BCS–BEC crossover in the $1/N$ expansion

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Outline

Introduction

General framework

- BCS–BEC crossover: Discussed in detail in other talks at this meeting.
- Fluctuation effects: Bosonize the theory and include one-loop corrections (NLO) to the mean-field approximation (LO). (Almost) the same machinery as in NSR theory—see talk by H. Abuki.

$1/N$ expansion

- Widely used in high-energy as well as condensed-matter physics.
- In the context of cold attractive Fermi gases introduced in:

  P. Nikolić and S. Sachdev, PRA 75 (2007) 033608 (NS)

  M. Y. Veillette, D. E. Sheehy, and L. Radzihovsky, PRA 75 (2007) 043614 (VSR)

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Scope of the work

Previous results

- NS: Fluctuation correction to T_c at unitarity.
- VSR: Formalism for the superfluid phase. Correction to T_c at unitarity; whole BCS–BEC range at $T = 0$.

This work

- Correction to T_c off unitarity, asymptotic behavior in the BCS limit.
- Fluctuation effects in relativistic superconductors (esp. CSC).

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Outline

Some standard formulas

- Euclidean Lagrangian for an attractive, balanced, two-component Fermi gas with large scattering length:

$$\mathcal{L} = \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi_\sigma - G \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow$$

- Decouple the interaction by introducing a pairing field, $\phi \sim G\psi_\downarrow\psi_\uparrow$, and the Nambu spinor $\Psi = (\psi_\uparrow, \psi_\downarrow^\dagger)^T$:

$$\mathcal{L} = \frac{|\phi|^2}{G} - \Psi^\dagger \mathcal{D}^{-1} \Psi, \quad \mathcal{D}^{-1} = \begin{pmatrix} -\partial_\tau + \frac{\nabla^2}{2m} + \mu & \phi \\ \phi^* & -\partial_\tau - \frac{\nabla^2}{2m} - \mu \end{pmatrix}$$

What is N?

- Make N copies of the spin-$\frac{1}{2}$ fermion.

$$(\psi_\uparrow, \psi_\downarrow) \to (\psi_{1\uparrow}, \psi_{1\downarrow}, \ldots, \psi_{N\uparrow}, \psi_{N\downarrow})$$

- The Euclidean Lagrangian generalizes to:

$$\mathcal{L} = \psi^\dagger_{i\sigma}\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\psi_{i\sigma} - \frac{G}{N}\psi^\dagger_{i\uparrow}\psi^\dagger_{i\downarrow}\psi_{j\downarrow}\psi_{j\uparrow}$$

- The spin $SU(2)$ symmetry extends to $Sp(2N)$. However, this remains unbroken by the condensate of the (generalized) pairing field.

$$\phi(x) \sim G\sum_{i=1}^{N}\psi_{i\downarrow}(x)\psi_{i\uparrow}(x)$$

- No unwanted NG bosons.

Counting factors of $1/N$

- Bosonized action:

$$\mathcal{S} = N \int_0^\beta d\tau \int d^3\mathbf{x} \frac{|\phi(\mathbf{x},\tau)|^2}{G} - N \operatorname{Tr} \log \mathcal{D}^{-1}[\phi(\mathbf{x},\tau)]$$

- Make formal expansion in $1/N$ and at the end set $N = 1$.
- Each boson propagator contributes $1/N$ and each fermion loop in the effective boson self-interaction gives $N \Rightarrow 1/N$ expansion equivalent to expansion in bosonic loops.

- LO in $1/N \Leftrightarrow$ MFA. NLO in $1/N \Leftrightarrow$ one-boson loop corrections.
- $1/1$ is not really a small expansion parameter, but at least gives a systematic ordering to corrections beyond MFA.

Outline

Solution of gap and number equations

- Direct self-consistent solution of NLO gap equation not possible.
- Avoid this by a systematic expansion of both gap and number equations in $1/N$.
- With $\Omega = \Omega^{(0)} + \frac{1}{N}\Omega^{(1)} + \cdots$, one finds explicit expressions for gap and chemical potential shifts.

$$\begin{pmatrix} \partial_{\mu\mu}\Omega^{(0)} & \partial_{\mu\Delta}\Omega^{(0)} \\ \partial_{\Delta\mu}\Omega^{(0)} & \partial_{\Delta\Delta}\Omega^{(0)} \end{pmatrix} \begin{pmatrix} \delta\mu \\ \delta\Delta \end{pmatrix} = - \begin{pmatrix} \partial_{\mu}\Omega^{(1)} \\ \partial_{\Delta}\Omega^{(1)} \end{pmatrix}$$

- Likewise, fixing $\Delta = 0$ yields the shifts of critical temperature and chemical potential.

$$\begin{pmatrix} \partial_{\mu\mu}\Omega^{(0)} & \partial_{\mu\beta}\Omega^{(0)} \\ \partial_{\Delta\mu}\Omega^{(0)} & \partial_{\Delta\beta}\Omega^{(0)} \end{pmatrix} \begin{pmatrix} \delta\mu \\ \delta\beta_c \end{pmatrix} = - \begin{pmatrix} \partial_{\mu}\Omega^{(1)} \\ \partial_{\Delta}\Omega^{(1)} \end{pmatrix}$$

- All expressions are evaluated at the MF solution!

Unitarity

$$\textcolor{red}{NS}: \frac{\epsilon_F}{T_c} = 2.014 + \frac{5.317}{N}$$
$$\frac{\mu_c}{T_c} = 1.504 + \frac{2.785}{N}$$

$$\textcolor{red}{VSR}: \frac{T_c}{\epsilon_F} = 0.496 - \frac{1.310}{N}$$
$$\frac{\mu_c}{\epsilon_F} = 0.747 - \frac{0.580}{N}$$
$$\frac{\Delta_0}{\epsilon_F} = 0.686 - \frac{0.163}{N}$$
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- Results for T_c, μ_c formally equivalent, but $1/N$ corrections are large $\Rightarrow$ ambiguities!
- $1/N$ corrections smaller at $T = 0$, but $\xi \equiv \frac{\mu_0}{\epsilon_F} = 0.28$ far from the expected value $\xi \approx 0.4$.

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Mid-summary

- The results are exact in the $N \to \infty$ limit, but extrapolation to $N = 1$ troublesome.
- Final predictions depend on which observable is chosen to perform the extrapolation.
- T_c useless at unitarity—negative value!
- $1/T_c$-based extrapolation yields $\frac{T_c}{\epsilon_F} = 0.14$, reasonably close to Monte Carlo simulations $(0.152(7)$ E. Burovski et al., PRL 96 (2006) 160402$)$.
- $1/T$ is the natural variable of Ω. However, more convincing justification would be welcome.
- No selfconsistency, extrapolation from mean-field solution $\Rightarrow$ $1/N$ will fail when molecular states dominate thermodynamics.
- For BEC, we expect the accuracy to become even worse. Still, $1/N$ expansion might be reasonable for BCS.

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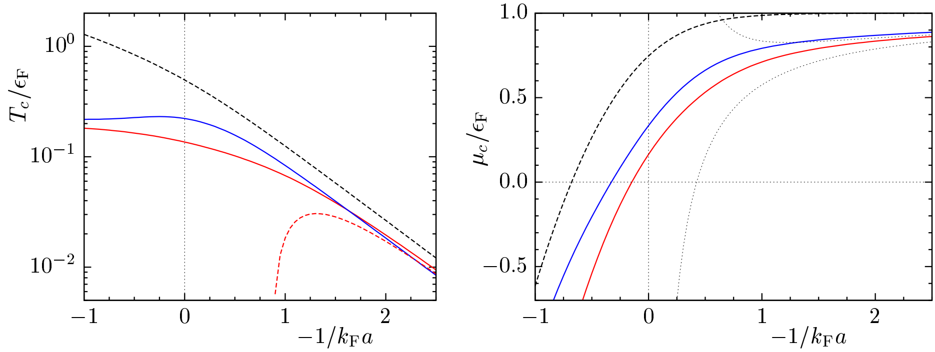
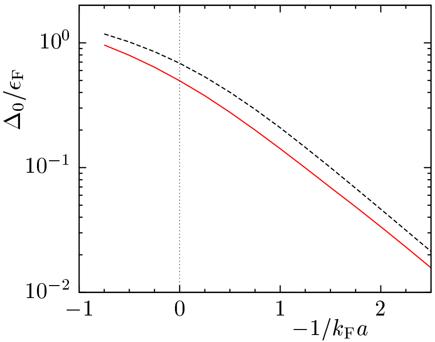
Off the unitarity: Critical temperature

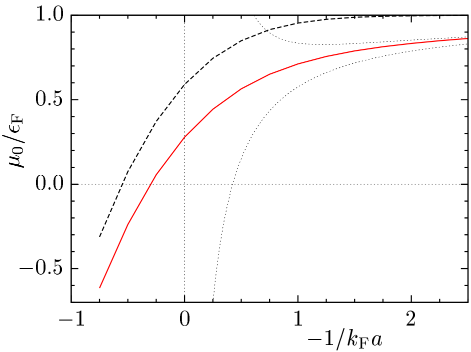


- T_c reduced by a constant factor in the BCS limit!
- Chemical potential in the BCS limit governed by perturbative corrections. Reproduces second-order analytic formula:

$$\frac{\mu}{\epsilon_F} = 1 + \frac{4}{3\pi} k_F a + \frac{4(11 - 2\log 2)}{15\pi^2} (k_F a)^2$$

Off the unitarity: $T = 0$





- $1/N$ corrections moderate even around unitarity.
- Gap reduced by a constant factor in the BCS limit. A different factor than for $T_c \Rightarrow$ departure from the BCS ratio $\frac{\pi}{e^{\gamma}}$!

Outline

Model definition

- Several relativistic fermion (quark) species with equal chemical potentials and masses.
- For simplicity, no quark–antiquark condensate, just the pairing channel, allowing for arbitrary flavor structure with total spin zero.

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ + \mu\gamma_0 - m)\psi + \frac{G}{4}\sum_a (\overline{\psi^{\mathcal{C}}}\gamma_5 Q_a \psi)(\overline{\psi}\gamma_5 Q_a^\dagger \psi^{\mathcal{C}})$$

- Identical formalism as for nonrelativistic Fermi gas, with the necessary generalization to include antiparticles, and modification of the fermion dispersion relation.
- At this stage, determine $1/N$ correction to T_c $\Rightarrow$ universal result up to a simple algebraic factor determined by the pairing pattern.

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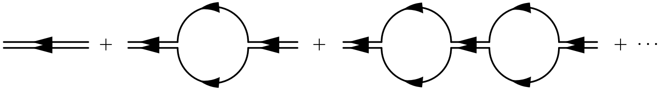
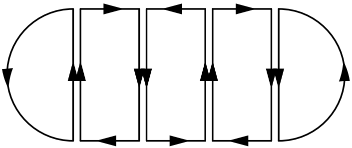
What is N?

- We already have three colors, how about extending to $SU(N)_c$? $1/3$ might be a reasonable expansion parameter.
- However:



- No trace over $SU(N)$ indices! Full RPA series not resummed at any finite order in $1/N$, unless coupling $\sim \mathcal{O}(1)$.
- In such a case, ladder contributions to Ω have increasing power of N with # of loops $\Rightarrow$ $1/N$ expansion even impossible.



N is not color!

- $1/N$ expansion based on extension of color $SU(3)$ will not lead to Cooper pairing: Resums different class of diagrams than needed.
- Solution: Introduce a new quantum number.

$$\phi_a \sim G \sum_{i=1}^{N} \psi_i C \gamma_5 Q_a \psi_i$$

- Global symmetry now $SU(3)_c \times SO(N) \times$ flavor group.
- $SO(N)$ again unbroken by Cooper pairs. Perform $1/N$ expansion $\leftrightarrow$ expansion in bosonic loops. Set $N = 1$ at the end.
- We thus lose the color expansion parameter $1/3$. On the other hand, this construction can be applied to any pattern of relativistic fermion pairing.

$1/N$ expansion

Inverse boson propagator at zero momentum (Thouless criterion): Universal nonrelativistic expression with necessary relativistic modifications and a pairing-dependent algebraic prefactor.

$$G^{-1}(0) = G_{(0)}^{-1}(0) + \frac{N_B}{N_F} \not{\int} dQ\, G_{(0)}(Q) \sum_{e,f=\pm} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[1 + ef \frac{m^2 + \mathbf{k}\cdot(\mathbf{k}+\mathbf{q})}{\epsilon_{\mathbf{k}}\epsilon_{\mathbf{k}+\mathbf{q}}}\right] I(e\xi^e_{\mathbf{k}}, f\xi^f_{\mathbf{k}+\mathbf{q}}; i\Omega_N)$$

$$I(a,b;i\Omega_N) = \frac{1}{8a^2}\frac{\tanh\frac{\beta a}{2} + \tanh\frac{\beta b}{2} - \beta a \cosh^{-2}\frac{\beta a}{2}}{i\Omega_N + b + a} + \frac{1}{8a^2}\frac{\tanh\frac{\beta a}{2} - \tanh\frac{\beta b}{2}}{i\Omega_N + b - a} + \frac{1}{4a}\frac{\tanh\frac{\beta a}{2} + \tanh\frac{\beta b}{2}}{(i\Omega_N + b + a)^2}$$

Fluctuations distinguish otherwise MF-identical patterns:

pairing	N_B	N_F	N_B/N_F
“BCS”	1	N	$1/N$
2SC	3	$6N$	$1/2N$
CFL	9	$9N$	$1/N$

Outline

High-density approximation

Physical idea

- In the far BCS region, the pairing and Fermi energy scales are well separated.
- Only the degrees of freedom close to Fermi surface are relevant for pairing physics.
- We want to avoid interference with irrelevant scales, in particular all vacuum UV divergences.

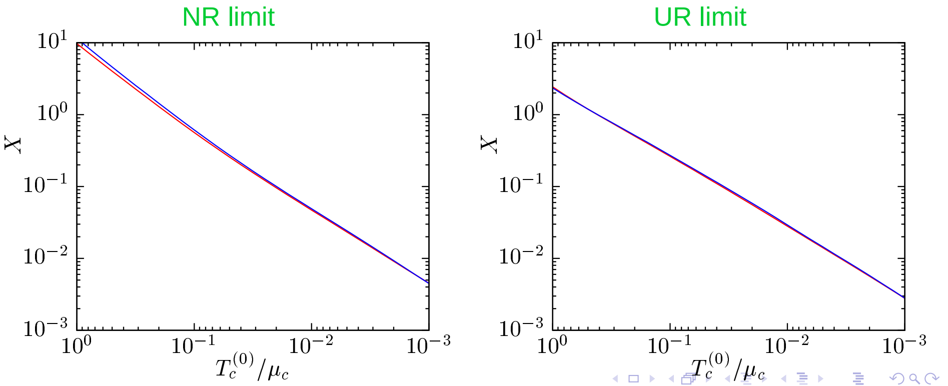
Technical realization

- Neglect antiparticle contributions.
- “Flatten” the Fermi surface—take constant density of states.
- Cut off the fermion momentum integration at the pairing scale.
- Expand fermion dispersion relations about the Fermi surface.

Numerical results

Correction to critical temperature given by a universal function with a model-dependent algebraic prefactor.

$$\log \frac{T_c}{T_c^{(0)}} = -\frac{N_B}{N_F} X\left(\frac{T_c^{(0)}}{\mu}\right)$$



Summary

General remarks on $1/N$ expansion

- Perturbative extrapolation based on MF values of $\Delta, T, \mu, \dots$
- Avoids problems with self-consistency, technically very easy.
- Only reliable when the NLO corrections are small.

Color-superconducting quark matter

- Strongly coupled, though more likely on the BCS side of the crossover.
- Fluctuation corrections non-negligible, may affect competition of various pairing patterns.
- Improvements necessary: Fermi surface mismatch (mass & chemical potential), color neutrality etc.
- Generalization below the critical temperature.

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