
Ultracold Gases far from Equilibrium

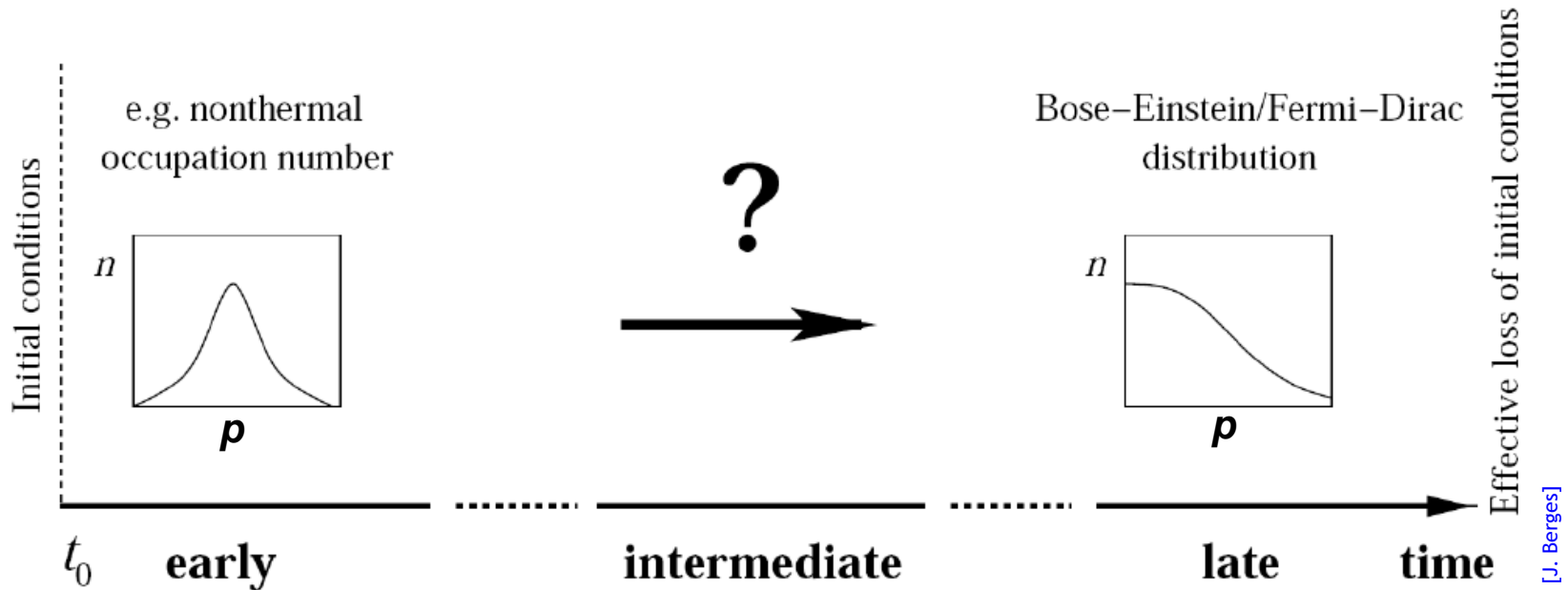


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Preface

Far-from-equilibrium dynamics



Thermal equilibrium: **Loss of information** about prior evolution.

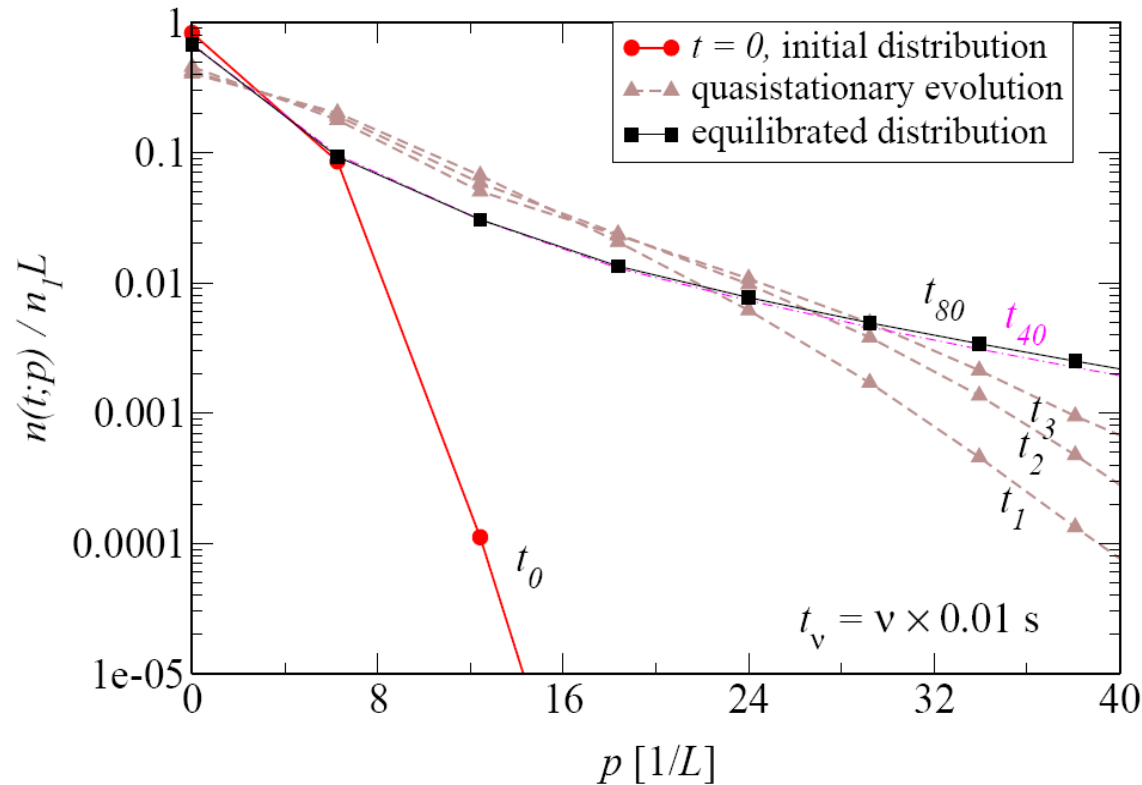


Only a **few conserved quantities** persist.



Equilibration of a 1D Bose gas

Momentum distribution for different times:



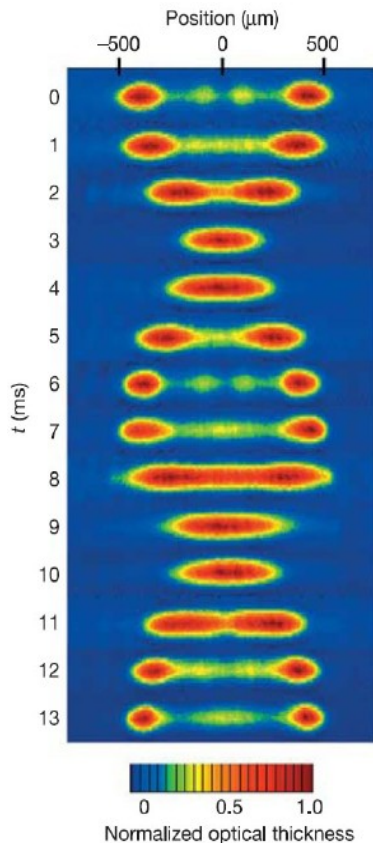
initial state: ^{23}Na atoms in 1D, $n_1 = 10^7 \text{ m}^{-1}$



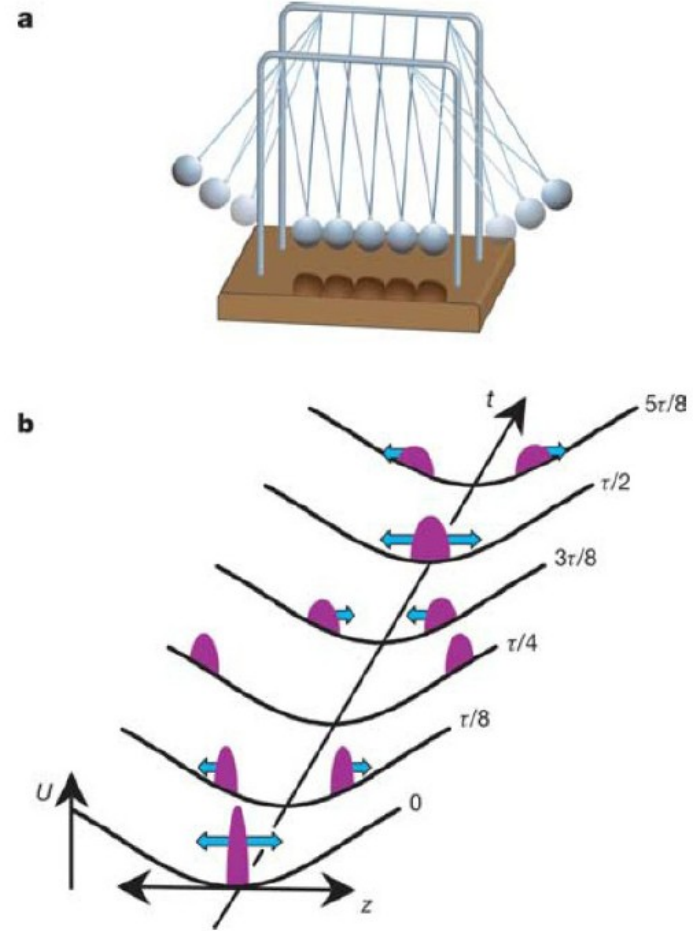
Long-time dynamics of ultracold gases

“A quantum Newton's cradle”

[T. Kinoshita et al. Nature 440 (06)]



Indication for strong suppression of damping



Effective-action for many-body dynamics

Dynamical Field Theory



We will be interested, in particular, in the explicit time dependence of the lowest-order **correlation functions**:

$$\phi_a(x) = \langle \Phi_a(x) \rangle \quad (\text{mean field})$$

$$G_{ab}(x, y) = \langle \mathcal{T} \Phi_a(x) \Phi_b(y) \rangle_c \quad (\text{density matrix, 2-point function, propagator})$$

where $x = (\mathbf{x}, t)$

connected, i.e., $= \langle \mathcal{T} \Phi_a \Phi_b \rangle - \phi_a \phi_b$



Path Integral Approach



Classical dynamics of φ from $\delta S[\varphi] = 0$.



Path Integral Approach



QM transition amplitude:

$$\langle t_{\text{fin}} | t_{\text{ini}} \rangle = \int \mathcal{D}\varphi e^{iS[\varphi]/\hbar}$$

$$\mathcal{D}\varphi = \prod_{x=x_{\text{ini}}}^{x_{\text{fin}}} d\varphi(x)$$

Classical dynamics of φ from $\delta S[\varphi] = 0$.



Path Integral Approach

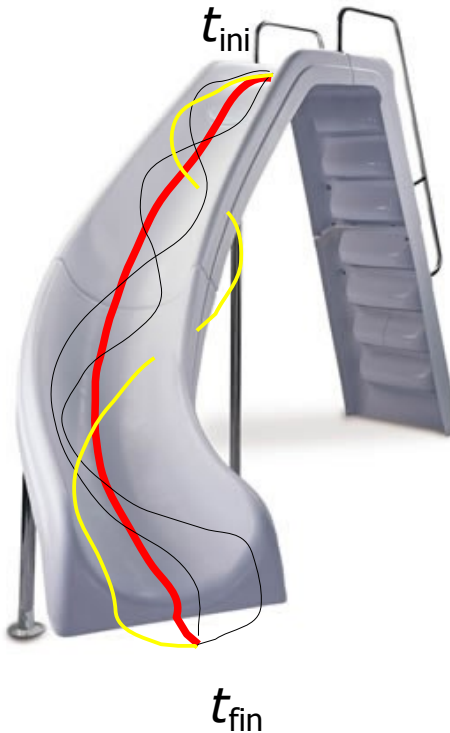


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Path Integral Approach...

...in QFT:

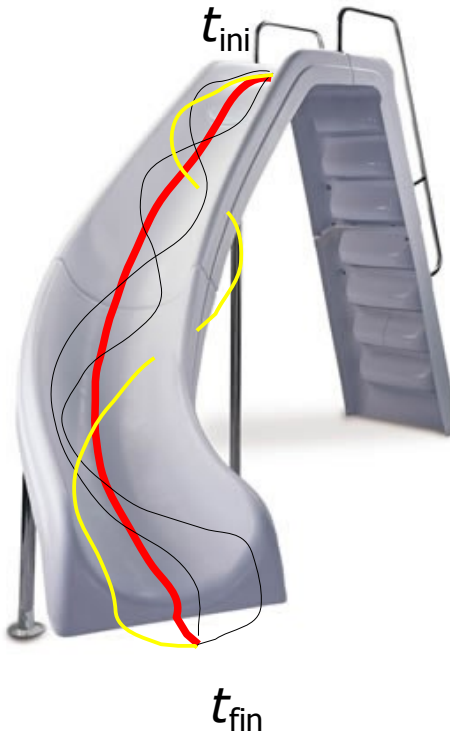


Generating functional:

$$Z[J] = \int \mathcal{D}\varphi e^{iS[\varphi]/\hbar - i \int J \varphi}$$

$$\phi = i \left. \frac{\delta \ln Z}{\delta J} \right|_{J=0} = Z^{-1} \int \mathcal{D}\varphi \varphi e^{iS[\varphi]/\hbar}$$

Classical dynamics of φ from $\delta S[\varphi] = 0$.



Effective Action



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Classical dynamics of φ from $\delta S[\varphi] = 0$.

Quantum dynamics of ϕ from variation of an **effective action**, $\delta \Gamma[\phi]/\delta \phi = -J$:

$$Z[J] = \int \mathcal{D}\varphi \delta[\varphi - \phi] e^{i\Gamma[\varphi]/\hbar - i \int J \varphi}$$

$$\Gamma[\phi] = -i \hbar \ln Z[J] - \int J \phi$$



Legendre transform



Effective Action



Generating functional:

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$$\begin{aligned} \Gamma[\phi] &= -i\hbar \ln Z[J] - \int J \phi \\ &= S[\phi] - i/2 \text{Tr} \ln G + \dots \end{aligned}$$



Gaussian integration



2PI Effective Action (Φ -Functional)

[Luttinger, Ward (60); Baym (62); Cornwall, Jackiw, Tomboulis (74)]

- Double **Legendre transform**:

$$\Gamma[\phi, G] = -i \ln Z[J, K] - \phi_i J_i - \frac{1}{2}(\phi_i \phi_j + G_{ij}) K_{ij},$$
$$-i \left. \frac{\delta \ln Z[J, K]}{\delta J_i} \right|_{J=K \equiv 0} = \phi_i = \langle \hat{\Phi}_i \rangle,$$
$$-2i \left. \frac{\delta \ln Z[J, K]}{\delta K_{ij}} \right|_{J=K \equiv 0} = \phi_i \phi_j + G_{ij} = \langle \mathcal{T} \hat{\Phi}_i \hat{\Phi}_j \rangle.$$

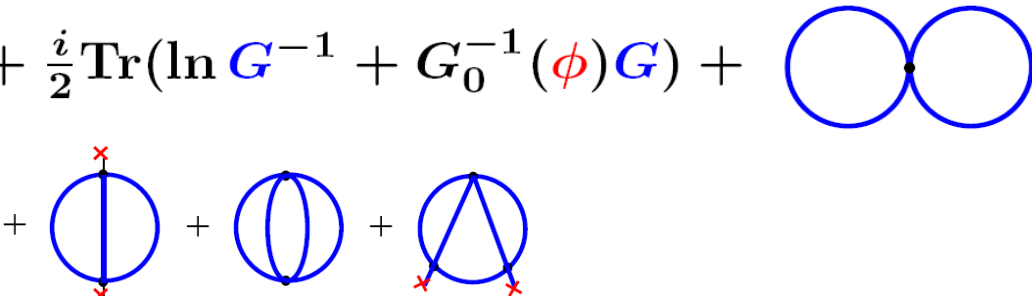


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$$\Gamma[\phi, G] = S[\phi] + \frac{i}{2} \text{Tr}(\ln G^{-1} + G_0^{-1}(\phi) G) +$$


The diagram shows four Feynman diagrams representing 2-particle irreducible (2PI) loop diagrams. The first diagram is a self-energy loop (a circle with a vertical line through the center). The second diagram is a bubble diagram (two circles connected by two vertical lines). The third diagram is a triangle diagram (a circle with a triangle inside). The fourth diagram is a sunset diagram (two circles connected by two vertical lines, with a horizontal line connecting the two vertices on the left). Each diagram has red 'x' marks at its external vertices.

Only 2-particle irreducible (2PI) loop diagrams

Conservation laws fulfilled

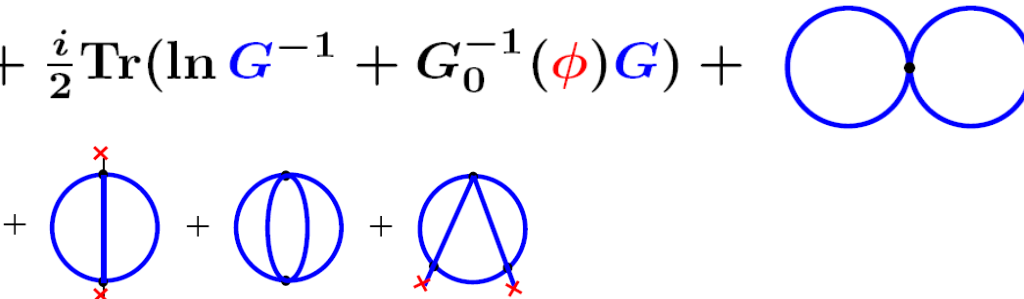


2PI Effective Action (Φ -Functional)

[Luttinger, Ward (60); Baym (62); Cornwall, Jackiw, Tomboulis (74)]

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- Dynamic equations: [closed for Gaussian initial conditions (only $\phi, G \neq 0$ @ $t = 0$)]

$$(@ J = 0, K = 0) \quad \frac{\delta \Gamma[\phi, G]}{\delta \phi_x} = 0, \quad \frac{\delta \Gamma[\phi, G]}{\delta G(x, y)} = 0$$

Conservation laws fulfilled



2PI $1/\mathcal{N}$ Expansion

[Berges, NPA **699** (02) 847; Aarts, Ahrensmaier, Baier, Berges, & Serreau, PRD **66** (02) 45008]



$$\Gamma_2^{\text{LO}}[\phi, G] = \text{Diagram 1} ,$$

$$\Gamma_2^{\text{NLO}}[\phi, G] = \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \dots$$

$$+ \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \dots$$

$$\text{Wavy line} = \text{Wavy line} + \text{Wavy line} \text{---} \text{Loop} \text{---} \text{Wavy line}$$

Vertex resummation

$$\text{Crossed lines} = \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10}$$

bare vertex

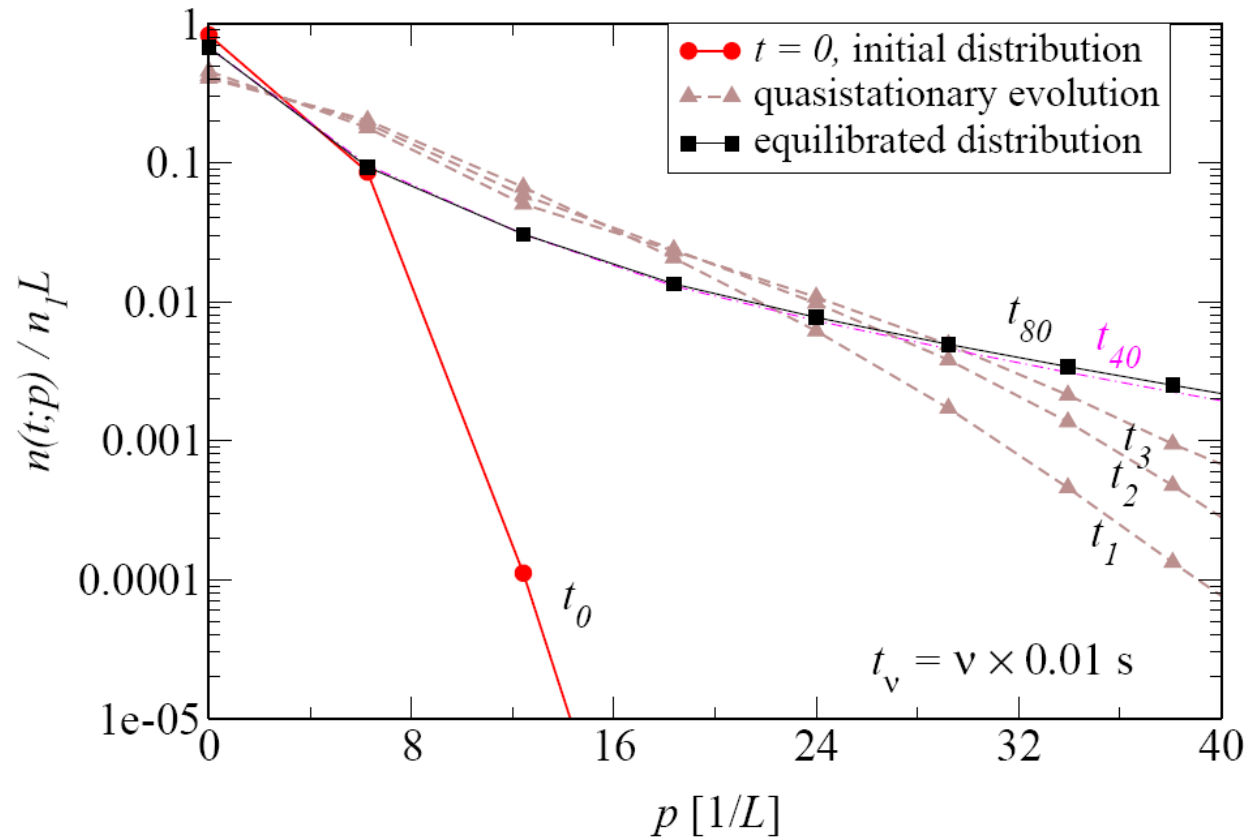
[Also: Mihaila, Dawson & Cooper (01); Cooper, Dawson & Mihaila (03); Berges & Serreau (03); Berges, Borsanyi & Serreau (03); Berges, Borsanyi & Wetterich (04); Alford, Berges & Cheyne (04); Arrizabalaga, Smit & Tranberg (04); ...; Rey, Hu, Calzetta (04); Baier & Stockamp (04); TG, Berges, Schmidt & Seco (05); ...]



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Momentum distribution for different times:



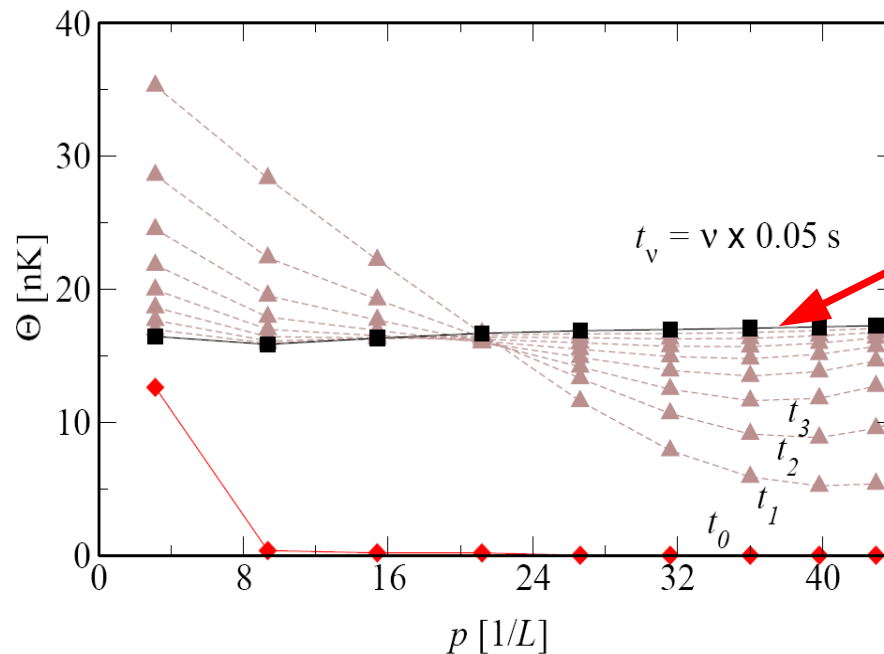
[TG, J. Berges, M. Seco & M.G.Schmidt, PRA 72 (05); J. Berges & TG, PRA 76 (07)]



Temperature appears

[also: Berges & Cox (01), Berges (01),
Berges, Borsanyi, & Serreau (03),
Berges, Borsanyi, & Wetterich (04)]

'Temperature' parameter $\Theta(p)$ at t_n



flat = temperature

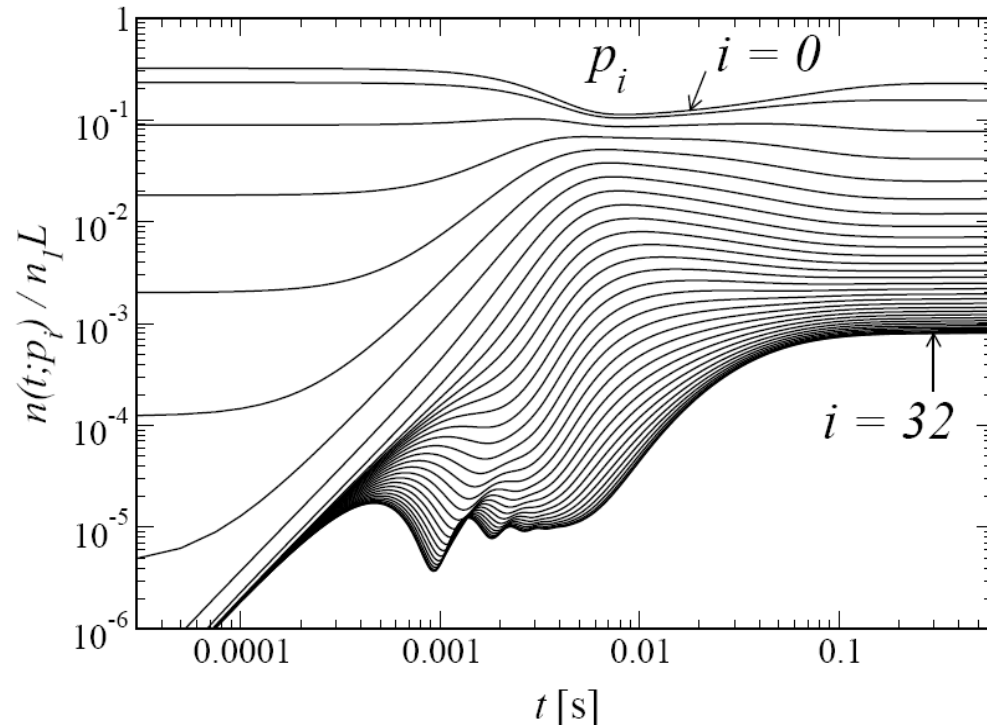
[J. Berges & TG, PRA 76 (07)]

$$n(t; p) = \frac{1}{e^{\frac{1}{k_B \Theta(p)} \left(\frac{p^2}{2m} - \mu \right)} - 1}$$



Far-from-equilibrium evolution

Time evolution of mode occupation n_i :



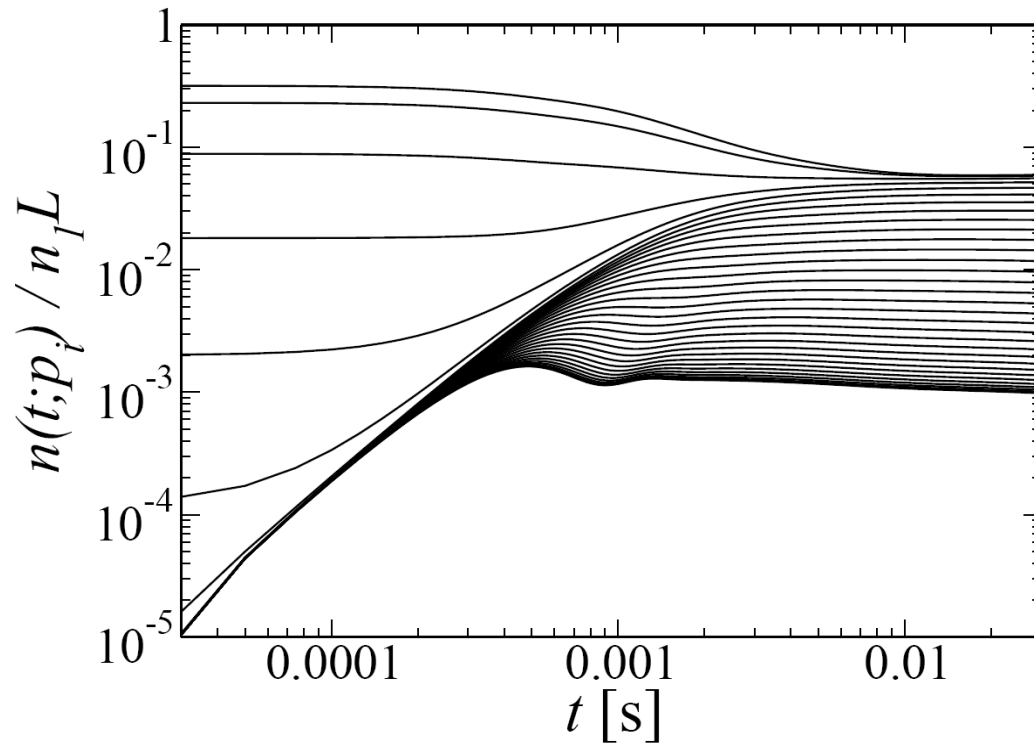
- initial state:
- ^{23}Na atoms in 1D, $n_1 = 10^7 \text{ m}^{-1}$
 - interaction parameter $\gamma = \lambda m / (\hbar^2 n_1) = 7.5 \cdot 10^{-4}$
 - Gaussian momentum distribution

[J. Berges & TG, PRA 76 (07)]



Far-from-equilibrium evolution

Time evolution of mode occupation n_0^s :



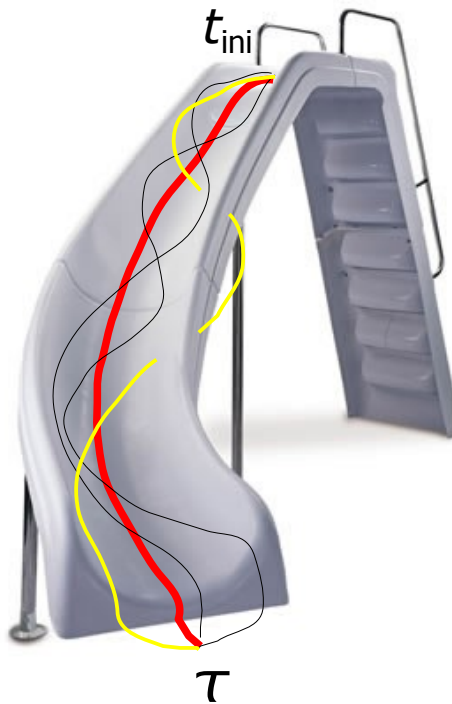
- initial state:
- ^{23}Na atoms in 1D, $n_1 = 10^5 \text{ m}^{-1}$
 - interaction parameter $\gamma = \lambda m / (\hbar^2 n_1) = 15$
 - Gaussian momentum distribution



Functional RG approach

RG approach to far-from-equilibrium dynamics

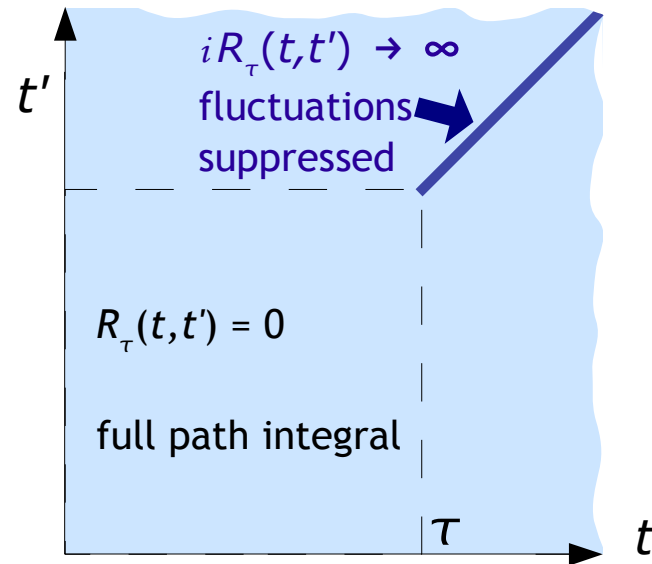
[TG & J.M. Pawłowski, cond-mat/0710.4627]



shift max. time τ
and watch Z_τ and Γ_τ evolve

$$Z[\mathbf{J}; \rho_0] = \int \mathcal{D}\varphi \rho_0 \exp \left\{ iS[\varphi] + i \int_{x, \mathbf{c}} \mathbf{J}_a \varphi_a \right\}$$

$$Z_\tau = \exp \left\{ i \int_{x, y, \mathbf{c}} \frac{\delta}{\delta \mathbf{J}_a(x)} R_{\tau, ab}(x, y) \frac{\delta}{\delta \mathbf{J}_b(y)} \right\} Z$$



Functional RG equation [C. Wetterich (93)]:

$$\partial_\tau \Gamma_\tau = \frac{i}{2} \int_{\mathbf{c}} \left[\frac{1}{\Gamma_\tau^{(2)} + R_\tau} \right]_{ab} \partial_\tau R_{\tau, ab}$$



RG approach to far-from-equilibrium dynamics

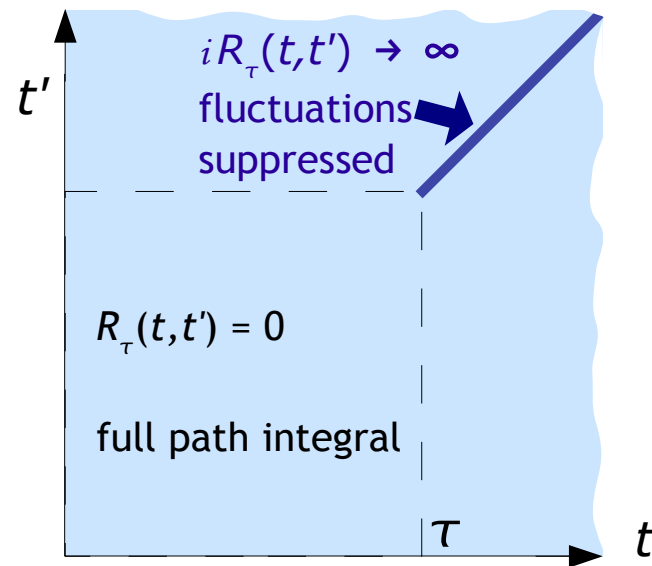
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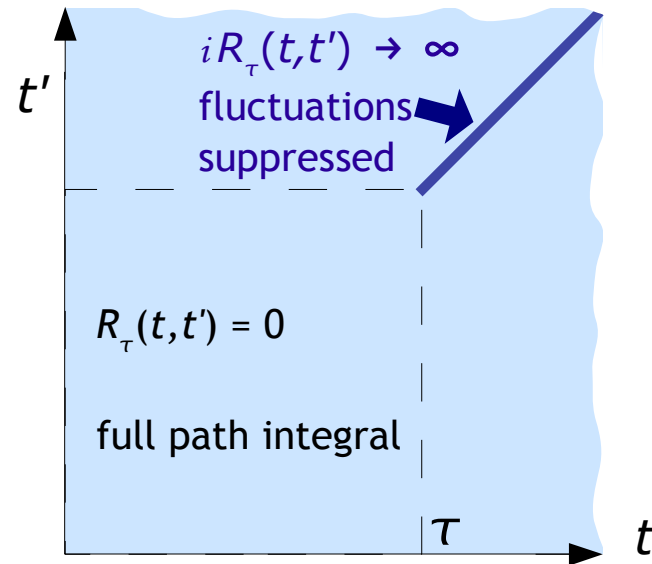
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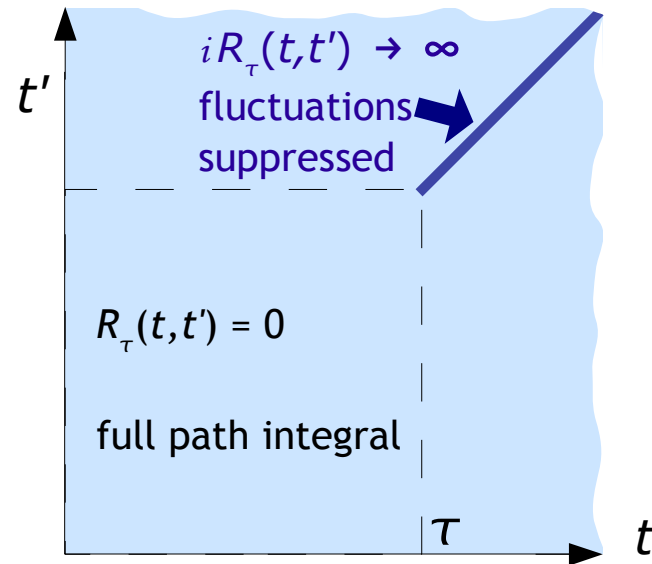
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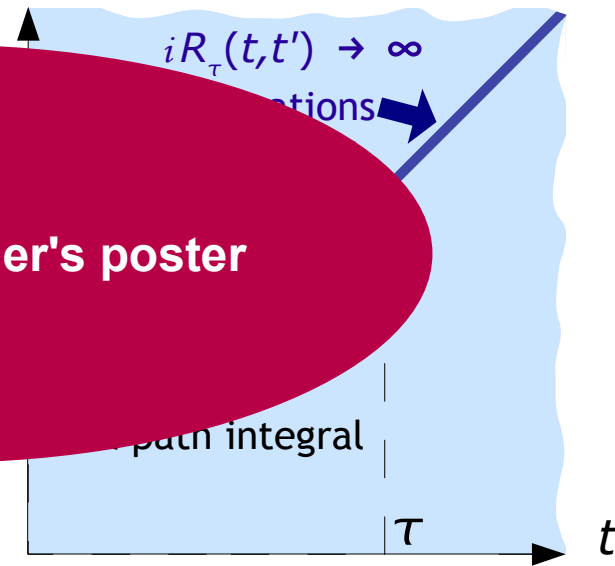
[TG & J.M. Pawłowski, cond-mat/0710.4627]



See Stefan Kessler's poster

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2PI $1/\mathcal{N}$ Expansion

[Berges, NPA **699** (02) 847; Aarts, Ahrensmaier, Baier, Berges, & Serreau, PRD **66** (02) 45008]

up
to
NLO

$$\Gamma_2^{\text{LO}}[\phi, G] = \text{Diagram 1} ,$$

Diagram 1: Two blue circles connected by a wavy line.

$$\Gamma_2^{\text{NLO}}[\phi, G] = \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \dots$$

Diagram 2: A blue circle with a wavy line entering from the left and exiting from the right.

Diagram 3: Two blue circles connected by two wavy lines (one top, one bottom).

Diagram 4: Two blue circles connected by two wavy lines, with each circle having a self-energy loop (a smaller blue circle) on its top and bottom edges.

Diagram 5: Two blue circles connected by two wavy lines, with each circle having a self-energy loop on its left and right edges.

Diagram 6: Two blue circles connected by two wavy lines, with each circle having a self-energy loop on its top and bottom edges, and a vertical wavy line segment on the right side of each circle.

$$\text{Wavy line} = \text{Wavy line} + \text{Wavy line} \text{ --- } \text{Blue circle} \text{ --- } \text{Wavy line}$$

Vertex resummation

$$\text{Crossed lines} = \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9}$$

Diagram 7: Two lines crossing at a point, with a wavy line segment connecting the two intersection points.

Diagram 8: Two lines crossing at a point, with a wavy line segment connecting the two intersection points, and a small circle at the intersection point.

Diagram 9: Two lines crossing at a point, with a wavy line segment connecting the two intersection points, and a small circle at the intersection point, and a small circle at the intersection point.

bare vertex

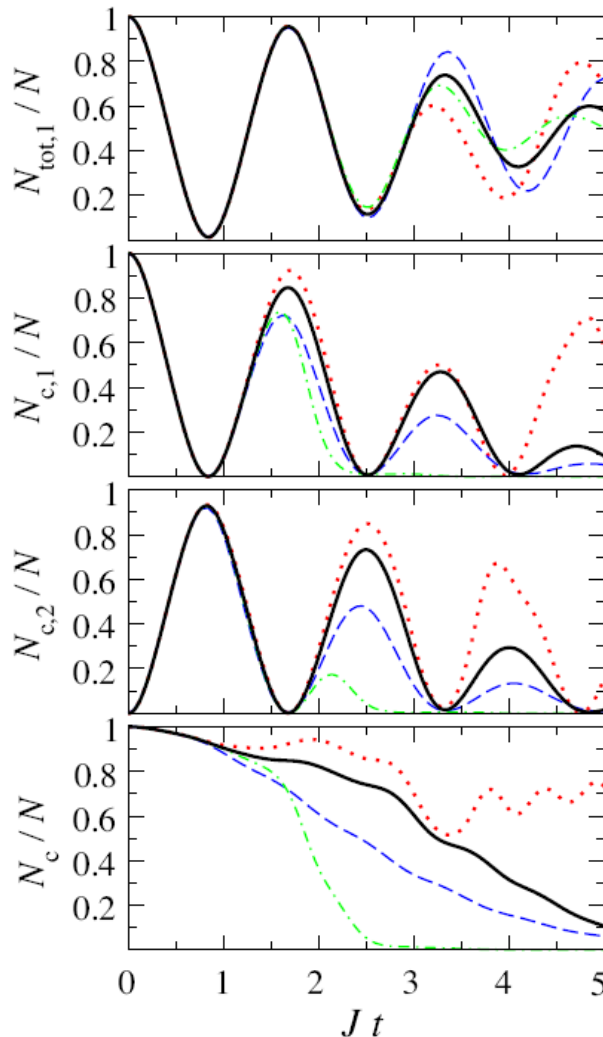
[Also: Mihaila, Dawson & Cooper (01); Cooper, Dawson & Mihaila (03); Berges & Serreau (03); Berges, Borsanyi & Serreau (03); Berges, Borsanyi & Wetterich (04); Alford, Berges & Cheyne (04); Arrizabalaga, Smit & Tranberg (04); ...;
Non-relativistic: Rey, Hu, Calzetta (04); Baier & Stockamp (04); TG, Berges, Schmidt & Seco (05); ...]



Benchmarks & applications

Dynamics in lattice potentials

Benchmark tests



total atom # in well 1

condensate well 1

condensate well 2

total condensate frac.



— $2\text{PI } 1/\mathcal{N}$

-- exact

... HFB

.-.- 2nd O. $1/\mathcal{N}$

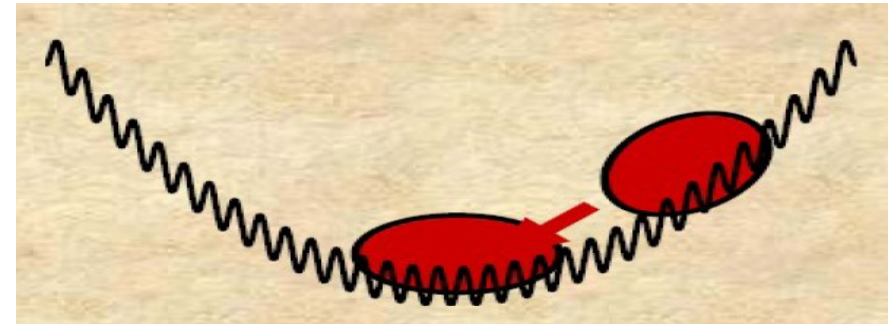
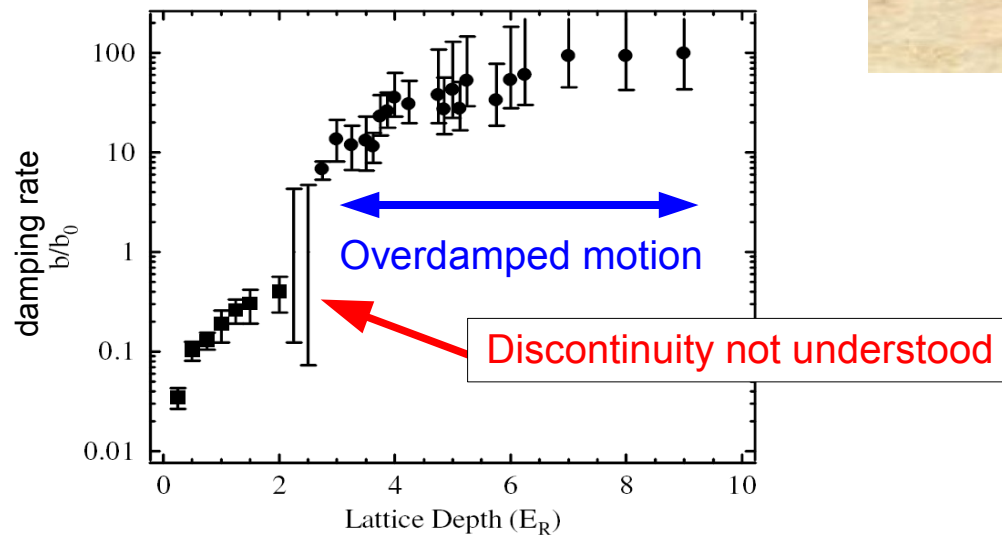
[Temme, TG (06)]



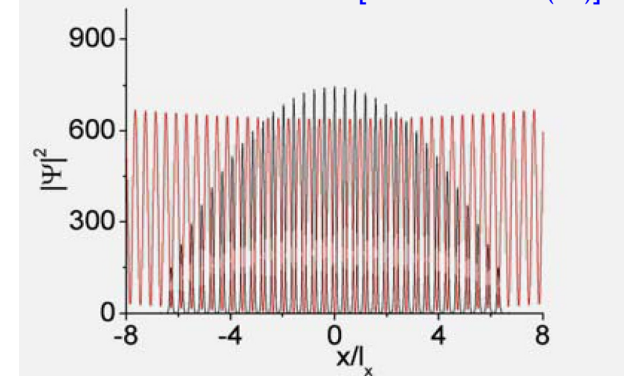
Nonequilibrium dynamics in lattices

Dipole oscillations in lattices:
Damping rates?

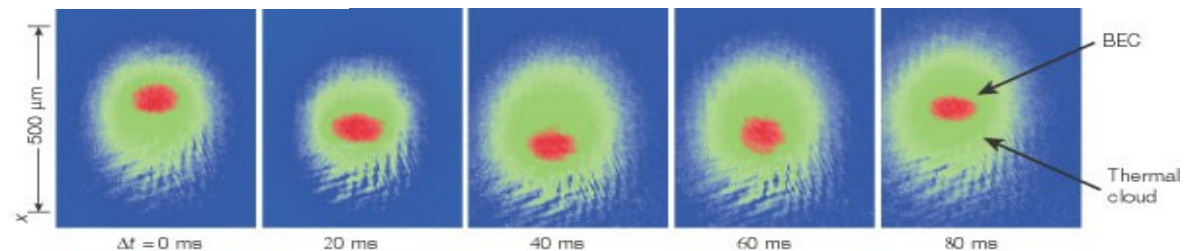
[Expt. @ NIST: Fertig et al. PRL94 (05)]



[J. Ruostekoski (07)]



[Anglin & Ketterle (02)]



Fermions

- 2PI dynamics: See [Matthias Kronenwett's](#) poster

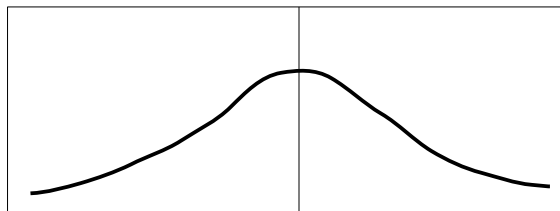


Exactly solvable model

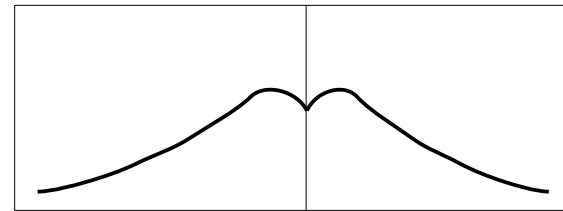
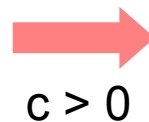
Lieb-Liniger gas in 1D: [PR 130, 1605 & 1616 (63)]

$$i \frac{\partial \psi_B}{\partial t} = - \sum_{i=1}^N \frac{\partial^2 \psi_B}{\partial x_i^2} + \sum_{1 \leq i < j \leq N} 2c \delta(x_i - x_j) \psi_B$$

Solution via Bethe-Ansatz with cusp conditions @ particle contact hyperplanes



$$x_i = x_j$$

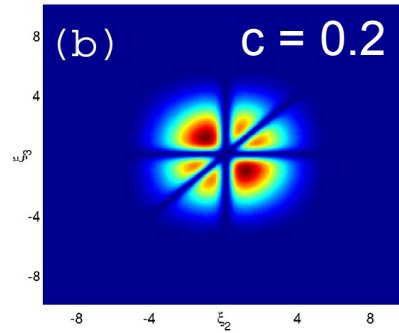
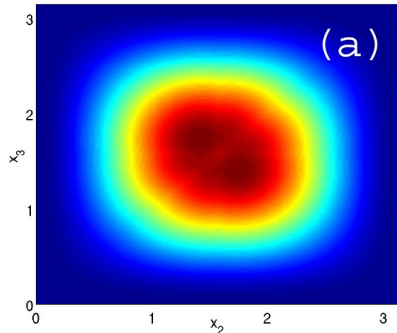


Dirichlet-von Neumann

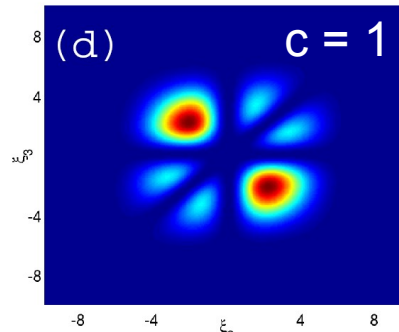
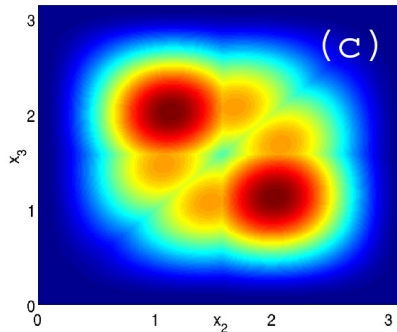
Dynamics: H. Buljan, R. Pezer, & TG, PRL 100 (2008)



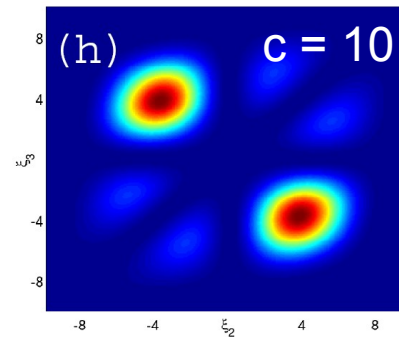
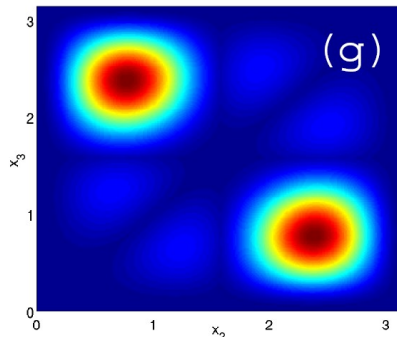
Free 1D expansion: Exact results



$$i\frac{\partial\psi_B}{\partial t} = -\sum_{i=1}^N \frac{\partial^2\psi_B}{\partial x_i^2} + \sum_{1\leq i<j\leq N} 2c\delta(x_i - x_j)\psi_B$$



$$|\psi_B(0, x_2, x_3, t)|^2$$



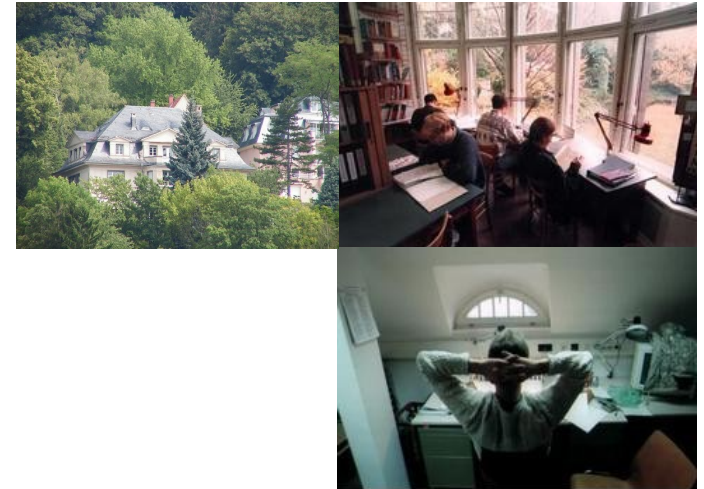
[H. Buljan, R. Pezer, & TG, PRL **100** (2008)
D. Jukic, H. Buljan, R. Pezer, & TG, cond-mat/0804.2580]



Thanks & credits to...

...my work group in Heidelberg:

Cédric Bodet
Alexander Branschädel (→ Karlsruhe)
Stefan Keßler
Matthias Kronenwett
Christian Scheppach
Philipp Struck (→ Konstanz)
Kristan Temme (→ Vienna)
Martin Trappe



...my collaborators

Jürgen Berges • Darmstadt
Hrvoje Buljan • Zagreb
Jan M. Pawłowski • Heidelberg
Robert Pezer • Sisak
Michael G. Schmidt • Heidelberg
Marcos Seco • Santiago de Compostela

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Keith Burnett • Oxford/Sheffield • David Hutchinson • Otago • Thorsten Köhler • UCL • Paul S. Julienne • NIST Gaithersburg • David Roberts • ENS Paris • Janne Ruostekoski • Southampton

