Simple Neutron Star Model

To give an estimate of the density inside a neutron star we want to use first a simple model.

We assume that

- It consists of neutrons only.
- The temperature is zero.
- The neutrons behave like an ideal Fermi gas.
- The star is spherically symmetric.

Force balance inside the star: Fermi pressure $\longleftrightarrow$ Gravity.

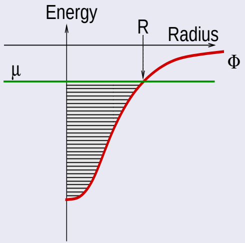
Force Balance Inside the Neutron Star

We derive the force balance by considering chemical and thermal equilibrium.

Chemical Potential

$$\mu = \epsilon_{\rm F}(r) + \Phi_{\rm gr}(r) = constant.$$

Here $\epsilon_{\rm F}(r)$ is the Fermi energy and $\Phi_{\rm gr}(r)$ the gravitational potential. We use relativistic dispersion for the neutrons with $k_{\rm F}(r)$ the local Fermi momentum



$$\epsilon_{\rm F}(r) = \sqrt{c^2\hbar^2 k_{\rm F}^2(r) + m^2c^4} \ ,$$

$$n(r) = 2\int \frac{d^3k}{(2\pi)^3}\theta(k_{\rm F}(r) - k) = \frac{k_{\rm F}^3(r)}{3\pi^2} \ .$$

Differential Equation for the Density

Newtonian Gravitational Potential

Since the neutron star is not homogeneous in the radial direction, the gravitational potential complicated, but from Gauss's law we know the force

$$-\Phi'_{\mathrm{gr}}(r) = -\frac{Gm^2}{r^2}\int_0^r \mathrm{d}r'\, 4\pi(r')^2\, n(r')\ . \tag{1}$$

Differential Equation

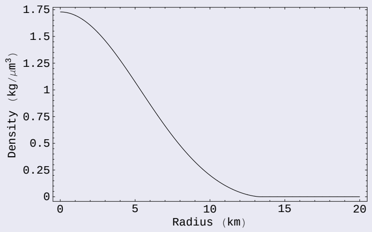
Taking the derivative of the chemical potential $\epsilon_{\mathrm{F}}(r) + \Phi_{\mathrm{gr}}(r) = \mu$ with respect to the radius gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\epsilon_{\mathrm{F}}(r) + \Phi'_{\mathrm{gr}}(r) = 0\ . \tag{2}$$

Notice that $-d\epsilon_{\mathrm{F}}(r)/dr$ is thus the force due to the Fermi pressure.

Numerical Solution

Density Profile

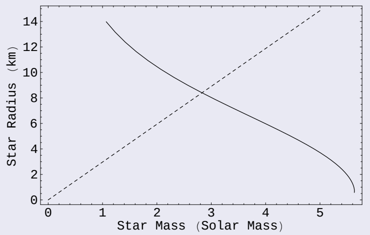


We solved the equation with the boundary conditions

$$n(0) \equiv n_c \quad \text{and} \quad n'(0) = 0 \ .$$

The total star mass 1.44 $M_\odot$

Maximum Mass



With different n_c we notice that

- The maximum mass is 5.7$M_\odot$.
- Above 2.8 $M_\odot$ the radius is smaller than the Schwarzschild radius.

General Relativity

This shows that general relativity is required.

General Spherical Symmetric Line Element

$$\mathrm{d}s^2 = -e^{2\phi(r)}(c\mathrm{d}t)^2 + e^{2\lambda(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \ . \tag{3}$$

Outside the Neutron Star

The stress-energy tensor is zero, the solution to the Einstein equations is the **Schwarzschild solution**.

Inside the Neutron Star

For the stress-energy tensor we take a **perfect fluid**, this is a general fluid without viscosity or heat conduction. With pressure P and energy density ρ

$$T_{\mu\nu} = \rho\frac{u_\mu u_\nu}{c^2} + P\left(g_{\mu\nu} + \frac{u_\mu u_\nu}{c^2}\right) \ . \tag{4}$$

Interior Solution: Tolman-Oppenheimer-Volkoff Equation

- In 1939 Tolman, Oppenheimer and Volkoff solved the Einstein equation for a perfect fluid.
- The solution gives a differential equation for the pressure and energy density as a function of radius.

Tolman-Oppenheimer-Volkoff (TOV) Equation

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -(P(r) + \rho(r)) \left(\frac{Gm(r) + 4\pi G r^3 P(r)/c^2}{r(c^2 r - 2Gm(r))} \right) , \tag{5}$$

with

$$m(r) = \frac{1}{c^2} \int_0^r \rho(r')\, 4\pi (r')^2 \,\mathrm{d}r' . \tag{6}$$

- To solve this equation we need the equation of state (EOS) $P(\rho)$.

Interior Solution

The EOS for relativistic neutrons can be derived from homogeneous matter. We thus have

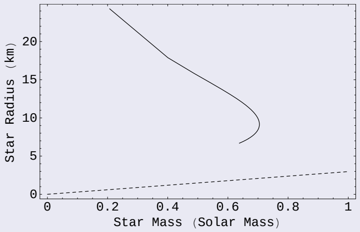
$$\rho = 2\int \frac{d^3k}{(2\pi)^3}\theta(k_F - k)\epsilon(k)$$

and the thermal relation

$$P = -\frac{\partial E}{\partial V} = n\,\mu - \rho\ .$$

We can express $P(r)$ and $\rho(r)$ in terms of the particle density $n(r)$.

Maximum Mass with TOV



- The maximum mass is $0.7\ M_\odot$.
- The central density in this star is about $1.5\ \text{fm}^{-3}$.

Baryonic Matter with Interactions through Mesons

The density is high enough to produce more massive baryons than neutrons.

The interaction between the baryons is relevant for the phase transition.

Effective Interaction Model for Baryons

- Model the interaction through meson exchange and use a mean-field approximation.
- Use three types of mesons (scalar, vector and isovector).
- Determine the couplings by comparison with data and extrapolation.
- Use 13 baryon types (n^0, p^+, Λ, $\Sigma^{+,0,-}$, $\Delta^{++,+,0,-}$, $\Xi^{0,-}$, Ω^-).
- Use two types of leptons, the electron and muon. Tauon is too heavy.

Deconfined Quarks in Neutron Star Core

The density in a neutron star can be much greater than the nuclear density and, therefore, there can possible be deconfined quarks inside.

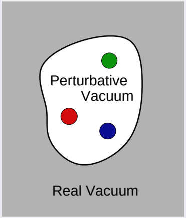
The MIT Bag Model

Description of hadrons in QCD is very complex. Can be modeled by giving the quark vacuum extra energy BV, with B the Bag constant. The energy density

$$\rho = \rho_f + B \ ,$$

with ρ_f the energy density of free quarks, and the pressure

$$P = -\frac{\partial E}{\partial V} = P_f - B \ .$$



The value of B is not known exactly, but about 200 MeV/fm^3.

Phase Transition with Two Conserved Quantities

Consider two phases, the **nuclear phase** and the **quark phase**.

Nuclear phase: many particles like neutron, proton, electron, ...
Quark phase: three relevant flavors: up, down and strange

Conservation of Electric Charge and Baryon Number

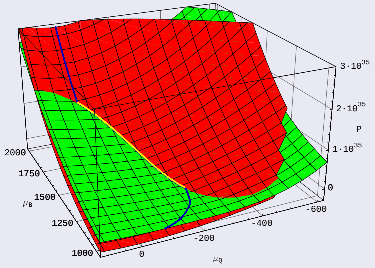
Two conserved charges: Electric charge and Baryon number $\longrightarrow$ two chemical potentials μ_{Q} and μ_{B}

- The particle chemical potentials $(\mu_{\mathrm{n}}, \mu_{\mathrm{p}}, \mu_{\mathrm{e}}, \mu_{\mathrm{up}}, \dots)$ are determined from μ_{Q} and μ_{B}.
- The EOS can be computed for both phases individually.
- The Coulomb force can be included by demanding the matter to be locally neutral.

The Mixed Phase

- The phase with lowest grand potential Ω is the one the system is in.
- For the grand potential we have: $\Omega = -PV$
- There does not always exist a single phase with neutral configuration and lowest Ω or highest P.
- This leads to a neutral mixed phase.

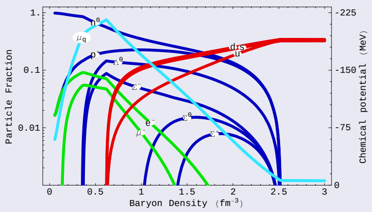
Pressure of the Mixed Phase



The blue lines are the charge neutral phases. The yellow line is the neutral mixed phase.

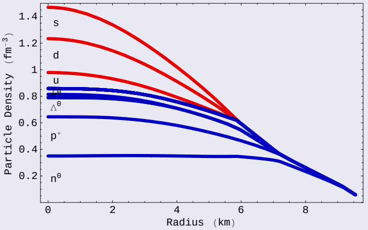
Solution of Mixed Baryon Quark Matter

Mixed Phase: Particle Fractions



- Quark bag constant B is 230 MeV/fm^3.
- The maximum star mass is 1.5 $M_\odot$ in this model.

Maximum Mass Star



- No pure quark phase in a neutron star.
- Possibility with correct geometrical structure.
- In center $\mu_{\mathrm{B}} = 1500$ MeV.

The Strong Interaction

Quarks interact with the strong interaction described by QCD.

The quarks can have three colors (red, green, blue).

QCD based on nonabelian SU(3) gauge transformation, with gauge bosons called gluons (A_μ^a).

The QCD Lagrangian

$$\mathscr{L}_{\mathrm{QCD}} = \bar{\psi}(\partial\!\!\!/ + m)\psi + \frac{1}{2}A_\mu^a \left(\partial_\mu\partial_\nu - \partial^2\delta_{\mu\nu}\right) A_\nu^a - ig\bar{\psi}\gamma_\mu t^a \psi A_\mu^a$$
$$+ \frac{1}{2}g \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) f^{abc} A_\mu^b A_\nu^c$$
$$+ \frac{1}{4}g^2 f^{abc} f^{ade} A_\mu^b A_\mu^c A_\nu^d A_\nu^e$$

Effective Quark-Quark Interaction

We can integrate out the gluon fields to get an effective interaction.

Effective Action

$$S_{\text{eff}} = \frac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y\ \bar{\psi}^{\alpha i}_f(x)\psi^{\beta j}_f(x)V^{\alpha\beta\gamma\delta}_{ijkl}(x-y)\bar{\psi}^{\gamma k}_g(y)\psi^{\delta l}_g(y) \tag{7}$$

with the potential

$$V^{\alpha\beta\gamma\delta}_{ijkl}(x-y) = g^2\gamma^{\alpha\beta}_\mu\gamma^{\gamma\delta}_\nu t^a_{ij}t^a_{kl}D_{\mu\nu}(x-y)$$

The effective interaction contains the $SU(3)$ group generators t^a_{ij}, the gamma matrices for the spinor structure, and the gluon propagator.

Attractive Quark-Quark Interaction

Effective Interaction

$$V_{ijkl}^{\alpha\beta\gamma\delta}(x-y) = g^2 \gamma_\mu^{\alpha\beta} \gamma_\nu^{\gamma\delta} t_{ij}^a t_{kl}^a D_{\mu\nu}(x-y) \tag{8}$$

- The interaction is similar to the effective electron-electron interaction in QED.
- The difference are the group generators $t_{ij}^a t_{kl}^a$.
- The sign of this part determines whether the interaction is attractive or repulsive.

Group structure

$$t_{ij}^a t_{kl}^a = -\frac{1}{3}\left(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj}\right) + \frac{1}{6}\left(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}\right)$$

The red part is the antisymmetric part in j and l and is **attractive**!

Conclusions

Neutron Star

- A neutron star is dense enough to have a deconfined quark phase.
- This deconfined phase is most likely not pure, but is mixed with the baryonic phase.
- The baryon chemical potential is above 1500 MeV in the center of the neutron star.

Superconductivity

- Quarks have a fundamental attractive interaction and because a neutron star is relatively cold, there can exist a superconducting phase. This is presumably a 2SC phase as in ultracold atomic gases.