RF BASICS

517. WE-Heraeus Seminar, Accelerator Physics for Intense Ion Beams 14-18 Oct. 2012, Bad Honnef F. Gerigk (CERN/BE/RF)

overview

- Maxwells Equations,
- Reminder to basic vector analysis,
- Ampères Law and Faradays Induction Law,
- What is displacement current?
- Boundary conditions for magnetic and electric fields.
- Wave equation and its complex notation.
- Skin depth, energy propagation, and losses.

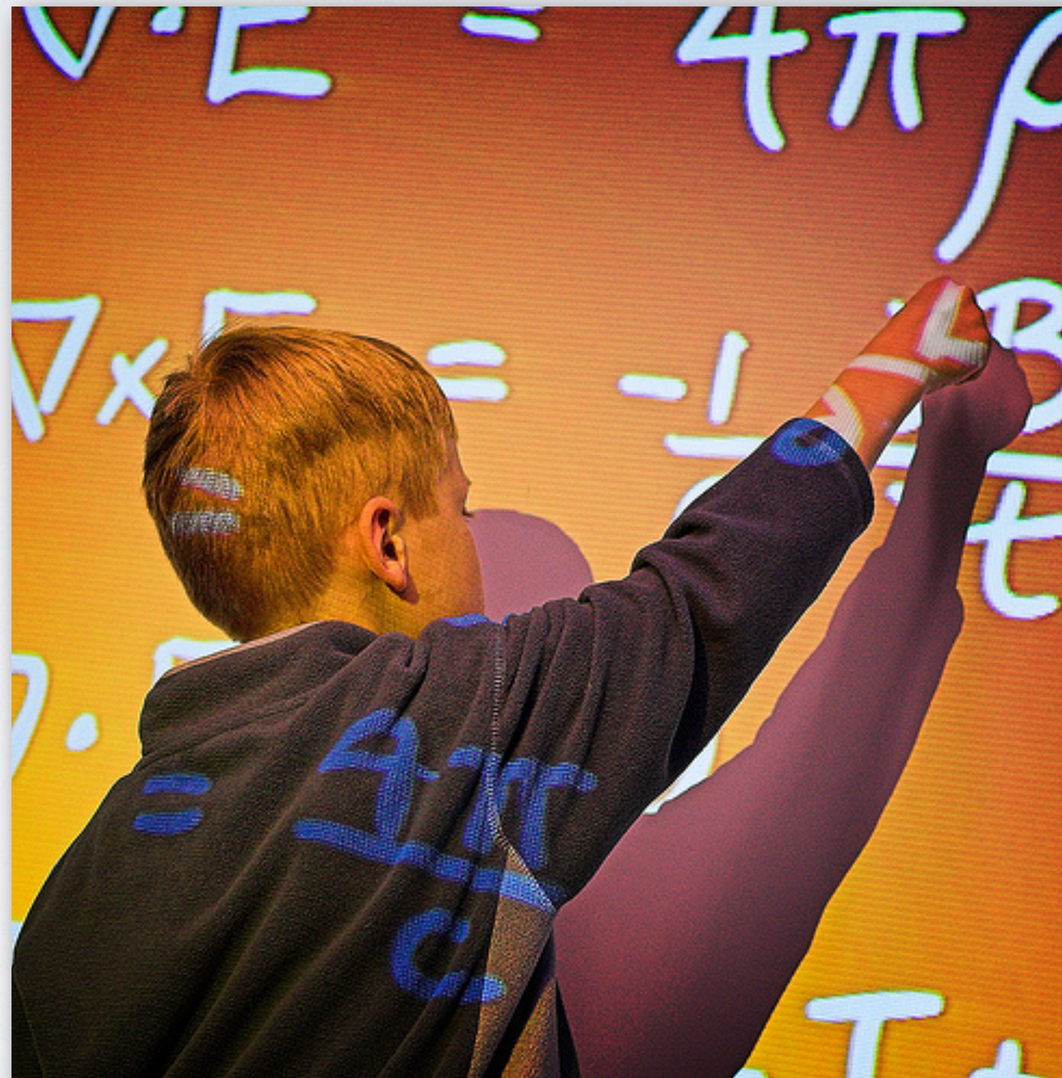
MAXWELLS EQUATIONS

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt} \quad (I)$$

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \quad (II)$$

$$\nabla \cdot \mathbf{D} = \rho_V \quad (III)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (IV)$$



and there was light...

Maxwells equations: components

$\mathbf{E}$	–	electric field [V/m]
$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$	–	dielectric displacement [$\mathrm{As/m^2}$]
$\mathbf{B}$	–	magnetic induction, magnetic flux density [T]
$\mathbf{H} = \frac{1}{\mu_0 \mu_r} \mathbf{B}$	–	magnetic field strength/field intensity [A/m]
$\mathbf{J} = \kappa \mathbf{E}$	–	electric current density [$\mathrm{A/m^2}$]
$\frac{d}{dt} \mathbf{D}$	–	displacement current [$\mathrm{A/m^2}$]

$\varepsilon_0 = 8.854 \cdot 10^{-12} \frac{F}{m}$	electric field constant
ε_r	relative dielectric constant
$\mu_0 = 4\pi \cdot 10^{-7} \frac{H}{m}$	magnetic field constant
μ_r	relative permeability constant
κ	electrical conductivity [S/m]

$\varepsilon = \varepsilon_0 \varepsilon_r \qquad \mu = \mu_0 \mu_r$

REMINDER OF VECTOR ANALYSIS AND SOME APPLICATIONS TO MAXWELLS EQUATIONS

differential operators in cartesian* coordinates

gradient of a potential

$$\nabla\Phi = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial z} \end{pmatrix}$$

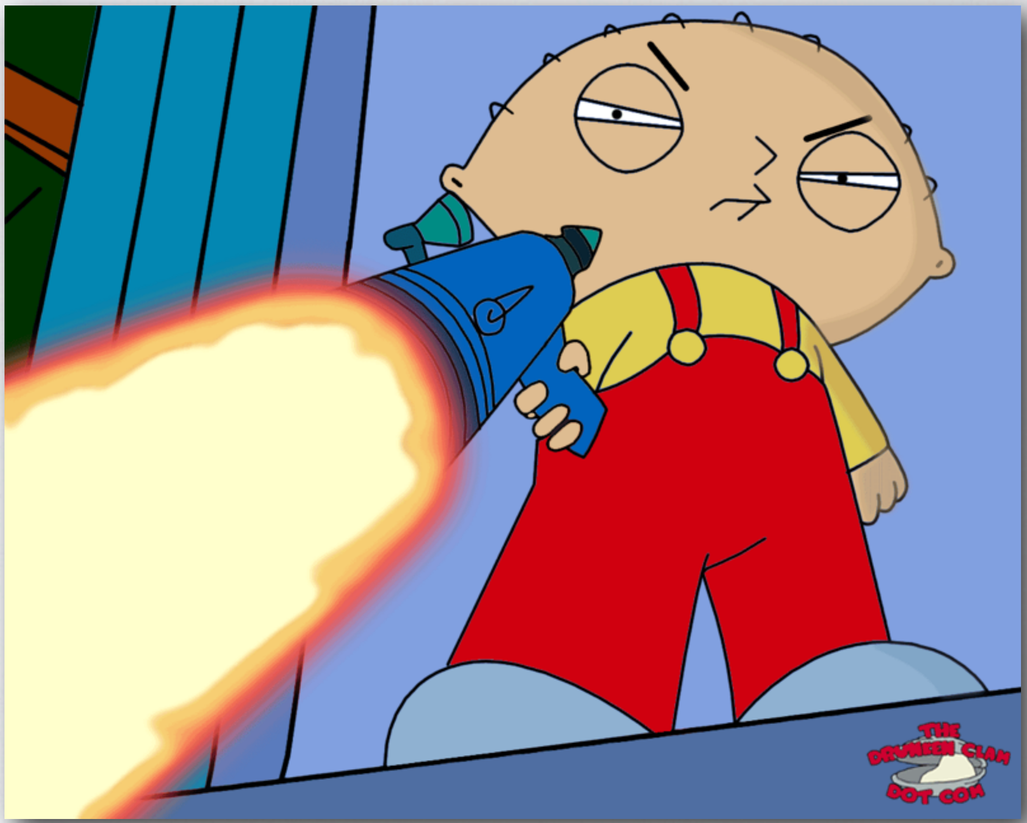
The result is a **vector** telling us how much the potential changes in x, y, z direction.

* see annex for cylindrical coordinates

differential operators in cartesian coordinates

divergence of a vector field

$$\nabla \cdot \mathbf{a} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$



The result is a **scalar**, telling us if the vector field has a source.

$$(\nabla \cdot \mathbf{a} \neq 0)$$

differential operators in cartesian coordinates

curl of a vector

$$\nabla \times \mathbf{a} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = det \begin{pmatrix} \mathbf{u_x} & \mathbf{u_y} & \mathbf{u_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{pmatrix} = \begin{pmatrix} \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \\ \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \\ \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \end{pmatrix}$$

Are there curls/eddies around the x, y, z axes? (Imagine a ball fixed around the x, y, z axis. If it starts rotating, then $\nabla \times \mathbf{a} \neq \mathbf{0}$)

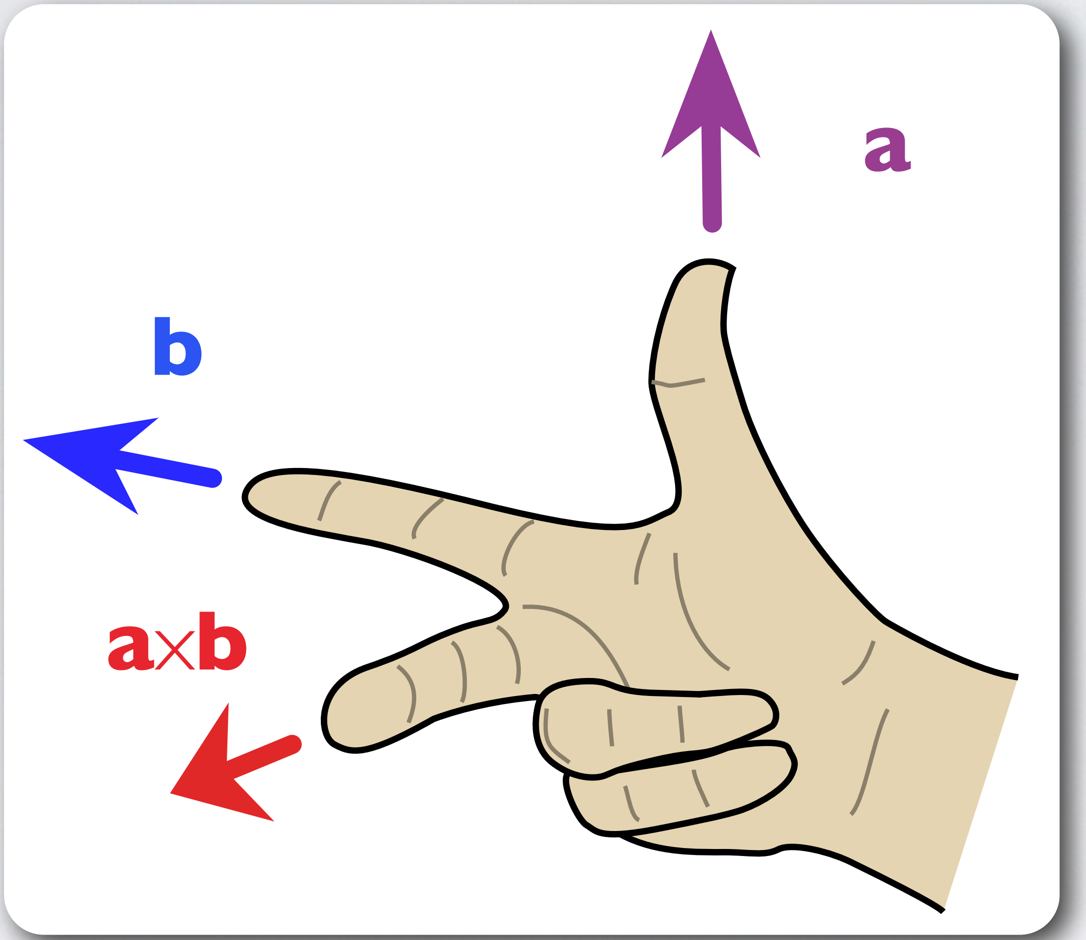
Laplace operator

$$\Delta\Phi = \nabla \cdot (\nabla\Phi) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

E.g: electrostatic potentials are defined everywhere in space, if they fulfill **$\Delta\Phi$**=0, and have the correct values at the boundaries.

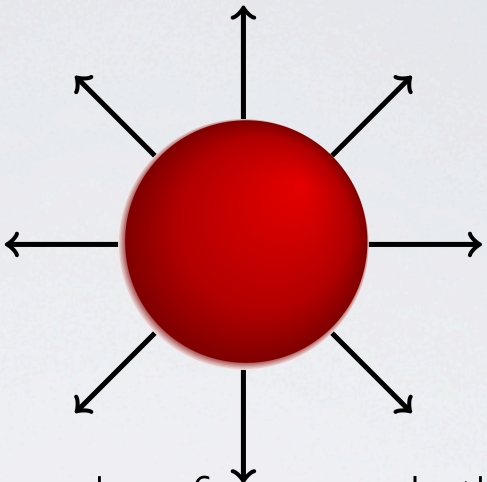
cross products

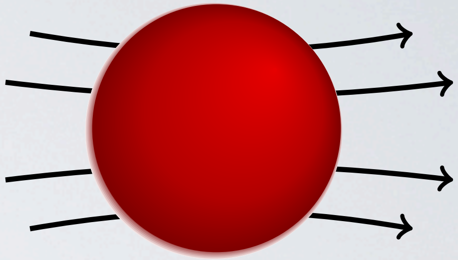
or the so-called “right hand rule”



Gauss' theorem

$$\int_V \underbrace{\nabla \cdot \mathbf{a}}_{\text{"sources"}} \, dV = \oint_S \mathbf{a} \cdot d\mathbf{S}$$





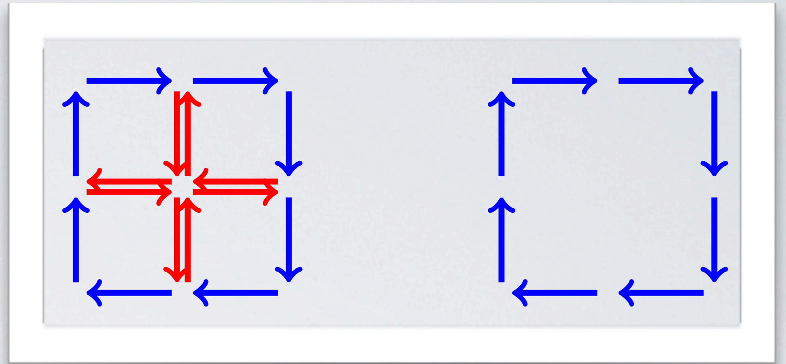
- The net vector flux through a closed surface equals the flux sources (e.g. charges), which are enclosed in the volume.
- If there are no sources, the amount of flux entering and leaving a volume must be equal.

- Electric field lines originate from el. charges. $\int_V \nabla \cdot \mathbf{E} dV = \oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon}$

- Since there are no "magnetic charges", magnetic field lines are always closed. $\int_V \nabla \cdot \mathbf{B} dV = \oint_S \mathbf{B} \cdot d\mathbf{S} = 0$

Stokes' theorem

$$\int_A (\nabla \times \mathbf{a}) \cdot d\mathbf{A} = \oint_C \mathbf{a} \cdot d\mathbf{l}$$



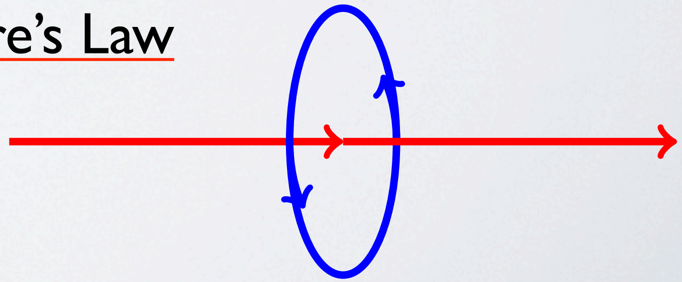
- Tells us that the area integral over the curls of a vector field can be calculated from a line integral along its closed border, or in other words, that,
- the field lines of a vector field with non-zero curls must be closed contours.

Applied to Maxwells equations:

$$\int_A (\nabla \times \mathbf{H}) \cdot d\mathbf{A} = \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_A \left(\mathbf{J} + \frac{d\mathbf{D}}{dt}\right) \cdot d\mathbf{A}$$

and in the electrostatic case $\left(\frac{d\mathbf{D}}{dt}\right) = 0$ we get Ampère's Law

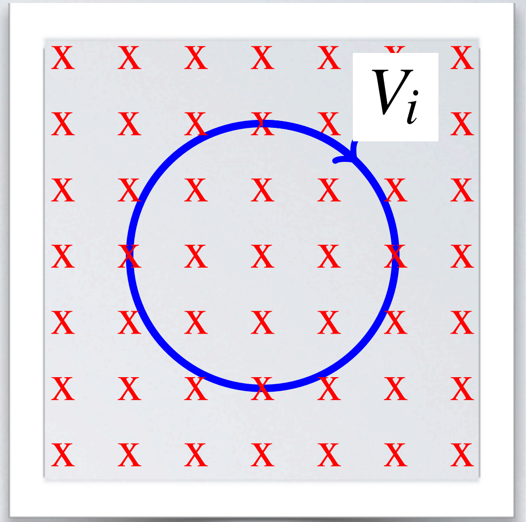
with a one-line derivation! $$\oint_C \mathbf{H} \cdot d\mathbf{l} = I$$



Stokes' theorem II

Applied once more to Maxwells equations:

$$\int_A (\nabla \times \mathbf{E}) \cdot d\mathbf{A} = \underbrace{\oint_c \mathbf{E} \cdot d\mathbf{l}}_{V_i} = \underbrace{-\frac{d}{dt} \int_A \mathbf{B} \cdot d\mathbf{A}}_{\frac{d\psi_m}{dt}}$$



we get **Faradays induction law**
(again in a one-line derivation):

$$V_i = -\frac{d\psi_m}{dt}$$

The electric field around a closed loop (the induced voltage) equals the rate of change of the magnetic flux penetrating that loop.
(The basis of every electric motor or generator.)

what is displacement current (d**D**/dt)?

we apply the divergence to $\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}$ apply a volume integral and make use of Gauss' theorem:

$$\underbrace{\nabla \cdot \nabla \times \mathbf{H}}_{\equiv 0} = \nabla \cdot \mathbf{J} + \underbrace{\nabla \cdot \frac{d\mathbf{D}}{dt}}_{\frac{d}{dt}\rho_v}$$

continuity equation → $\nabla \cdot \mathbf{J} = -\frac{d}{dt}\rho_v$

$$\int_V \nabla \cdot \mathbf{J} dV = \oint_S \mathbf{J} \cdot d\mathbf{S} = \sum I_n = -\frac{d}{dt}\int_V \rho_v dV$$

continuity equation: "*electric charges cannot be destroyed: if the amount of charges in a volume is changing a current needs to flow*"

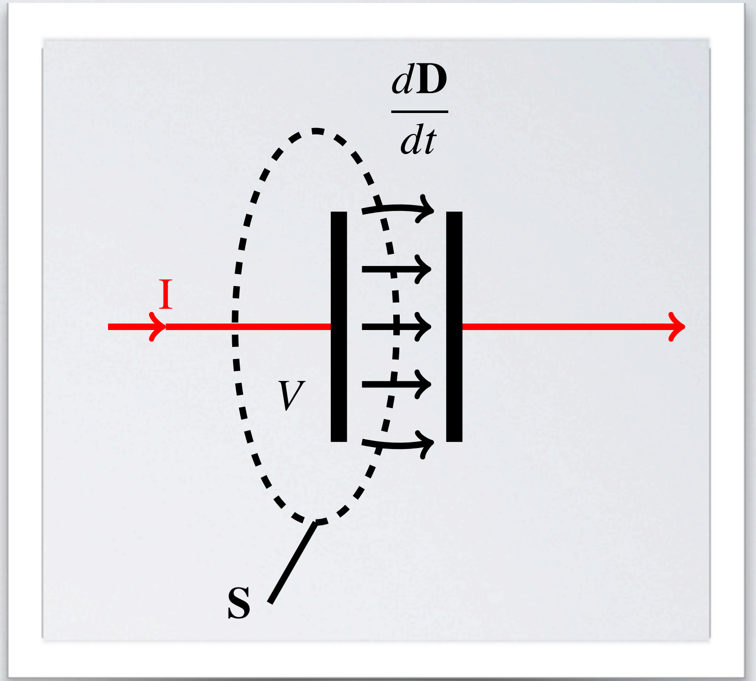
further interpretation: "*the sources of the displacement current are time varying charges*", or: "*curls of the magnetic field are excited either by static currents, or by displacement currents* (which are a consequence of time-varying charges, which is equal to current...)"

example of displacement current (d**D**/dt)

e.g: charging a capacitor

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = I = -\frac{d}{dt}\underbrace{\int_V \rho_v dV}_{\frac{d}{dt}Q_C}$$

$$= -\frac{d}{dt}\oint_S \mathbf{D} \cdot d\mathbf{S}$$

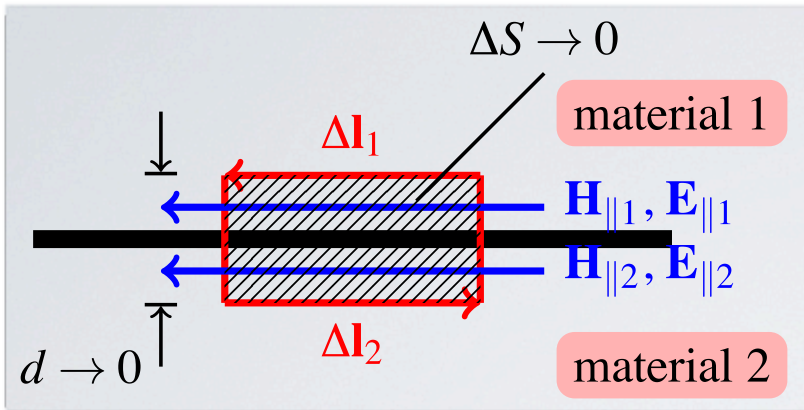


The current used to charge the capacitor equals the rate of change of the charge on a capacitor plate and equals the displacement current between the capacitor plates.
or ***"In case we don't have a conductor, we can use the displacement current to transport energy instead of using moving charges"*** or ***"the current charging the capacitor plate equals the displacement current between the plates"***

To design RF equipment for accelerators we need to understand:

- what are electromagnetic waves and how do they propagate in free space,
- how can we guide these wave (e.g. in wave-guides),
- or even trap them in a resonator (an accelerating cavity),
- energy density and energy flux in electromagnetic waves,
- standing waves,
- boundary conditions, and losses on electric boundaries.

Boundary conditions (ll to a surface)



$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt} \quad (I)$$

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \quad (II)$$

$$\Rightarrow \oint_C \mathbf{H} \cdot d\mathbf{l} = \underbrace{\int_A \mathbf{J} \cdot d\mathbf{A}}_{=i'\Delta l} + \underbrace{\frac{d}{dt}\int_A \mathbf{D} \cdot d\mathbf{A}}_{\to 0 \text{ for } \mathbf{A} \to 0} \quad \Rightarrow \quad H_{\|1} - H_{\|2} = i'$$

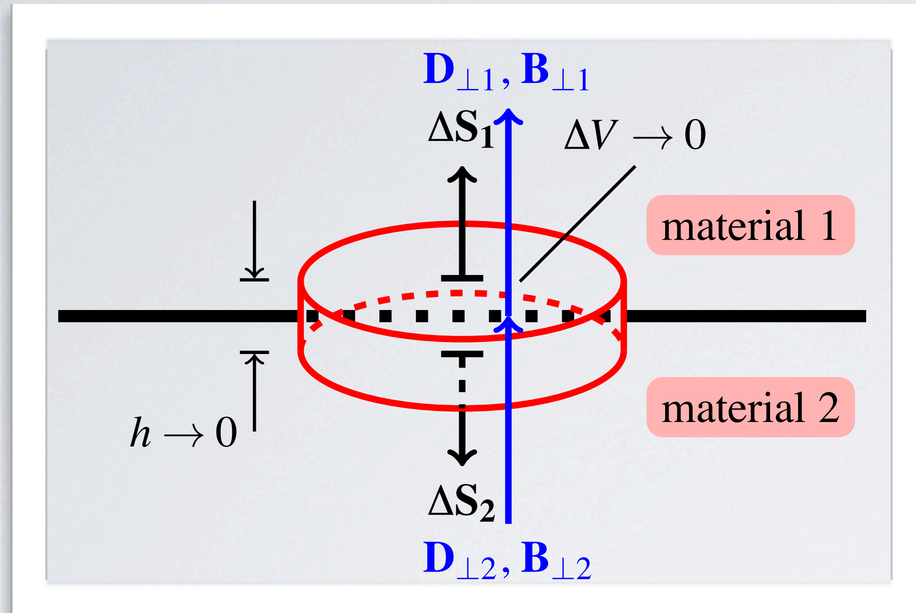
Stokes' theorem

$$\Rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = \underbrace{-\frac{d}{dt}\int_A \mathbf{B} \cdot d\mathbf{A}}_{\to 0 \text{ for } \mathbf{A} \to 0} \quad \Rightarrow \quad E_{\|1} = E_{\|2}$$

In case material 2 is an ideal conductor:

$$H_{\|1} = i' \qquad E_{\|1} = 0$$

Boundary conditions (⊥ to a surface)



$$\nabla \cdot \mathbf{D} = \rho_V \qquad (III)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (IV)$$

Gauss' theorem

$$\Rightarrow \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V q_V \, dV \quad \Rightarrow \quad D_{\perp 1} - D_{\perp 2} = q_S$$

$$\Rightarrow \oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad \Rightarrow \quad B_{\perp 1} = B_{\perp 2}$$

In case material 2 is an ideal conductor:

$$D_{\perp 1} = q_S \qquad B_{\perp 1} = 0$$

Wave equation

We consider homogenous media, meaning media in which the electromagnetic fields "see" the same material conditions ($\kappa, \ \varepsilon, \ \mu$) in all directions. In that case we can write Maxwells Equations as:

$$\nabla \times \mathbf{H} = \kappa \mathbf{E} + \varepsilon \frac{d\mathbf{E}}{dt} \quad (I) \qquad \nabla \cdot \mathbf{E} = \frac{\rho_V}{\varepsilon} \quad (III)$$

$$\nabla \times \mathbf{E} = -\mu \frac{d\mathbf{H}}{dt} \quad (II) \qquad \nabla \cdot \mathbf{H} = 0 \quad (IV)$$

Curl of (II) together with (I), and curl of (I) together with (II) and (III) results in the general wave equations in homogenous media:

$$\nabla^2 \mathbf{E} - \nabla \left(\nabla \cdot \mathbf{E} \right) = \mu\kappa \frac{d}{dt} \mathbf{E} + \mu\epsilon \frac{d^2}{dt^2} \mathbf{E} \qquad \nabla^2 \mathbf{H} = \mu\kappa \frac{d}{dt} \mathbf{H} + \mu\epsilon \frac{d^2}{dt^2} \mathbf{H}$$

In most cavities and wave-guides we consider electromagnetic field in non-conducting media ($\kappa = 0$) and charge free volumes ($\nabla \cdot \mathbf{E} = 0$):

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{d^2}{dt^2} \mathbf{E} \qquad \nabla^2 \mathbf{H} = \mu\epsilon \frac{d^2}{dt^2} \mathbf{H}$$

Complex notation for time-harmonic fields

In Radio Frequency we are usually dealing with sine-waves, which are sometimes modulated in phase or in amplitude. This means we will concentrate on time-harmonic solutions of Maxwells Equations. For this purpose we introduce the time-harmonic notation, which can be used for all linear processes. (Electric and magnetic fields can be linearly superimposed.)

Let us assume a time-harmonic electric field with amplitude E_0 and phase **φ**:

$$E(t) = E_0 \cos(\omega t + \varphi)$$

this corresponds to the Real part of:

$$E(t) = \Re\left\{E_0 e^{i\varphi} e^{i\omega t}\right\} = \Re\left\{\cos(\omega t + \varphi) + i \sin(\omega t + \varphi)\right\}$$

by defining a complex amplitude (or phasor): $\tilde{E} = E_0 e^{i\varphi}$

we can write: $E(t) = \Re\left\{\tilde{E} e^{i\omega t}\right\}$

from now on we will only use complex amplitudes and write them without tilde:

$$E_0 \cos(\omega t + \varphi) \longrightarrow \tilde{E} e^{i\omega t} \longrightarrow E$$

why do we do that?

- our expressions become considerably shorter,
- we can simplify all time derivations,

$$\frac{d}{dt}E(t) \longrightarrow \frac{d}{dt}\tilde{E}e^{i\omega t} = i\omega\tilde{E}e^{i\omega t}$$

$$\longrightarrow \frac{d}{dt}E = i\omega E$$

Complex notation of Maxwells Equations

The use of phasors yields the following form:

$$\nabla \times \mathbf{H} = i\omega\varepsilon\left(1 - i\frac{\kappa}{\omega\varepsilon}\right)\mathbf{E} \quad (I) \qquad \nabla \cdot \mathbf{E} = \frac{\rho_V}{\varepsilon} \quad (III)$$

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H} \quad (II) \qquad \nabla \cdot \mathbf{H} = 0 \quad (IV)$$

Consequently the **general wave equations** become:

$$\nabla^2\mathbf{E} - \nabla\left(\nabla \cdot \mathbf{E}\right) = -k^2\mathbf{E}$$

$$\nabla^2\mathbf{H} = -k^2\mathbf{H}$$

with the wavenumber

$$k^2 = \omega^2\mu\underline{\varepsilon} = \omega^2\mu\varepsilon\left(1 - i\frac{\kappa}{\omega\varepsilon}\right)$$

$\underline{\varepsilon} \rightarrow \varepsilon' - i\varepsilon''$

Remark: in conducting media k becomes complex. In non-conducting charge-free media the **wave equations** simplify to:

$$\nabla^2\mathbf{E} = -k^2\mathbf{E}$$

$$\nabla^2\mathbf{H} = -k^2\mathbf{H}$$

with the **wave number**

$$k^2 = \omega^2\mu\varepsilon = \frac{\omega^2}{c^2}$$

and c being the speed of light

plane waves

- homogenous, isotropic, linear medium,
- no space charge distribution, no currents,
- fields vary only in one direction (e.g. z).

The solution of the harmonic wave equation then becomes:

$$E_x(z) = \underline{C}_1 e^{-\underline{\gamma} z} + \underline{C}_2 e^{+\underline{\gamma} z}$$

$$H_y(z) = \frac{1}{\underline{Z}} \left(\underline{C}_1 e^{-\underline{\gamma} z} + \underline{C}_2 e^{+\underline{\gamma} z} \right)$$

→ propagation in positive and negative z direction

with the **propagation constant** gamma

$$\underline{\gamma} = \alpha + i\beta = j\underline{k} = i\omega\sqrt{\mu\underline{\varepsilon}}$$

α - attenuation, β - phase shift

and the **wave impedance** Z

$$\underline{Z} = \frac{E_y}{H_z} = \sqrt{\frac{\mu}{\underline{\varepsilon}}}$$

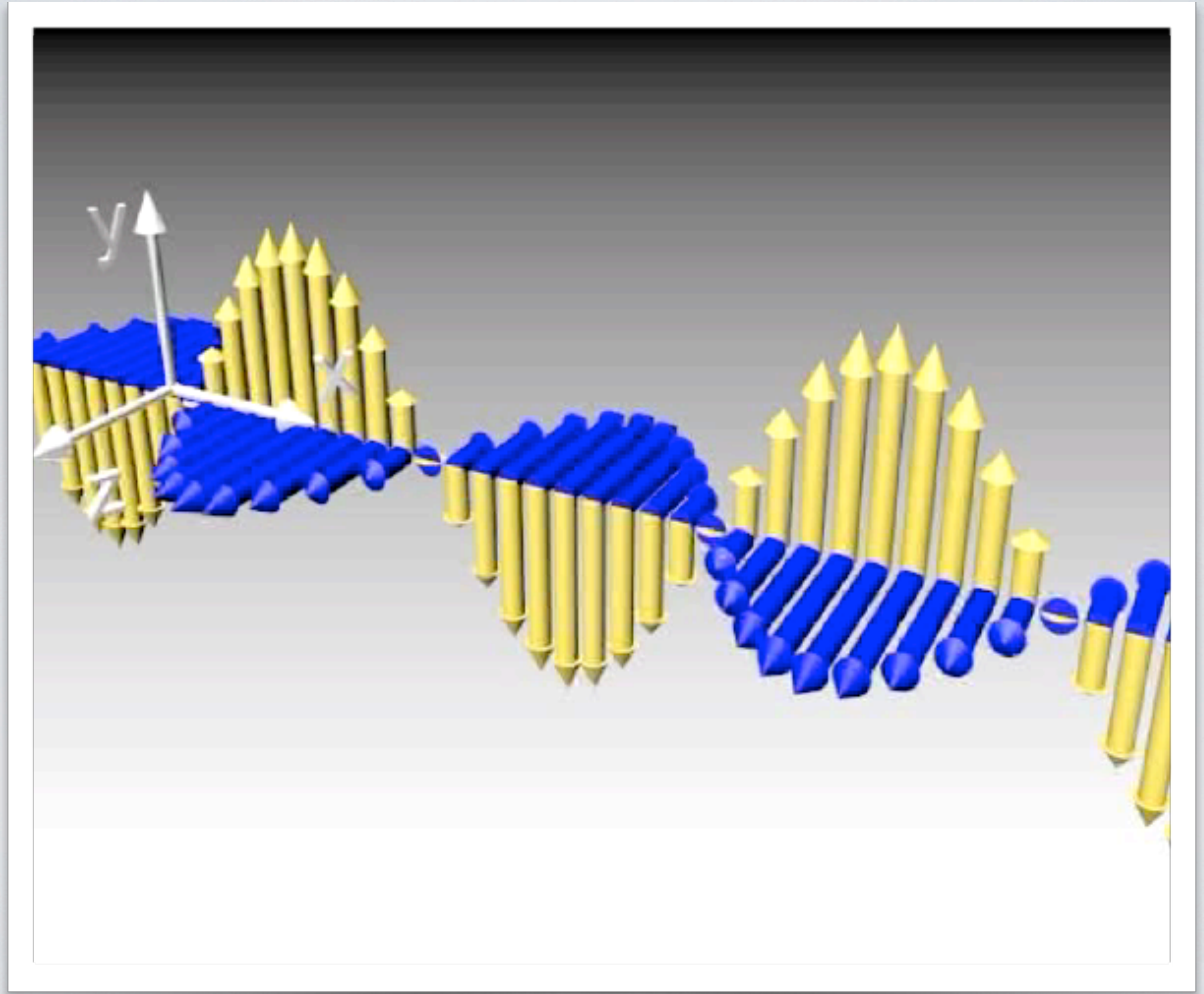
in vacuum →

$$Z = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\Omega$$

the velocity with which the maxima travel along x is called phase velocity v_{ph} (exact definition later)

From MIT Open Courseware

the velocity with which the maxima travel along x is called phase velocity v_{ph} (exact definition later)



From MIT Open Courseware

skin depth

In conducting material RF waves are strongly attenuated, which means that:

$$\frac{\kappa}{\omega\varepsilon} \gg 1 \quad \text{or} \quad \underline{\varepsilon} \approx -i\varepsilon'' = -i\frac{\kappa}{\omega}$$

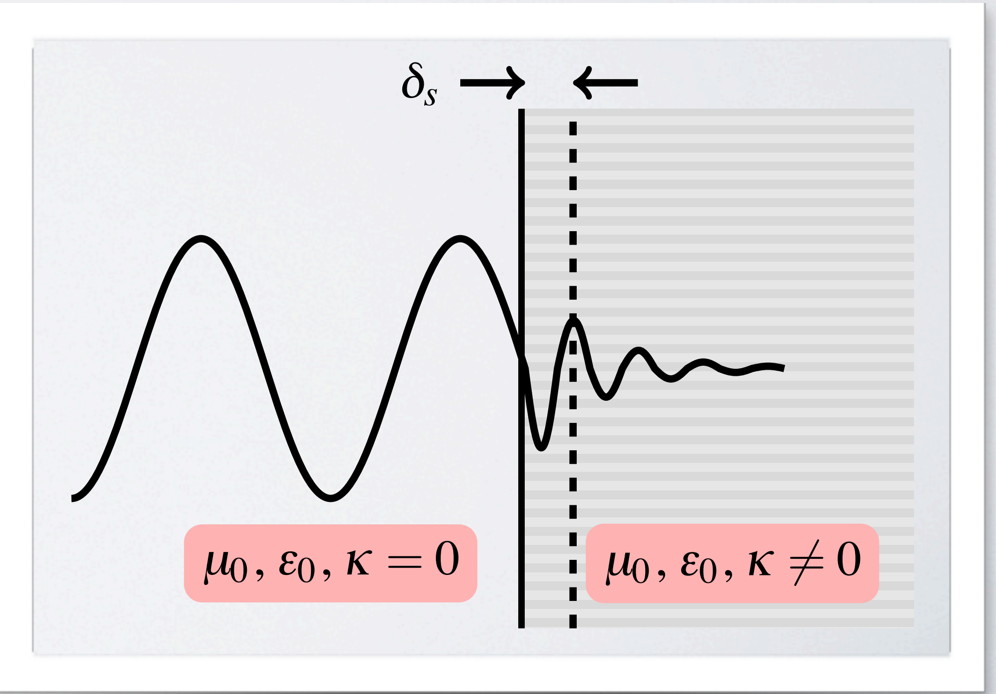
equivalent to neglecting the displacement current!

then the propagation constant becomes:

$$\gamma = \alpha + i\beta = i\omega\sqrt{\frac{-i\mu\kappa}{\omega}} = (1+i)\sqrt{\frac{\kappa\mu\omega}{2}}$$

the skin depth is defined as the distance after which the wave is attenuated by $1/e \approx 36.8\%$

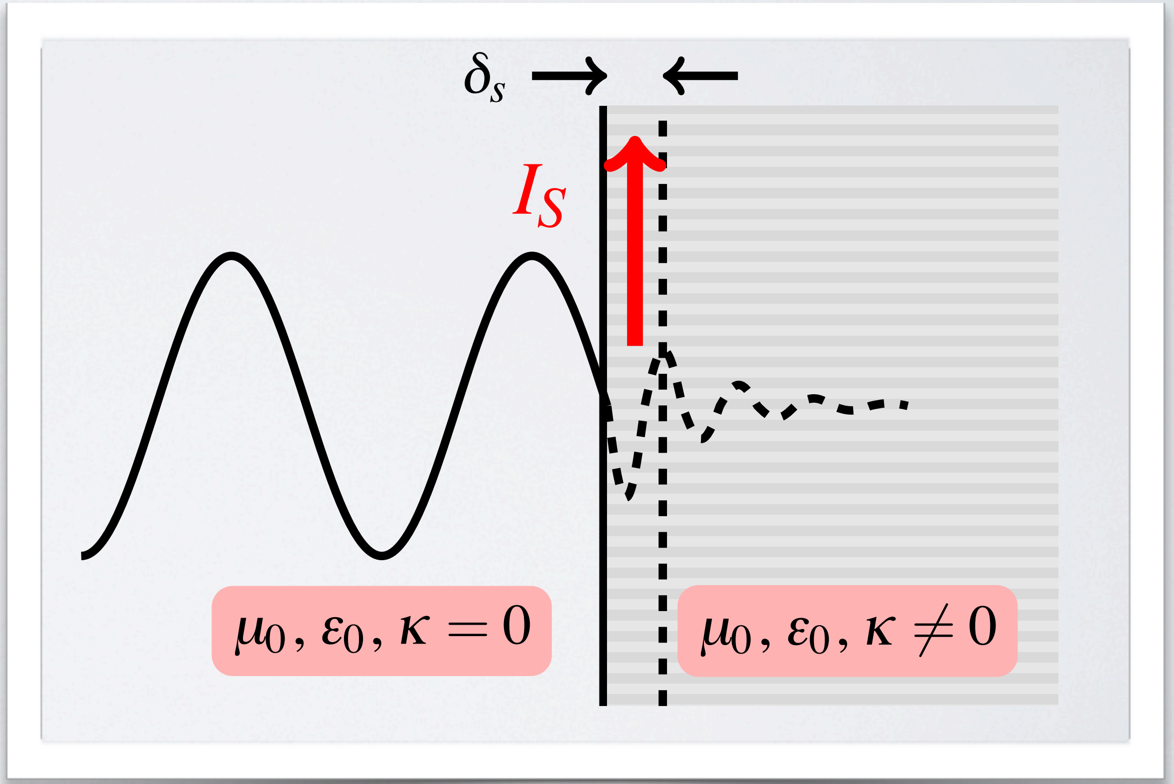
$$\delta_s = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\kappa}}$$



skin depth II

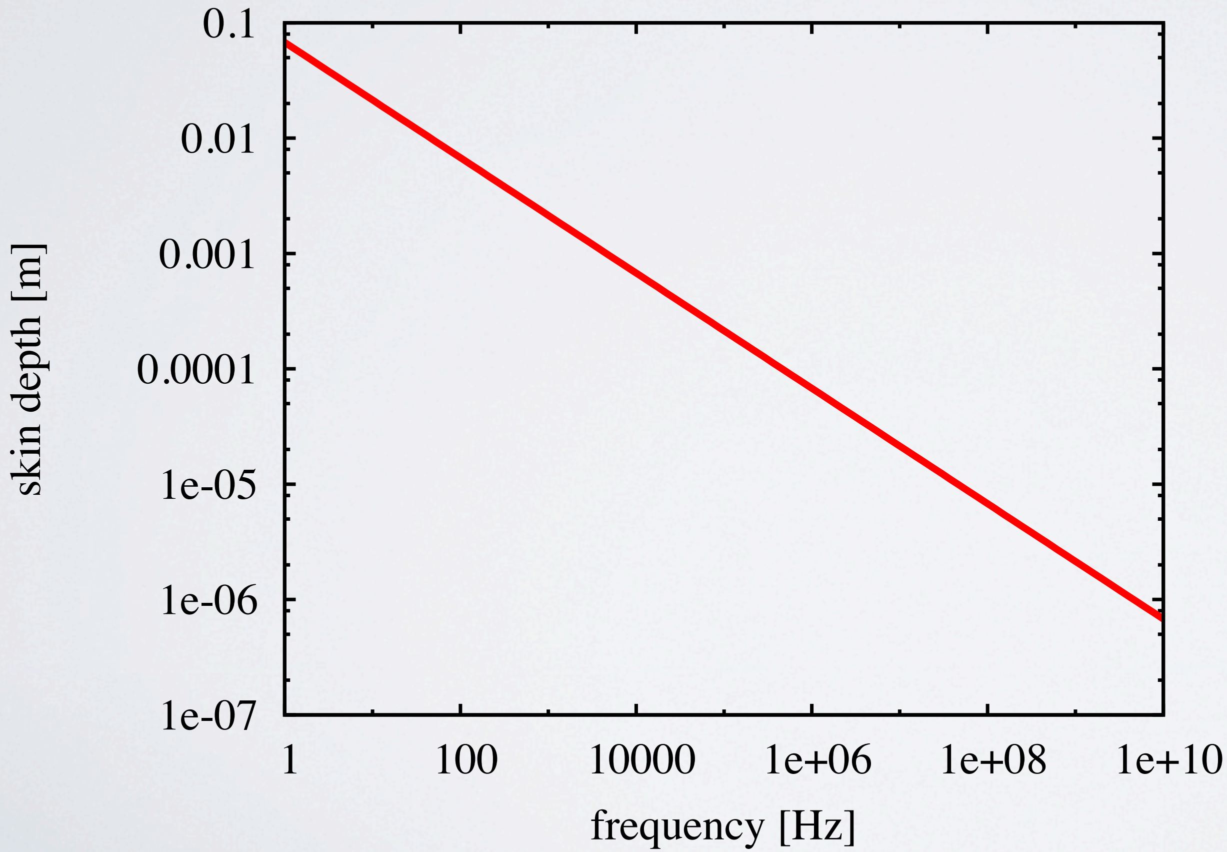
If a wave travels along a conducting surface, then we can calculate the surface resistance by assuming a constant current density within a material layer equivalent to the skin depth.

$$R_{surf} = \frac{1}{\kappa \delta_s} \left[\frac{\Omega}{m} \right]$$



skin depth III

For cavities (usually copper) and wave-guides (usually aluminum) and typical frequencies for accelerators (100 MHz - 10 GHz) the skin depth is usually in the μm range, **which is why we can build cavities out of steel and copper plate them with just a few 10s of um.**



e.g. copper

$$\kappa_{Cu} = 55 \cdot 10^6 \frac{1}{\Omega m}$$

$$\mu_0 = 4\pi 10^{-7} \frac{Vs}{Am}$$

Energy density and energy flow

The **energy density of electric and magnetic fields** is defined as (without derivation):

$$w_e = \frac{1}{2}\mathbf{E} \cdot \mathbf{D}$$
$$w_m = \frac{1}{2}\mathbf{H} \cdot \mathbf{B}$$

for time-harmonic fields in complex notation

$$w_{e,k} = \frac{1}{4}\mathbf{E} \cdot \mathbf{D}^*$$
$$w_{m,k} = \frac{1}{4}\mathbf{H} \cdot \mathbf{B}^*$$

Integration of the total energy density (magnetic + electric) and a bit of vector analysis gives us **Poynting's Law:**

$$-\frac{d}{dt}\int_V w dV = \oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} + \int_V (\mathbf{E} \cdot \mathbf{J})\, dV$$

reduction of total energy in V

Energy leaving V (through **S**) per time unit

work on charges in V per time unit

Poynting vector

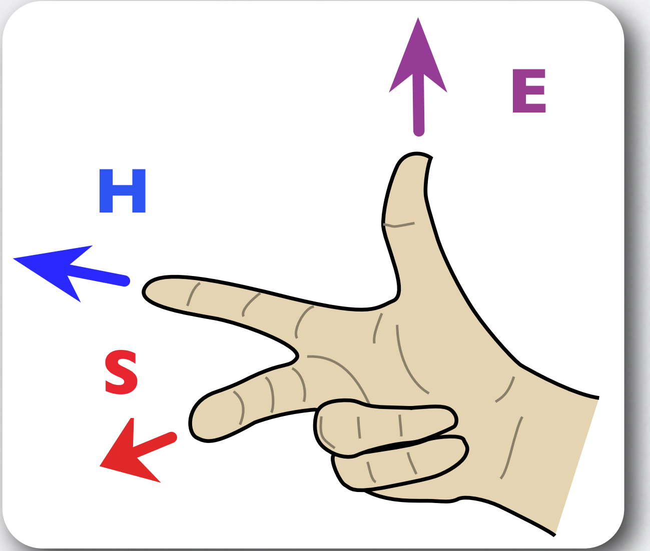
From Poynting's Law we get the definition of the Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

for time-harmonic fields in complex notation

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$$

which defines the direction of the energy propagation of an electromagnetic wave. It also tells us that the propagation direction of the energy transport is perpendicular to the directions of the electric and magnetic field components.



Solution of the wave equation

To find the electric and magnetic fields in free space, in wave-guides or in cavities, one needs to solve the wave equation in the appropriate coordinate system (cartesian, cylindric, spherical).

A common approach to **solve the wave equation for wave guides is to define a vector potential for TE and TM waves**, so that electric and magnetic fields can be calculated from:

$$\mathbf{E}^{TE} = \nabla \times \mathbf{A}^{TE} \quad \text{and} \quad \mathbf{H}^{TM} = \nabla \times \mathbf{A}^{TM}$$

$$\mathbf{H}^{TE} = \nabla \times (\nabla \times \mathbf{A}^{TE}) \quad \text{and} \quad \mathbf{E}^{TM} = \nabla \times (\nabla \times \mathbf{A}^{TM})$$

In both cases the vector potential fulfills the wave equation,

$$\nabla^2 \mathbf{A} = -k^2 \mathbf{A} \quad \text{with} \quad k^2 = \omega^2 \mu \varepsilon$$

which can then be solved for different coordinate systems for TE and TM waves and which has just one vector component: $\mathbf{A} = A_z \mathbf{e}_z$

Nomenclature of modes in cavities/wave guides

TM_{mnp}-mode = E_{mnp}-mode → E-field parallel to axis, B_z =0, only transverse magn. (TM) components

TE_{mnp}-mode = H_{mnp}-mode → B-field parallel to axis, E_z =0, only transverse el. (TE) components

in a circular cavity this means:

- number of full-period variations of the field components in the azimuthal-direction
- number of zeros of the axial field component in radial direction.
- number of half-period variations of the field components in the longitudinal-direction

$$\mathbf{E} \text{ or } \mathbf{B} \propto \cos(m\phi) \text{ or } \sin(m\phi)$$

$$E_z \text{ or } B_z \propto J_m(x_{mn} r/R_c)$$

$$\mathbf{E} \text{ or } \mathbf{B} \propto \cos(p\pi z/l) \text{ or } \sin(p\pi z/l)$$

Solution of the wave equation

For circular wave guides we obtain (without derivation):

$$A_z^{TM/TE} = C\,\mathrm{J}_m(k_c r)\cos(m\varphi)e^{-ik_z z} \qquad \text{with} \qquad k_z = \sqrt{k^2 - k_c^2}$$

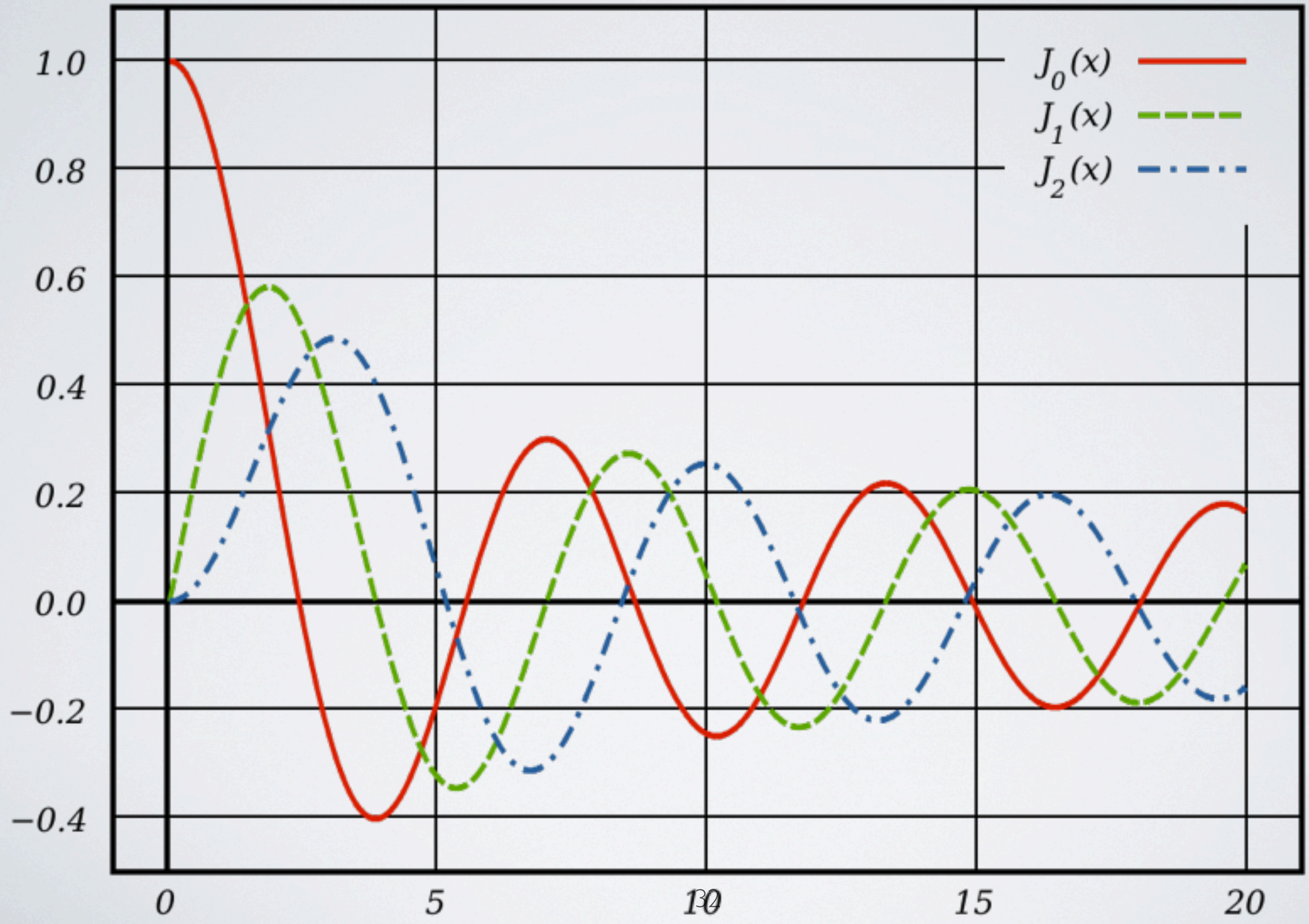
using $\mathbf{H}^{TM} = \nabla \times \mathbf{A}$ and $\mathbf{E}^{TM} = \nabla \times (\nabla \times \mathbf{A})$

results in the following **field components for TM waves**:

$$\left.\begin{aligned}
E_r &= \frac{i}{\omega\varepsilon}\frac{\partial H_\varphi}{\partial z} &&= -C\frac{k_z k_c}{\omega\varepsilon}\mathrm{J'}_m(k_c r)\cos(m\varphi)\\
E_\varphi &= -\frac{i}{\omega\varepsilon}\frac{\partial H_r}{\partial z} &&= C\frac{m k_z}{\omega\varepsilon r}\mathrm{J}_m(k_c r)\sin(m\varphi)\\
E_z &= \frac{ik_c^2}{\omega\varepsilon}A_z &&= C\frac{ik_c^2}{\omega\varepsilon}\mathrm{J}_m(k_c r)\cos(m\varphi)\\
H_r &= \frac{1}{r}\frac{\partial A_z}{\partial\varphi} &&= -C\frac{m}{r}\mathrm{J}_m(k_c r)\sin(m\varphi)\\
H_\varphi &= -\frac{\partial A_z}{\partial r} &&= -Ck_c\mathrm{J'}_m(k_c r)\cos(m\varphi)
\end{aligned}\right\} e^{-ik_z z}$$

J_m are Bessel functions of the first kind and of m'th order

Bessel functions of the first kind

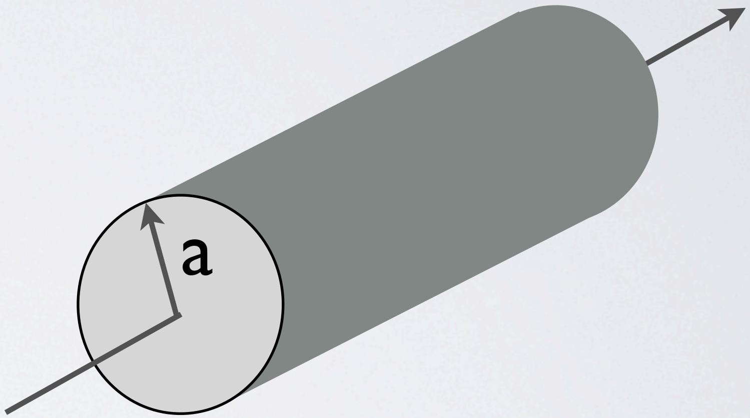


wave propagation in a cylindrical pipe

let us consider the simplest accelerating mode (electric field in z-direction): m=0, n=1, TM_{01}

using $J_0'(r) = -J_1(r)$

$$\left.\begin{aligned} E_r &= C\tfrac{k_z k_c}{\omega\varepsilon}\mathrm{J}_1(k_c r) \\ E_z &= -C\tfrac{ik_c^2}{\omega\varepsilon}\mathrm{J}_0(k_c r) \\ H_\varphi &= C k_c \mathrm{J}_1(k_c r) \end{aligned}\right\} e^{-ik_z z}$$



propagation constant: $k_z^2 = k^2 - k_c^2$ **wave number:** $k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$

k_c is determined by the boundary conditions of the wave-guide

$$\mathbf{E}_{||} = 0 \Rightarrow E_z(r=a) = 0 \Rightarrow J_0(k_c a) = 0 \Rightarrow k_c a = 2.405$$

wave propagation in a cylindrical pipe

and from $k_c = \frac{2\pi}{\lambda_c} = \frac{\omega_c}{c}$ we can calculate the **cut-off frequency** for the TM_{01} mode in a cylindrical conducting pipe

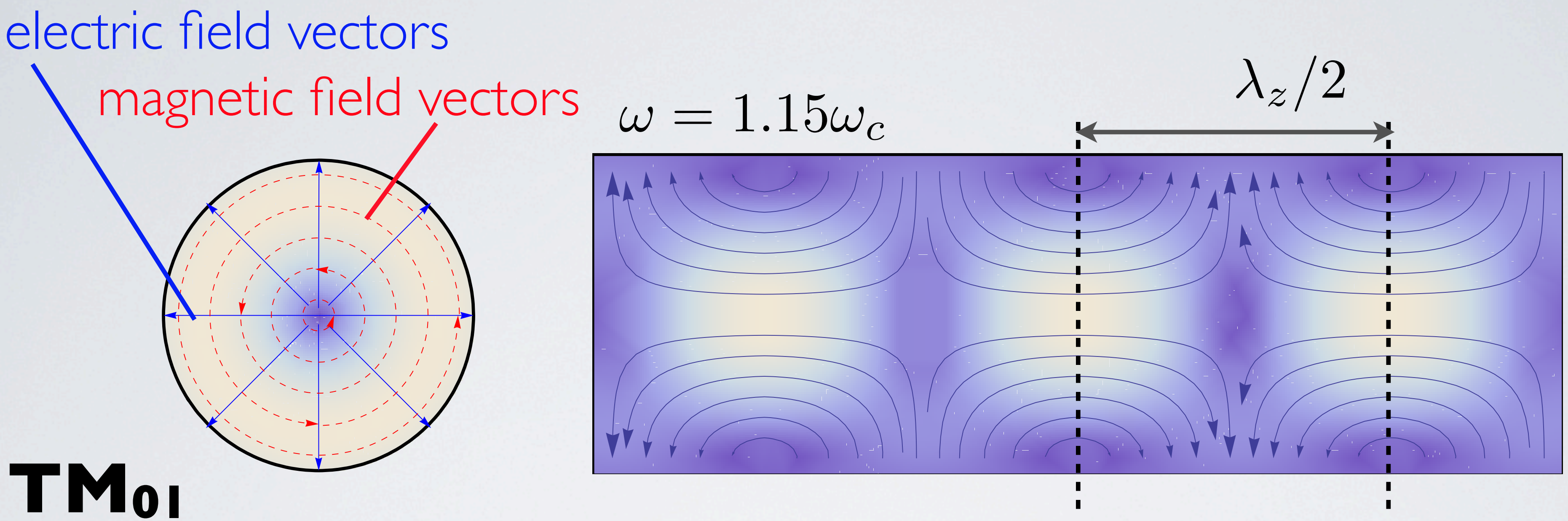
$$\omega_c = \frac{2.405c}{a}$$

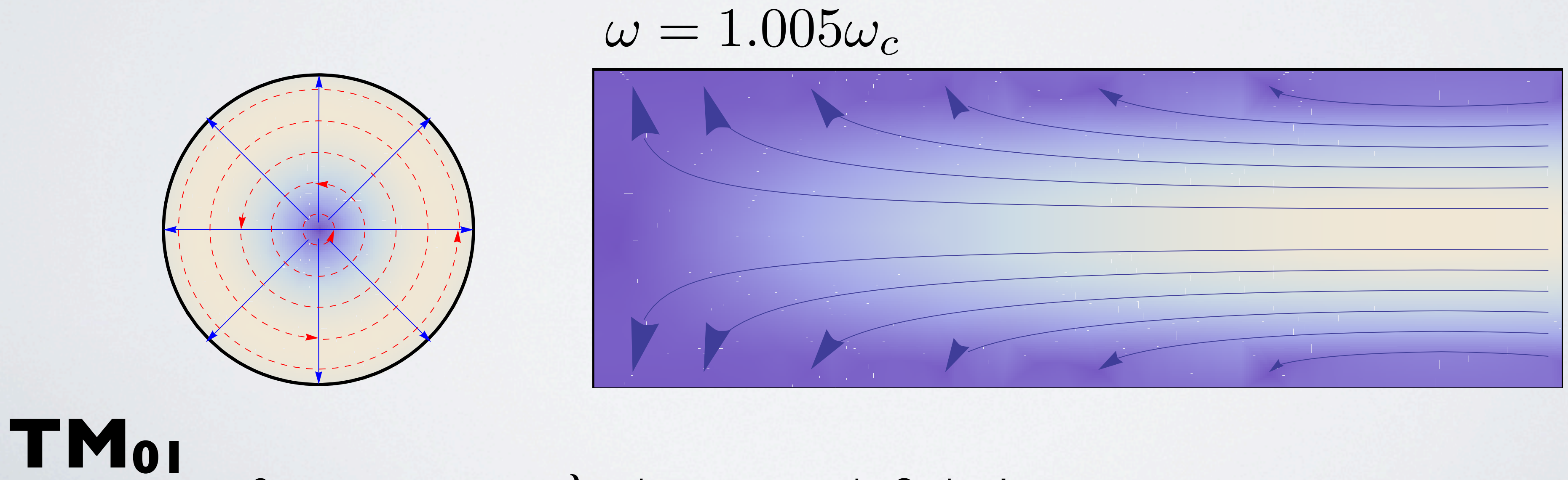
from $k_z^2 = k^2 - k_c^2$ we also get the **dispersion relation**

$$k_z^2 = \frac{\omega^2 - \omega_c^2}{c^2} = \frac{\omega^2}{v_{ph}^2}$$

- TM_{01} waves propagate for: $\omega > \omega_c$
- and are exponentially damped for: $\omega < \omega_c$
- the phase velocity is: $v_{ph} = \frac{\omega}{k_z}$

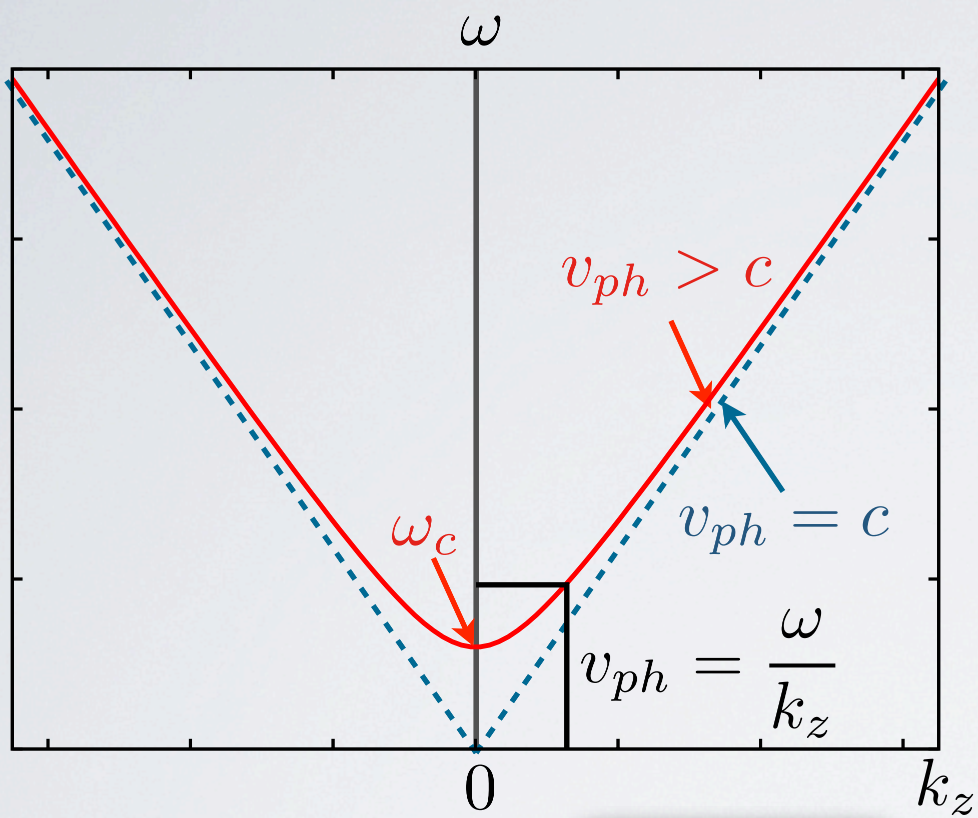
wave propagation in a cylindrical pipe





for $\omega \to \omega_c$, λ_z becomes infinite!

dispersion relation (Brillouin diagram)



group velocity: $v_{gr} = \frac{d\omega}{dk_z}$

phase velocity: $v_{ph} = \frac{\omega}{k_z}$

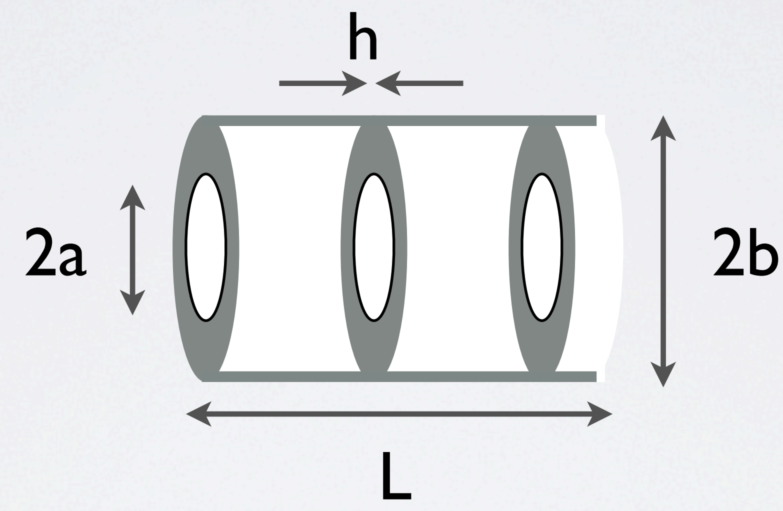
- each frequency corresponds to a certain phase velocity,
- the phase velocity is always larger than c! (at $\omega=\omega_c$: $k_z=0$ and $v_{ph}=\infty$),

$$v_{ph}^2 = c^2 \frac{\omega^2}{\omega^2 - \omega_c^2}$$

- **synchronism with RF** (necessary for acceleration) **is impossible** because a particle would have to travel at $v=v_{ph}>c$!
- energy (and therefore information) travels at the group velocity $v_{gr}<c$,

How can we slow down the phase velocity?

put some obstacles into the wave-guide: e.g: discs



Only then can we achieve synchronism between the particles and the phase velocity of the RF wave.

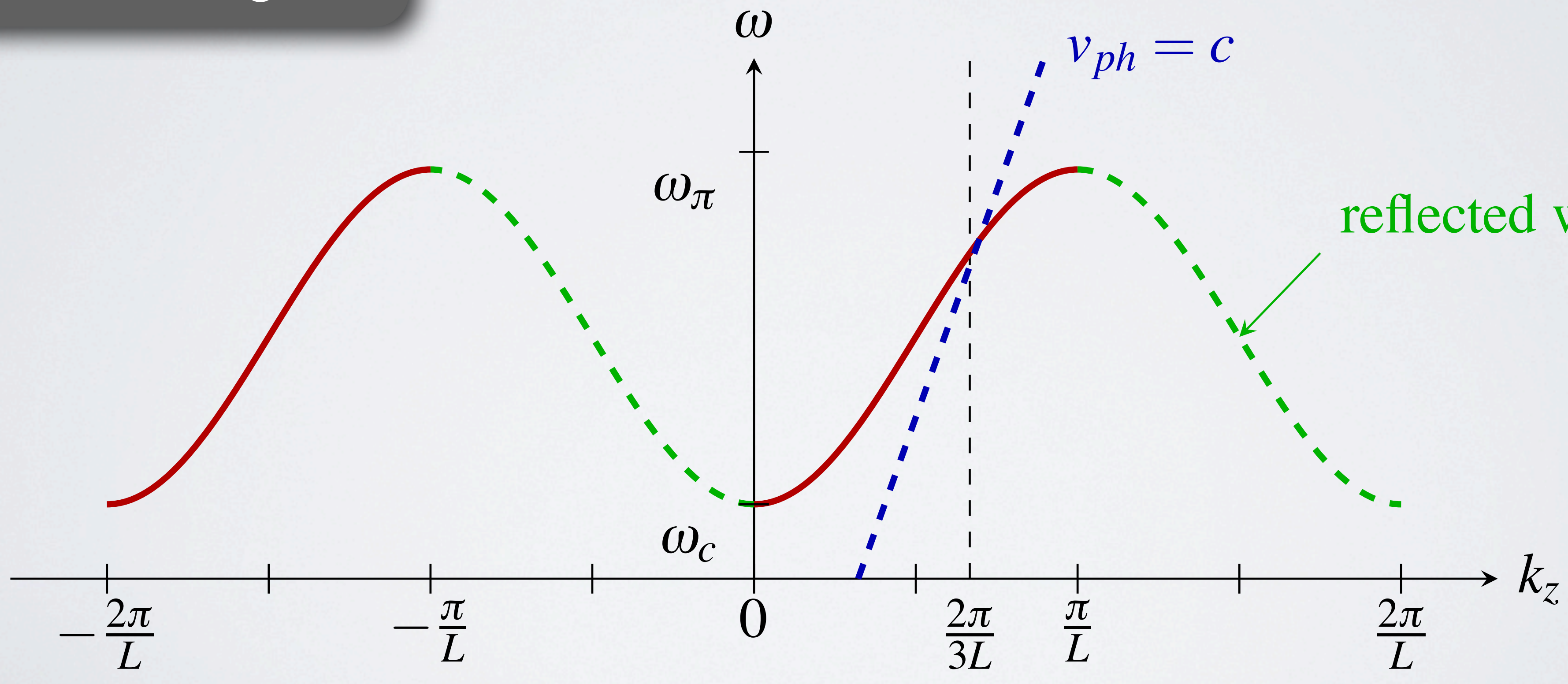
Dispersion relation for disc-loaded circ. wave guides

$$\omega = \frac{2.405c}{b}\sqrt{1 + \kappa(1 - \cos(k_z L)e^{-\alpha h})}$$

$$\kappa = \frac{4a^3}{3\pi J_1^2(2.405)b^2 L} \ll 1$$

damping: $\alpha \approx \dfrac{2.405}{a}$

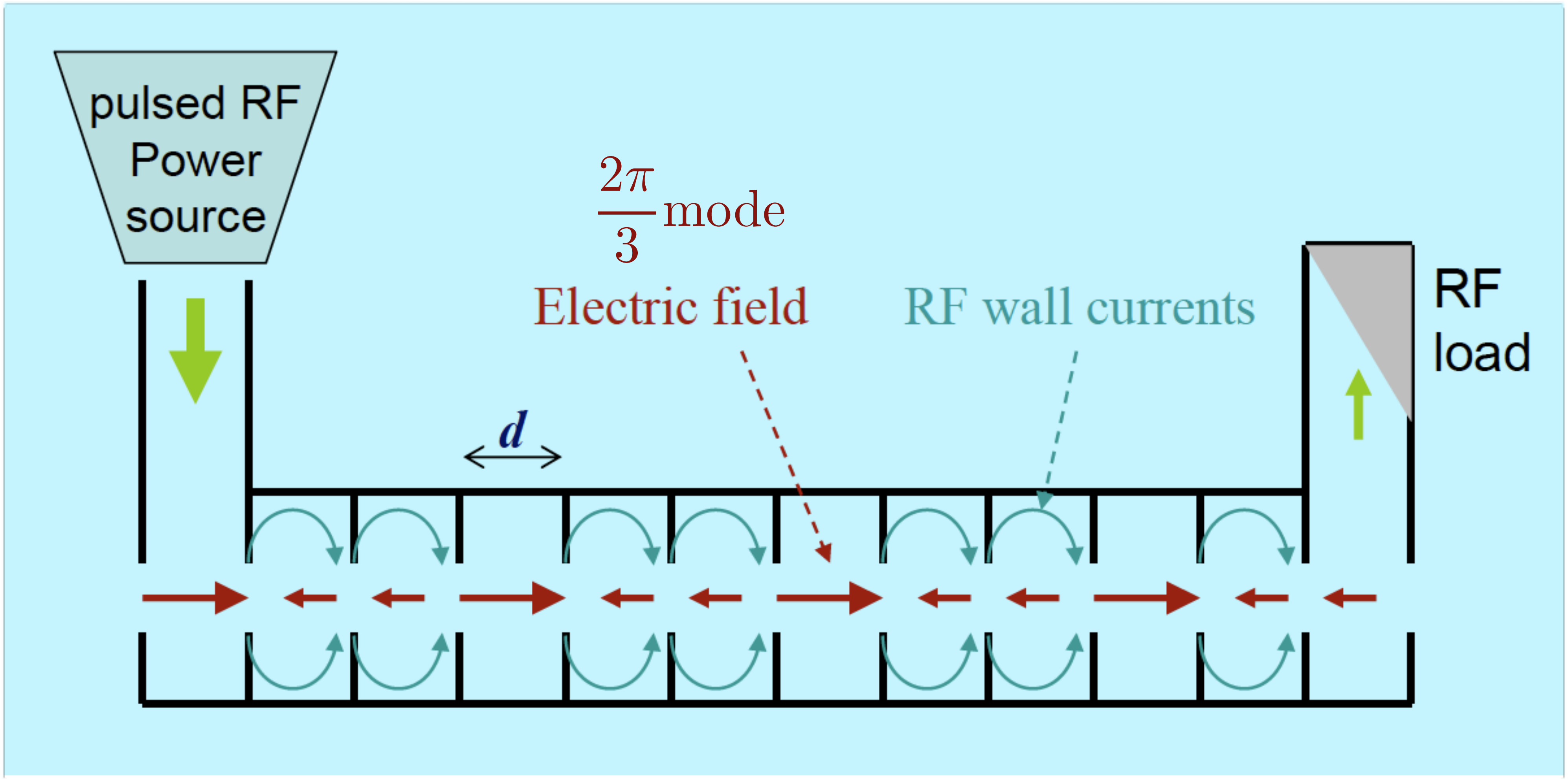
Brioullin diagram



typical operating point

Example of a 2/3π traveling wave structure

synchronism condition: $d = \frac{(\beta)\lambda}{3}$ with $\beta \approx 1$



Traveling wave structures

- Since the particles gain energy the EM-wave is damped along the structure ("**constant impedance structure**"). But by changing the bore diameter one can decrease the group velocity from cell to cell and obtain a **"constant-gradient"** structure. Here one can operate in all cells near the break-down limit and thus achieve a higher average energy gain.
- Traveling wave structures are often used for very short (us) pulses, and can reach high efficiencies, and high accelerating gradients (up to 100 MeV/m, CLIC).
- are generally used for electrons at $\beta \approx 1$,
- difficult to use for ions with $\beta < 1$: i) constant cell length does not allow for synchronism, ii) long structures do not allow for sufficient transverse focusing,

let us see if we can apply all of this to calculate the **resistive** damping in a circular wave guide for the TM_{01} mode

to do this we need:

- Boundary conditions on electric surfaces (or Ampère's Law);
- The complex notation of EM fields;
- The solution of the harmonic wave equation for the TM_{01} mode;
- Poynting's Law;
- A bit of common sense;
- and the Power-Loss Method, which we will learn on the way.

attenuation of waves (power loss method)

We assume highly conductive wave guide boundaries, which means that we have a small skin depth and that the electric fields will basically be orthogonal to the surface. We can therefore assume that the fields in the lossless wave guide (infinite conductivity) are basically identical to the fields in the lossy wave guide.
It seems therefore reasonable to calculate the surface currents from the ideal fields, and then to apply the surface resistance to calculate the losses.

We will proceed in 3 steps:

1. **Definition of the attenuation constant**
2. **Calculation of the power transported in the wave guide**
3. **Calculation of the losses in the wave guide surface**

1. Attenuation constant

We start by defining the power lost per longitudinal distance:

$$P' = -\frac{dP}{dz}$$

from $E, H \propto e^{-\alpha z} \quad \Rightarrow \quad P \propto e^{-2\alpha z}$

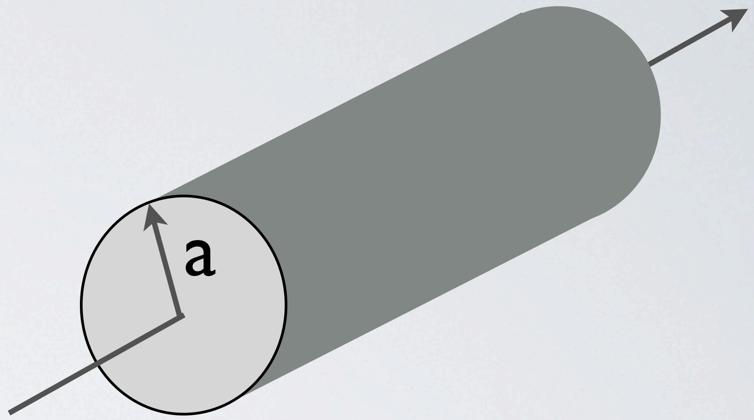
we get immediately $P' = -\frac{dP}{dz} = 2\alpha P$

and thus the definition of the attenuation constant

$$\alpha = \frac{P'}{2P}$$

2. Power transport in a wave-guide

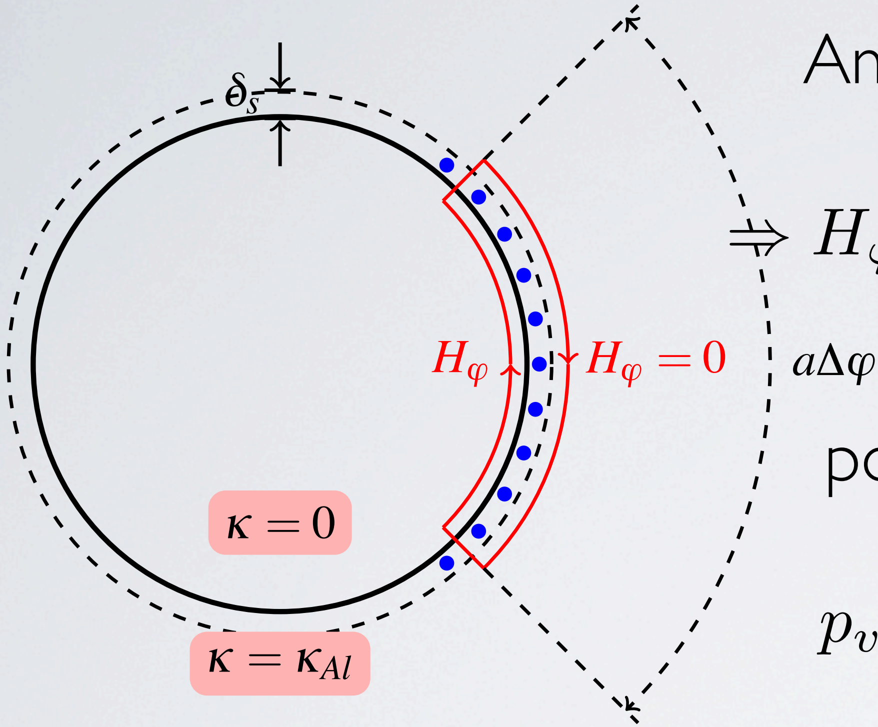
$$\left.\begin{aligned} E_r &= C\frac{k_z k_c}{\omega\varepsilon}\mathrm{J}_1(k_c r) \\ E_z &= -C\frac{ik_c^2}{\omega\varepsilon}\mathrm{J}_0(k_c r) \\ H_\varphi &= C k_c \mathrm{J}_1(k_c r) \end{aligned}\right\} e^{-ik_z z}$$



$$P = \frac{1}{2}\int_A (\mathbf{E}\times\mathbf{H}^*)\cdot d\mathbf{A} = \frac{1}{2}\int_0^a\int_0^{2\pi} E_r H_\varphi^* r dr d\varphi = \frac{C^2 k_z k_c^2 \pi a^2 J_1^2(k_c a)}{\omega\varepsilon}$$

$$\int_0^a J_1^2(k_c r) r dr = \frac{a^2}{2} J_1^2(k_c a)$$

3. Losses on wave-guide surface



Ampère's Law: $\oint_c \mathbf{H} \cdot d\mathbf{l} = I = \oint_c \mathbf{J} \cdot (\delta_s d\mathbf{l})$

$$\Rightarrow H_\varphi(r = a, z) = C k_c \mathrm{J}_1(k_c a) e^{-ik_z z} = J_z(z)\delta_s$$

power density in the wall [W/m^3]:

$$p_v = \frac{1}{2}\mathbf{E} \cdot \mathbf{J}^* = \frac{1}{2\kappa} J_z J_z^* = \frac{\partial^3 P}{(\partial r)(r\partial\varphi)(\partial z)}$$

power loss per meter along z [W/m]:

$$P' = \frac{\partial P}{\partial z} = \int_a^{a+\delta_s} \int_0^{2\pi} p_v r dr d\varphi = \frac{\pi a C^2 k_c^2 J_1^2(k_c a)}{\kappa \delta_s}$$

$\delta_s \ll a$

attenuation per unit length

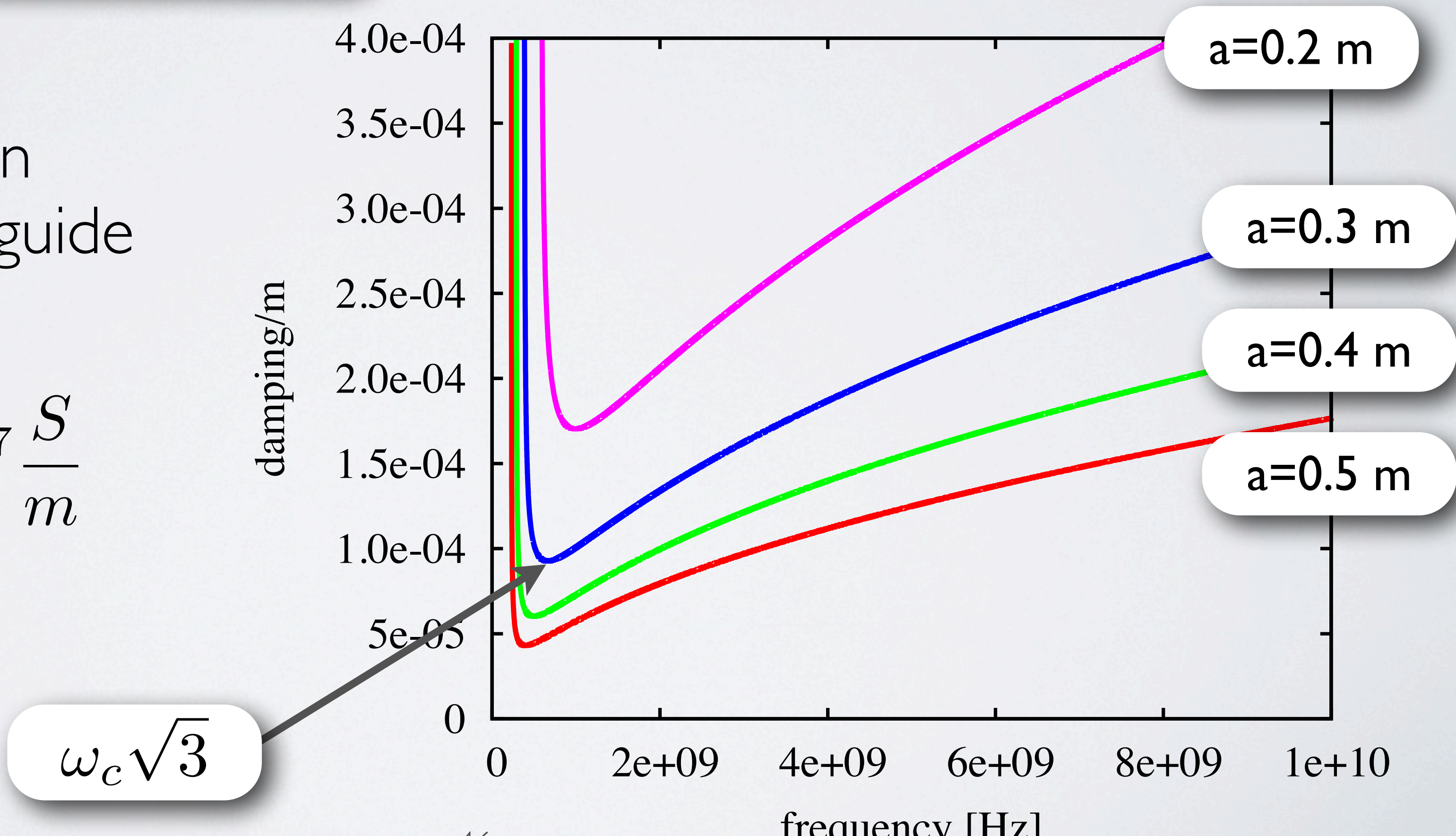
$$\alpha = \frac{P'}{2P} = \frac{R_{surf}}{Z_0 a \sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

surface resistance $R_{surf} = \frac{1}{\kappa \delta_s}$

wave impedance (vacuum) $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$

e.g. damping in an aluminum wave guide with

$$\kappa_{Al} = 3.66 \cdot 10^7 \frac{S}{m}$$



what we should know by now

- The power of Maxwells Equations,
- Vector analysis: Gauss' Law, Stokes' Law
- Ampère's Law, Faraday's Law
- Displacement current
- Boundary conditions for electric and magnetic fields
- Wave equation/plane waves and the complex form of time-harmonic fields
- Skin effect
- Poynting's law
- Solution of the wave equation for circular geometries
- Dispersion relation, group velocity, traveling wave structures
- Power-loss method and damping in wave guides
- write-up of this lecture: soon available as CAS proceedings of Bilbao 2011 school

Annex: differential operators in cylindrical coordinates

$$\nabla\phi = \begin{pmatrix} \frac{\partial\phi}{\partial r} \\ \frac{\partial\phi}{r\partial\varphi} \\ \frac{\partial\phi}{\partial z} \end{pmatrix}$$

$$\nabla\times\mathbf{a} = \begin{pmatrix} \frac{\partial a_z}{r\partial\varphi} - \frac{\partial a_\varphi}{\partial z} \\ \frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r} \\ \frac{\partial(r a_\varphi)}{r\partial r} - \frac{\partial a_r}{r\partial\varphi} \end{pmatrix}$$

$$\nabla\cdot\mathbf{a} = \frac{\partial(r a_r)}{r\partial r} + \frac{\partial a_\varphi}{r\partial\varphi} + \frac{\partial a_z}{\partial z}$$

$$\Delta\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{\partial\phi}{r\partial r} + \frac{\partial^2\phi}{r^2\partial\phi^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$x = r\cos\varphi \qquad \text{with } 0 \le r \le \infty$$

$$y = r\sin\varphi \qquad \text{with } 0 \le \varphi \le 2\pi$$

$$z = z$$

$$d\mathbf{l} = \begin{pmatrix} dr \\ r d\varphi \\ dz \end{pmatrix}$$

$$dV = r dr d\varphi dz$$