Momentum integration

$$\begin{aligned} d^4l &= dl^0 l^2 d|l| d\Omega_l \,, \\ (q-l)^2 - m_\pi^2 &\stackrel{l^2=m_\pi^2}{\longrightarrow} s - 2\sqrt{s}l^0 \,, \end{aligned} \tag{2}$$

where $d\Omega_l$ is the solid angle of the $\pi\pi$ subsystem

$$(p-p')_\mu \mathrm{disc} F_\pi(s) = \frac{i}{8\pi^2} F_\pi(s) \int \frac{l^2 d|l| d\Omega_l}{2l^0} \delta(s - 2\sqrt{s}l^0) T_{\pi\pi}^{I*}(s, z_l)(q-2l)_\mu \,, \tag{3}$$

- using $|l|d|l| = l^0 dl^0$, with $(l^0)^2 = l^2 + m_\pi^2$

$$(p-p')_\mu \mathrm{disc} F_\pi(s) = \frac{i}{16\pi^2} F_\pi(s) \int \sqrt{(l^0)^2 - m_\pi^2} dl^0 d\Omega_l \delta(s - 2\sqrt{s}l^0) T_{\pi\pi}^{I*}(s, z_l)(q-2l)_\mu \,, \tag{4}$$

$$(p-p')_\mu \mathrm{disc} F_\pi(s) = \frac{i}{64\pi^2} \sigma_\pi(s) F_\pi(s) \int d\Omega_l T_{\pi\pi}^{I*}(s, z_l)(q-2l)_\mu \tag{5}$$

- **Check!**

$$\int d\Omega_l T_{\pi\pi}^{I}(s, z_l)(q-2l)_\mu = 2\pi(p-p')_\mu \int_{-1}^{1} dz_l \, z_l T_{\pi\pi}^{I}(s, z_l) \,. \tag{6}$$

Calculation of the discontinuity

- Using:
  - the partial-wave expansion of the $\pi\pi$ scattering amplitude

$$T^I_{\pi\pi}(s, z_l) = 32\pi \sum_{\ell=0}^{\infty} (2\ell + 1) t^I_\ell(s) P_\ell(z_l)\,, \tag{7}$$

    $t^I_\ell(s)$ is the amplitude of the ℓ-th partial wave
  - the orthogonality condition

$$(2\ell + 1) \int_{-1}^{1} dz_l P_\ell(z_l) P_{\ell'}(z_l) = 2\delta_{\ell\ell'}\,, \tag{8}$$

    Note: $P_0(z) = 1\,, \quad P_1(z) = z$
- Only $\ell = 1$ is projected out, we arrive at:

$$\mathrm{disc} F_\pi(s) = 2i\sigma_\pi(s) F_\pi(s) t^{1*}_1(s) \theta(s - 4m^2_\pi)\,, \tag{9}$$

Discontinuity and Watson's theorem

- **Rewriting** $t_1^1(s)$ via the P-wave $\pi\pi$ scattering phase shift $\delta_1^1(s)$

$$t_1^1(s) = \frac{1}{\sigma_\pi(s)} \sin \delta_1^1(s) e^{i\delta_1^1(s)}\,, \tag{10}$$

- We arrive at the **final result!**

$$\mathrm{Im} F_\pi(s) = \frac{\mathrm{disc} F_\pi(s)}{2i} = F_\pi(s) \sin \delta_1^1(s) e^{-i\delta_1^1(s)} \theta(s - 4m_\pi^2)\,, \tag{11}$$

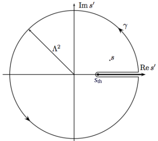
- **Watson's** theorem [Watson, Phys.Rev. 95, 228 (1954)]:

$$\mathrm{Im} F_\pi(s) = |F_\pi(s)| e^{i\delta_{F_\pi}(s)} \sin \delta_1^1(s) e^{-i\delta_1^1(s)} \theta(s - 4m_\pi^2)\,, \tag{12}$$

$$\Rightarrow \quad \delta_{F_\pi}(s) = \delta_1^1(s) \tag{13}$$

Dispersion relation

- Let's derive an analytic solution for $F_\pi(s)$!
- Invoke Cauchy integral



$$F_\pi(s) = \frac{1}{2\pi i}\oint_\gamma \frac{F_\pi(s')}{s'-s}ds' \,, \tag{14}$$

- Assume $F_\pi(s) \to 0$ when $\Lambda^2 \to \infty$
- Only the integral over the cut remains

$$\begin{aligned} F_\pi(s) &= \frac{1}{2\pi i}\int_{\text{cut}} \frac{F_\pi(s')}{s'-s}ds' = \frac{1}{2\pi i}\left\{\int_{4m_\pi^2}^{\infty} \frac{F_\pi(s'+i\varepsilon)}{s'-s}ds' - \int_{4m_\pi^2}^{\infty} \frac{F_\pi(s'-i\varepsilon)}{s'-s}ds'\right\}, \\ F_\pi(s) &= \frac{1}{2\pi i}\int_{4m_\pi^2}^{\infty} \frac{\text{disc}F_\pi(s')}{s'-s}ds' \,, \end{aligned} \tag{15}$$

- We know $\text{disc}F_\pi(s)$, we can reconstruct $F_\pi(s)$ in the whole complex plane!

Omnès equation

- We begin with the following manipulations:

$$
\begin{aligned}
\operatorname{disc} f(s) &= 2i f(s+i\varepsilon)\sin\delta_1^1(s) e^{-i\delta_1^1(s)}\,, && (16)\\
f(s+i\varepsilon) - f(s-i\varepsilon) &= f(s+i\varepsilon)\left(1 - e^{-2i\delta_1^1(s)}\right)\,, && (17)\\
f(s+i\varepsilon) &= f(s-i\varepsilon) e^{2i\delta_1^1(s)}\,, && (18)\\
\operatorname{disc}\log f(s) &= = 2i\delta_1^1(s)\,, && (19)
\end{aligned}
$$

Omnès equation

- Write a dispersion relation for $\log f(s)$

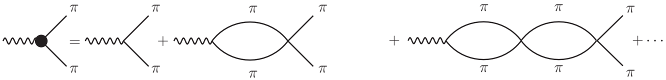
$$\begin{aligned}
\log f(s) &= \frac{1}{2\pi i}\int_{4m_\pi^2}^{\infty} ds' \frac{\mathrm{disc}\log f(s')}{s'-s}\,, &(20)\\
&= \frac{1}{2\pi i}\int_{4m_\pi^2}^{\infty} ds' \frac{(s'-s+s-s_0)\mathrm{disc}\log f(s')}{(s'-s_0)(s'-s)}\,, &(21)\\
&= \frac{1}{2\pi i}\int_{4m_\pi^2}^{\infty} ds' \frac{\mathrm{disc}\log f(s')}{s'-s_0} + \frac{s-s_0}{2\pi i}\int_{4m_\pi^2}^{\infty} ds' \frac{\mathrm{disc}\log f(s')}{(s'-s_0)(s'-s)}\,, &(22)\\
&= \log f(s_0) + \frac{s-s_0}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{(s'-s_0)(s'-s)}\,, &(23)
\end{aligned}$$

- Omnès solution [Omnès, Nuovo Cimento 8, 316 (1958)]

$$\Omega_1^1(s) = \frac{f(s)}{f(0)} = \exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\}\,, \qquad (24)$$

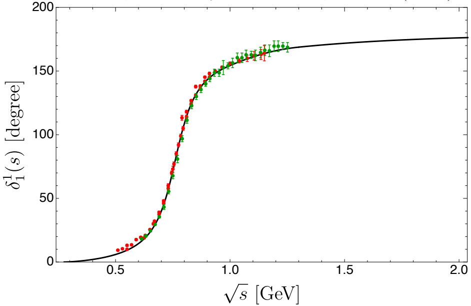
Omnès equation

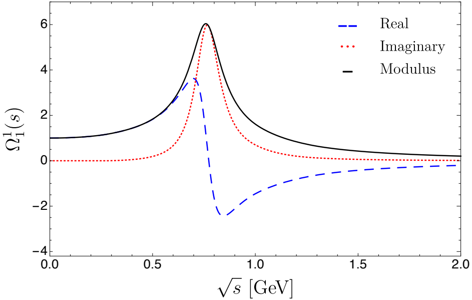
- Diagrammatic interpretation



- Solution: depends solely on the P-wave phase shift of $\pi\pi$

Garcia-Martin, *et.al.* Phys. Rev. D83, 074004 (2011)





Polynomial ambiguity

- Most general solution: $F_\pi(s) = P(s)\Omega_1^1(s)$
- Polynomial $P(s)$, not fixed by unitarity: matched to EFT, or fitted to data
- To find the constraint on $P(s)$, we need $\lim_{s\to\infty} \Omega_1^1(s)$ (excercise)
- Assume $\delta_1^1(s > \Lambda^2) = n\pi$

$$\Omega_1^1(s) = \exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\Lambda^2} ds' \frac{\delta_1^1(s')}{s'(s'-s)} + \frac{s}{\pi}\int_{\Lambda^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\}, \tag{25}$$

$$= \exp\left\{-\frac{1}{\pi}\int_{4m_\pi^2}^{\Lambda^2} ds' \frac{\delta_1^1(s')}{s'} + n\int_{\Lambda^2}^{\infty} ds' \left(\frac{1}{s'-s} - \frac{1}{s'}\right)\right\}, \tag{26}$$

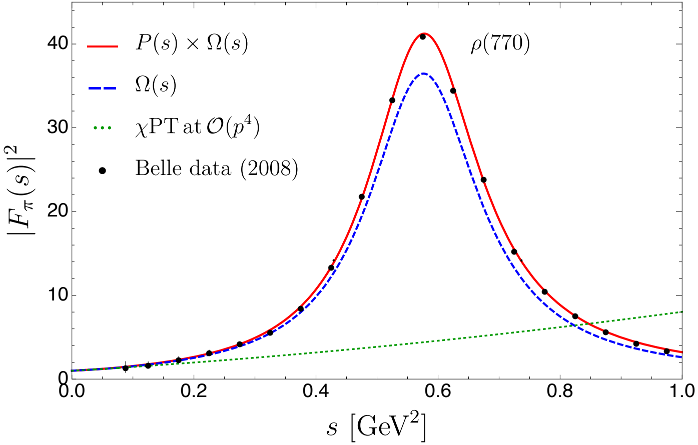
$$= \exp\left\{\text{constant} - n\log\left(\frac{\Lambda^2 - s}{\Lambda^2}\right)\right\} \Rightarrow \lim_{s\to\infty} \Omega_1^1(s) \propto s^{-n}, \tag{27}$$

$$\tag{28}$$

- Assume $\lim_{s\to\infty} F_\pi(s) \propto s^m \Rightarrow$ order of $P(s)$

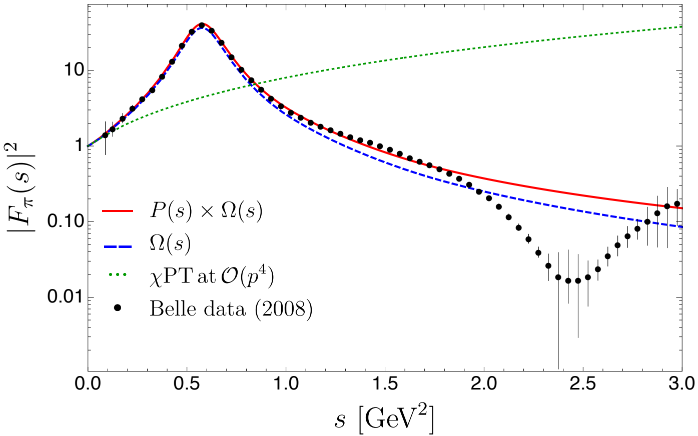
Dispersive pion form factor

- $P(s) = 1 + \alpha s$, with $\alpha = 0.11\ \text{GeV}^{-2}$ due to inelasticities



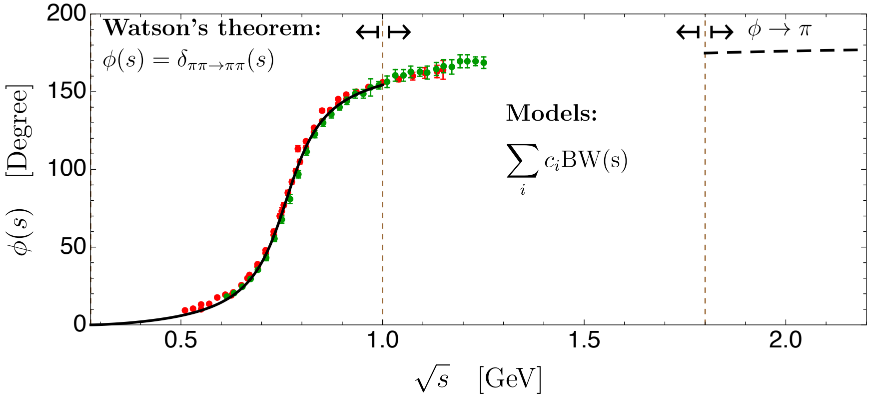
Beyond the elastic region

- $P(s) = 1 + \alpha s$, with $\alpha = 0.11\ \text{GeV}^{-2}$ due to inelasticities



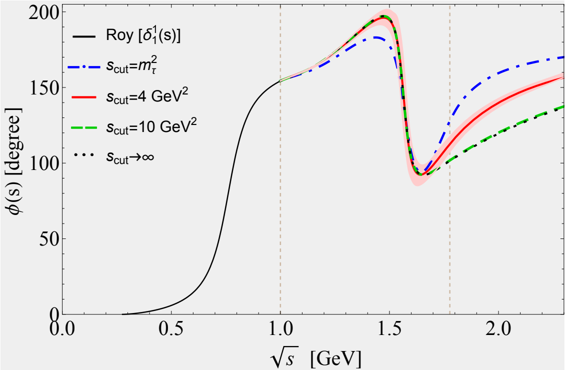
Beyond the elastic approximation

- Built an effective phase:

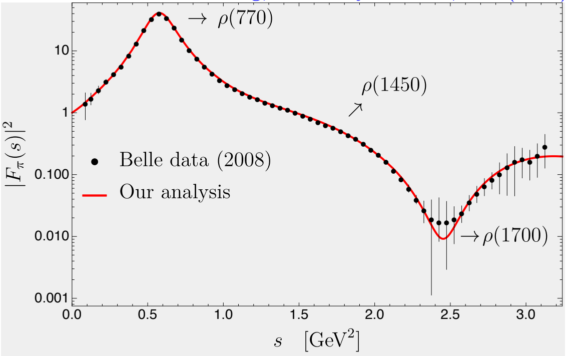


Beyond the elastic approximation

- Fit to data:



S.G-S and P. Roig, Eur. Phys. J C79, 436 (2019)



Summary

- **Low-energy** ("strong") QCD not-well understood, perturbative treatment impossible ($\alpha_s \gtrsim 1$)
- Study this regime by using fundamental principles: **analyticity, unitarity and crossing symmetry** $\Rightarrow$ **Dispersive** formalism:
  - Model-independent
  - 2-particle Final-State Interactions
  - Input: phase shifts, *e.g.* $\pi\pi$ scattering $\delta_1^1(s)$ for $F_\pi(s)$
  - Predictive power (subtraction constants), experimental data well described
- Formalism can be extend to 3-particle FSI (Khuri-Treiman equations)