Beam Dynamics in ring accelerators

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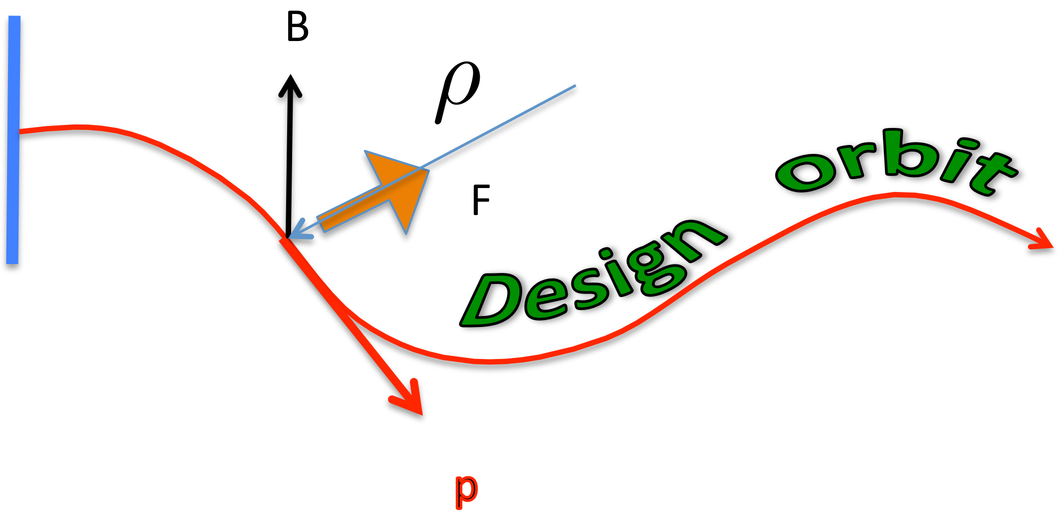
Topics

1) Theory of the linear beam dynamics
2) Nonlinear beam dynamics
3) Resonances
4) High intensity on beams
5) High intensity beams and resonances
6) Single and periodic crossing of resonances by high intensity beams
7) Intensity limitations: space charge and beyond

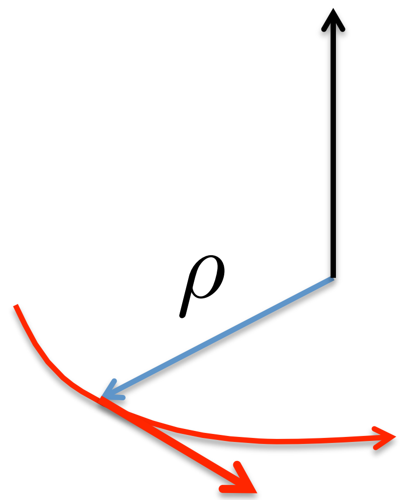
Reference orbit

Lorentz Force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$



Rigidity



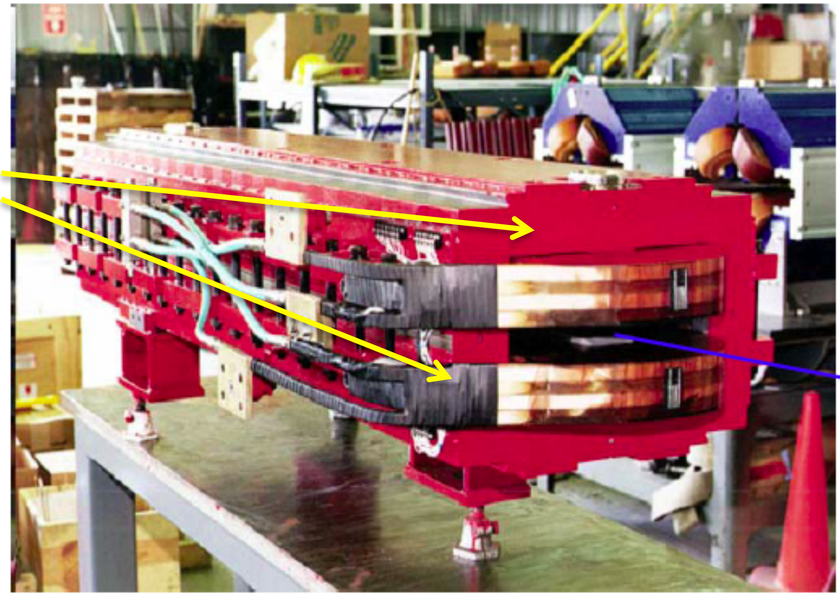
$$p = m\gamma v$$

$$p = \rho q B$$

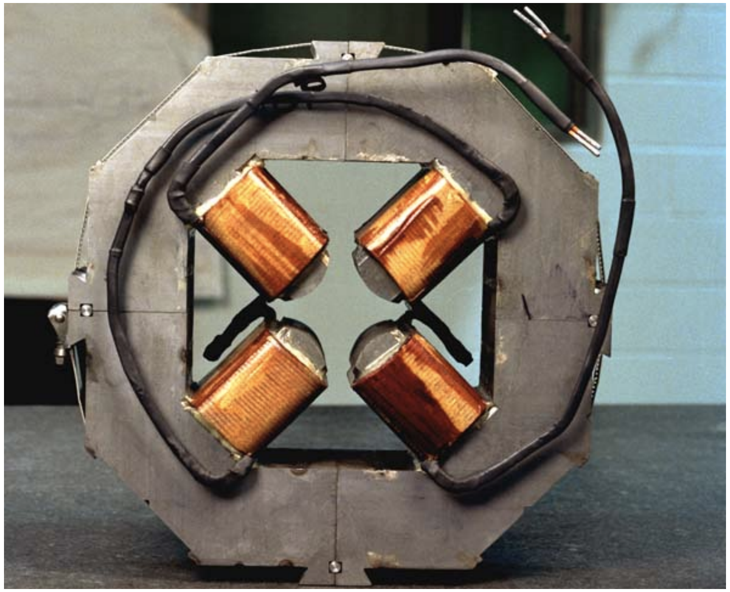
$B\rho$ is called beam rigidity

Particle energy sets Bρ, hence it is established a relation that connects particle energy with magnetic field, in order to keep a reference particle on the reference orbit

Bending magnets



Quadrupole magnet



Field expansion

In absence of current Maxwell equations become

$$\nabla \times B = 0$$

$$\nabla \cdot B = 0$$

therefore there is a potential V

$$B = -\nabla V$$

$$\nabla^2 V = 0$$

In 2 dimensions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

General solution in the complex notation

$$V = Re \sum_{m=0}^{\infty} C_m (x + iy)^m$$

Therefore the magnetic field reads

$$B_x = -\frac{\partial V}{\partial x} = -Re \sum_{m=1}^{\infty} C_m m(x+iy)^{m-1}$$

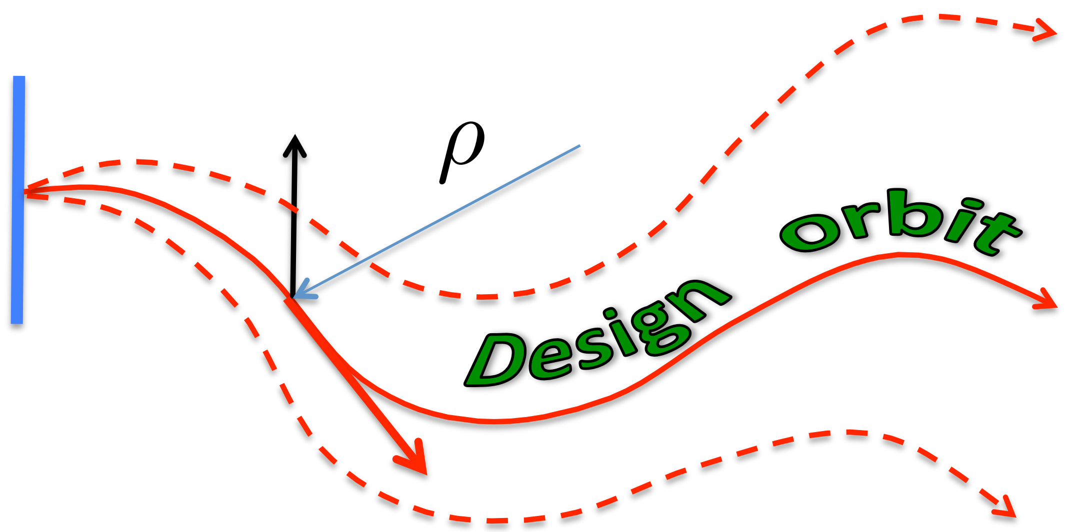
$$B_y = -\frac{\partial V}{\partial y} = -Re \sum_{m=1}^{\infty} C_m mi(x+iy)^{m-1}$$

Standard multipolar expansion

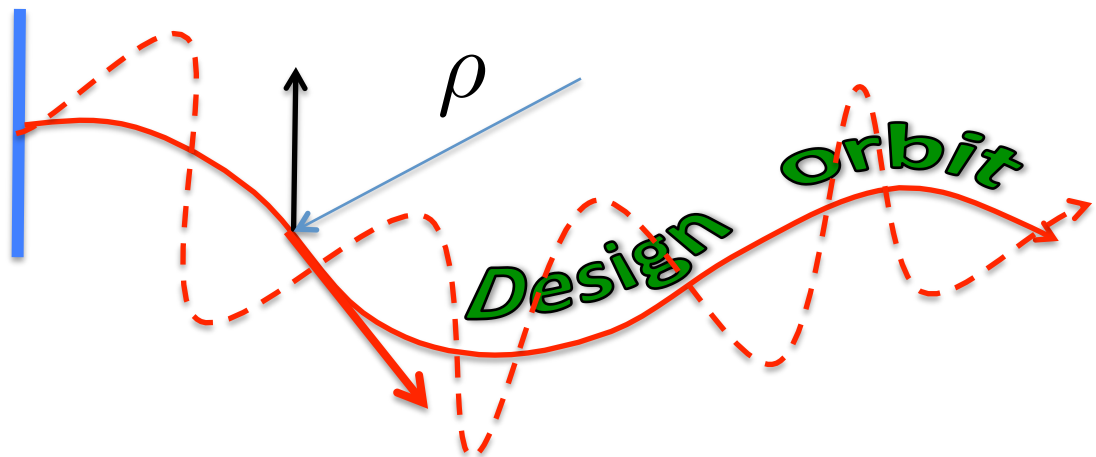
$$B_y + iB_x = B\rho \sum_{n=0}^{\infty} (k_n + ij_n) \frac{(x+iy)^n}{n!}$$

Equation of motion

What happen to a particle that is not the reference particle ?

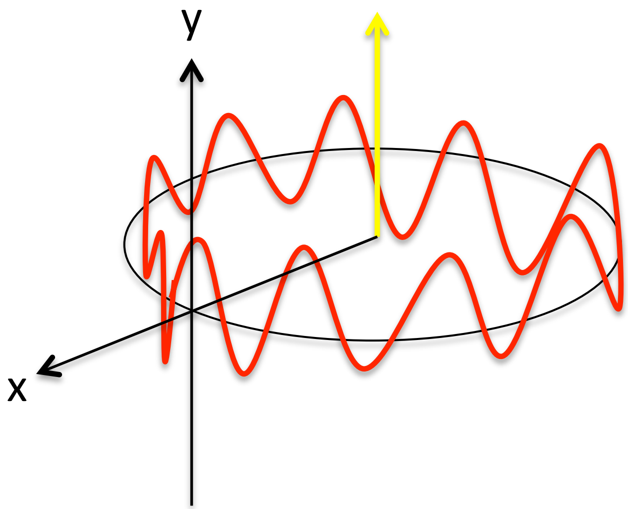


Focusing around the design orbit



equation of motion on the frame of the design orbit

Equation of motion



valid in paraxial approximation

$$x'' - \left(k - \frac{1}{\rho^2}\right) x = \frac{1}{\rho}\frac{\Delta p}{p_0}$$

$$y'' + ky = 0$$

$$x(s) = x_h(s) + x_i(s)$$

$$x(s) = C(s)x_0 + S(s)x_0' + D(s)\frac{\Delta p}{p_0}$$

Dispersion

$$D(s) = \frac{x_i(s)}{\Delta p/p_0}$$

$$D''(s) + k(s)D(s) = \frac{1}{\rho}$$

$$D(s) = S(s)\int_{s_0}^{s} \frac{1}{\rho(t)} C(t)dt - C(s)\int_{s_0}^{s} \frac{1}{\rho(t)} S(t)dt$$

Map approach to transport

$$\begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p_0} \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p_0} \end{pmatrix}_{s_0}$$

Examples

Drift

$$M_x = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Quadrupole

$$M_x = \begin{pmatrix} \cosh\psi & \frac{1}{\sqrt{|k|}}\sinh\psi & 0 \\ \sqrt{|k|}\sinh\psi & \cosh\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_z = \begin{pmatrix} \cos\psi & \frac{1}{\sqrt{|k|}}\sin\psi & 0 \\ -\sqrt{|k|}\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\psi = \sqrt{k}\Delta s$$

Courant-Snyder theory

$$x''(s) + k_x(s)x(s) = 0$$

$$k_x(s) = k_x(s+L)$$

$$x(s) = \sqrt{\epsilon_x \beta_x(s)}\,\sin(\psi_x(s) + \delta_x)$$

Parameterization of the solution

$$\psi'_x \beta'_x + \psi''_x \beta_x = 0$$

$$\frac{\beta''_x}{2} - \frac{(\beta'_x)^2}{4\beta_x} - \beta_x(\psi'_x)^2 + \beta_x k_x = 0$$

$$\psi_x(s) = c_x \int_0^s \frac{ds'}{\beta_x(s')}$$

Twiss parameters

Choice → $c_x = 1$

The requirement of periodicity is also a choice

Tune → $Q_x = \dfrac{\psi_x(L)}{2\pi}$

Matrix formulation $\mathbf{x}(s) = T(s)R(s)\mathbf{w}$

$$\mathbf{w} = \begin{pmatrix} \sqrt{\epsilon_x}\sin\delta_x \\ \sqrt{\epsilon_x}\cos\delta_x \\ \sqrt{\epsilon_y}\sin\delta_y \\ \sqrt{\epsilon_y}\cos\delta_y \end{pmatrix} \quad T(s) = \begin{pmatrix} \sqrt{\beta_x(s)} & 0 & 0 & 0 \\ -\frac{\alpha_x(s)}{\sqrt{\beta_x(s)}} & \frac{1}{\sqrt{\beta_x(s)}} & 0 & 0 \\ 0 & 0 & \sqrt{\beta_y(s)} & 0 \\ 0 & 0 & -\frac{\alpha_y(s)}{\sqrt{\beta_y(s)}} & \frac{1}{\sqrt{\beta_y(s)}} \end{pmatrix}$$

$$\alpha_x = -\frac{1}{2}\beta_x'$$

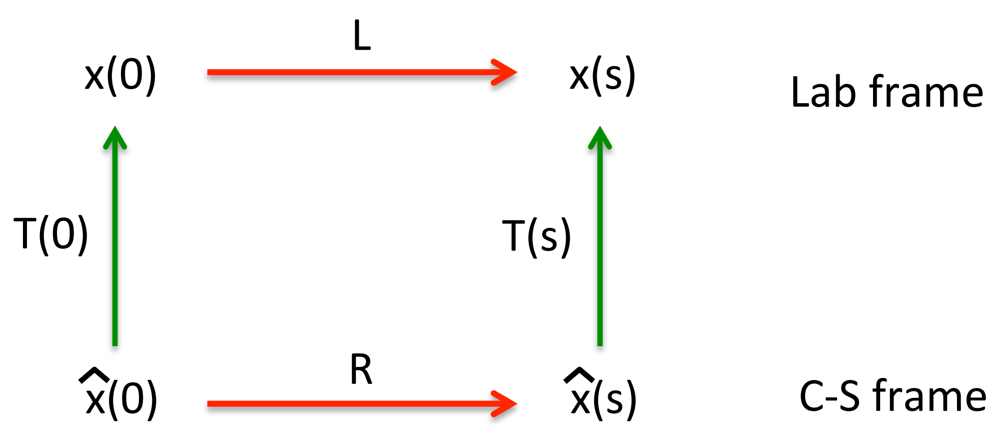
$$R(\psi(s)) = \begin{pmatrix} \cos\psi_x(s) & \sin\psi_x(s) & 0 & 0 \\ -\sin\psi_x(s) & \cos\psi_x(s) & 0 & 0 \\ 0 & 0 & \cos\psi_y(s) & \sin\psi_y(s) \\ 0 & 0 & -\sin\psi_y(s) & \cos\psi_y(s) \end{pmatrix}$$

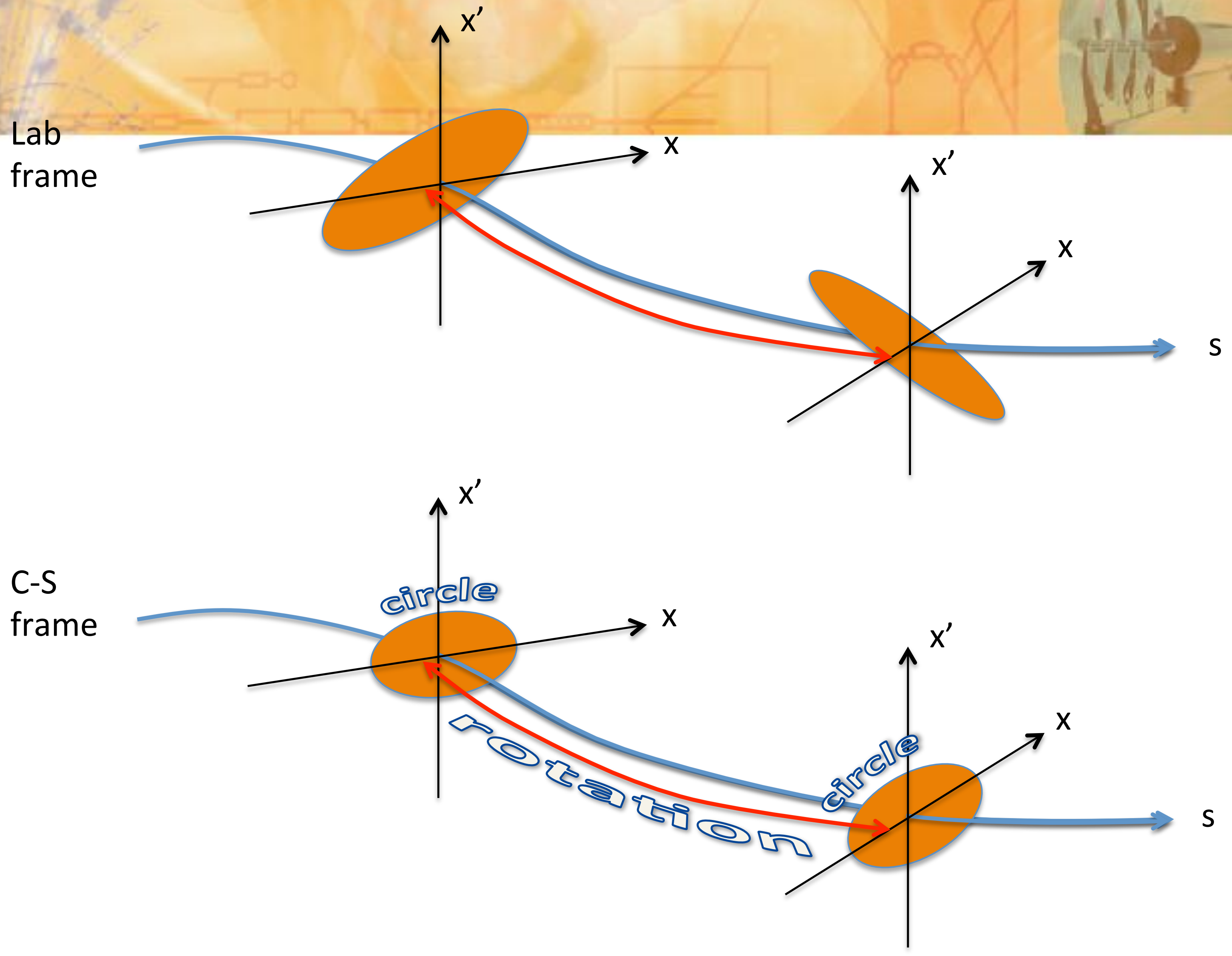
Lab frame, Courant-Snyder frame

$$\mathbf{x}(s) = T(s)R(\psi(s))T(0)^{-1}\mathbf{x}(0)$$

$$\hat{\mathbf{x}}(s) = T(s)^{-1}\mathbf{x}(s) \longrightarrow \hat{\mathbf{x}}(s) = R(\psi(s))\hat{\mathbf{x}}(0)$$

$$\mathbf{x}' = L\mathbf{x} \Longleftrightarrow L = TRT^{-1}$$





Lab
frame
x'
x
x'
x
s
C-S
frame
circle
x'
x
x'
x
circle
rotation
s

One turn map

It is the map after that tracks particles for one turn

$$L = \begin{pmatrix} L_x & 0 \\ 0 & L_y \end{pmatrix}$$

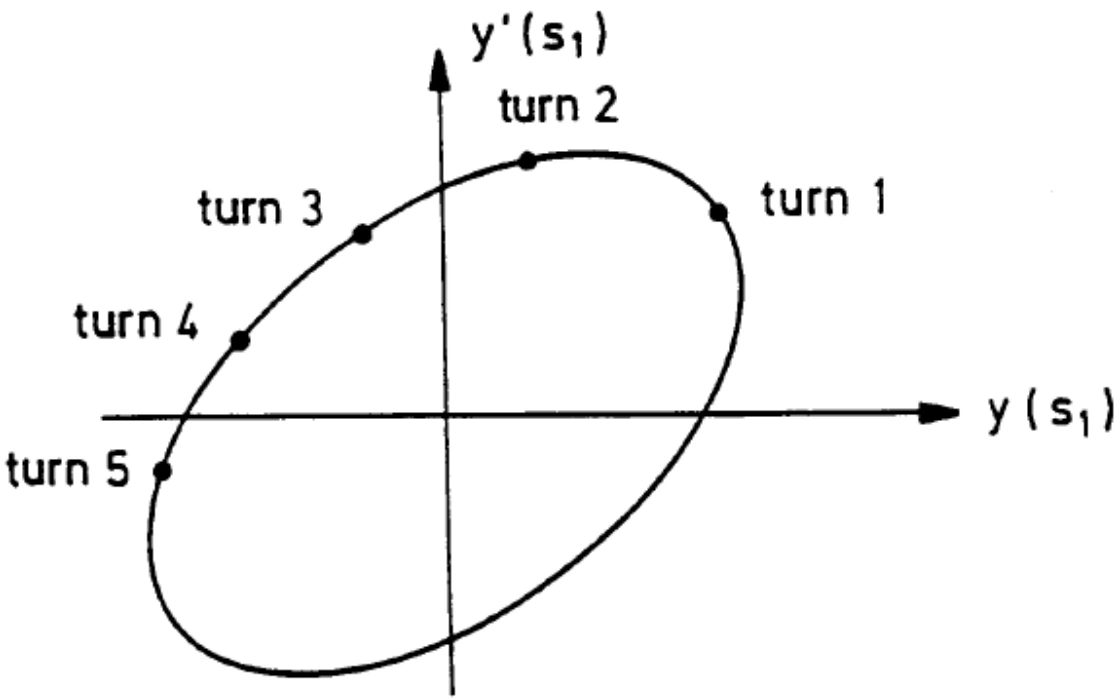
$$L_x = \begin{pmatrix} \cos(2\pi Q_x) + \alpha_x \sin(2\pi Q_x) & \beta_x \sin(2\pi Q_x) \\ -\gamma_x \sin(2\pi Q_x) & \cos(2\pi Q_x) - \alpha_x \sin(2\pi Q_x) \end{pmatrix}$$

$$\gamma_x = \frac{1 + \alpha_x^2}{\beta_x}$$

Dynamics at fix position

In C-S the x-invariant is $\epsilon_x = \hat{x}^2 + \hat{p}_x^2$

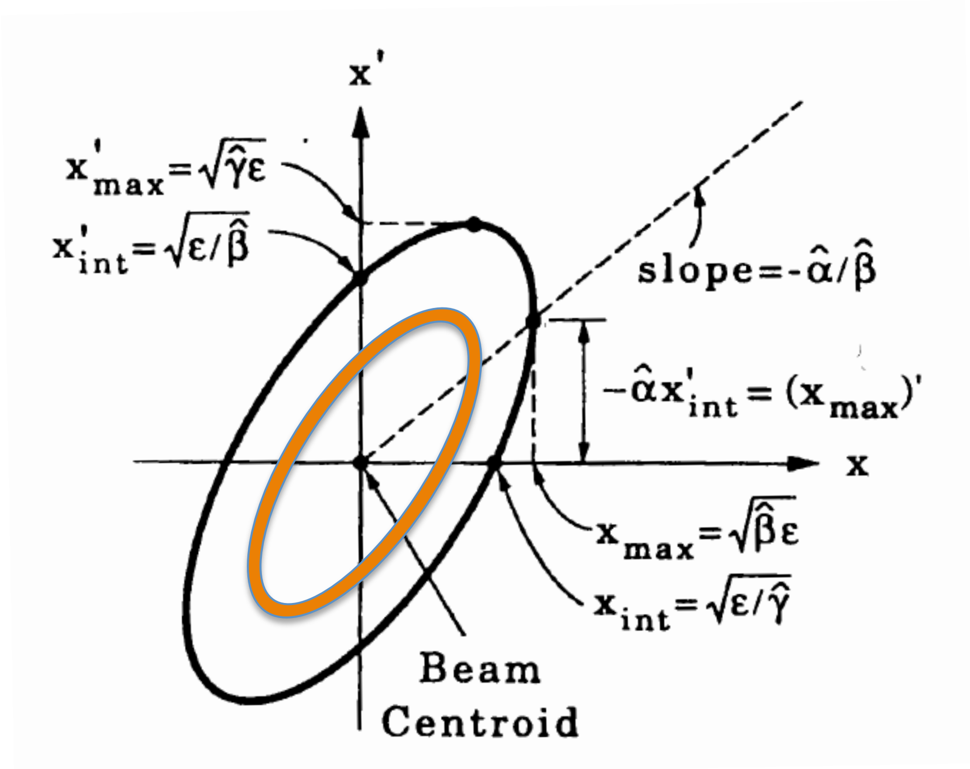
In the Laboratory frame the x-invariant is $\epsilon_x = \gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2$



The collection of all the iterates is called orbit

Beam distribution: matched beam

In a circular accelerator matched means that after one turn the beam is seen exactly the same as before

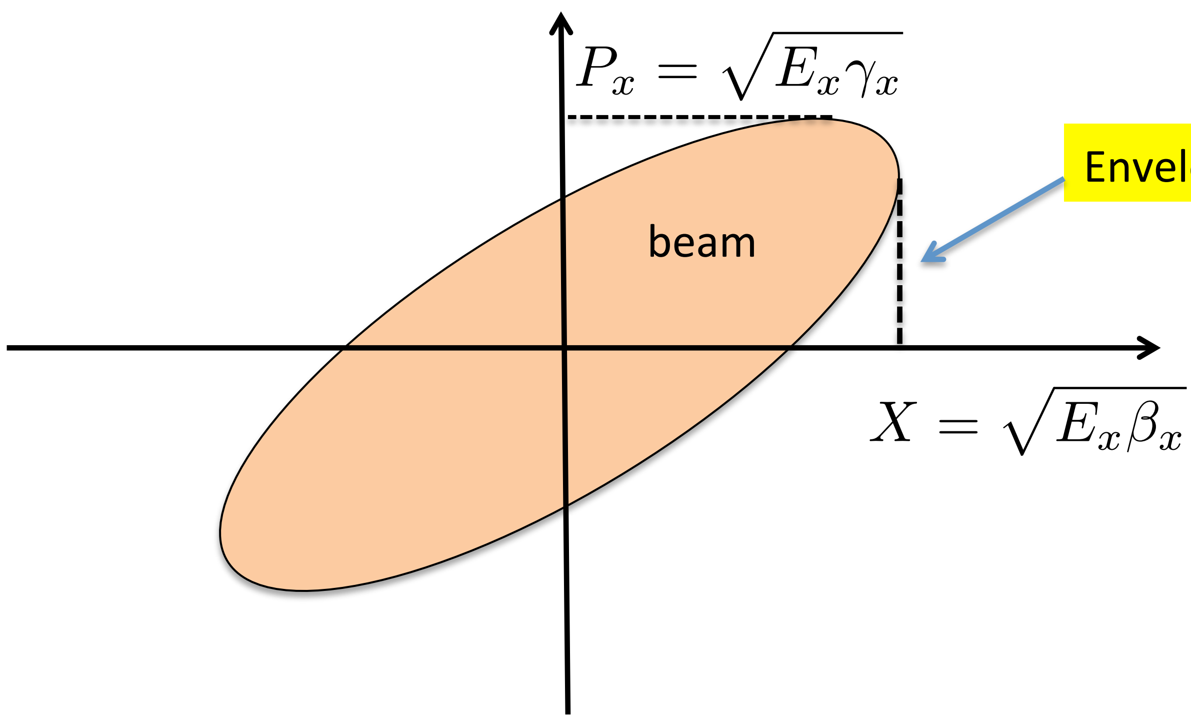


Method: the Courant-Snyder ellipses are uniformly populated

$$\rho(x, p_x, y, p_y) = f(\epsilon_x, \epsilon_y)$$

Beam envelope

For a matched beam, beam envelopes are easy to define



RMS emittance, emittance

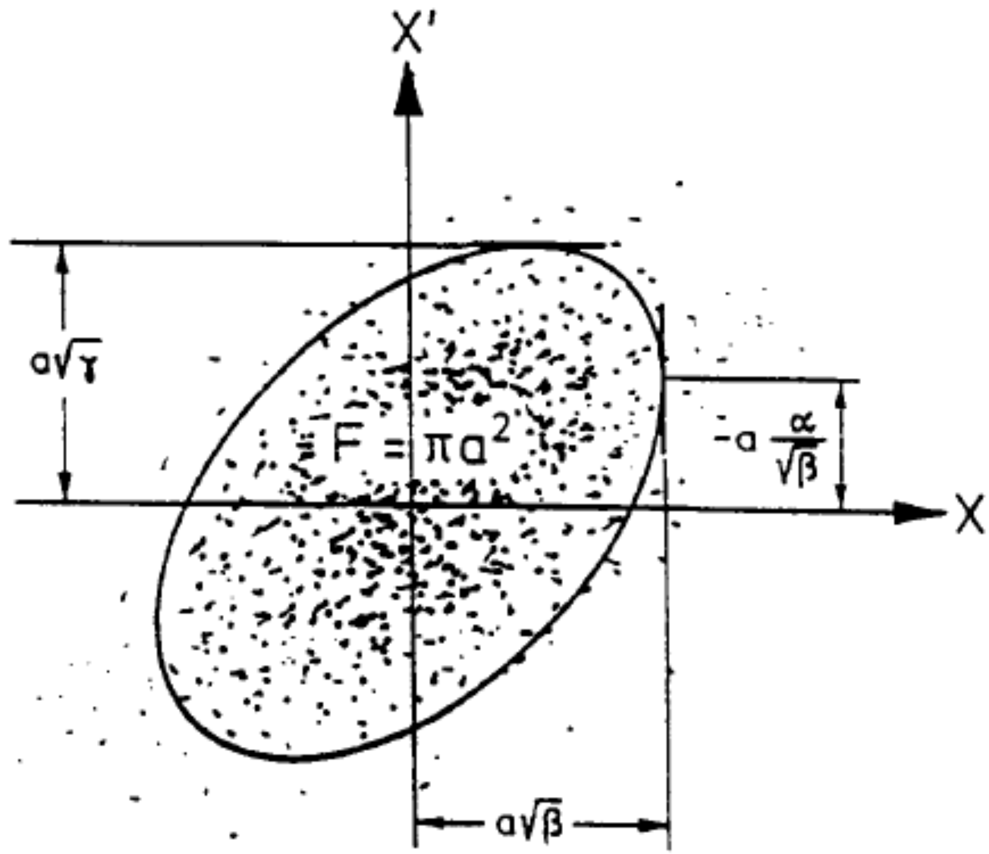
$$\langle x^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i^2$$

$$\langle p_x^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} p_{x,i}^2$$

$$\langle x p_x \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i \, p_{x,i}$$

$$E_x^2 = \langle x^2 \rangle \langle p_x \rangle - \langle x p_x \rangle$$

RMS emittance depends on the beam distribution



RMS Envelope equation

From the equation of motion

$$x + k_x(s)x = 0$$

$$X = \sqrt{\langle x^2 \rangle}$$

$$X'' + k_x(s)X - \frac{E^2}{X^3} = 0$$

Here the RMS emittance E_x remains constant

Perfectly equivalent to

$$\frac{1}{2}\beta_x\beta_x'' - \frac{1}{4}{\beta_x'}^2 + k_x(s)\beta_x^2 = 1$$

Beam distributions

Distribution Function	Definition (Normalized), $f(r_4)$	Ratio of Total Emittance to rms Emittance, $\epsilon_t/\bar{\epsilon}$	Particle Density in Real Space, $r^2 = x^2 + y^2$
Kapchinsky–Vladimirsky (K–V)	$\frac{1}{2\pi^2 a^3}\delta(r_4 - a)$	4	$\frac{1}{\pi a^2}$
Waterbag (WB)	$\frac{2}{\pi^2 a^4}$	6	$\frac{2}{\pi a^2}\left(1 - \frac{r^2}{a^2}\right)$
Parabolic (PA)	$\frac{6}{\pi^2 a^4}\left(1 - \frac{r_4^2}{a^2}\right)$	8	$\frac{10}{3\pi a^2}\left(1 - 3\frac{r^2}{a^2} + 2\frac{r^3}{a^3}\right)$
Gaussian (GA)	$\frac{1}{4\pi^2\delta^4}\exp\left(-\frac{r_4^2}{2\delta^2}\right)$ $\delta^2 = \overline{x^2}$	$\simeq n^2$ if truncated at $n\delta$, $n \geq 4$	$\frac{1}{2\pi\delta^2}\exp\left(-\frac{r^2}{2\delta^2}\right)$

Equation of motion: nonlinearities

A derivation

$$\frac{d}{dt} m\gamma v = qv \times B + qE$$

Change variable t → s

$$\frac{d}{dt} v_x = v \frac{d}{ds} v_x = v^2 \frac{d^2 x}{ds^2}$$

valid in paraxial approximation

$$\frac{d^2 x}{ds^2} = -\frac{q}{m\gamma v} B_y \qquad \frac{d^2 y}{ds^2} = \frac{q}{m\gamma v} B_x$$

But $$\frac{q}{m\gamma} = \frac{v}{\rho B}$$

→

$$\frac{d^2x}{ds^2} = -\frac{B_y}{B\rho}$$

$$\frac{d^2y}{ds^2} = \frac{B_x}{B\rho}$$

Now remember the field expansion

$$B_y + iB_x = B\rho \sum_{n=0}^{\infty} (k_n + ij_n) \frac{(x+iy)^n}{n!}$$

Nonlinear dynamics

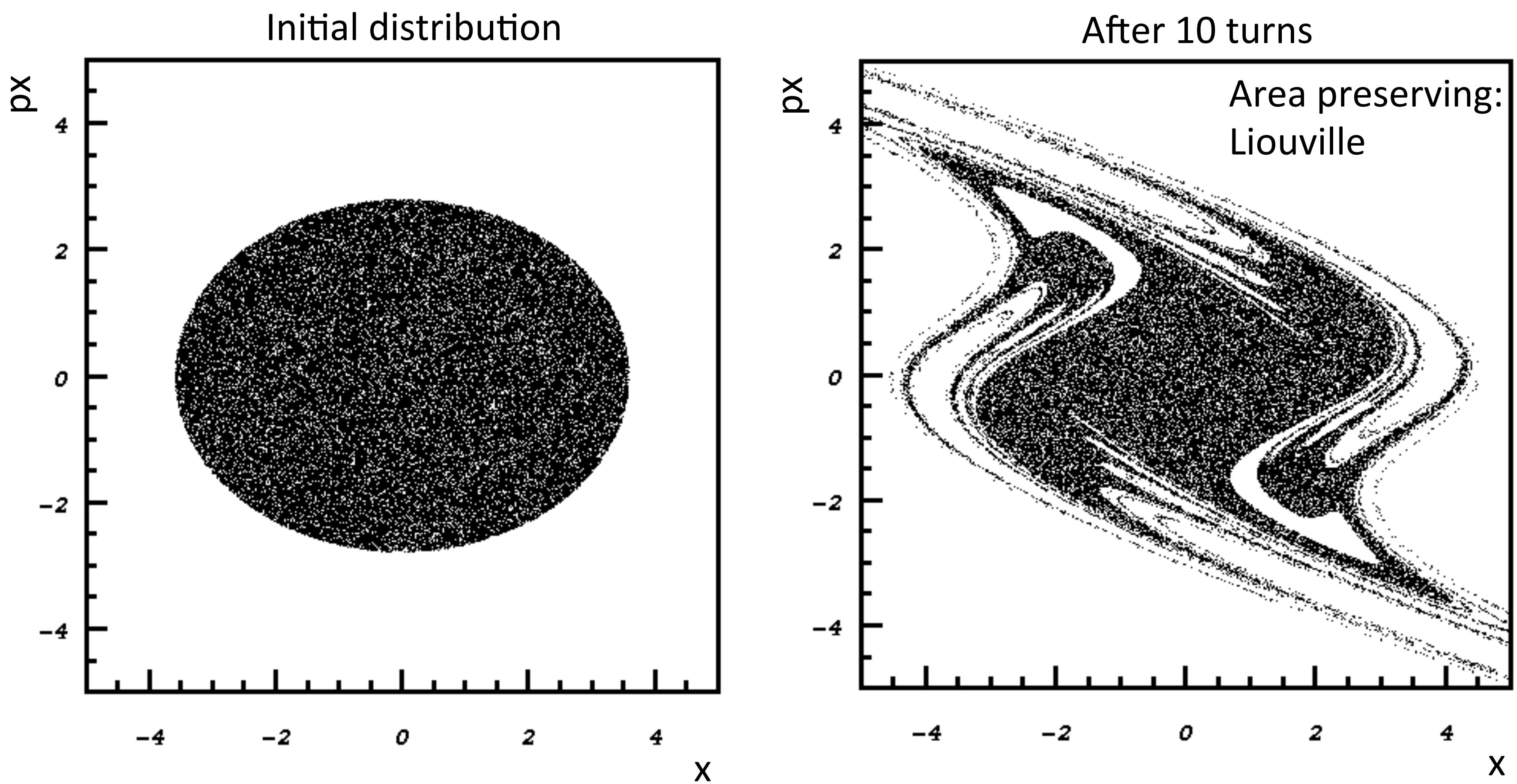
Equation of motion

$$\frac{d^2x}{ds^2} - k_x x = Re\left[\sum_{n=2}^{M}(k_n(s) + ij_n(s))\frac{(x+iy)^n}{n!}\right]$$

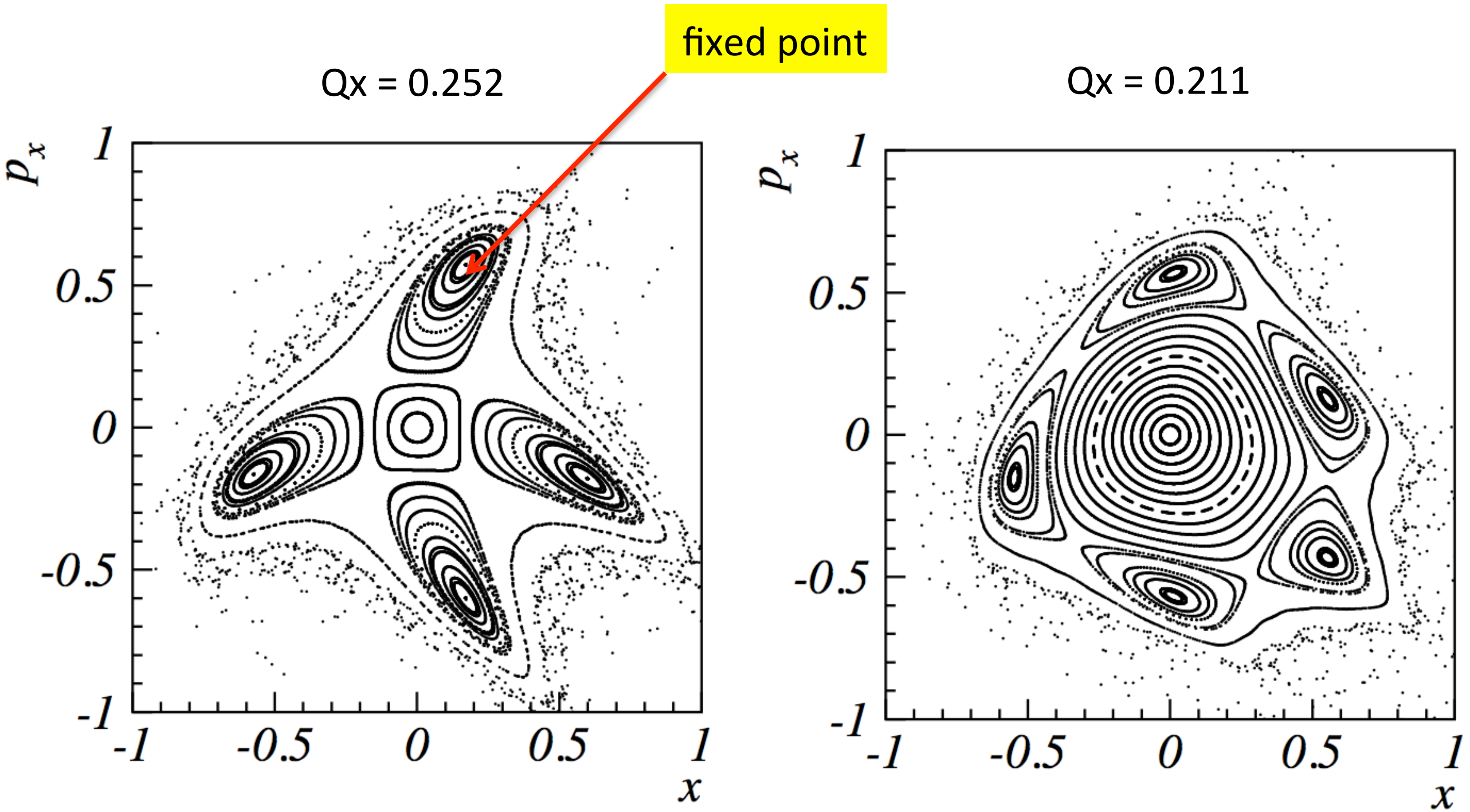
$$\frac{d^2y}{ds^2} - k_y y = -Im\left[\sum_{n=2}^{M}(k_n(s) + ij_n(s))\frac{(x+iy)^n}{n!}\right]$$

Beam in phase space

Example: Qx = 4.252, R =34.4 m, 1 octupole k_3 = 0.1 m^{-4}

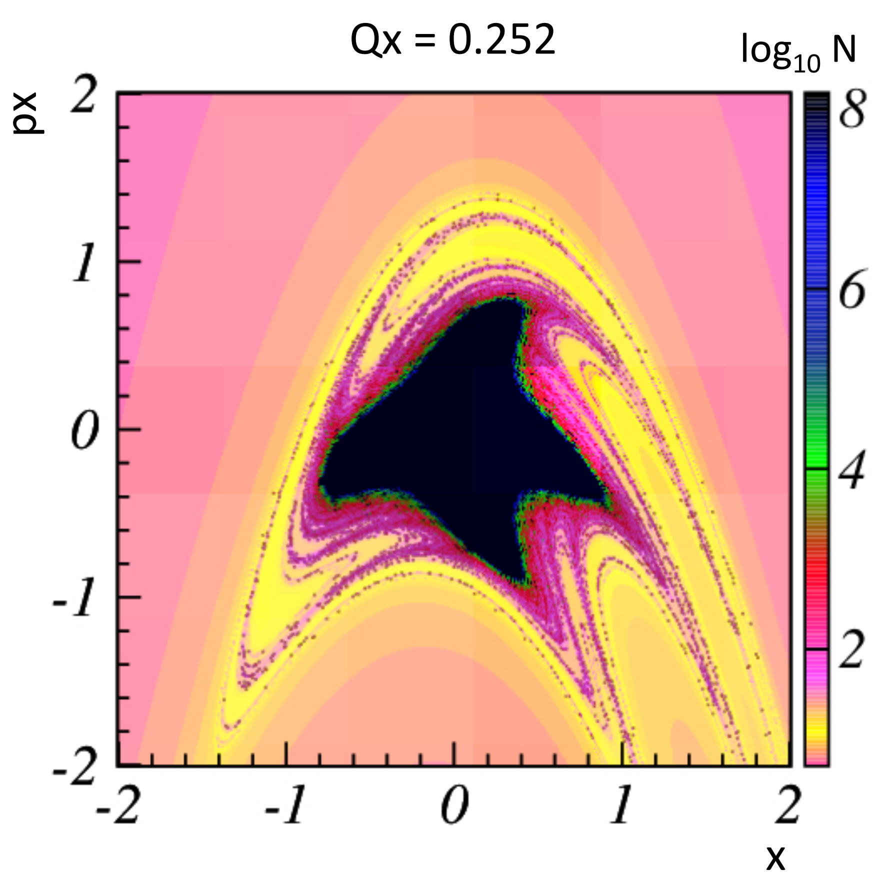


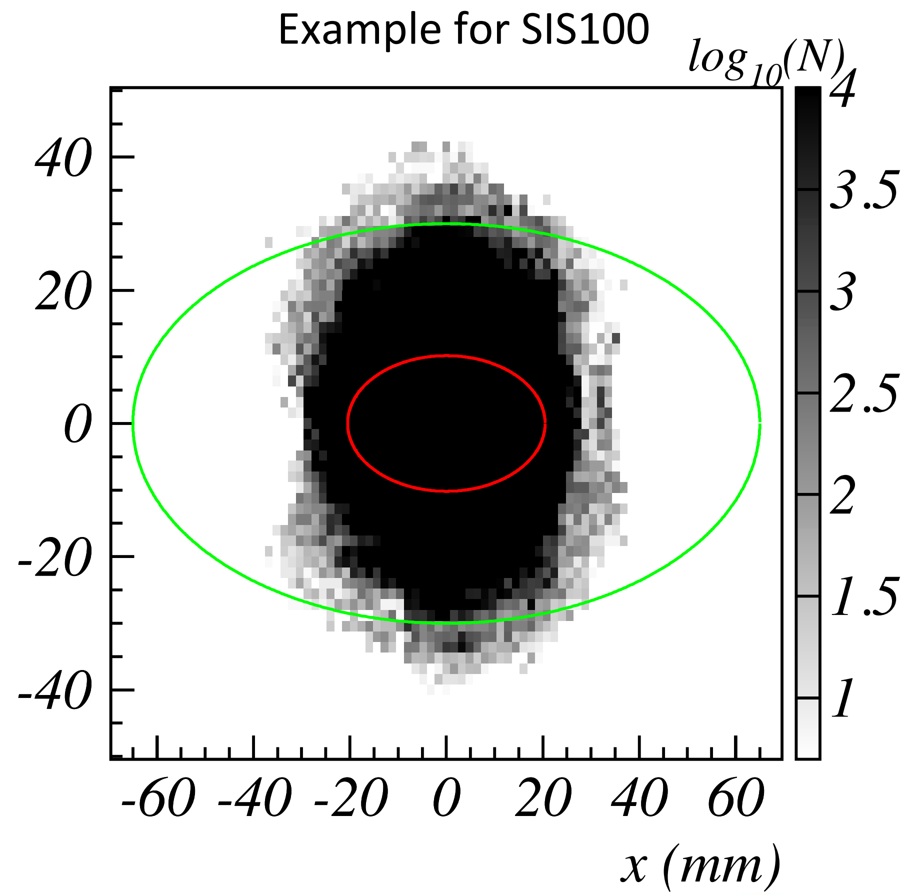
Fixed points, Islands



Dynamic aperture

In a nonlinear system particles at large amplitude the motion becomes unstable





Resonances

$$\frac{d^2x}{d\theta^2} + Q_H^2 x = \epsilon \cos(m\theta)$$

harmonic oscillator

driving force

frequency → Q_H

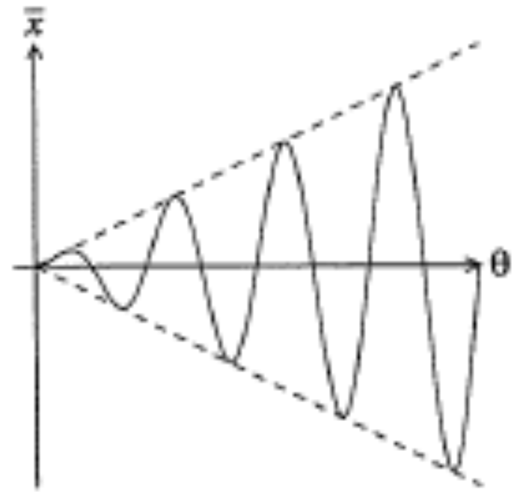
frequency → m

Resonance: $Q_H = m$

Resonances

Integer resonance

$$\frac{d^2x}{d\theta^2} + Q_H^2 x = \epsilon \cos(m\theta)$$



Particular solution

$$\tilde{x} = \frac{\epsilon\theta}{Q_H + m} \sin\left(\frac{Q_H + m}{2}\theta\right) \frac{2}{(Q_H - m)\theta} \sin\left(\frac{Q_H - m}{2}\theta\right)$$

Resonance condition $$Q_H \pm m = 0$$

General treatment

Systematic approach with a perturbative theory. Take this simple Hamiltonian:

$$x = a_1 u\, e^{iQ_x\theta} + \bar{a}_1 \bar{u}\, e^{-iQ_x\theta}$$

$$z = a_2 v\, e^{iQ_z\theta} + \bar{a}_2 \bar{v}\, e^{-iQ_z\theta}$$

u, v are the Floquet's function

a1, a2
are variable invariant!

$$U(a_n,\theta) = \sum_{\nu} \sum_{q+s=\nu} h^{(2\nu)}_{qqss\,0} (a_1\bar{a}_1)^q (a_2\bar{a}_2)^s + \sum_N \sum_{\substack{j,k,\ell,m,p \\ j+k+\ell+m=N \\ j-k=n_x \\ \ell-m=n_z \\ |n_x|+|n_z|=N^* \\ N^*\le N}} \Big\{ h^{(N)}_{jk\ell m\,-p}\, a_1^j \bar{a}_1^k a_2^\ell \bar{a}_2^m \exp\left[i\left(n_x Q_x + n_z Q_z - p\right)\theta\right] + h^{(N)}_{kjm\ell\,p}\, a_1^k \bar{a}_1^j a_2^m \bar{a}_2^\ell \exp\left[-i\left(n_x Q_x + n_z Q_z - p\right)\theta\right] \Big\}.$$

Driving terms

Defining

$$a_1 = r_1 e^{i\phi_1}$$

$$a_2 = r_2 e^{i\phi_2}$$

Equation of motion

$$\frac{dr_1}{d\theta} = n_x|\kappa| r_1^{(|n_x|-1)} r_2^{|n_z|} \sin\psi$$

$$\frac{d\phi_1}{d\theta} = \sum_{\nu} \sum_{q+s=\nu} q h^{(2\nu)}_{qqss\,0}\, r_1^{2(q-1)}\, r_2^{2s} + |n_x||\kappa| r_1^{(|n_x|-2)} r_2^{|n_z|} \cos\psi$$

$$\frac{dr_2}{d\theta} = n_z|\kappa| r_1^{|n_x|} r_2^{(|n_z|-1)} \sin\psi$$

$$\frac{d\phi_2}{d\theta} = \sum_{\nu} \sum_{q+s=\nu} s\, h^{(2\nu)}_{qqss\,0}\, r_1^{2q} r_2^{2(s-1)} + |n_z||\kappa| r_1^{|n_x|} r_2^{(|n_z|-2)} \cos\psi$$

Resonance driving term

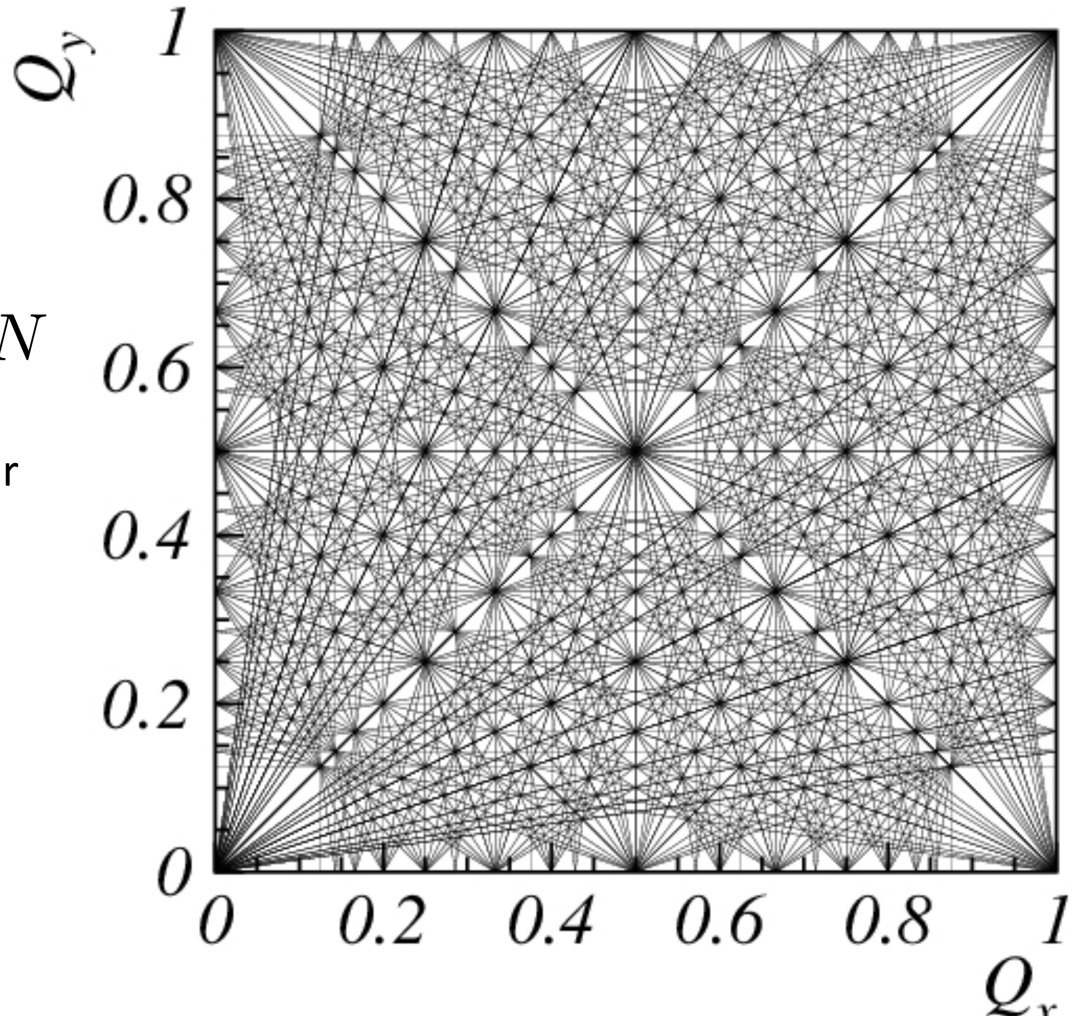
$$\kappa = \frac{1}{2\pi(2R)^{(N/2)}|n_x|!|n_z|!} \int_0^{2\pi} d\theta\, \beta_x^{|n_x|/2}\, \beta_z^{|n_z|/2} \times$$

$$\times \exp\left\{i\left[n_x\mu_x + n_z\mu_z - (n_x Q_x + n_z Q_z - p)\theta\right]\right\} \begin{cases} (-1)^{(|n_z|+2)/2} K_z^{(N-1)} & \text{for } n_z \text{ even} \\ (-1)^{(|n_z|-1)/2} K_x^{(N-1)} & \text{for } n_z \text{ odd} \end{cases}$$

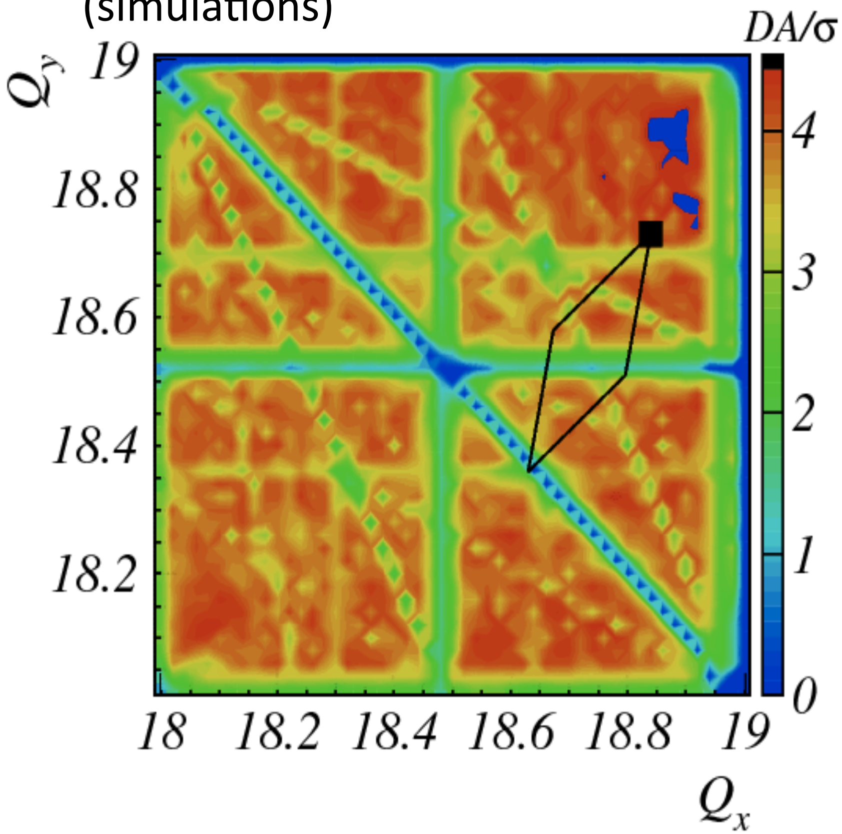
Resonances on working diagram

$$nQ_x + mQ_y = N$$

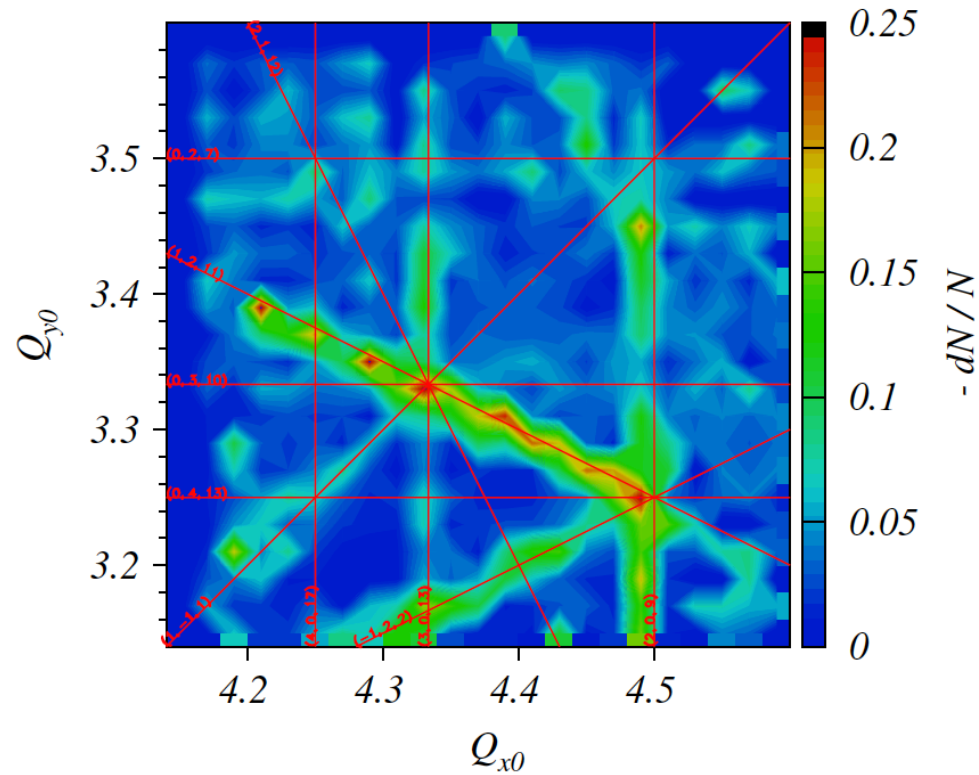
$$|n| + |m| =$$ order



Resonances for SIS100 seen in a DA scan (simulations)

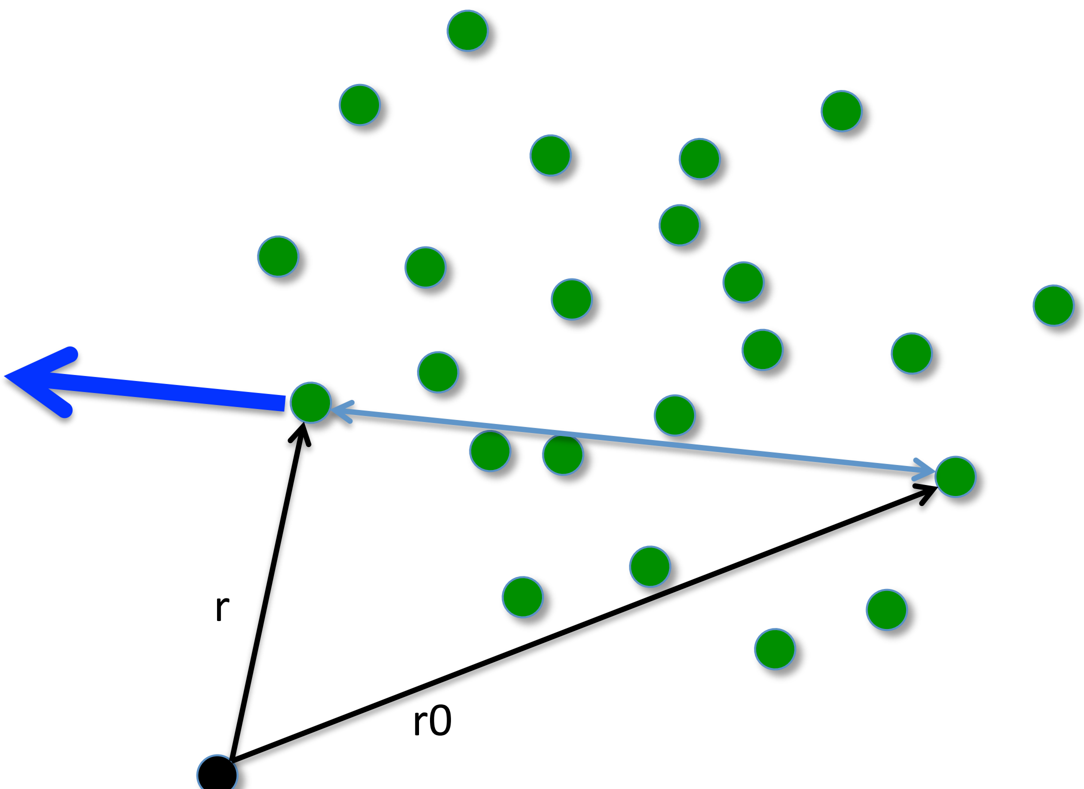


Measured resonances in SIS18



Coulomb forces

$$\vec{\mathcal{E}} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{|\vec{r}-\vec{r}_0|^3}(\vec{r}-\vec{r}_0)$$

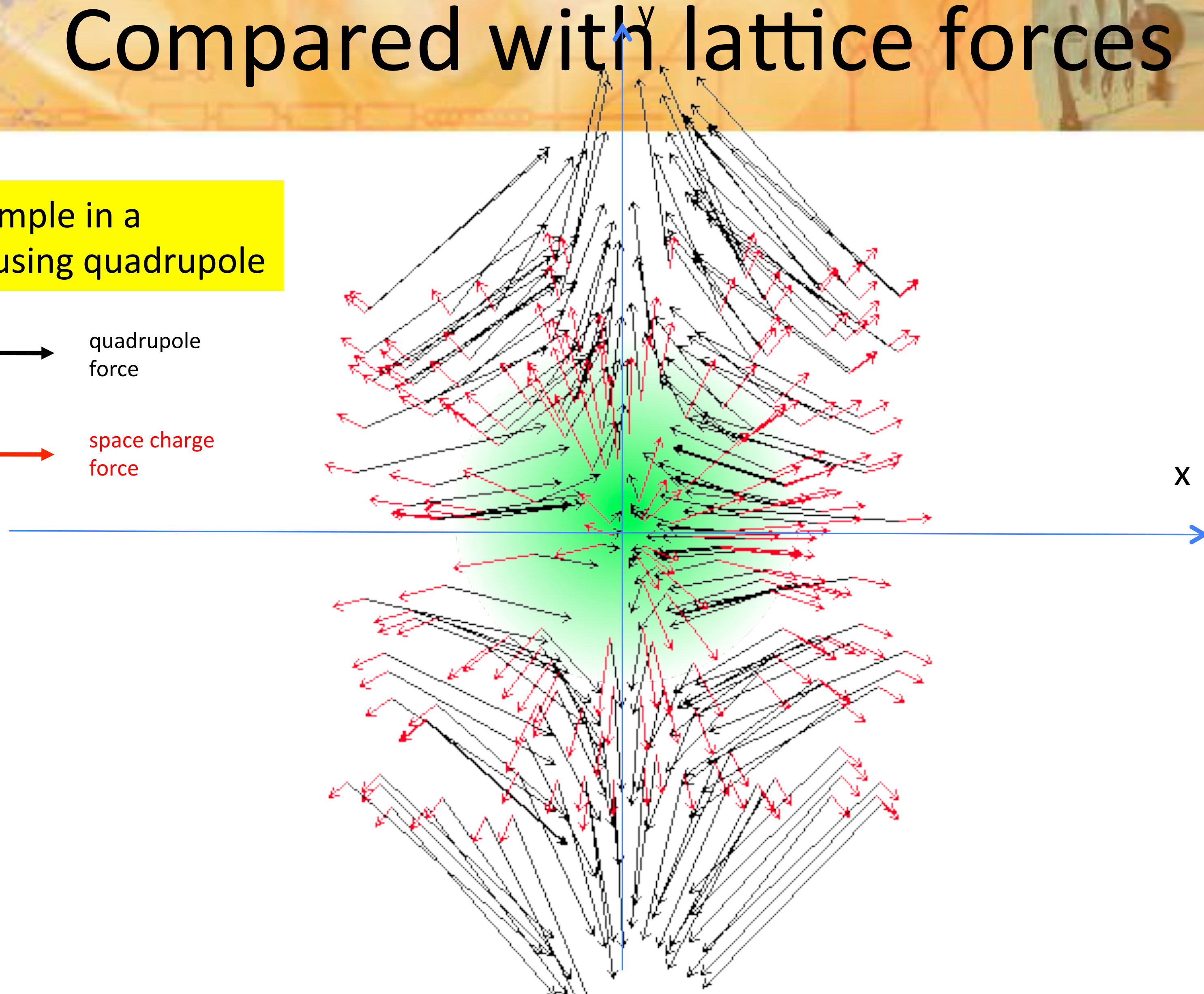


Compared with lattice forces

Example in a focusing quadrupole

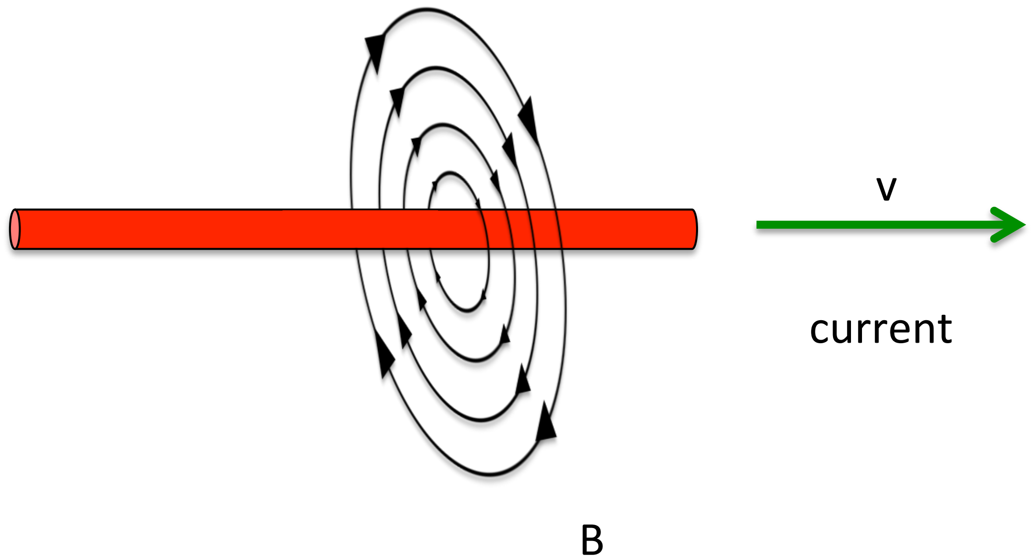
quadrupole force

space charge force



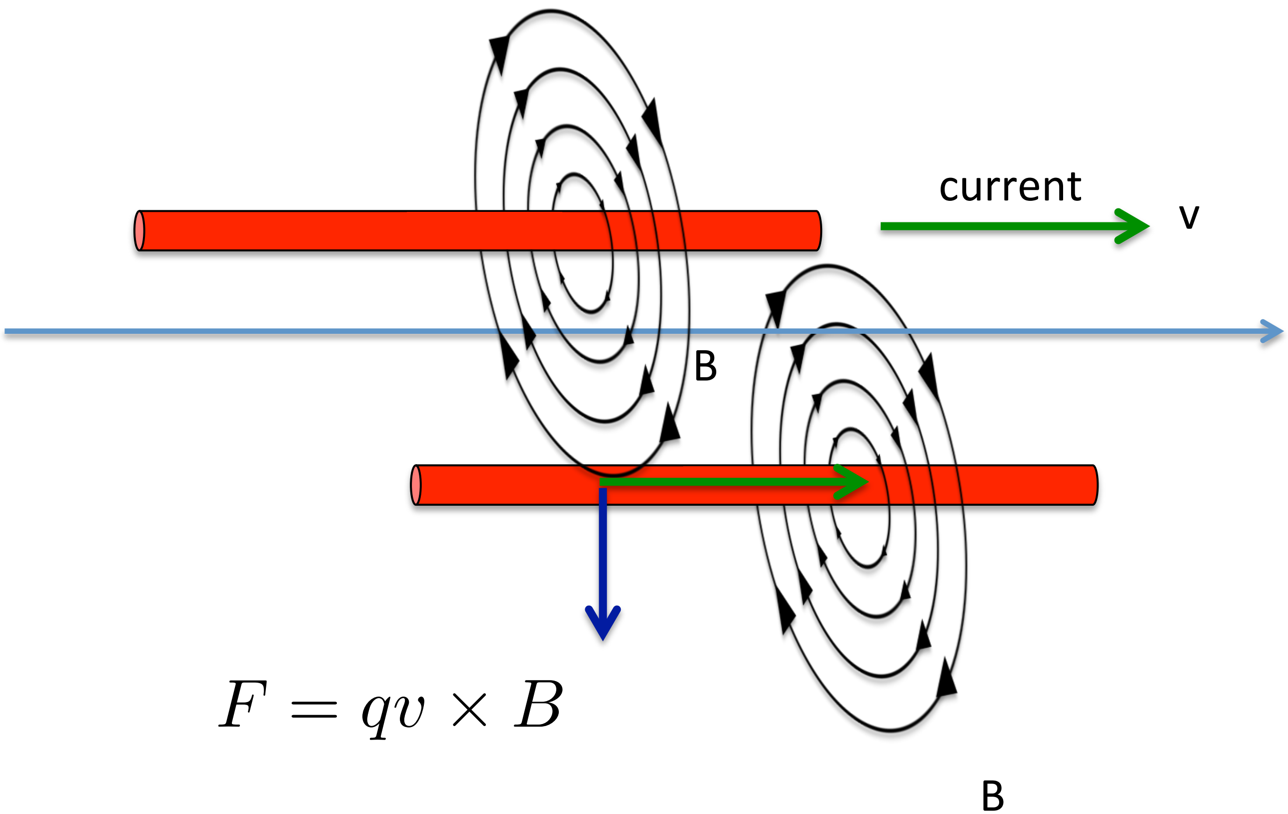
Giuliano Franchetti

High intensity beams: space charge



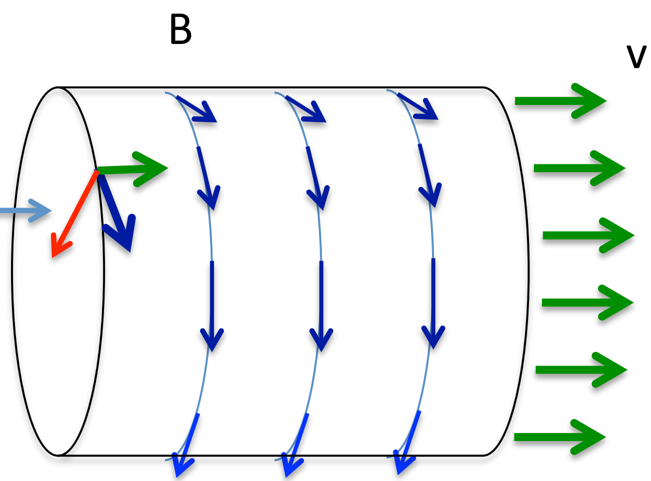
$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Self-field



$$F = qv \times B$$

self-field is parallel to space charge force



$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

Self-field

$$E_r = \frac{q}{\epsilon_0}\frac{1}{r}\int_0^r rn(r)dr \qquad B_\theta = \frac{q\beta}{c\epsilon_0}\frac{1}{r}\int_0^r rn(r)dr$$

Total field on a particle

$$E_r = \left(\frac{q}{\epsilon_0} - \frac{q\beta^2}{\epsilon_0}\right)\frac{1}{r}\int_0^r rn(r)dr = \frac{q}{\gamma^2\epsilon_0}\frac{1}{r}\int_0^r rn(r)dr$$

Effect of the self-field
it damps the space charge field as γ^2

For a Gaussian coasting beam

$$E_r = \frac{\lambda q}{2\pi\epsilon_0\gamma^2}\frac{1}{r}\left[1 - \exp\left(-\frac{r^2}{2\sigma_r^2}\right)\right]$$

Equation of motion

$$\frac{d}{dt}\frac{v_x}{v} = -\frac{B_y}{B\rho} + \frac{q}{m\gamma v}E_x \qquad \frac{d}{dt}\frac{v_y}{v} = \frac{B_x}{B\rho} + \frac{q}{m\gamma v}E_y$$

Linear lattice

$$B_y + iB_x = B\rho k_1(x+iy)$$

$$B_x = B\rho k_1 y$$

$$B_y = B\rho k_1 x$$

$$\frac{d^2x}{ds^2} = -k_1 x + \frac{q}{m\gamma v^2}E_x \qquad \frac{d^2y}{ds^2} = k_1 y + \frac{q}{m\gamma v^2}E_y$$

In one plane

$$\frac{d^2x}{ds^2} + k_1 x = \frac{q^2\lambda}{2\pi\epsilon_0 m\gamma^3\beta^2c^2}\frac{x}{r^2}\left[1 - \exp\left(-\frac{r^2}{2\sigma_r^2}\right)\right]$$

For small amplitudes

Usually this is a perturbation

$$\frac{d^2x}{ds^2} + k_1 x = \frac{q^2\lambda}{2\pi\epsilon_0 m\gamma^3\beta^2c^2}\frac{x}{2\sigma_r^2}$$

Perveance

$$K = \frac{qI}{2\pi\epsilon_0 m\gamma^3\beta^3c^3}$$

RMS equivalence

Envelope equation including space charge

$$X'' + k_x X - \frac{E_x^2}{X^3} - \frac{e}{mN}\frac{\langle x\mathcal{E}_x\rangle}{Y} = 0$$

$$Y'' + k_y Y - \frac{E_y^2}{Y^3} - \frac{e}{mN}\frac{\langle y\mathcal{E}_y\rangle}{Y} = 0$$

For ellipsoidal 2D beams

$$\mathcal{E}_x(x,y) = 2\pi eabx \int_0^\infty \frac{n(T)ds}{(a^2+s)^{3/2}(b^2+s)^{1/2}}$$

$$\mathcal{E}_y(x,y) = 2\pi eaby \int_0^\infty \frac{n(T)ds}{(a^2+s)^{1/2}(b^2+s)^{3/2}}$$

$$T = \frac{x^2}{a^2+s} + \frac{y^2}{b^2+s}$$

It is proved that

$$\langle x\mathcal{E}_x\rangle = \frac{eN^2 X}{X+Y}$$

for any distribution (matched!)!

$$X'' + k_x X - \frac{E_x^2}{X^3} - \frac{eN}{m}\frac{1}{X+Y} = 0$$

$$Y'' + k_y Y - \frac{E_y^2}{Y^3} - \frac{eN}{m}\frac{1}{X+Y} = 0$$

RMS equivalence

2D beams with the same RMS sizes have the same evolution of RMS sizes

It is again assumed that
the beam emittance are preserved

Evolution of beam distribution

The evolution of a beam distribution is determined by Vlasov equation

$$\frac{\partial f}{\partial t} + \dot{x}\frac{\partial f}{\partial x} + \dot{p}_x \frac{\partial f}{\partial p_x} = 0$$

For example, for a distribution

$$f(x, p_x) = F(\beta_x p_x^2 + 2\alpha_x x p_x + \gamma_x x^2)$$

for the equation of motion $\ddot{x} = -k(s)x$

Satisfies the Vlasov equation for any F(). (Why not prove it?)

Note that matching and self-consistency are separate issues

The KV distribution

The KV distribution is defined as

$$f(x,p_x,y,p_y,s)=\delta\left(\frac{\beta_x p_x^2+2\alpha_x x p_x+\gamma_x x^2}{E_x}+\frac{\beta_y p_y^2+2\alpha_y y p_y+\gamma_x x^2}{E_y}-1\right)$$

This distribution satisfies Vlasov equation (using the self-consistent Twiss parameters)

Exercise: prove that the projection in any plane is uniform

The space charge force is always linear !

Twiss parameters with space charge

For a KV distribution the space charge acts like a linear force that can be included in the Courant-Snyder theory

$$F_x = \frac{Kx}{\sqrt{\beta_x E_x}(\sqrt{\beta_x E_x} + \sqrt{\beta_y E_y})}$$

$$\frac{d^2x}{ds^2} + k_x(s)x = \frac{Kx}{\sqrt{\beta_x E_x}(\sqrt{\beta_x E_x} + \sqrt{\beta_y E_y})}$$

Now the beta function is function of the intensity

$$X(s) = \sqrt{\beta_x(s) E_x}$$
$$Y(s) = \sqrt{\beta_y(s) E_y}$$

$$\beta_x(s)$$
$$\beta_y(s)$$

Now the Twiss parameters that include space charge satisfy the condition

$$\sigma_r(s) = \sqrt{\beta_x(s)\epsilon_x} \quad \Rightarrow \quad \beta_x(s)$$

The same beta function

A method

$$\beta_{x,0}(s) \to \beta_{x,1}(s) \to \beta_{x,2}(s) \to \ldots . \to \beta_{x,\infty}(s)$$

Self-consistent Twiss

Self-consistent distributions

The evolution of the distribution is given by Vlasov equation

BUT

Vlasov equation has inside the space charge force

$f(\varepsilon_x, \varepsilon_y, t_0)$

↓

$E(x, y, f(\varepsilon_x, \varepsilon_y, t_0))$

→

Vlasov uses

k_x, k_y

$E(x, y, f(\varepsilon_x, \varepsilon_y, t_0))$

→

$f(\varepsilon_x, \varepsilon_y, t_1)$

If f() is independent on t, f() is said "self-consistent"

The KV distribution is constructed "t" independent → it is "self-consistent" (prove it)

Tune-shift

Detuning
$$\Delta Q_x = \frac{1}{4\pi}\oint \beta_x(s) k_{sc,x}(s) ds$$

Prove this relation using the matrix formalism

Therefore
$$\Delta Q_x = \frac{K}{4\pi}\oint \frac{\beta_x(s)}{2\sigma_r(s)^2} ds = \frac{K}{4}\frac{R}{\epsilon_x^2}$$

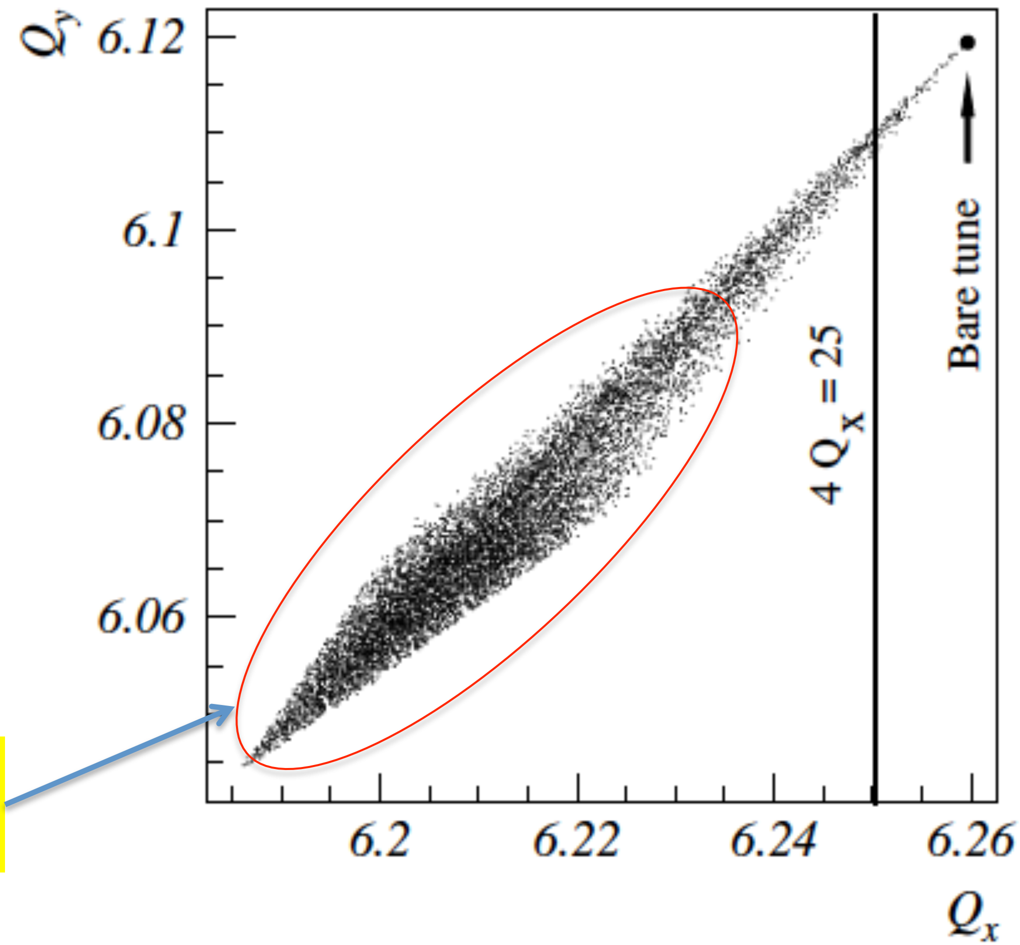
If the beam is not axi-symmetric

$$\Delta Q_x = \frac{\overline{\beta}_x}{2} R \frac{K}{\sqrt{\overline{\beta}_x \epsilon_x}\left(\sqrt{\overline{\beta}_x \epsilon_x} + \sqrt{\overline{\beta}_y \epsilon_y}\right)}$$

Space charge tune-spread

Each particle experiences a different tune-shift

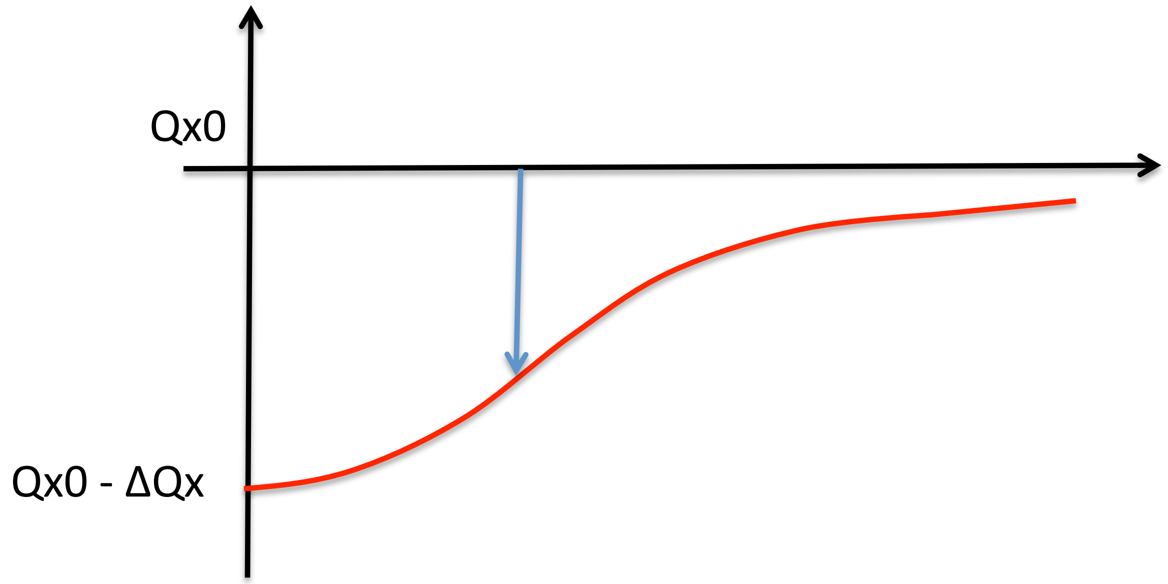
The space charge creates a tune-spread



this distribution depends on the beam distribution

Amplitude dependent detuning

Space charge create an amplitude dependent detuning



single particle
emittance
(or action)

Space charge vs. nonlinearities

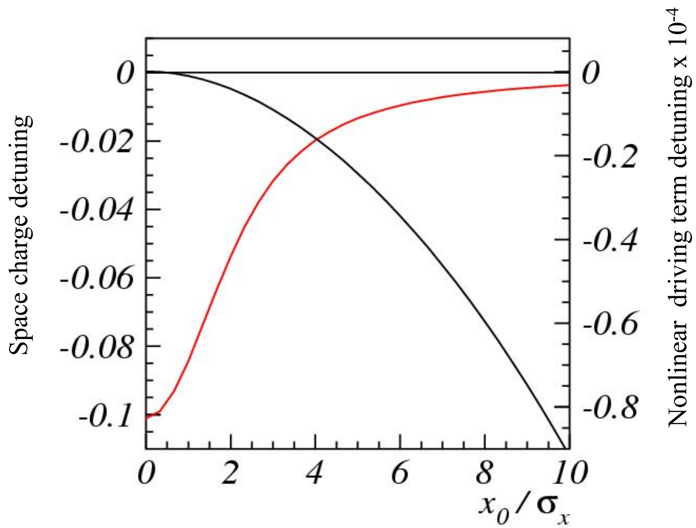
Standard nonlinear components $\Delta Q_a(\epsilon_x) = a_1\epsilon_x + a_2\epsilon_x^2 + O(\epsilon_x^3)$

The space charge $\Delta Q_x \propto \dfrac{1}{1+[x_m/(2\sigma_x)]^2}$

Prove that this is wrong within 4%

The space charge detuning has a different nature from the lattice nonlinear errors induced detuning

	small amplitudes	large amplitudes
Lattice nonlinear error	zero	Large
Space charge	Maximum	zero



Consequence: when the bare tune is set near a resonance, the particle amplitude evolves as

Amplitude growth → detuning → Exit from the resonance → Amplitude Stops growing

Space charge as driving term

Space charge is a nonlinear force as any other can be. Example in 1D

$$\frac{d^2x}{ds^2} + k_1 x = K\frac{1}{x}\left[1 - \exp\left(-\frac{x^2}{2\sigma_r^2}\right)\right]$$

Expanding space charge force

Driving terms

$$\frac{d^2x}{ds^2} + k_1 x = \frac{K}{2\sigma_r^2}x - \frac{K}{8\sigma_r^4}x^3 + \frac{K}{48\sigma_r^6}x^5 - \dots$$

order 2nd 4th 6th

Difficulties

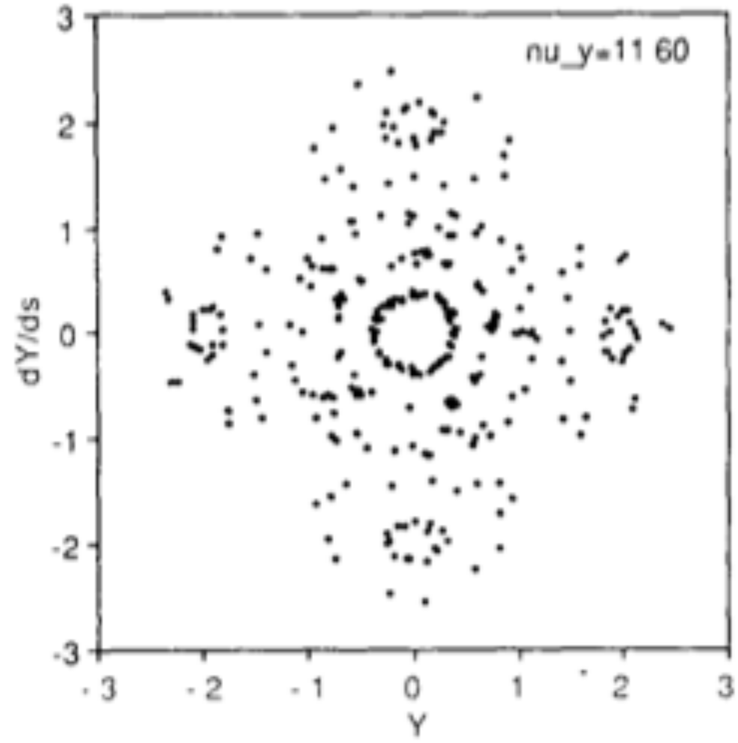
If all particles grow of amplitudes, then the rms size growth as well…

But then the strength of the resonance diminishes….

On the other hand not all particles are resonant, because each particle has different tune.

怎么回事？

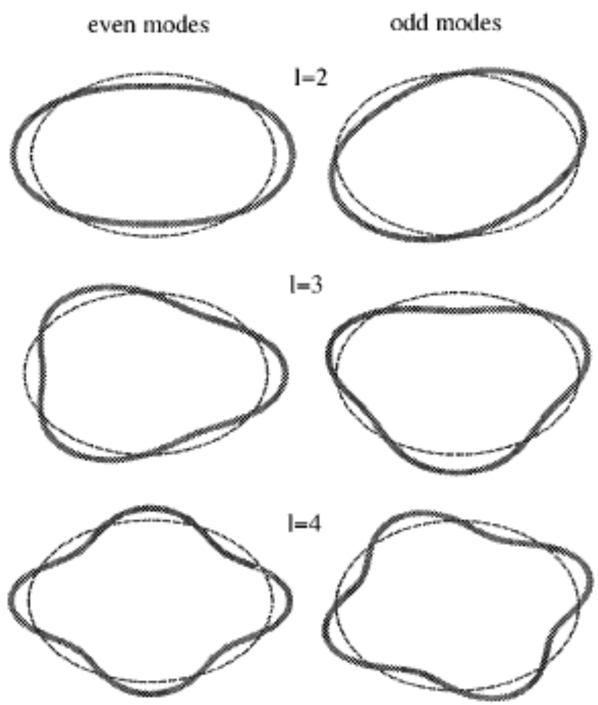
4th order structure resonance induced by space charge



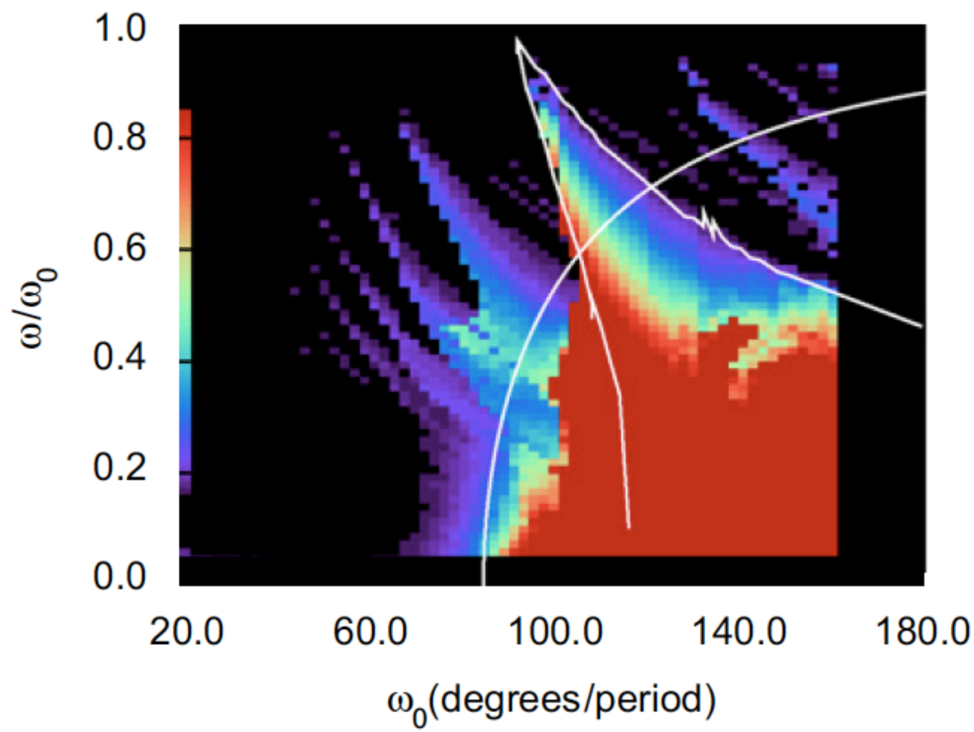
S. Machida 1991

More difficulties

Due to space charge beam edge oscillates following a mode-decomposition



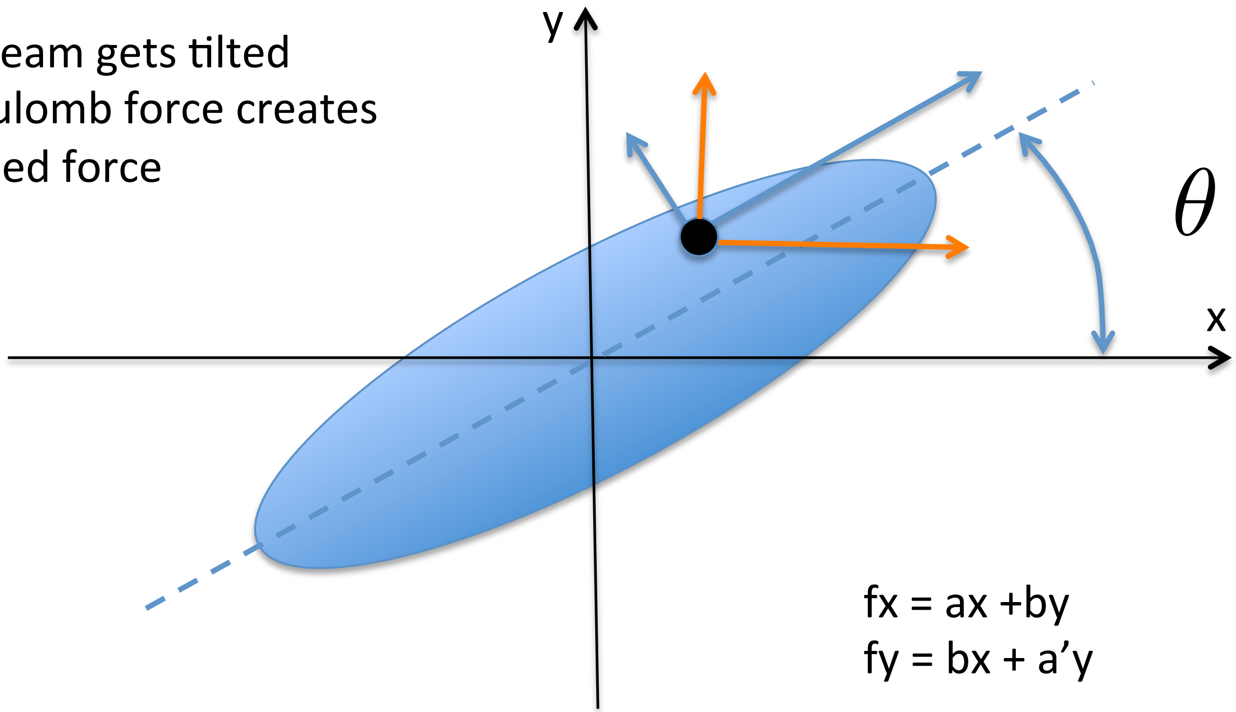
In certain condition modes becomes unstable.
PIC simulation of a FODO cell (C. Benedetti)



Stability chart first made by Ingo Hofmann (PRE, 1998)

Troubles

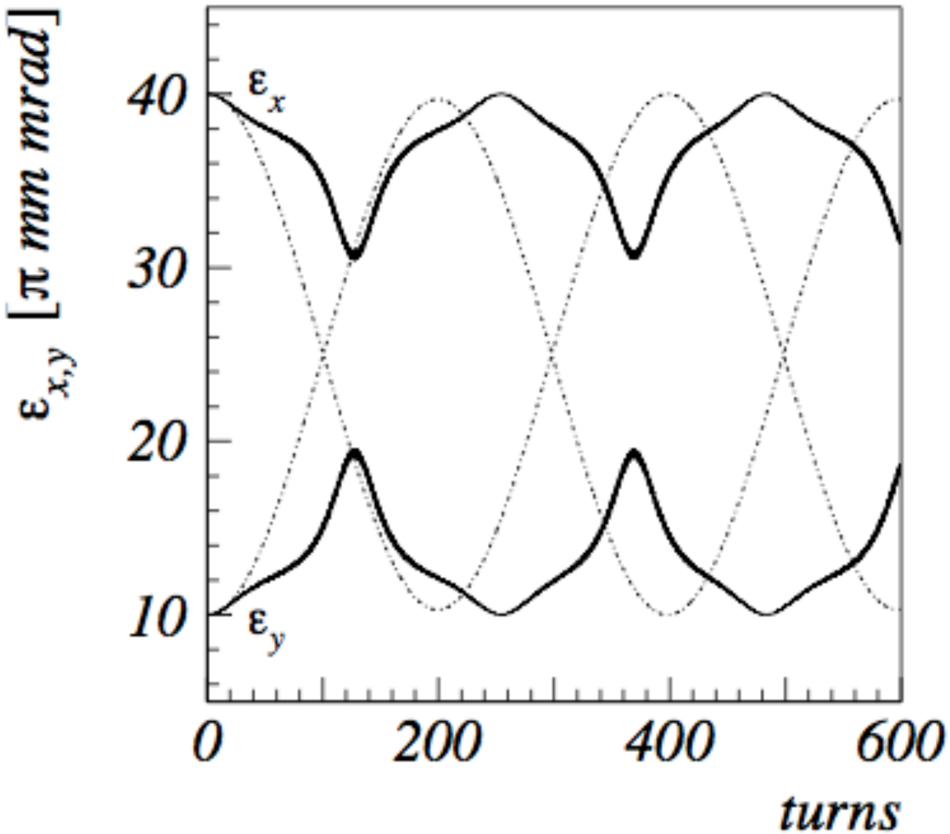
If the beam gets tilted
the Coulomb force creates
a coupled force



fx = ax +by
fy = bx + a'y

KV → the beam creates a linear coupling that makes the beam to resonate with the beam it-self

Example: of the space charge effect on the linearly coupled motion

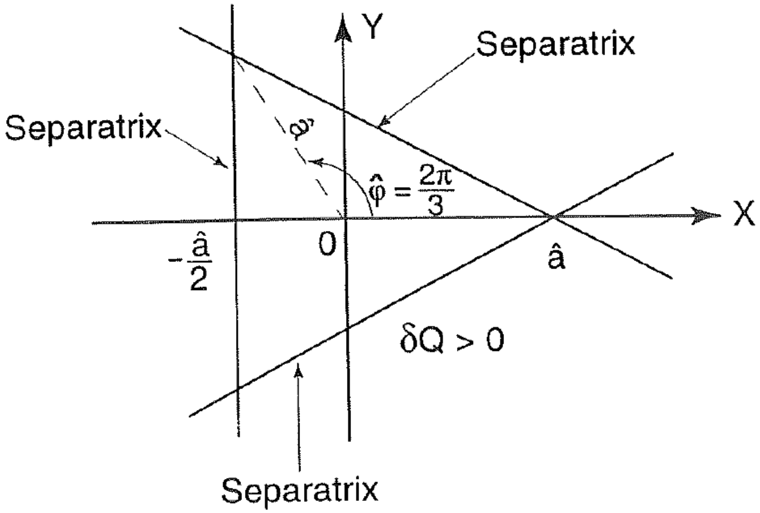


Space charge and magnet resonances (Troubles continuation)

Resonances appears shifted by the tune-shift

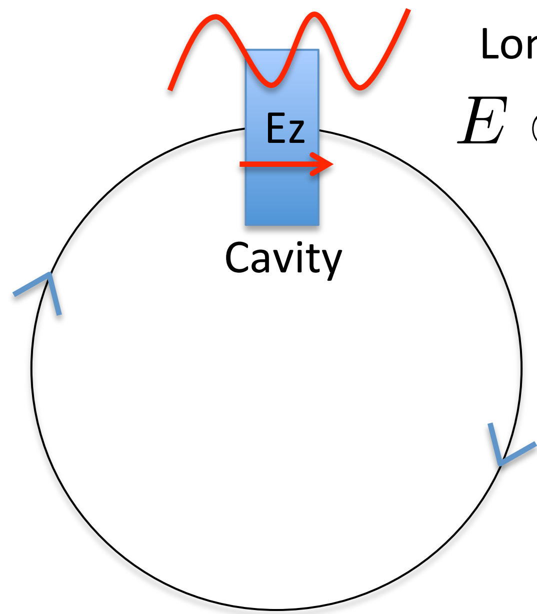
When particles increase amplitude they get out of the resonance

Stabilization of otherwise unstable phase space



We all live in a 6D phase space

The longitudinal plane



Longitudinal force

$$E \propto \sin(\omega t)$$

Longitudinal equation "type"

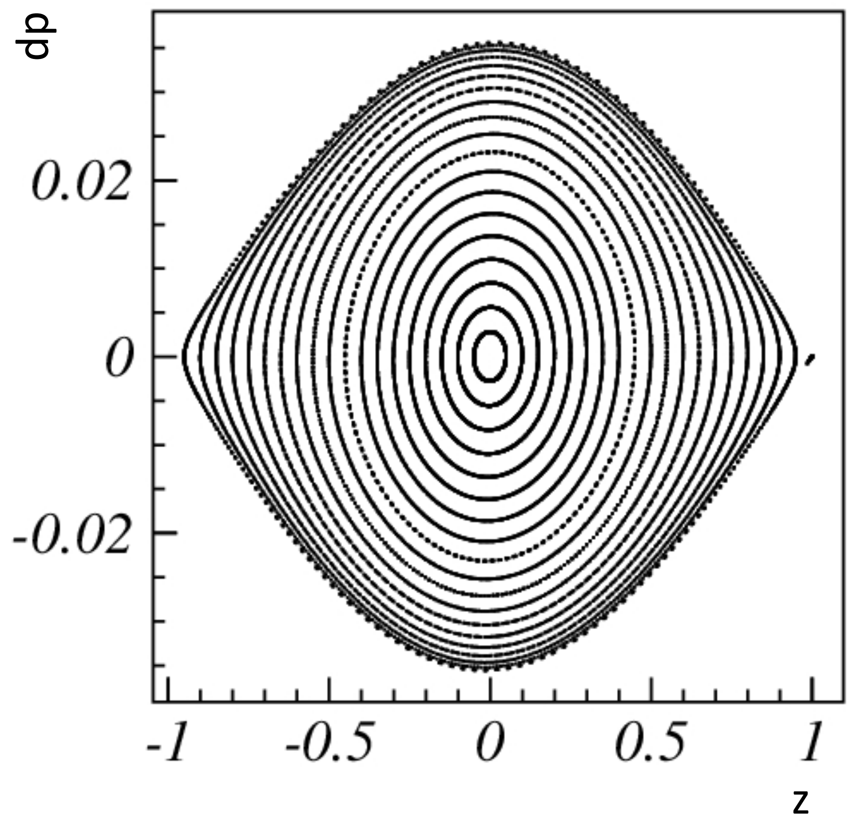
$$z'' + A \sin \pi z = 0$$

(No acceleration)

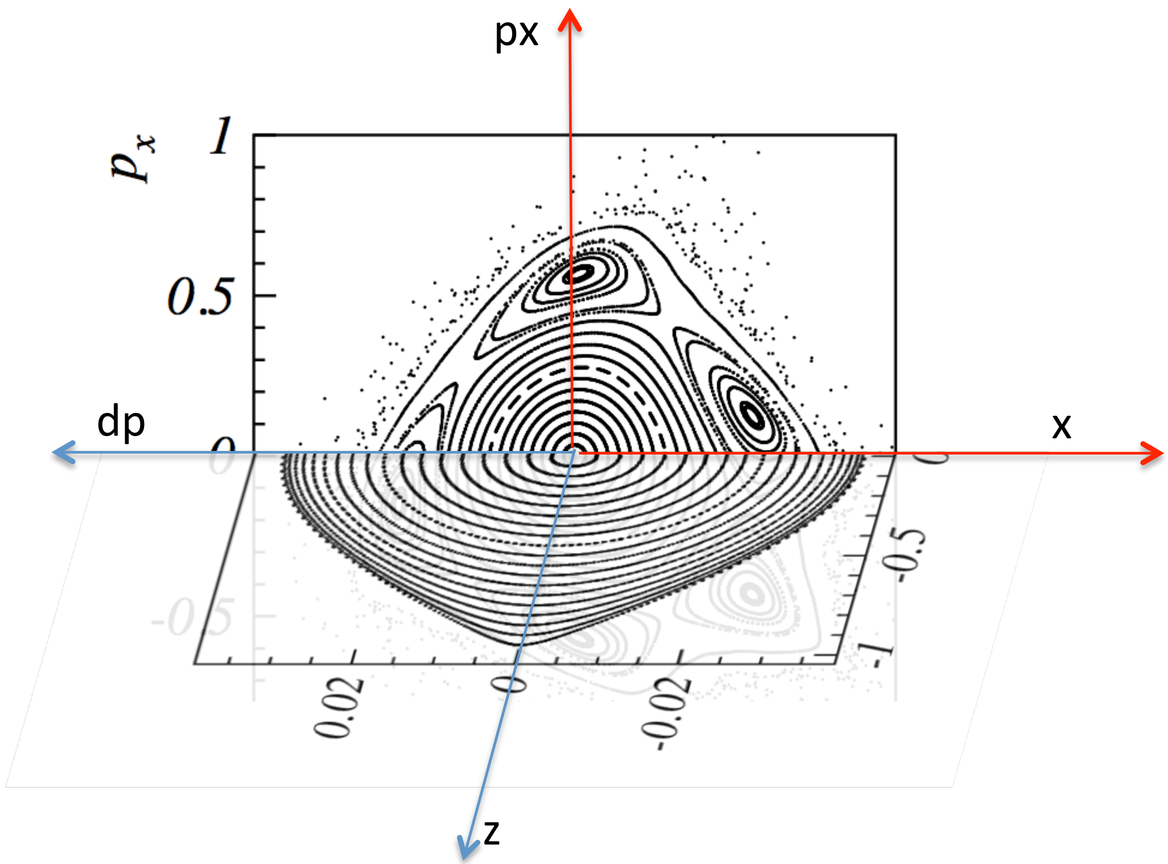
A, depends on many parameters (Voltage, frequency, energy, slip factor)

α_p momentum compaction ⟷ **Transition Energy**

Longitudinal phase space

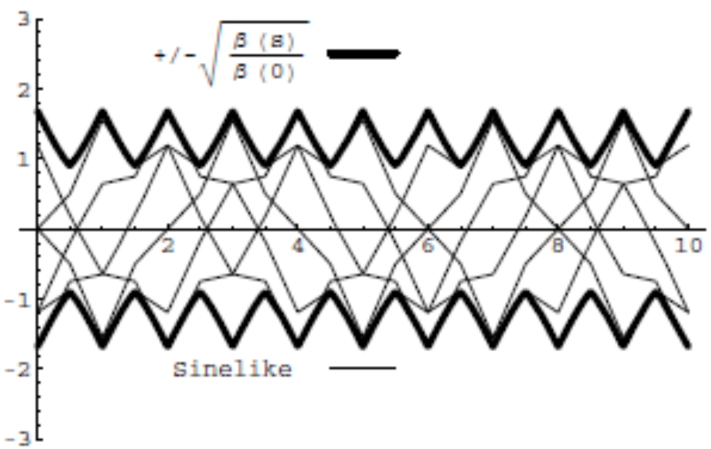


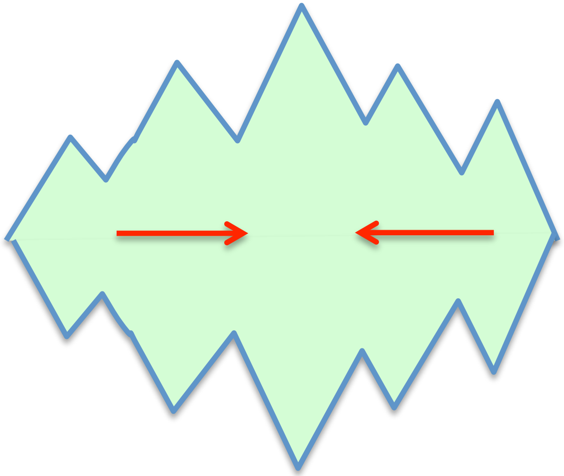
In an ideal world



Coasting beams and bunched beams

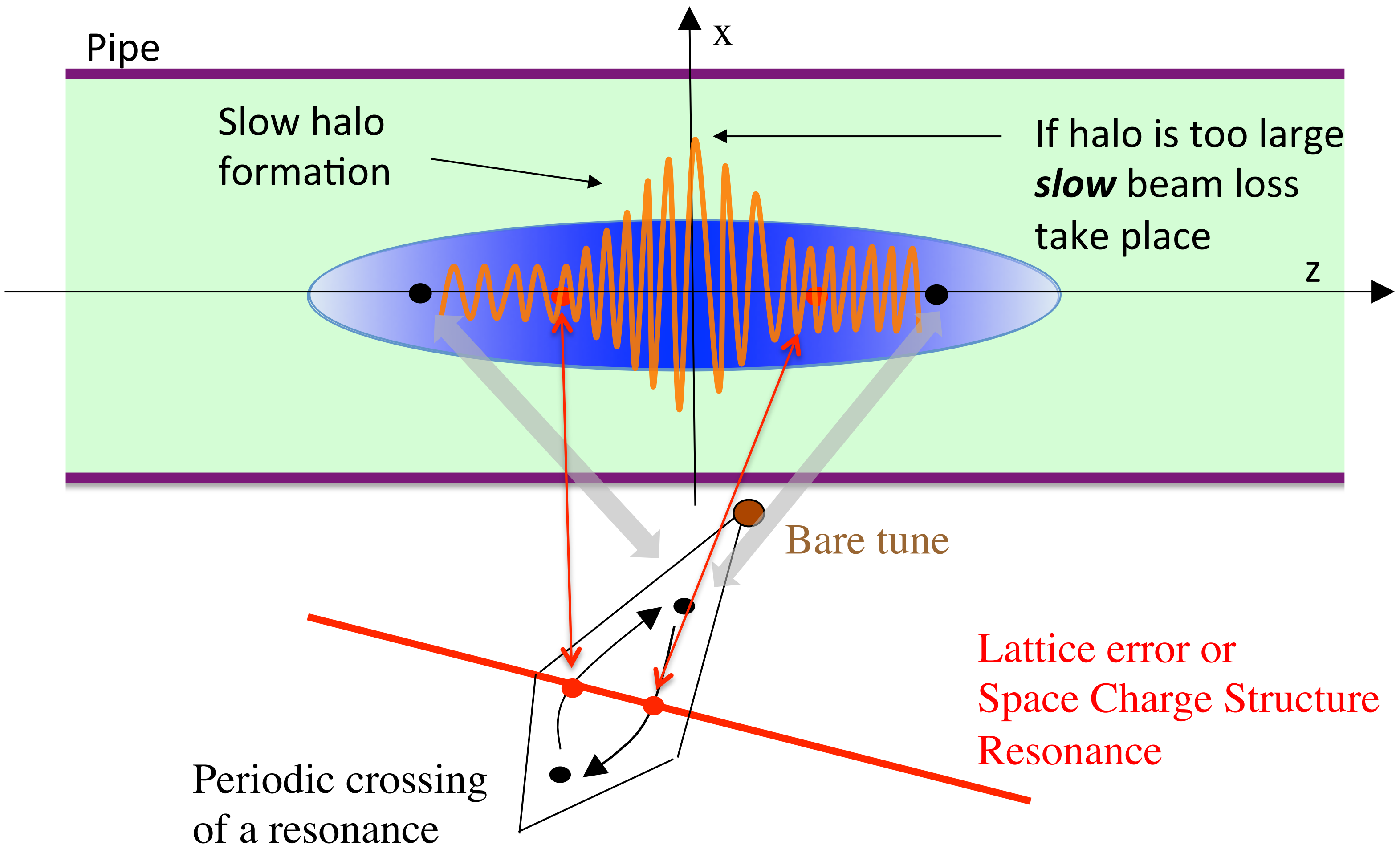
In a coasting beam the particle density does not change



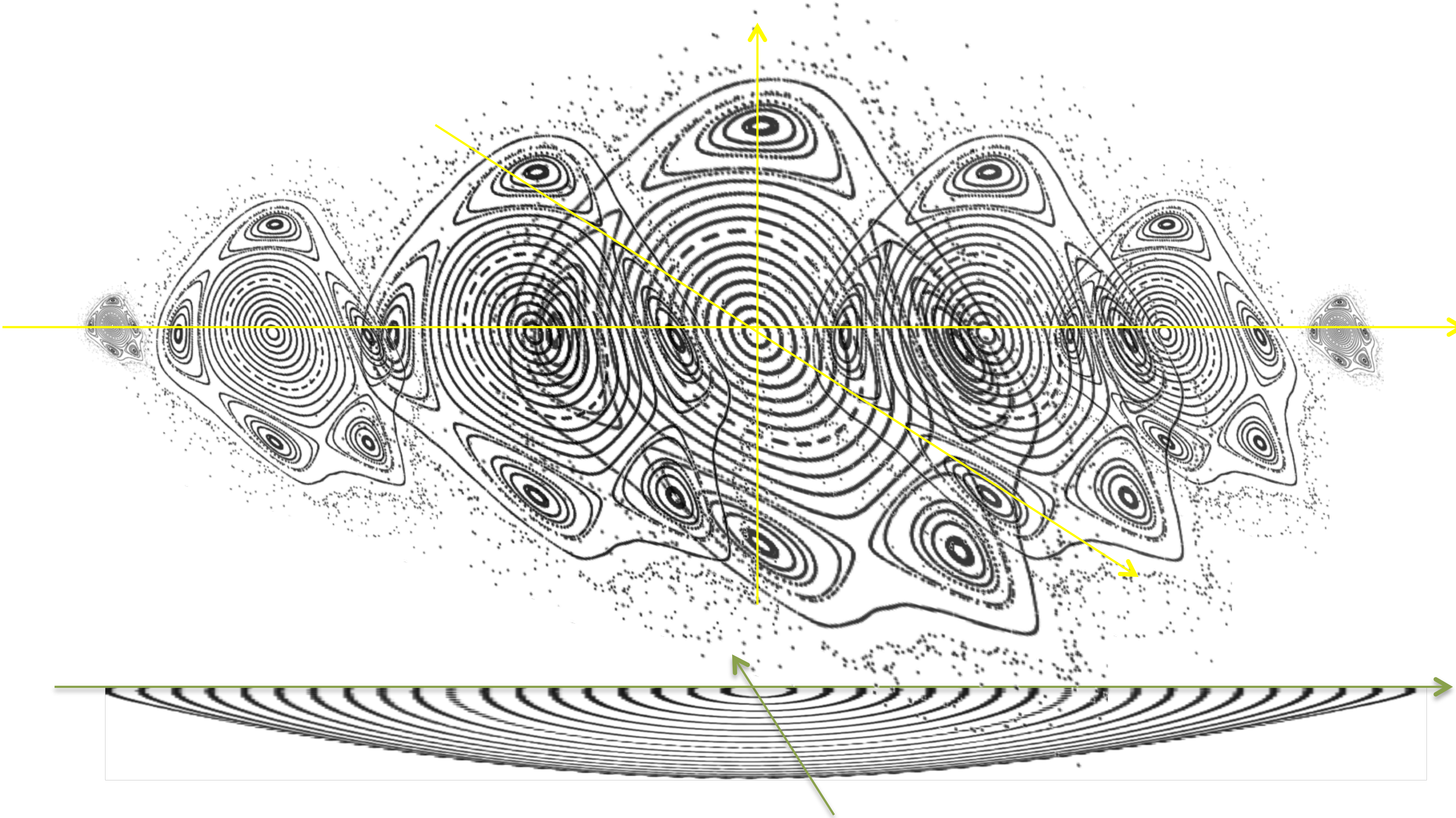


Bunch beam

Single and periodic crossing of resonances by high intensity beams

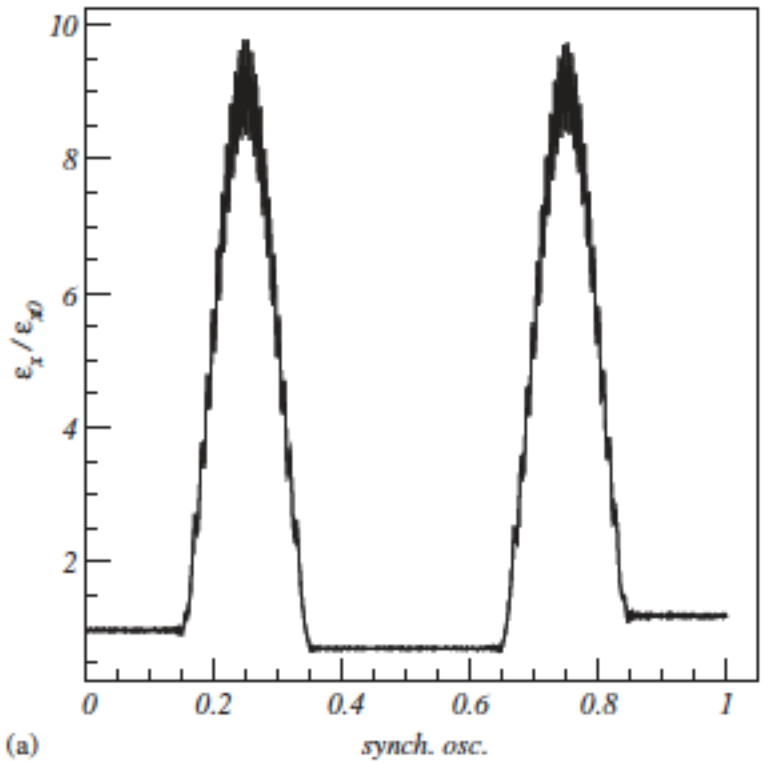


Periodic resonance crossing induced by space charge

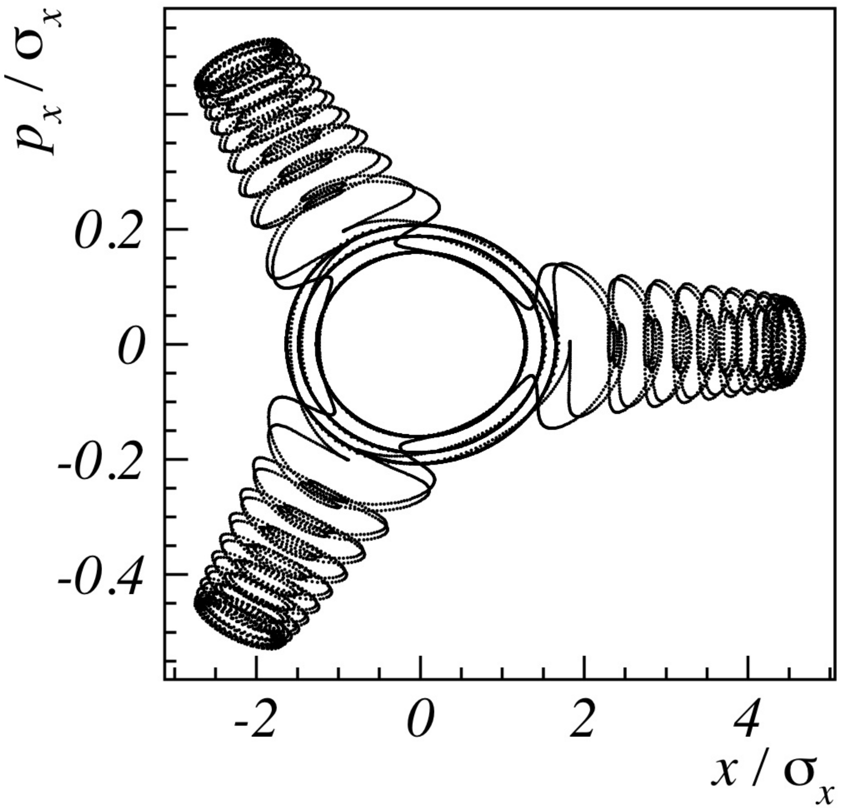


Trapping

Adiabatic longitudinal motion

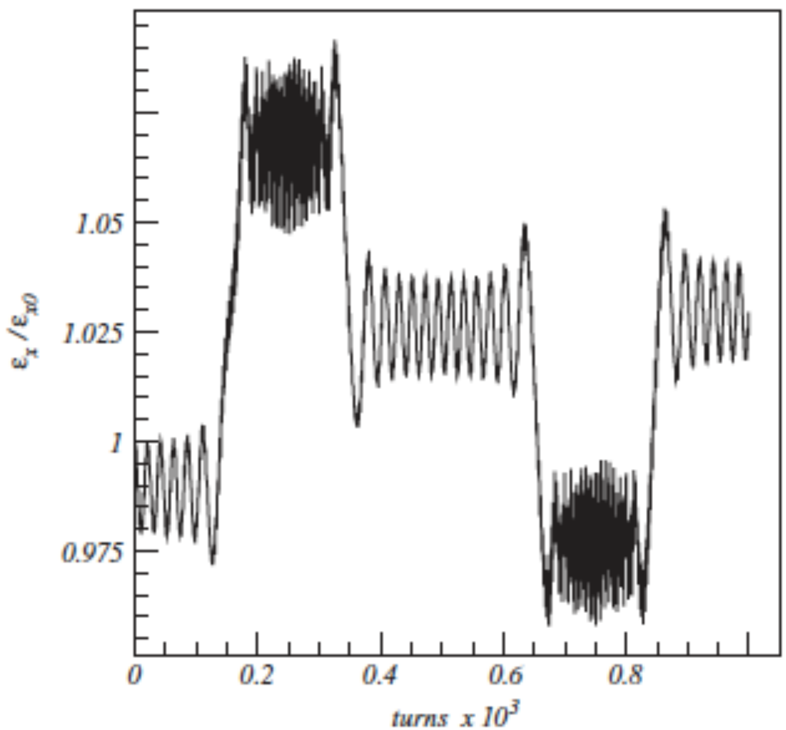


In the phase space

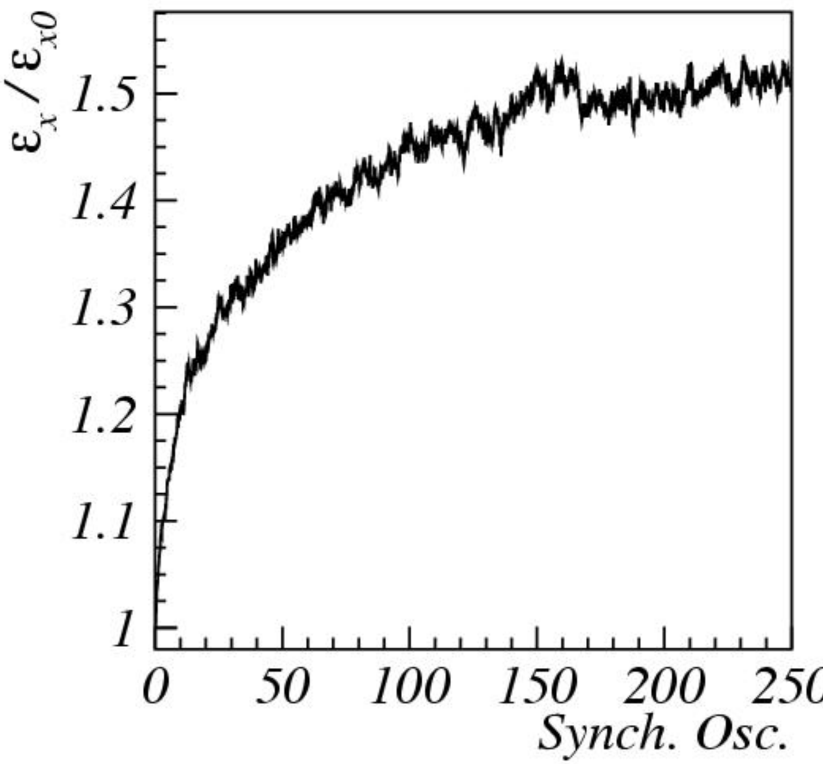


Scattering

Very fast longitudinal motion

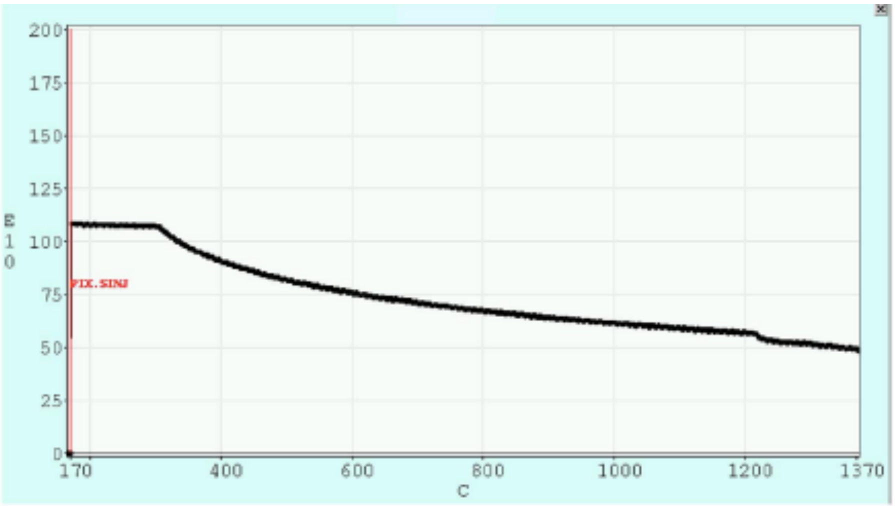


Slow emittance growth



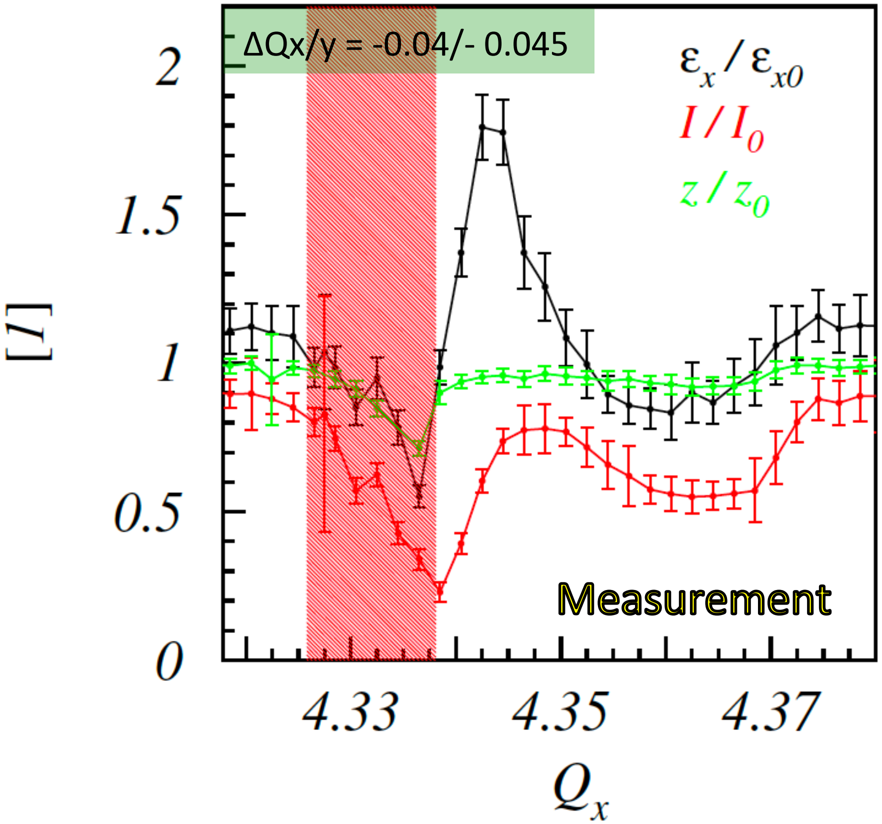
Diffusive beam loss, emittance growth, beam shortening

CERN-PS 2003 → PRSTAB



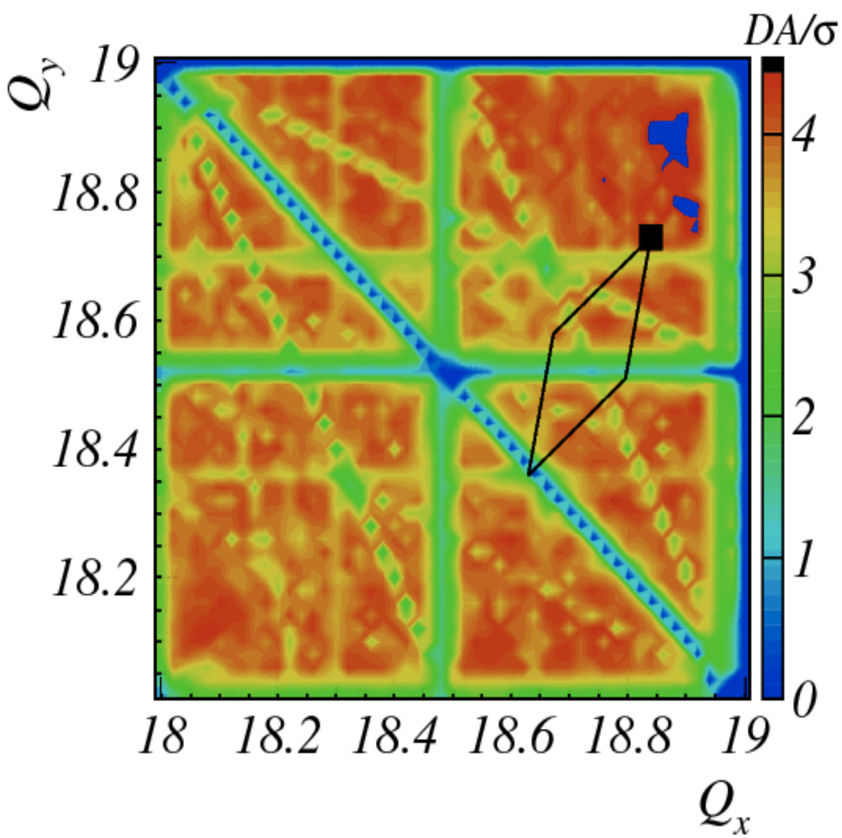
The dynamics is very complex as space charge is varying due to beam loss: it is referred to as the "self-consistent" problem

bunched beam high intensity

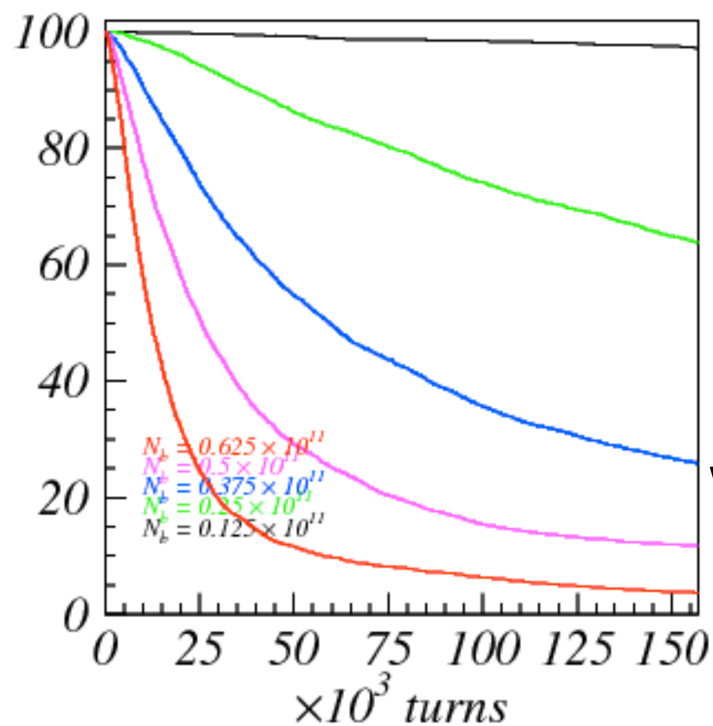


GSI 2008 → somewhere in PRSTAB

Nightmare for projects

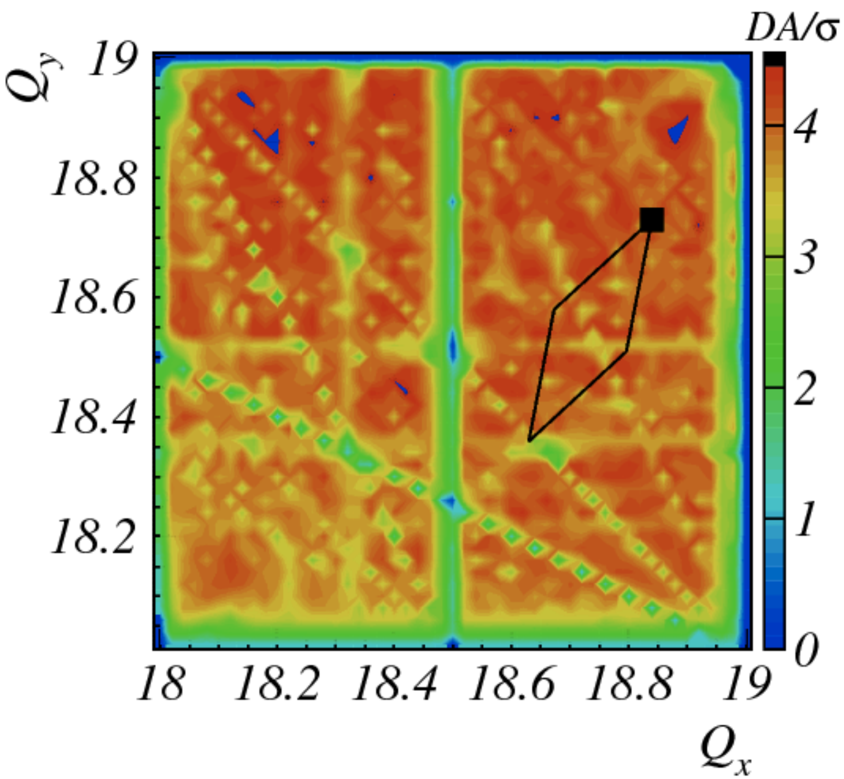


First bunch

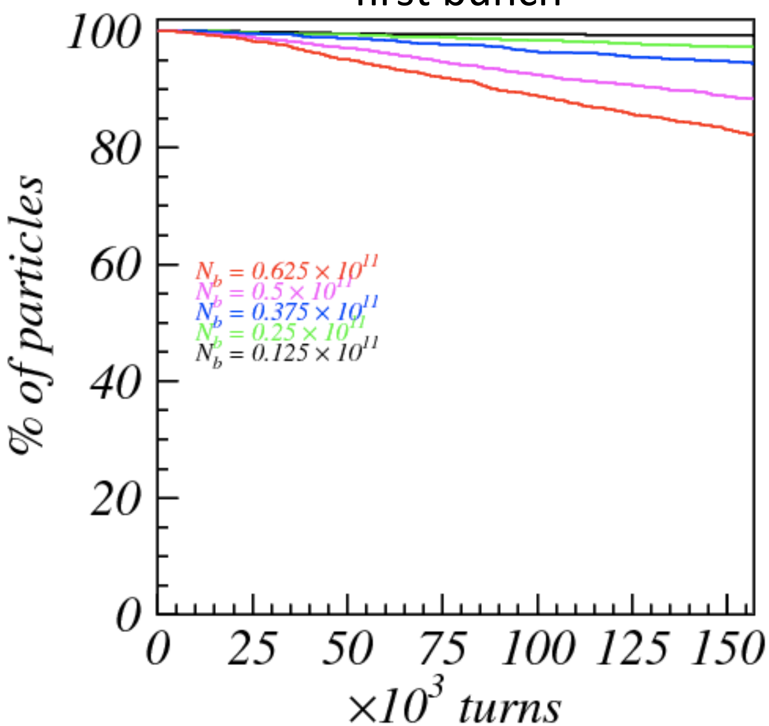


Wrong result!!

Removing lattice resonances



first bunch

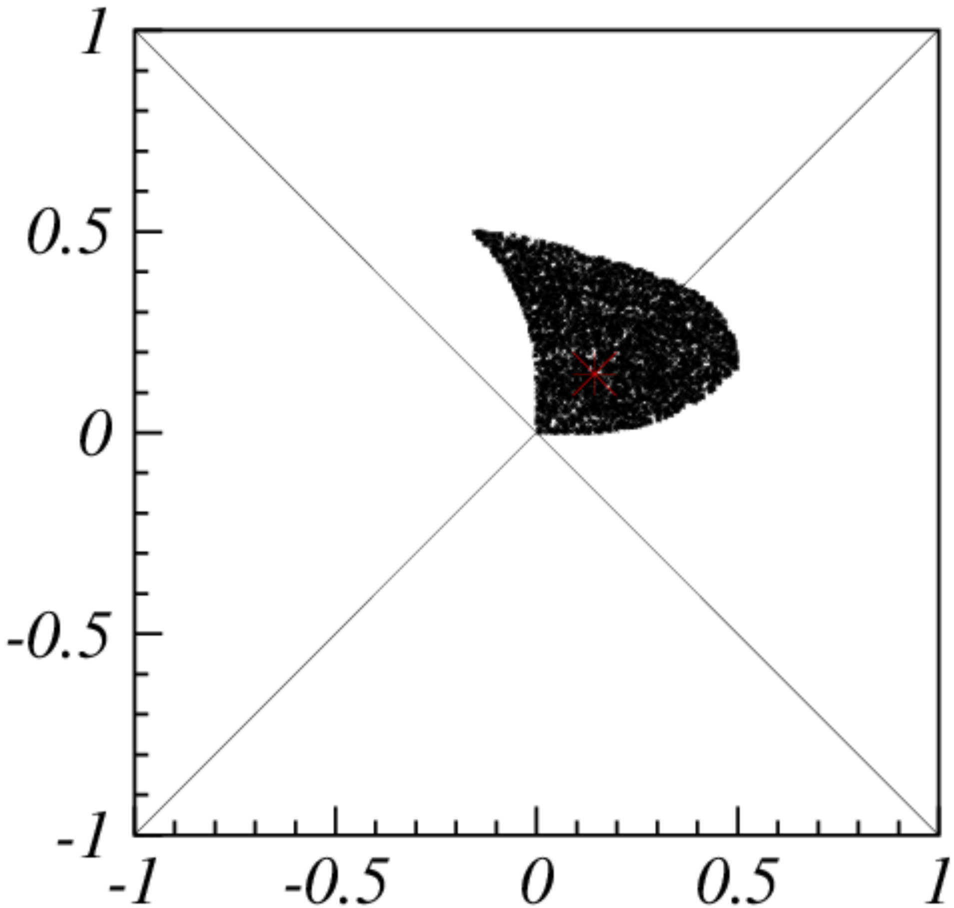


(but is it true? → experimental tests)

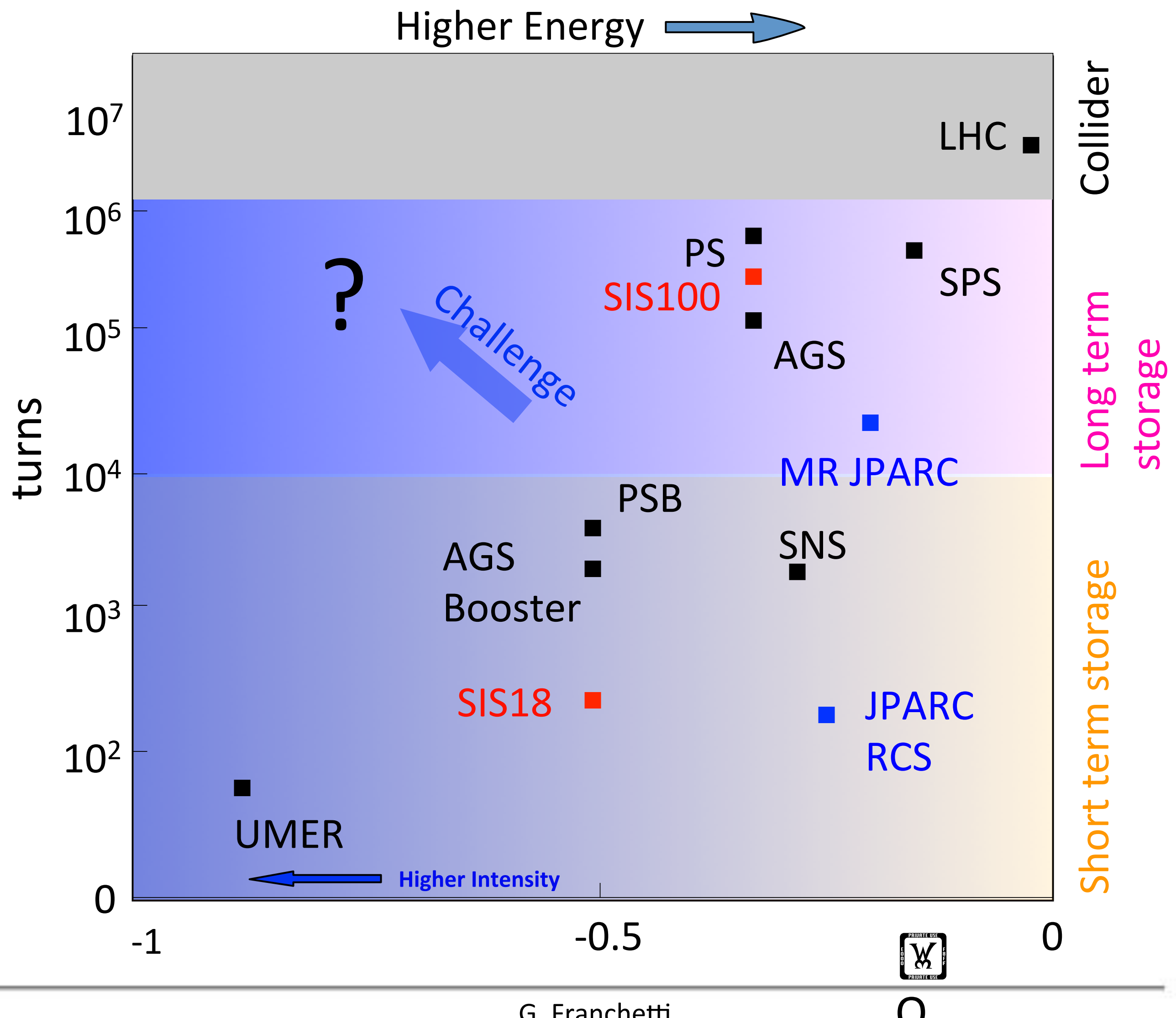
Scaling from first principle ?

Maybe!
The scaling exists if it exists an attraction point

(accepted and some day on PRL)



Intensity limitation



Summary

Nonlinear Resonances

- single particle motion (incoherent)
- orbit deformations
- long term effects: resonances and dynamic aperture

High intensity

+

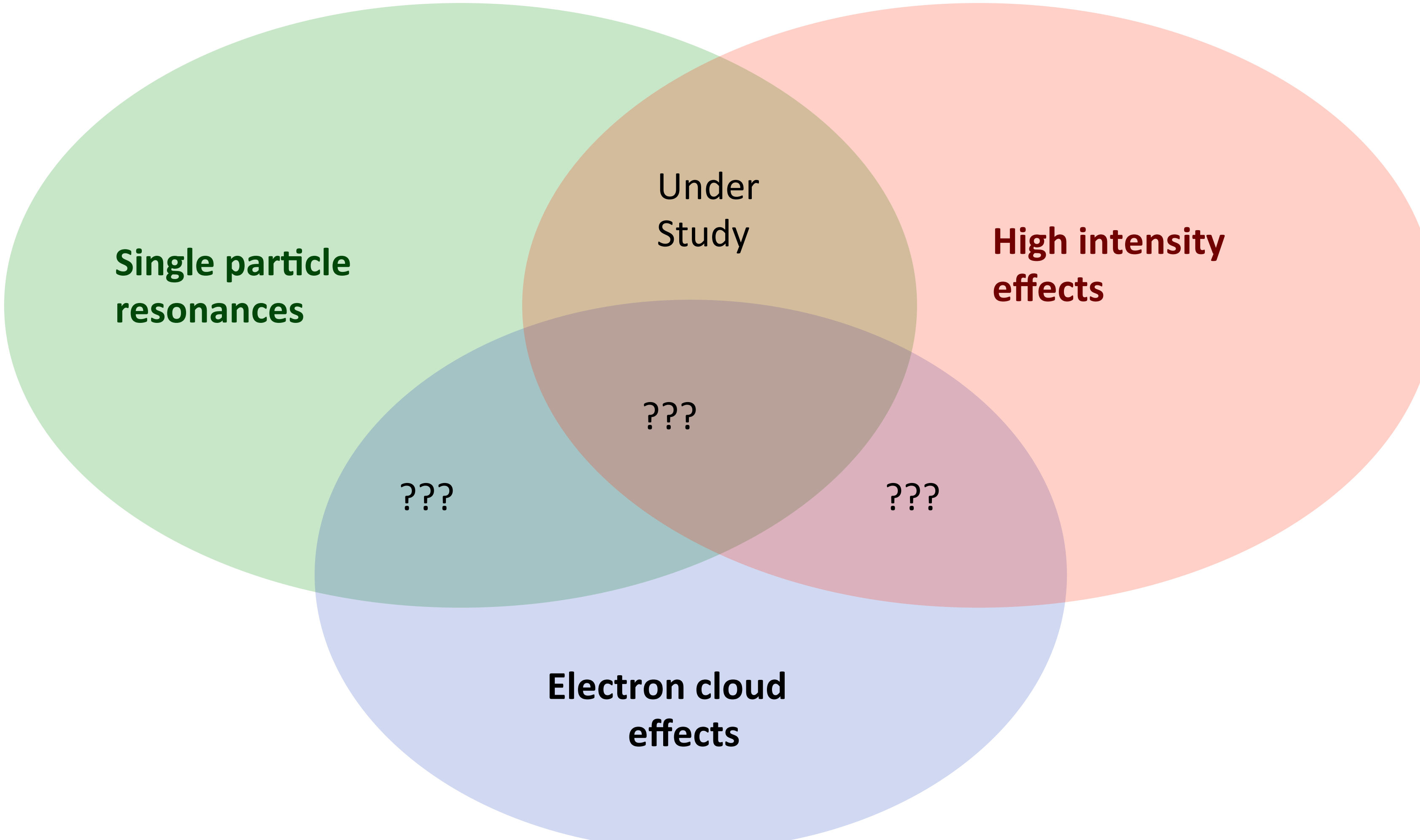
Nonlinear errors

Long storage

High Intensity effects

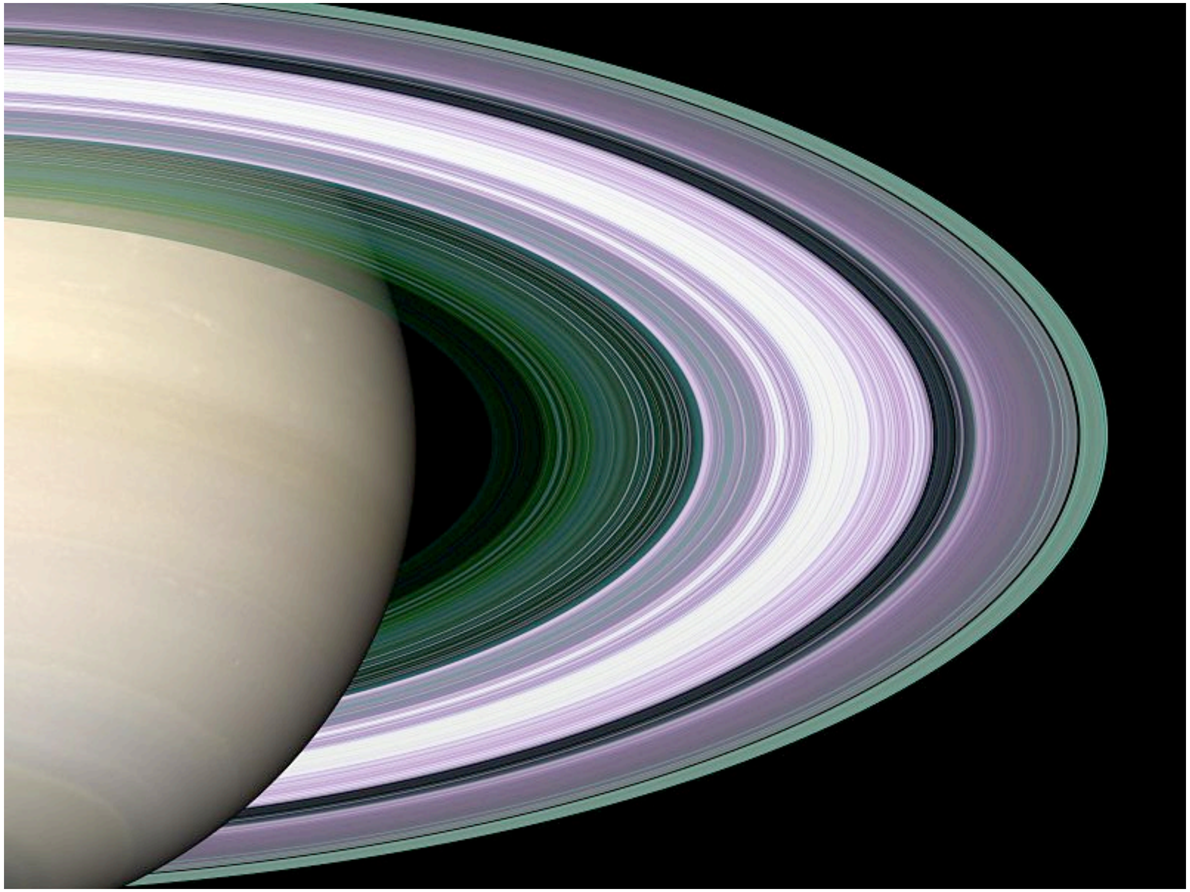
- many particle force (coherent)
- short term effects
- coherent beam motion
- strong in linac

Beyond...



Under Study
Single particle resonances
High intensity effects
???
???
???
Electron cloud effects

Very Large Storage Ring



Prove that the momentum compaction of a particle in the Saturn ring is

$$\alpha_p = -2$$

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