Lattice QCD: Finite-volume spectrum

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OLD DOMINION
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EXOHAD
EXOTIC HADRONS TOPICAL COLLABORATION

Finite-volume symmetry: Recap

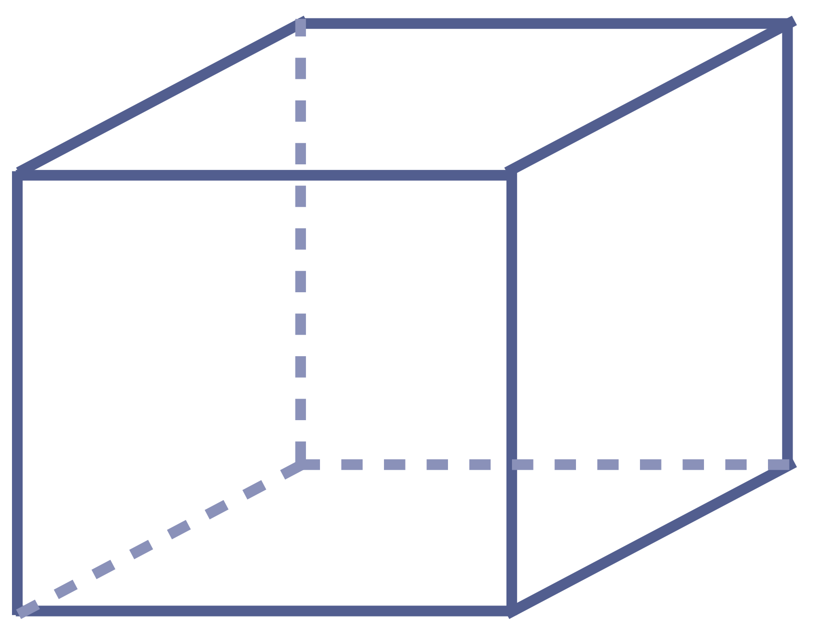
So, how does angular momentum subduce into O irreps?

	A_1	A_2	E	T_1	T_2	# Rows
$J=0$	1					*1*
$J=1$				1		*3*
$J=2$			1		1	*5*
$J=3$		1		1	1	*7*
$J=4$	1		1	1	1	*9*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

We can invert the table

Λ	Dimension	J
A_1	1	$0, 4, \ldots$
A_2	1	$3, 5, \ldots$
E	2	$2, 4, \ldots$
T_1	3	$1, 3, \ldots$
T_2	3	$2, 3, \ldots$

Finite-volume symmetry: Recap



O

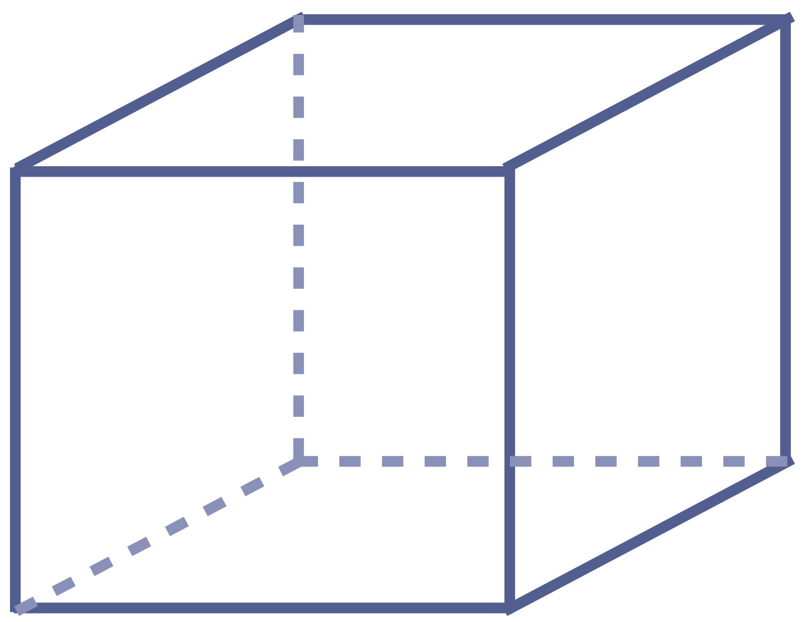
Symmetry operations on the octahedral group O

Operation	No.	Class Label
identity	1	$\mathbf{1}$
90° about axes through centres of opposite faces	6	C_4
180° about the same axes	3	C_4^2
120° about diagonals connecting opposite vertices	8	C_3
180° about axes through centers of opposite edges	6	C_2
	24	

Character table

$\mathbf{O}$	$\mathbf{1}$	$8C_3$	$6C_2$	$6C_4$	$3\,(C_4)^2$
A_1	+1	+1	+1	+1	+1
A_2	+1	+1	−1	−1	+1
E	+2	−1	0	0	+2
T_1	+3	0	−1	+1	−1
T_2	+3	0	+1	−1	−1

Finite-volume symmetry: In-flight lattices

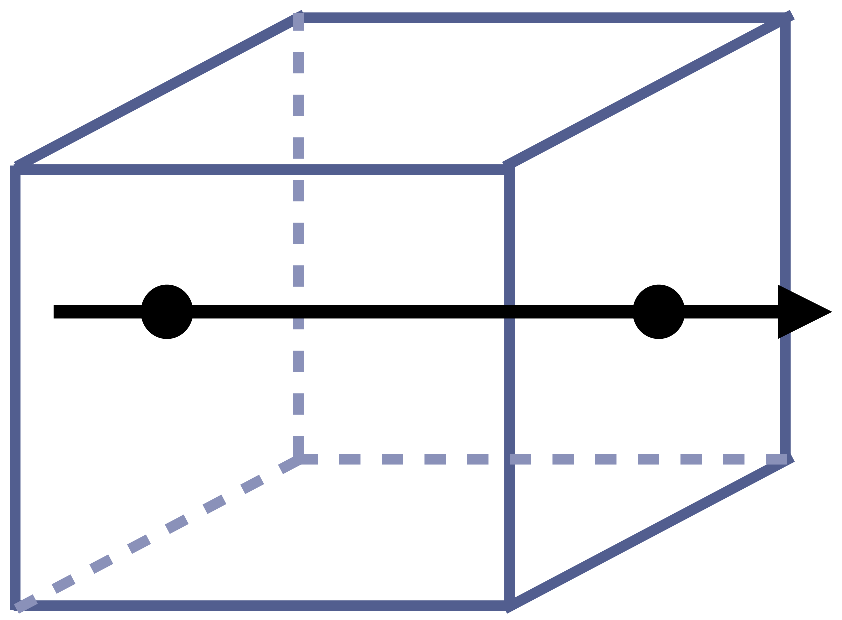


O_h

O_h **is the symmetry group of a lattice at rest, only**

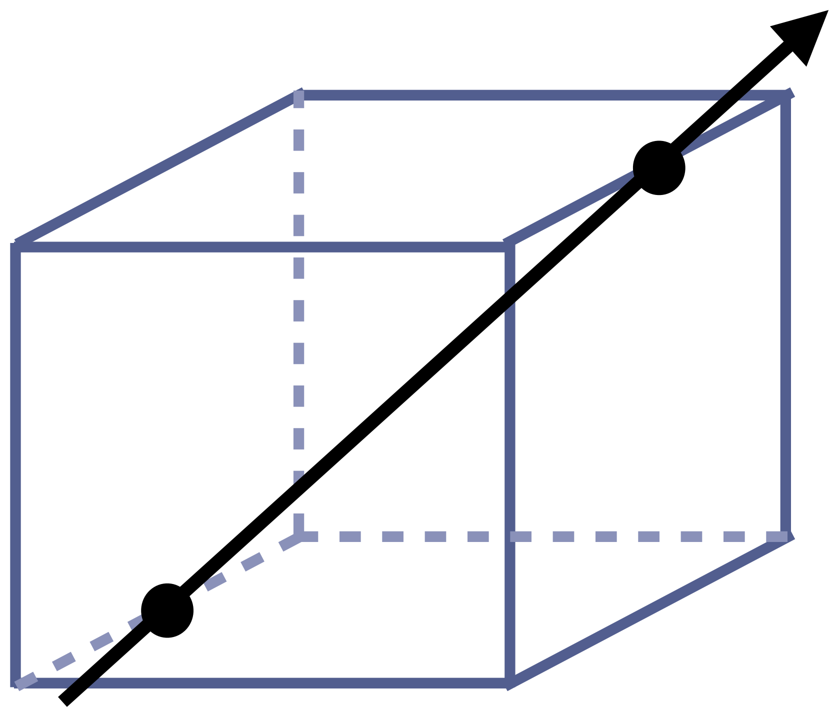
Lattice in flight (momenta $\neq 0$) have different reduced symmetry groups (subgroups of O_h)

$p = (n, 0, 0)$



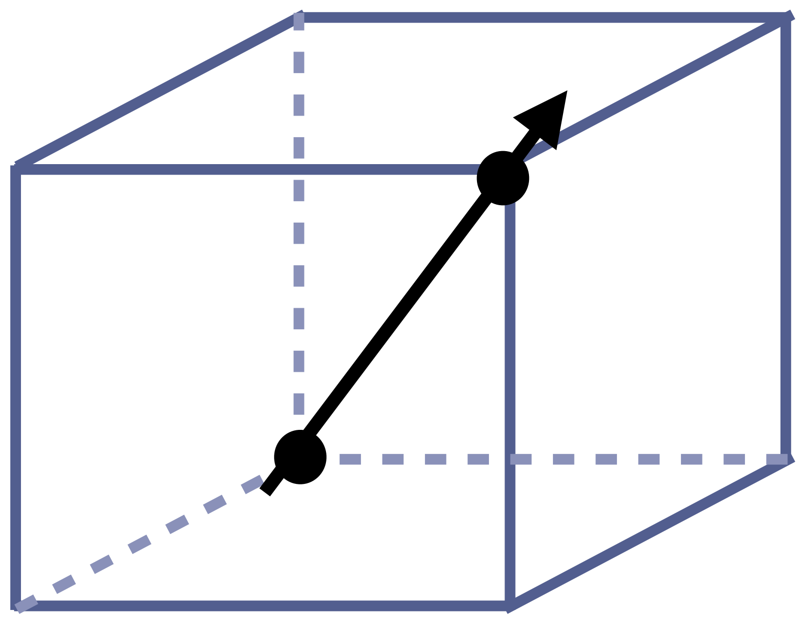
C_{4v}

$p = (0, n, n)$



C_{2v}

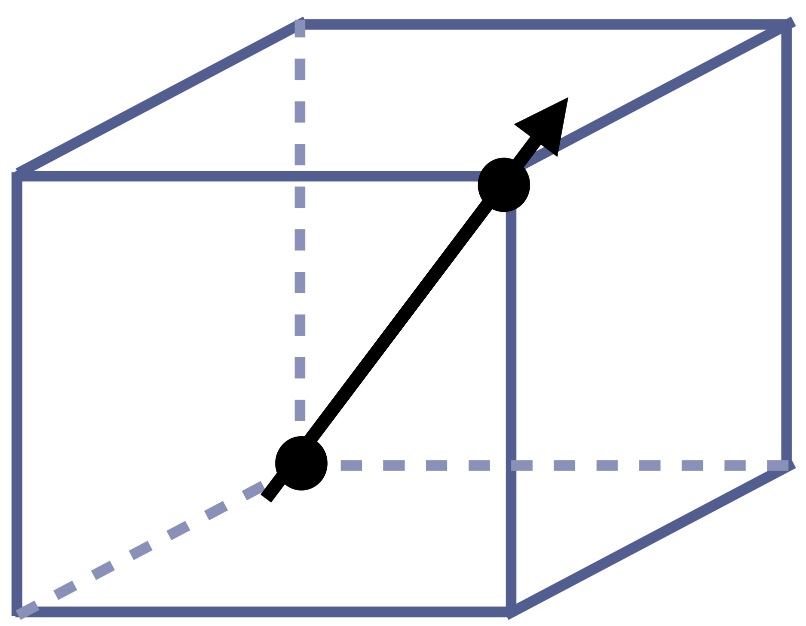
$p = (n, n, n)$



C_{3v}

Finite-volume symmetry: In-flight lattices

$p = (n, n, n)$



C_{3v}

Let's take a look at the little group C_{3v}

Operation	No.	Class Label
identity	1	$\mathbf{1}$
120° about diagonals connecting opposite vertices	2	C_3
Plane of symmetry	3	σ_v
	6	

Simpler character table

C_{3v}	E	$2C_3$	$3\sigma_v$
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

Finite-volume symmetry: In-flight lattices

Different combinations of momenta and corresponding irreps

p	Little Group	Irreps $\left(\Lambda^P\right)$
$(0,0,0)$	O_h	A_1, A_2, E, T_1, T_2
$(n,0,0)$	C_{4v}	A_1, A_2, B_1, B_2, E
$(n,n,0)$	C_{2v}	A_1, A_2, B_1, B_2
(n,n,n)	C_{3v}	A_1, A_2, E
$(n,m,0)$	C_4	A_1, A_2
(n,n,m)	C_4	A_1, A_2
(n,m,k)	C_2	A

Lattices in flight do not preserve parity, they mix both positive and negative combinations

Finite-volume symmetry: Subductions/reductions

There are an infinite number of irreps J in the continuum, but just a few on a lattice

To identify which continuum states can occur in a particular irrep note that O is a subgroup of $SO(3)$

Restricting the irreps of $SO(3)$ labelled by J to rotations allowed by the lattice generates representations that are reducible

J is reducible under O or O_h

We generate these representations by subducing $O(3)$ into O_h irreps

How?

Reducible

$$n_J^{(\alpha)} = \frac{1}{N_G} \sum_k n_k \chi_k^{(\alpha)} \chi_k^{(J)}$$

Irreducible

Number of operations

Finite-volume symmetry: Subductions/reductions

How?

$$n_J^{(\alpha)} = \frac{1}{N_G} \sum_k n_k \chi_k^{(\alpha)} \chi_k^{(J)}$$

Reducible ($\chi_k^{(\alpha)}$), *Irreducible* ($\chi_k^{(J)}$), *Number of operations* (N_G)

$\mathbf{O}$	$\mathbf{1}$	$8C_3$	$6C_2$	$6C_4$	$3\left(C_4\right)^2$
A_1	$+1$	$+1$	$+1$	$+1$	$+1$
A_2	$+1$	$+1$	-1	-1	$+1$
E	$+2$	-1	0	0	$+2$
T_1	$+3$	0	-1	$+1$	-1
T_2	$+3$	0	$+1$	-1	-1

	$\mathbf{1}$	$8C_3$	$6C_2$	$6C_4$	$3\left(C_4\right)^2$
J=0	**1**	**1**	**1**	**1**	**1**
J=1	**3**	**0**	**-1**	**1**	**-1**
J=2	**5**	**-1**	**1**	**-1**	**1**

	A_1	A_2	E	T_1	T_2
$J=0$	1				
$J=1$				1	
$J=2$			1		1
$J=3$		1		1	1
$J=4$	1		1	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Finite-volume symmetry: Subductions/reductions

How?

Start with the meson operator in the continuum

Old friend

$$\mathcal{O}^{Jm}_{F\nu}(\vec{p},t) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \mathcal{C}\begin{pmatrix} \overline{3} & 3 & F \\ \nu_1 & \nu_2 & \nu \end{pmatrix}_{SU(3)} \bar{\psi}_\mu(\vec{x},t)\Gamma^{Jm}(t)\psi_\mu(\vec{x},t)$$

SU(3) Clebsch-Gordan coefficients

Subduce it according to the irrep

$$\mathcal{O}^{\Lambda,\mu;[J]}_{F\nu} = \sum_{M'} \mathcal{S}^{\Lambda,\mu;[J]}_{M'} \mathcal{O}^{JM'}_{F\nu}$$

Subduction/reduction coefficients

Group projection formula

$$\mathcal{O}^{\Lambda,\mu;[J]}_{F\nu} = \frac{d_\Lambda}{g_G} \sum_{R\in G} \Gamma^{\Lambda\,\mu}_{M}(R) \sum_{M'} R_{MM'} \mathcal{O}^{JM'}_{F\nu}$$

Finite-volume symmetry: Subductions/reductions

Group projection formula

Irrep d

$$\mathcal{O}^{\Lambda,\mu;[J]}_{F\nu} = \frac{d_\Lambda}{g_G}\sum_{R\in G}\Gamma^{\Lambda\,\mu}_{M}(R)\sum_{M'}R_{MM'}\mathcal{O}^{JM'}_{F\nu}$$

order (24/48) *Representation of R in* Λ

Where $R_{MM'} = D^{(J)}_{M'M}(R)$

(α,β,γ)

Trivial example, $J=0$ **into** A_1 **irrep:**

$$\mathcal{O}^{A_1,\mu;[0]}_{F\nu} = \frac{1}{24}\times 24\times 1\,\mathcal{O}^{JM'}_{F\nu} = \mathcal{O}^{JM'}_{F\nu}$$

Semi-trivial example, $J=1$ **into** T_1 **irrep:**

$\mathbf{O}$	$\mathbf{1}$	$8C_3$	$6C_2$	$6C_4$	$3\,(C_4)^2$
T_1	$+3$	0	-1	$+1$	-1

$$\mathcal{O}^{\Lambda,\mu;[J]}_{F\nu} = \sum_{M'}\mathcal{S}^{\Lambda,\mu;[J]}_{M'}\mathcal{O}^{JM'}_{F\nu} \qquad \mathcal{S}^{\Lambda,\mu;[J]}_{M'} = \frac{d_\Lambda}{g_G}\sum_{R\in G}\Gamma^{\Lambda\,\mu}_{M}(R)R_{MM'}$$

Finite-volume symmetry: Subductions/reductions

Group projection formula

$$\mathcal{O}_{F\nu}^{\Lambda,\mu;[J]} = \frac{d_\Lambda}{g_G} \sum_{R \in G} \Gamma^{\Lambda\,\mu}_{M}(R) \sum_{M'} R_{MM'} \mathcal{O}_{F\nu}^{JM'}$$

Irrep d (d_Λ); *order (24/48)* (g_G); *Representation of R in Λ* ($\Gamma^{\Lambda\,\mu}_{M}(R)$)

Where $R_{MM'} = D^{(J)}_{M'M}(R)$, with R given by (α, β, γ)

Trivial example, $J = 0$ **into** A_1 **irrep:**

$$\mathcal{O}_{F\nu}^{A_1,\mu;[0]} = \frac{1}{24} \times 24 \times 1\, \mathcal{O}_{F\nu}^{JM'} = \mathcal{O}_{F\nu}^{JM'}$$

Semi-trivial example, $J = 1$ **into** T_1 **irrep:**

$\mathbf{O}$	$\mathbf{1}$	$8C_3$	$6C_2$	$6C_4$	$3\,(C_4)^2$
T_1	$+3$	0	-1	$+1$	-1

$$\mathcal{O}_{F\nu}^{\Lambda,\mu;[J]} = \sum_{M'} \mathcal{S}^{\Lambda,\mu;[J]}_{M'} \mathcal{O}_{F\nu}^{JM'} \qquad \mathcal{S}^{\Lambda,\mu;[J]}_{M'} = \frac{d_\Lambda}{g_G} \sum_{R \in G} \Gamma^{\Lambda\,\mu}_{M}(R) R_{MM'}$$

$$\mathcal{S}^{T_1,\mu;[1]}_{M'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Finite-volume symmetry: Subductions/reductions

Group projection formula

$$\mathcal{O}_{F\nu}^{\Lambda,\mu;[J]} = \sum_{M'} \mathcal{S}_{M'}^{\Lambda,\mu;[J]} \mathcal{O}_{F\nu}^{JM'} \qquad \mathcal{S}_{M'}^{\Lambda,\mu;[J]} = \frac{d_\Lambda}{g_G} \sum_{R \in G} \Gamma_M^{\Lambda\,\mu}(R) R_{MM'}$$

ALTERNATIVE METHOD: combine lower momenta to produce the higher subduction coefficients

$$S_{\Lambda,\lambda}^{J,M} = N \sum_{\lambda_1,\lambda_2} \sum_{m_1,m_2} \underbrace{C \begin{pmatrix} \Lambda & \Lambda_1 & \Lambda_2 \\ \lambda & \lambda_1 & \lambda_2 \end{pmatrix}_{\mathbf{O}}}_{\Lambda_1 \otimes \Lambda_2 \to \Lambda} S_{\Lambda_1,\lambda_1}^{J_1,m_1} S_{\Lambda_2,\lambda_2}^{J_2,m_2} \underbrace{\langle J_1, m_1; J_2, m_2 \mid J, M \rangle}_{SO(3)}$$

$\Lambda_1 \otimes \Lambda_2 \to \Lambda$ *Clebsch-Gordan*

$SO(3)$ *Clebsch-Gordan*

$(J=2) \to T_2$

λ \ M	2	1	0	-1	-2
1	0	1	0	0	0
2	$\frac{1}{\sqrt{2}}$	0	0	0	$-\frac{1}{\sqrt{2}}$
3	0	0	0	1	0

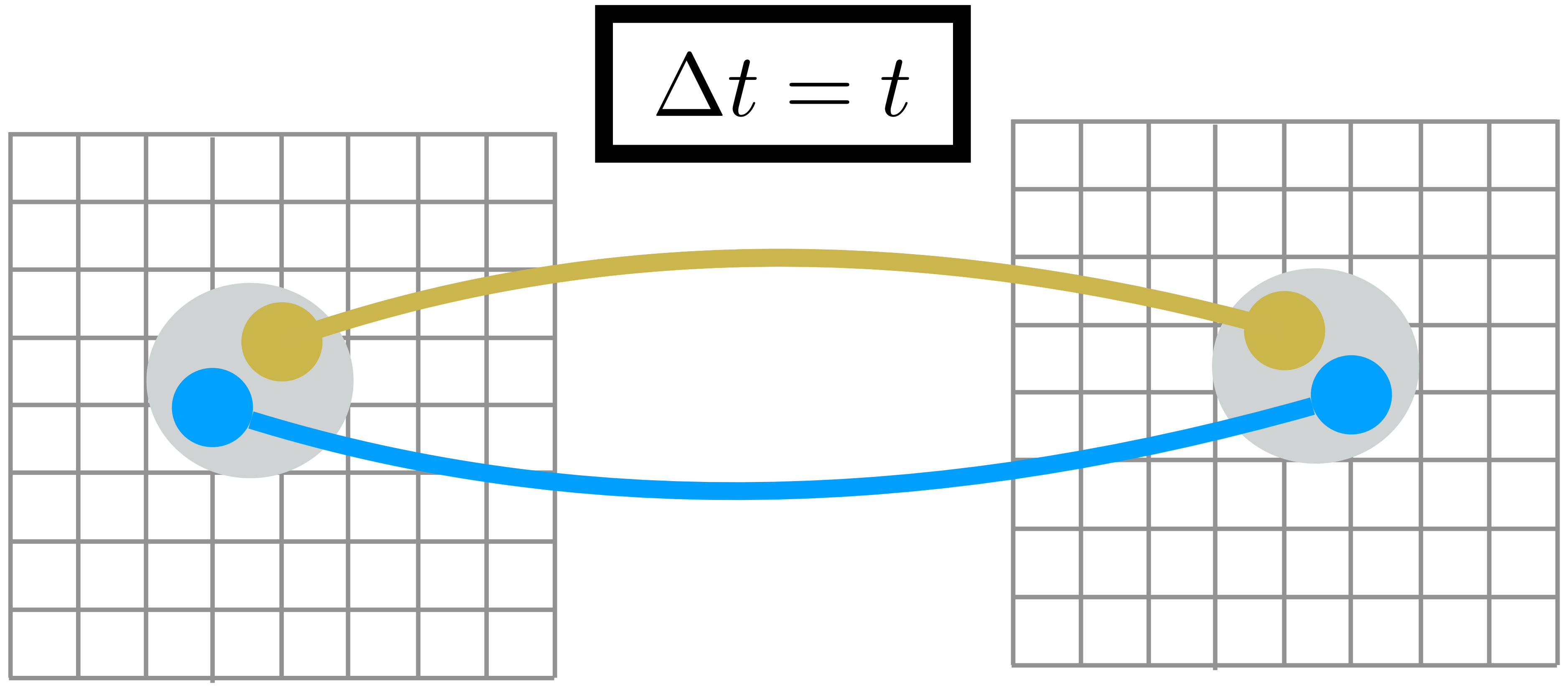
$(J=2) \to E$

λ \ M	2	1	0	-1	-2
1	0	0	1	0	0
2	$\frac{1}{\sqrt{2}}$	0	0	0	$\frac{1}{\sqrt{2}}$

Questions? — Some water?

Next: The GEVP!!

Pions on the lattice



Lets study the temporal evolution of a single particle

$$C(t) \equiv \left\langle 0 \left| \mathcal{O}(t)\, \mathcal{O}^{\dagger}(0) \right| 0 \right\rangle$$

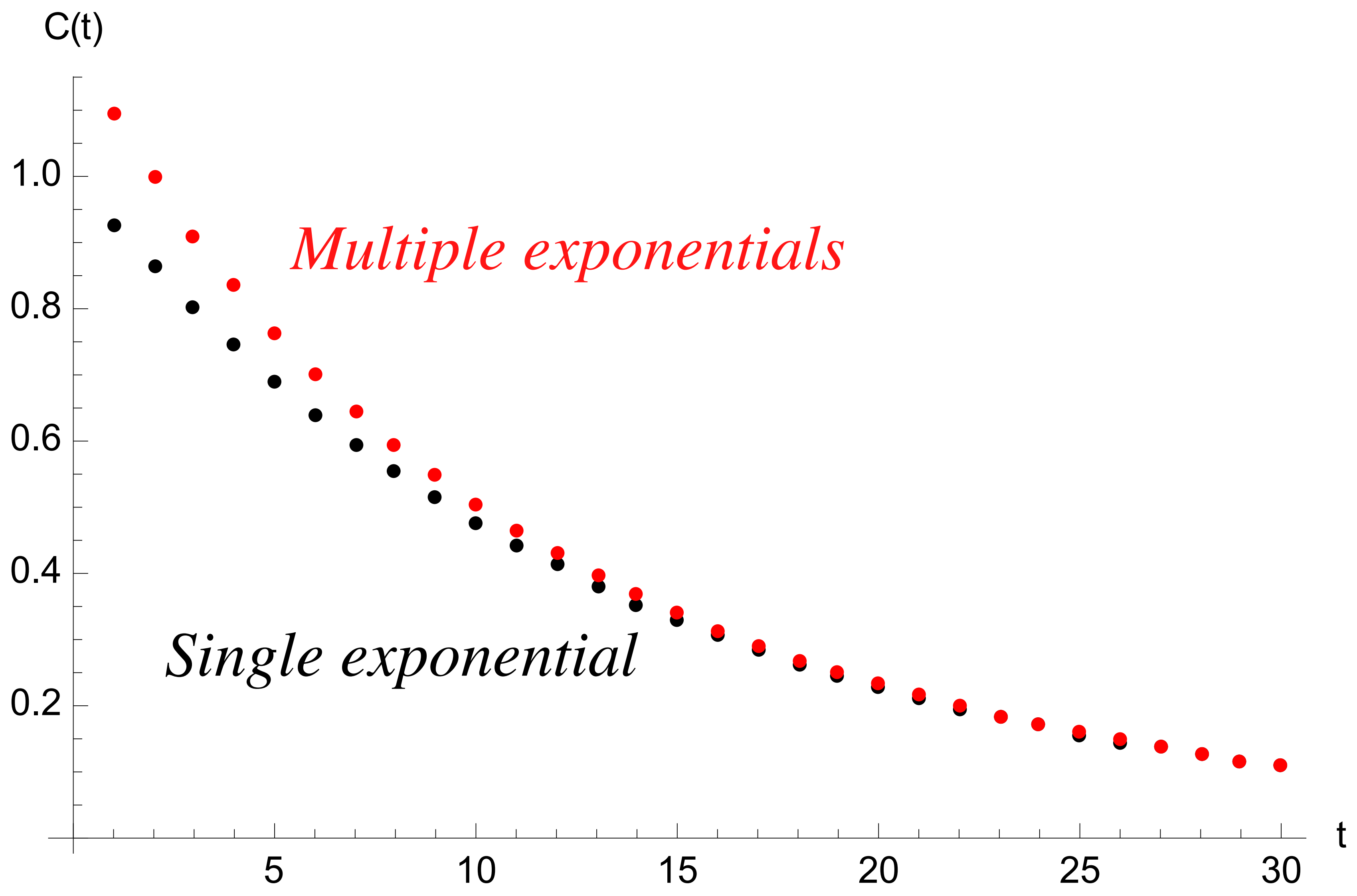
Basis

$$= \sum_n \left\langle 0 \left| \mathcal{O}(t) \right| n \right\rangle \left\langle n \left| \mathcal{O}^{\dagger}(0) \right| 0 \right\rangle$$

$$e^{-iHt_M/\hbar}$$

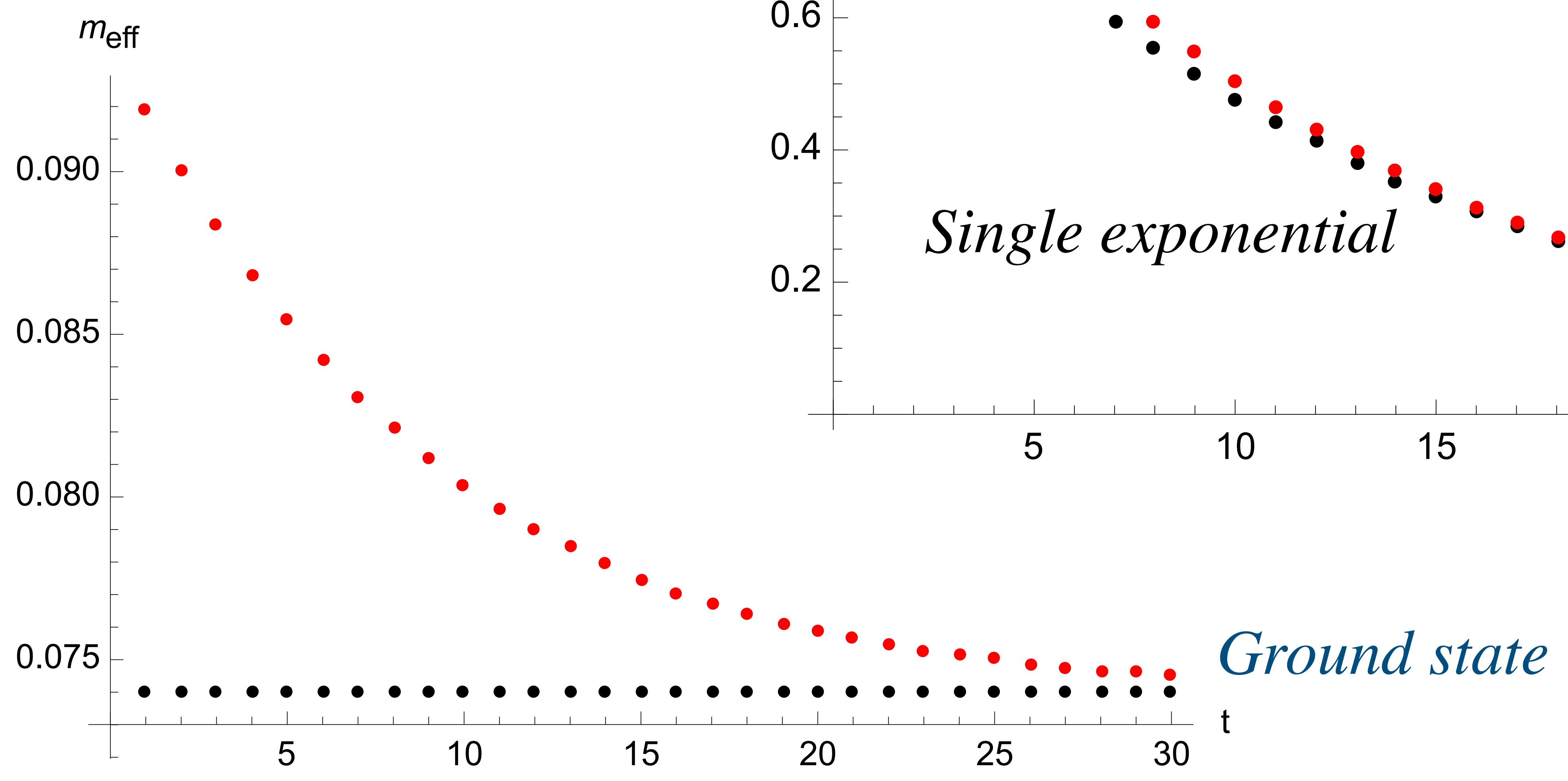
Euclidean time

$$= \sum_n A_n e^{-E_n t}$$



We determine these energies from fitting the temporal evolution of the system

$$m_{\text{eff}} = log\left[\frac{C(t)}{C(t+1)}\right]$$



Generalized eigenvalue problem

What is a pion??

$$u\gamma_5\bar{d} \quad u\gamma_0\gamma_5\bar{d} \quad u\gamma_5\Delta\bar{d} \quad |\varepsilon_{ijk}|\,|\varepsilon_{klm}|\,u\gamma_j\nabla_l\nabla_m\bar{d}$$

All of these combinations couple to the "pion state"

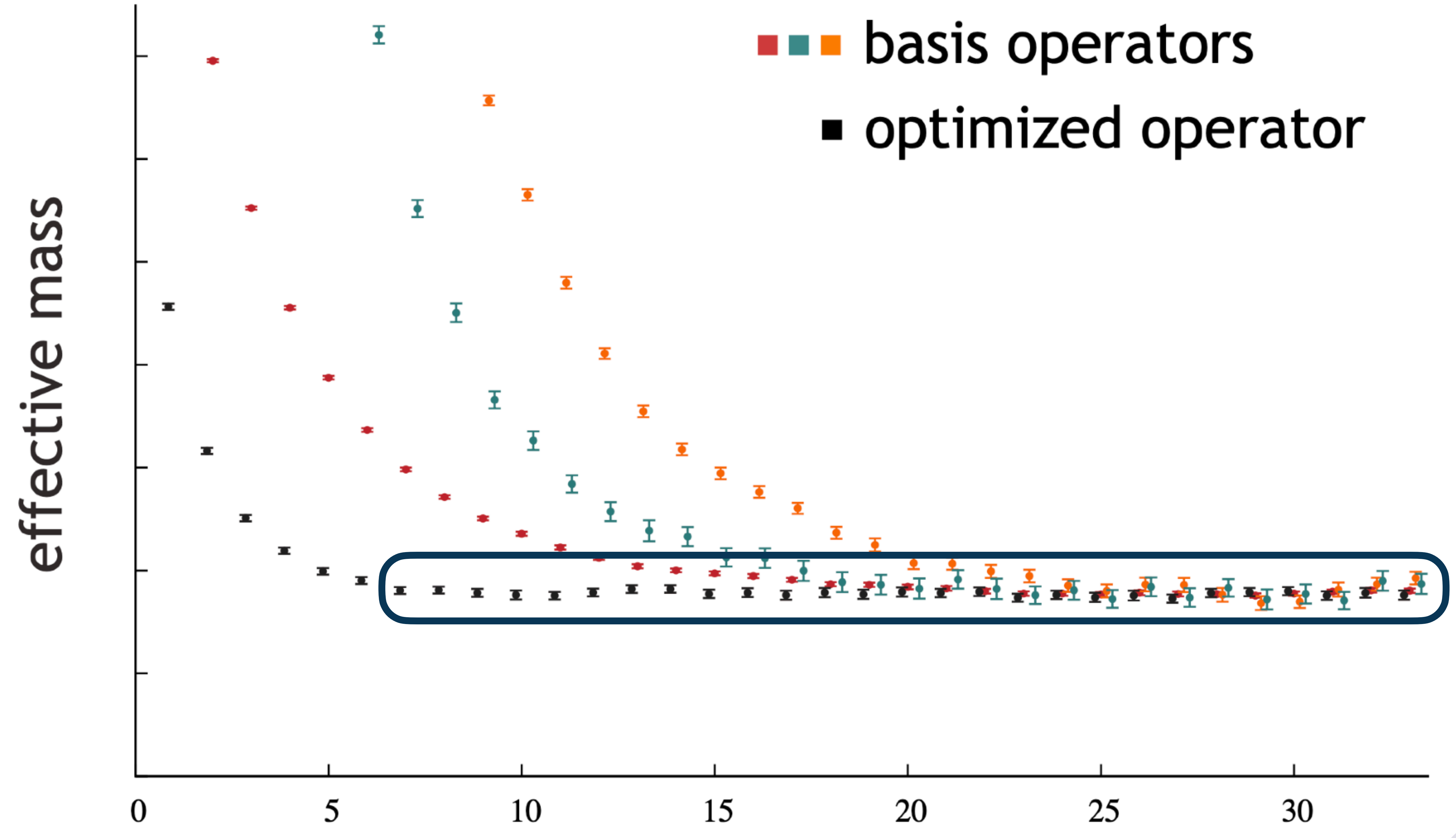
On the same lattice, the same operator couples to different states

$$\langle\eta|\mathcal{O}^\dagger|0\rangle \neq 0 \longrightarrow \langle\eta'|\mathcal{O}^\dagger|0\rangle \neq 0$$

What is the optimal combination of states to create a pion?

The one that flattens the effective mass calculation the most

The faster we reach plateau, the more precise it is



Generalized eigenvalue problem

First, given N operator, we produce the NxN matrix containing all possible combinations

Remember $$C_{ij}(t) = \sum_n \langle 0 \,|\mathcal{O}_i(0)|\, n\rangle \langle n \,|\mathcal{O}_j(0)|\, 0\rangle \, e^{-E_n t}$$

Our goal is to construct an ideal operator that couplings mostly to the state of interest

Linear combination of operators $$\Omega^n = \sum_i v_i^{n\,*} \mathcal{O}_i$$

The modulus of this operator can be calculated as

$$\left\langle 0 \left| \Omega^n(t) \Omega^{n\,\dagger}(0) \right| 0 \right\rangle = \sum_n \langle 0|\Omega^n(0)|n\rangle \left\langle n \left| \Omega^{n\,\dagger}(0) \right| 0 \right\rangle e^{-E_n t} = \sum_n \left| \left\langle n \left| \Omega^{n\,\dagger}(0) \right| 0 \right\rangle \right|^2 e^{-E_n t}$$

If $$\left| \left\langle n \left| \Omega^{n\dagger}(0) \right| 0 \right\rangle \right|^2 = \delta_{nm} \longrightarrow \left\langle 0 \left| \Omega^n(t) \Omega^{n\dagger}(0) \right| 0 \right\rangle \propto e^{-E_m t}$$

Desired behavior

Generalized eigenvalue problem

The modulus of this operator can be calculated as

$$\left\langle 0 \left| \Omega^m(t) \Omega^{m\dagger}(0) \right| 0 \right\rangle = \sum_n \langle 0 | \Omega^m(0) | n \rangle \left\langle n \left| \Omega^{m\dagger}(0) \right| 0 \right\rangle e^{-E_n t} = \sum_n \left| \left\langle n \left| \Omega^{m\dagger}(0) \right| 0 \right\rangle \right|^2 e^{-E_n t}$$

If $\left| \left\langle n \left| \Omega^{m\dagger}(0) \right| 0 \right\rangle \right|^2 = \delta_{nm} \longrightarrow \left\langle 0 \left| \Omega^m(t) \Omega^{m\dagger}(0) \right| 0 \right\rangle \propto e^{-E_m t}$

Desired behavior

Our local minima in the vector space of linear combinations produce the desired operator

Note that $\left| \left\langle n \left| \Omega^{m\dagger}(0) \right| 0 \right\rangle \right|^2 \geq 0$

The must normalize our states so that

$$\sum_{i,j} v_i^{n\,*} \langle 0 | \mathcal{O}_i(t_0) \mathcal{O}_j(0) | 0 \rangle v_j^n = N^n$$

Generalized eigenvalue problem

Therefore we have an optimization problems with constraints → Lagrange multipliers

Minimize $$\left\langle 0 \left| \Omega^n(t) \Omega^{n\,\dagger}(0) \right| 0 \right\rangle = \sum_{i,j} v_i^{n\,*} \langle 0 | \mathcal{O}_i(t) \mathcal{O}_j(0) | 0 \rangle v_j^n = \sum_{i,j} v_i^{n\,*} C_{ij}(t) v_j^n$$

Constraint $$\sum_{i,j} v_i^{n\,*} \langle 0 | \mathcal{O}_i(t_0) \mathcal{O}_j(0) | 0 \rangle v_j^n = N^n$$

$$\Lambda(v_1^n, \cdots, v_m^n, \cdots, \lambda^n) = \sum_{i,j} v_i^{n*} C_{ij}(t) v_j^n - \lambda^n \left[\sum_{i,j} v_i^{n*} C_{ij}(t_0) v_j^n - N^n \right] = \sum_{i,j} v_i^{n*} \left[C_{ij}(t) - \lambda^n C_{ij}(t_0) \right] v_j^n + \lambda^n N^n$$

These are functions of t and t_0

Minimize with respect to the coefficients to get

$$0 = \frac{\partial \Lambda}{\partial v_i^{n\,*}} = \sum_j \left[C_{ij}(t) - \lambda^n C_{ij}(t_0) \right] v_j^n \longrightarrow C(t) v^n = \lambda^n(t, t_0) C(t_0) v^n$$

Properties of the GEVP

Lets move back to the spectral decomposition of the correlates

$$C_{ij}(t) = \langle 0 \,|\mathcal{O}_i(t)\mathcal{O}_j(0)|\, 0\rangle = \sum_n \frac{Z_i^{n*} Z_j^n}{2E_n} e^{-E_n t}$$

Using the GEVP equivalence

$$C_{ij}(t)v_j^m = \sum_n \frac{1}{2E_n} Z_i^{n*} Z_j^n v_j^m e^{-E_n t} = \lambda_m(t, t_0) C_{ij}\left(t_0\right) v_j^m = \sum_n \frac{1}{2E_n} Z_i^{n*} Z_j^n v_j^m \lambda_m(t, t_0) e^{-E_n t_0}$$

Sums of exponentials

Assume that our projection onto the desired state is almost perfect

We have our main exponential plus some small contamination

$$C(t)v^n \sim A_0 e^{-E_n t} + A_1 e^{-E_m t}$$

Properties of the GEVP

Using again the GEVP relation

$$A_0 e^{-E_n t} + A_1 e^{-E_m t} = \lambda^n(t, t_0)(A_0 e^{-E_n t_0} + A_1 e^{-E_m t_0})$$

We get

$$\lambda^n(t, t_0) = \frac{A_0 e^{-E_n t} + A_1 e^{-E_m t}}{A_0 e^{-E_n t_0} + A_1 e^{-E_m t_0}}$$

Remember now that $A_1 << A_0$

$$\lambda^n(t, t_0) = \frac{A_0 e^{-E_n t} + A_1 e^{-E_m t}}{A_0 e^{-E_n t_0} + A_1 e^{-E_m t_0}} \sim (1 - A_1')e^{-E_n(t-t_0)} + A_1' e^{-E_m(t-t_0)}$$

Where (homework)

$$A_1' = \frac{A_1}{A_0} e^{-(E_m - E_n)t_0}$$

Contamination term decreases exponentially with t_0

GEVP optimized operators

Although the should be constants in time, in practice, our vectors exhibit some noise around the plateau

$$u_i^n = \frac{v_i^n}{\sqrt{\mathcal{N}^n}}$$

We first must fit these values to a constant, once our problem has converge to the plateau region

We can then define our optimized operator as

$$\Omega^n = e^{-E_n t_0/2} \sum_i u_i^{n*} \mathcal{O}_i$$

We remove the residual t_0 dependence

Properties of the GEVP

In theory, if t_0 is large enough then *Blossier et al. (2009) 0902.1265*

$$E_n^{\text{eff}}(t, t_0) = E_n + \varepsilon_n(t, t_0)$$

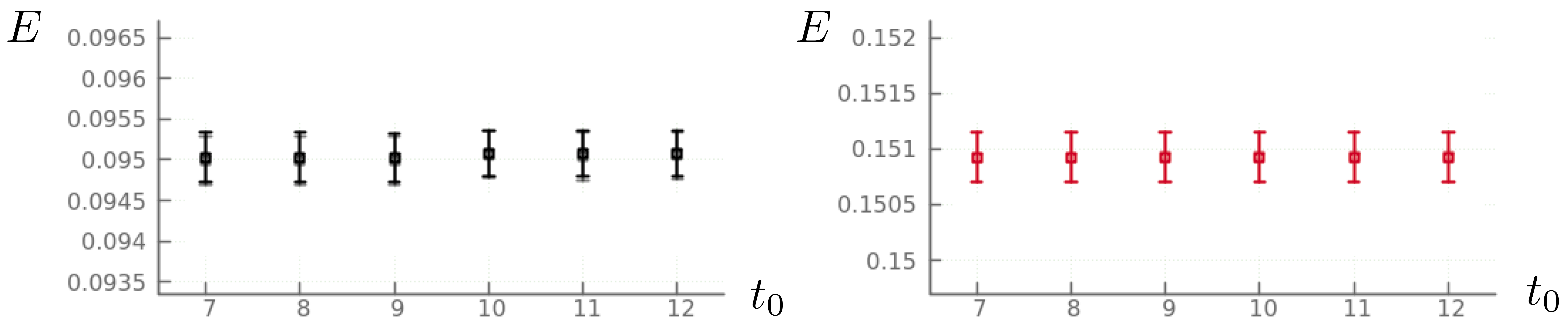
$$\varepsilon_n(t, t_0) = \mathrm{O}\left(\mathrm{e}^{-\Delta E_{N+1,n} t}\right) \qquad \Delta E_{m,n} = E_m - E_n$$

Given N contiguous states in our basis of operators, the contamination should come from higher up

$$E_n^{\text{eff}}(t, t_0) = a^{-1} \log \frac{\lambda^n(t, t_0)}{\lambda^n(t+a, t_0)} = E_n + \mathrm{O}\left(\mathrm{e}^{-\Delta E_{N+1,n} t}\right)$$

The mass can therefore be obtained with a very high degree of accuracy

In practice, if the basis of operators is pretty large (redundant) then results should not depend much on t_0



Solving the GEVP

1- First, assume $C(t_0)$ is positive definite, we then define

$$Q\left(t_0\right) Q^{\dagger}\left(t_0\right) = C\left(t_0\right)$$

2- We can now transform the problem into a simpler, ordinary eigenvalue problem for a fixed t_0 value

$$Q\left(t_0\right)^{-1} C\left(t\right) Q^{\dagger}\left(t_0\right)^{-1} v^n = \lambda^n\left(t, t_0\right) v^n$$

3- We can now solve the eigenvalue problem numerically to produce data for $\lambda(t, t_0)$, for every t

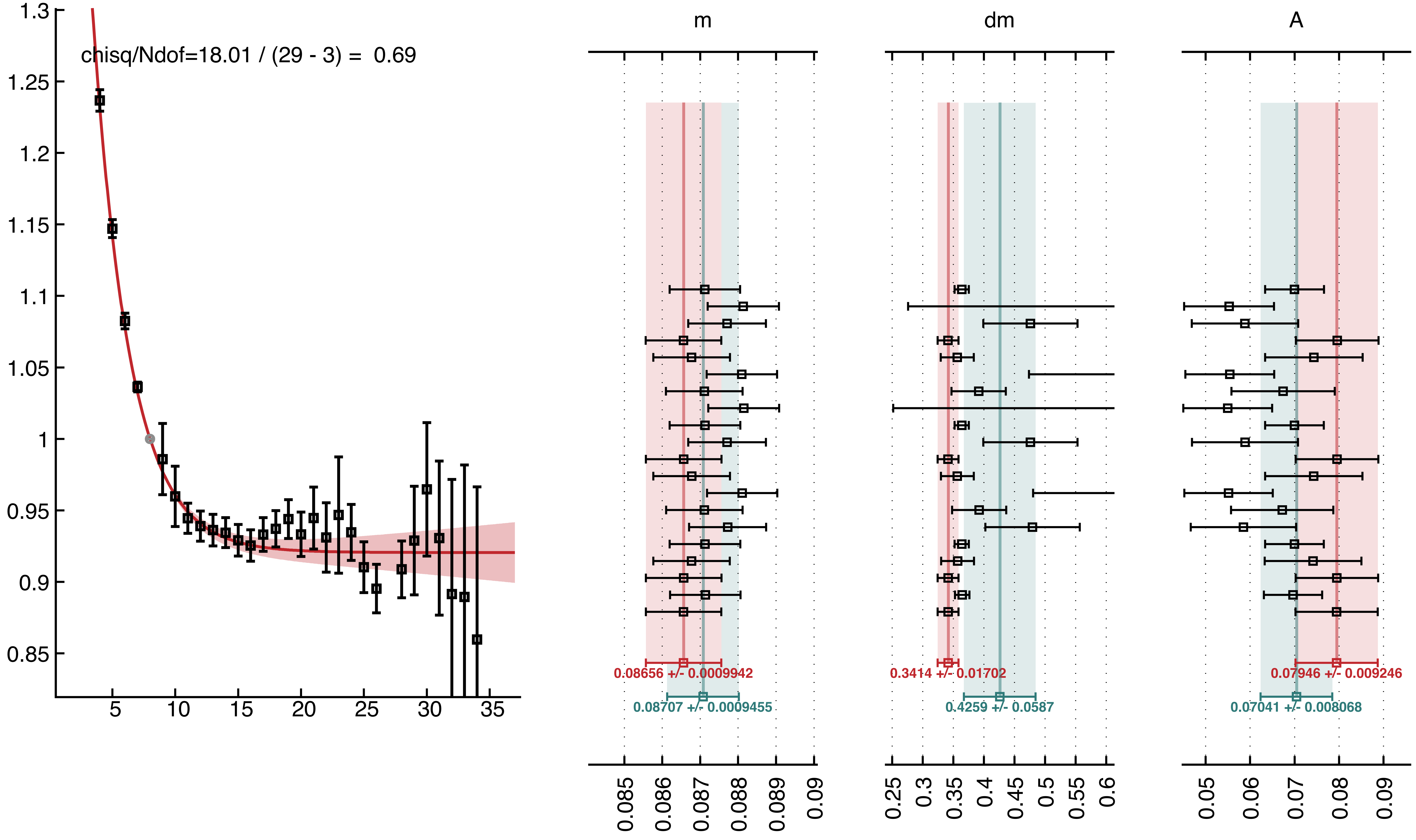
4- We fit these data using a correlated penalty function

$$\chi_n^2 = \sum_{i,j} (\widehat{\lambda(t_i, t_0)}_n - \lambda(t_i, t_0)) \Sigma_{ij}^{-1} (\widehat{\lambda(t_j, t_0)}_n - \lambda(t_j, , t_0))$$

Solving the GEVP

4- We fit these data using a correlated penalty function

$$\chi_n^2 = \sum_{i,j} (\widehat{\lambda(t_i, t_0)}_n - \lambda(t_i, t_0)) \Sigma_{ij}^{-1} (\widehat{\lambda(t_j, t_0)}_n - \lambda(t_j, , t_0))$$



Is $C(t_0)$ always invertible?

No!!

$C(t_0)$ **can have eigenvalues that are compatible with zero for two reasons**

1- Numerical noise: Remember that our signal-to-ratio decreases exponentially (typically) with increasing t

2- Linearly dependent vectors: This is produced when we use two operators in the basis that are highly correlated

Is $C(t_0)$ always invertible?

No!!

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1- Numerical noise: Remember that our signal-to-ratio decreases exponentially (typically) with increasing t

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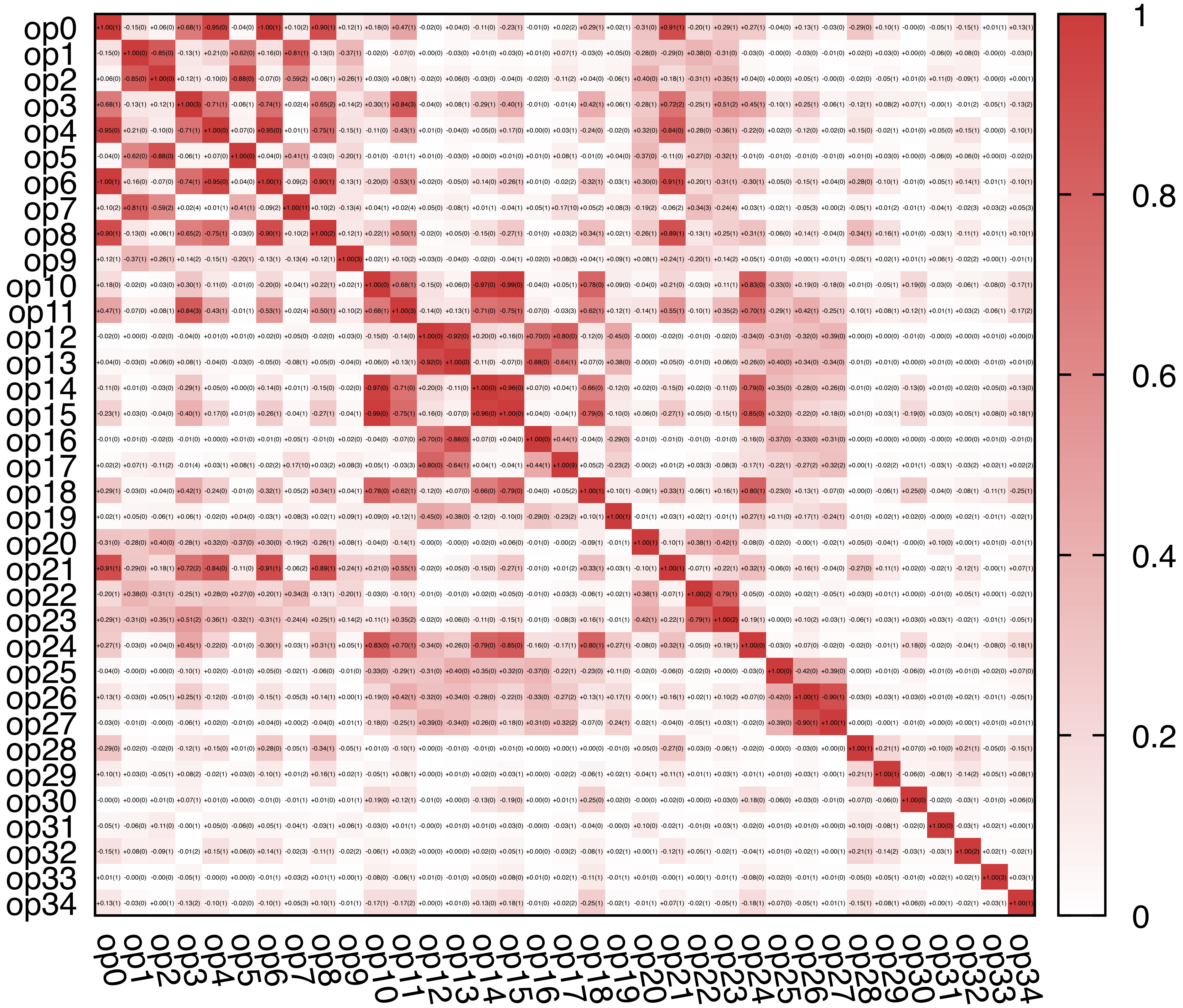
Notice the redundancy in this case

I=0 $\pi\pi$ scattering



GEVP solution as variational method

Given a collection of operators, how do we pick the best combination

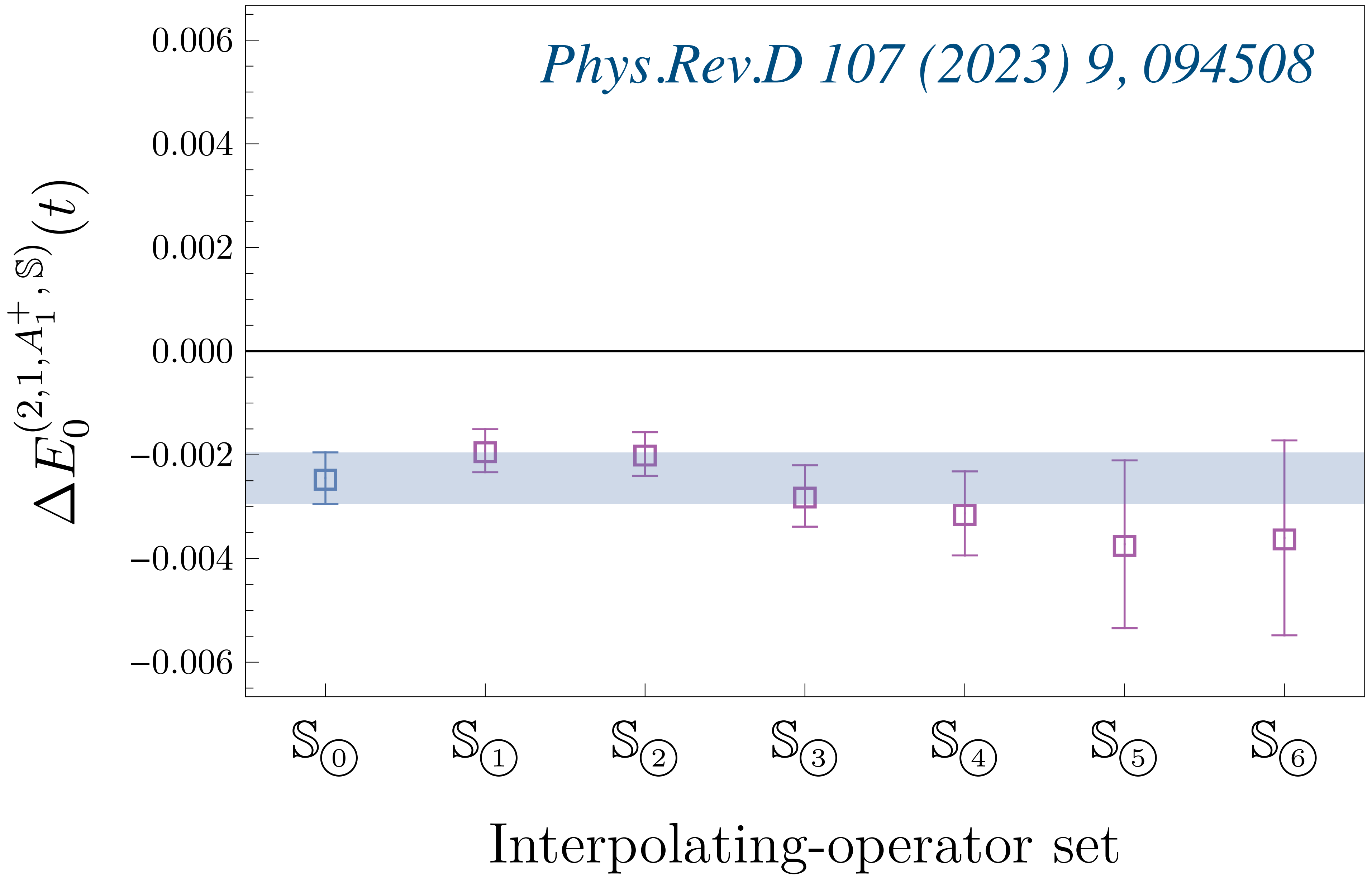
Remember the variational method, for any given state $|\psi\rangle$ of the hamiltonian

$$E_0 \geq \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

In principle, we look for bases that can saturate this bound from above

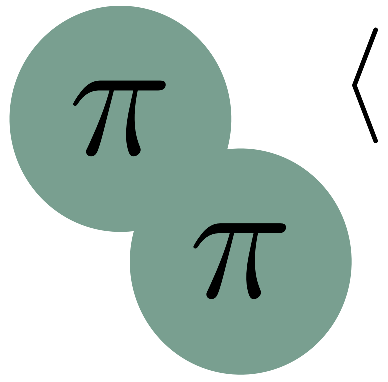
We also need some redundancy to decrease the contamination effects

Good bases will show some stability for the levels produced



Spectroscopy in lattice QCD

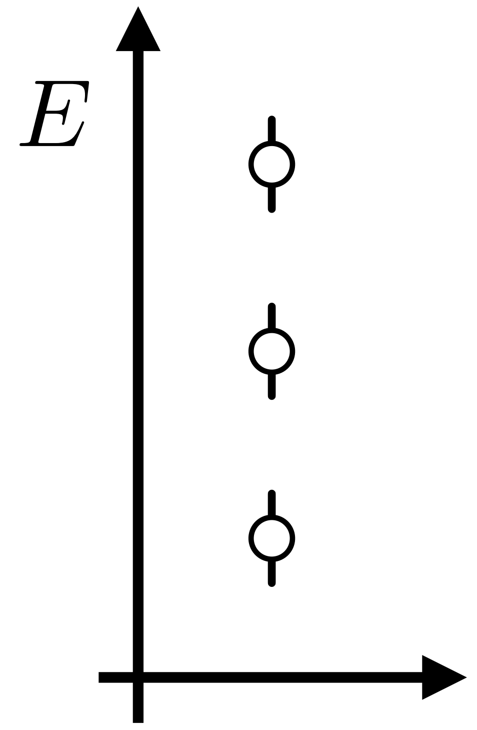
Quantum mechanical time evolution



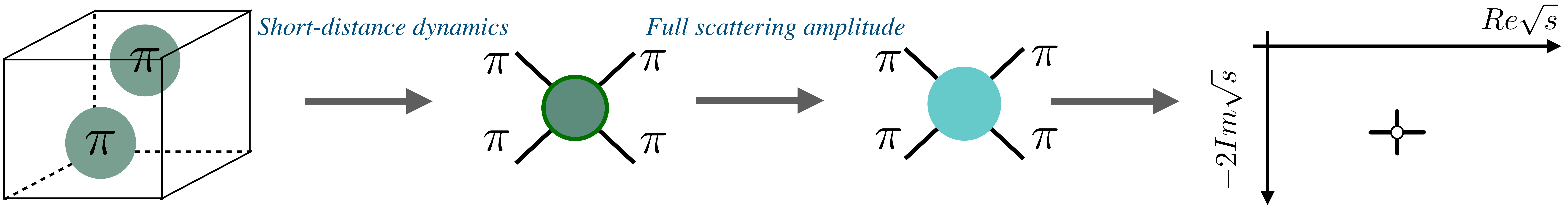
$$\langle O_f(t) O_i^\dagger(0) \rangle \sim \sum_{\mathrm{n}} \underline{e^{-E_{\mathrm{n}} t}} \langle 0 | O_{\mathrm{f}}(0) | \mathrm{n} \rangle \langle \mathrm{n} | O_{\mathrm{i}}^\dagger(0) | 0 \rangle$$

Time is imaginary

Extract $E_n \to s_n$



What's next!!



General $\det\left[\underline{F^{-1}(E_n, L)} + K(s_n)\right] = 0$ $\qquad t_\ell^I(s) = \dfrac{K(s)}{1 - i\rho(s)K(s)}$

Known kinematic function

Questions?

Two-particle operators for lattice at rest

Consider two identical particles, with back-to-back momentum

Momentum directions related by rotations are classified by $\Omega(p)$

Can make a set of operators, $\{\phi(p)\}$ **from** Ω **and these form a (reducible) representation of** O_h

$$\phi = \{\phi(1,0,0), \phi(0,1,0), \phi(0,0,1)\}$$

Reduces into $A_1 \oplus E$

p	irreducible content
$(0,0,0)$	A_1
$(1,0,0)$	$A_1 \oplus E$
$(1,1,0)$	$A_1 \oplus E \oplus T_2$
$(1,1,1)$	$A_1 \oplus T_2$

We can, once again, build operators based on group construction and inductions+subductions

$$\mathcal{O}^{\Lambda\mu}_{\boldsymbol{F}\nu}\left(\vec{P}, t; [p_1, p_2]\right) = \sum_{\substack{\nu_1,\nu_2 \\ \mu_1,\mu_2}} C\begin{pmatrix} \boldsymbol{F}_1 & \boldsymbol{F}_2 & \boldsymbol{F} \\ \nu_1 & \nu_2 & \nu \end{pmatrix}_{\mathrm{SU}(3)} C\begin{pmatrix} \Lambda_1^{\vec{p}_1} & \Lambda_2^{\vec{p}_2} & \Lambda^{\vec{P}} \\ \mu_1 & \mu_2 & \mu \end{pmatrix}_{\mathbf{O}} \sum_{\substack{\vec{p}_i \in \{\vec{p}_i\}^* \\ \vec{p}_1+\vec{p}_2=\vec{P}}} \mathcal{O}^{\Lambda_1\mu_1}_{1F_1\nu_1}(\vec{p}_1, t)\, \mathcal{O}^{\Lambda_2\mu_2}_{2F_2\nu_2}(\vec{p}_2, t)$$

Properties of the GEVP

Completeness

$$C^{-1}\left(t_0\right) = \sum_n v^n v^{n^\dagger}$$

Vectors corresponding to different eigenvalue solutions are orthogonal in the GEVP sense

$$v^{n^\dagger} C\left(t_0\right) v^m = N^n \delta_{nm}$$

If $C(t) = C^\dagger(t)$, the principal values are real functions

$$\lambda_n(t) v^{m^\dagger} C\left(t_0\right) v^n = v^{m^\dagger} C(t) v^n = \left(v^{n^\dagger} C(t) v^m\right)^* = \left(\lambda_m(t) v^{n^\dagger} C\left(t_0\right) v^m\right)^* = \lambda_m^*(t) v^{m^\dagger} C\left(t_0\right) v^n$$

GEVP overlap factors

Start by defining a normalized vector

$$u_i^n = \frac{v_i^n}{\sqrt{\mathcal{N}^n}}$$

Using the orthogonality relation we get

$$\sum_m \left(u_i^{n*} Z_i^m\right) \frac{e^{-E_m t_0}}{2E_m} \left(Z_j^{m*} u_j^k\right) = \delta_{nk}$$

$$\left|u_i^{n*} Z_i^m\right| = \sqrt{2E_n} e^{E_n t_0/2} \delta_{nm}$$

We finally arrive at our Ansatz

$$Z_i^n = \sqrt{2E_n} e^{E_n t_0/2} C_{ij}\left(t_0\right) u_j^n$$