Antiprotonic Helium Spectroscopy
Toward better than 10 digit precision

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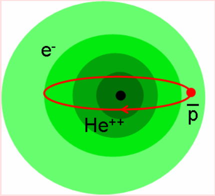


LEAP-13, June, 2013

Introduction.

The last year state of art in precision spectroscopy of $He^{+}\bar{p}$.

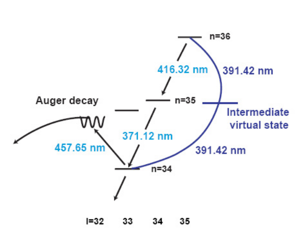
Was presented in March, 2012, at the JPS Meeting.

The two-photon $(36,34) \to (34,32)$ transition

$$
\begin{aligned}
\Delta E_{nr} &= 1\,522\,150\,208.13 \\
\Delta E_{\alpha^2} &= -50\,320.63 \\
\Delta E_{\alpha^3} &= 7\,069.5(0.3) \\
\Delta E_{\alpha^4} &= 113.1 \\
\Delta E_{\alpha^5} &= -11.3(2.1) \\
\hline
\Delta E_{total} &= 1\,522\,107\,058.8(2.1)(0.3)
\end{aligned}
$$

Transition $(36,34) \to (34,32)$ (in MHz).

Two-photon spectroscopy. Year 2010.



[M. Hori *et al.* Nature **475**, 484 (2011)]

Isotope	Transition $(n,l)\to(n-2,l-2)$	Transition frequency (MHz) Experiment	Theory
$\bar{p}^4He^+$	$(36,34)\to(34,32)$	1,522,107,062(4)(3)(2)	1,522,107,058.9(2.1)(0.3)
	$(33,32)\to(31,30)$	2,145,054,858(5)(5)(2)	2,145,054,857.9(1.6)(0.3)
$\bar{p}^3He^+$	$(35,33)\to(33,31)$	1,553,643,100(7)(7)(3)	1,553,643,100.7(2.2)(0.2)

Experimental values show respective total, statistical and systematic 1-s.d. errors in parentheses; theoretical values (ref. 3 and V. I. Korobov, personal communication) show respective uncertainties from uncalculated QED terms and numerical errors in parentheses.

$$A_r(e) = 0.000\,548\,579\,909\,1(7)[1.4\times10^{-9}]$$

Major problems to be solved

- The nonrelativistic Bethe logarithm ($m\alpha^5$ order) has been calculated within the Feshbach formalism only and the accuracy is limited by at most 5 significant digits.
- Direct numerical calculations of the $m\alpha^7$ order one-loop self-energy contribution had not yet been completed that brings the largest theoretical uncertainty.
- The theoretical relative uncertainty of $8 \cdot 10^{-10}$ for the transition frequencies is still larger than the one for the proton-to-electron mass ratio.

The nonrelativistic Bethe Logarithm

CCR Resonances

The Coulomb Hamiltonian is analytic under dilatation transformations

$$(U(\theta)f)(\mathbf{r}) = e^{m\theta/2} f(e^{\theta}\mathbf{r}),$$

for real θ and can be analytically continued to the complex plane.

The CCR method "rotates" the coordinates of the dynamical system

$$r_{ij} \to r_{ij} e^{i\varphi}, \qquad \theta = i\varphi$$

where φ is the parameter of the complex rotation.

Under this transformation the Hamiltonian changes as a function of φ

$$\boxed{H_\varphi = Te^{-2i\varphi} + Ve^{-i\varphi},}$$

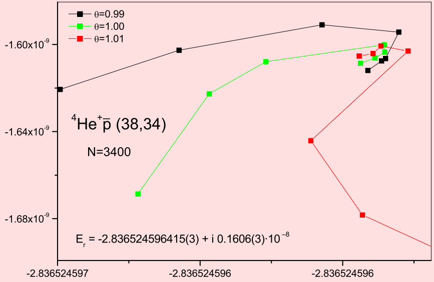
and the continuum spectrum of H_φ rotates on the complex plane around branch points ("thresholds") to "uncover" resonant poles.

CCR Resonances

The resonance energy is determined by solving the complex eigenvalue problem for the "rotated" Hamiltonian

$$(H_\varphi - E)\Psi_\varphi = 0,$$

The eigenfunction Ψ_φ obtained from this equation is square-integrable and the corresponding complex eigenvalue $E = E_r - i\Gamma/2$ defines the energy E_r and the width of the resonance, Γ.



To evaluate the nonrelativistic Bethe logarithm for CCR states a second order perturbation theory is required.

Theorem (B. Simon)

Let H be a three-body Hamiltonian with Coulomb pairwise interaction, and $W(\theta)$ be a dilatation analytic perturbation. Let E_0 be an isolated simple resonance energy (discrete eigenvalue of $H(\theta)$). Then for β small, there is exactly one eigenstate of $H(\theta)+\beta W(\theta)$ near E_0 and

$$E(\beta)=E_0+a_1\beta+a_2\beta^2+\dots$$

is analytic near $\beta=0$. In particular,

$$a_1=E'(0)=\left\langle\Psi_0^*(\theta)\left|W(\theta)\right|\Psi_0(\theta)\right\rangle,$$

$$a_2=\sum_{n\neq 0}\frac{\left\langle\Psi_0^*(\theta)\left|W(\theta)\right|\Psi_n(\theta)\right\rangle\left\langle\Psi_n^*(\theta)\left|W(\theta)\right|\Psi_0(\theta)\right\rangle}{E_0-E_n(\theta)} \tag{1}$$

It is assumed that the wavefunction are normalized as $\langle\Psi_\theta^*,\Psi_\theta\rangle=1$. Coefficients a_1, a_2, etc **do not depend** *on θ if only branches uncover E_0 and vicinity.*

The nonrelativistic Bethe logarithm

A new set of calculations is needed for

$$\beta(L,v)=\frac{\langle \mathbf{J}(H_\varphi-E_0)\ln\left((H_\varphi-E_0)/R_\infty\right)\mathbf{J}\rangle}{\langle[\mathbf{J},[H_\varphi,\ \mathbf{J}]]/2\rangle}$$

where H_φ is a rotated Hamiltonian
In other terms Bethe logarithm may be expressed as

$$\beta(L,v)=\frac{\displaystyle\int_0^{E_h}k\,dk\left\langle\mathbf{J}\left(\frac{1}{E_0-H-k}+\frac{1}{k}\right)\mathbf{J}\right\rangle+\int_{E_h}^{\infty}\frac{dk}{k}\left\langle\mathbf{J}\frac{(E_0-H)^2}{E_0-H-k}\mathbf{J}\right\rangle}{\left\langle[\mathbf{J}\,[H,\mathbf{J}]]\right\rangle/2}.$$

via basic integrand

$$J(k)=\left\langle\mathbf{J}\,(E_0-H_\varphi-k)^{-1}\,\mathbf{J}\right\rangle.$$

Nonrelativistic Bethe log. Numerical scheme.

Basis set for intermediate states [$\exp(-\alpha r_1 - \beta r_2 - \gamma r_{12})$]:

1. A regular basis set with regular values of parameters (α, β, γ).
2. A special basis set with exponentially growing parameters for a particular r_{ij}

$$\begin{cases} A_1^{(0)} = A_1, & A_2^{(0)} = A_2 \\ A_1^{(n)} = \tau^n A_1, & A_2^{(n)} = \tau^n A_2 \end{cases}$$

   where $\tau = A_2/A_1$.
   Typically $[A_1, A_2] = [2.5, 4.5]$, and $n_{\max} = 5-7$, that corresponds to the photon energy interval $k \in [0, 10^4]$.
3. For other pairs of (i, j) we take the similar basis sets as in 2.

Then the Hamiltonian H_I is diagonalized to get a set of (pseudo)state energies, E_m, and $\langle 0|i\mathbf{J}|m\rangle$.

These two sets of data are enough to restore $J(k)$.

Leading order radiative corrections

- The one-loop self-energy correction ($R_\infty\alpha^3$):

$$E_{se}^{(3)} = \alpha^3\frac{4}{3}\left[\ln\frac{1}{\alpha^2} - \beta(L,v) + \frac{5}{6} - \frac{3}{8}\right]\left\langle Z_{\rm He}\delta(\mathbf{r}_{\rm He}) + Z_{\bar{p}}\delta(\mathbf{r}_{\bar{p}})\right\rangle,$$

where

$$\beta(L,v) = \frac{\langle \mathbf{J}(H_0-E_0)\ln\left((H_0-E_0)/R_\infty\right)\mathbf{J}\rangle}{\langle[\mathbf{J},[H_0,\ \mathbf{J}]]/2\rangle}$$

is the Bethe logarithm of the three-body state.

- Recoil correction of order $R_\infty\alpha^3(m/M)$:

$$E_{recoil}^{(3)} = \sum_{i=1,2}\frac{Z_i\alpha^3}{M_i}\left\{\frac{2}{3}\left(-\ln\alpha - 4\beta(L,v) + \frac{31}{3}\right)\langle\delta(\mathbf{r}_i)\rangle - \frac{14}{3}\langle Q(r_i)\rangle\right\},$$

where $Q(r)$ is the so-called Araki-Sucher term.

- One-loop vacuum polarization:

$$E_{vp}^{(3)} = \frac{4\alpha^3}{3}\left[-\frac{1}{5}\right]\left\langle Z_{\rm He}\delta(\mathbf{r}_{\rm He}) + Z_{\bar{p}}\delta(\mathbf{r}_{\bar{p}})\right\rangle.$$

Relativistic corrections to the Bethe logarithm in the two-center problem

One-loop radiative correction in order $m\alpha^7$

The finite expression in a general form may be written

$$
\begin{aligned}
\Delta E^{(7)}_{\rm se} = \frac{\alpha}{\pi}\Biggl\{ & (Z\alpha)^6\mathcal{L} + \left(\frac{5}{9}+\frac{2}{3}\ln\left[\frac{1}{2}(Z\alpha)^{-2}\right]\right)\left\langle 4\pi\rho\, Q(E-H)^{-1}Q\, H_B\right\rangle_{\rm fin} \\
& +2\left\langle H_{so}\, Q(E-H)^{-1}Q\, H_B\right\rangle + \left(\frac{779}{14400}+\frac{11}{120}\ln\left[\frac{1}{2}(Z\alpha)^{-2}\right]\right)\left\langle \boldsymbol{\nabla}^4 V\right\rangle_{\rm fin} \\
& + \left(\frac{23}{576}+\frac{1}{24}\ln\left[\frac{1}{2}(Z\alpha)^{-2}\right]\right)\left\langle 2i\sigma^{ij}p^i\boldsymbol{\nabla}^2 V p^j\right\rangle \\
& + \left(\frac{589}{720}+\frac{2}{3}\ln\left[\frac{1}{2}(Z\alpha)^{-2}\right]\right)\left\langle \boldsymbol{\mathcal{E}}^2\right\rangle_{\rm fin} + \frac{3}{80}\left\langle 4\pi\rho\,\mathbf{p}^2\right\rangle_{\rm fin} - \frac{1}{4}\left\langle \mathbf{p}^2 H_{so}\right\rangle \\
& +(Z\alpha)^2\left[-\ln^2\left[(Z\alpha)^{-2}\right] + \left[\frac{16}{3}\ln 2-\frac{23}{60}\right]\ln\left[(Z\alpha)^{-2}\right] - 0.81971202(1)\right]\langle\pi\rho\rangle\Biggr\}
\end{aligned}
$$

It has been obtained from [U. Jentschura, A. Czarnecki, and K. Pachucki, Phys. Rev. A **72**, 062102 (2005)] and comparison with the hydrogen ground state result [K Pachucki, Ann. Phys. (N.Y.) **226**, 1 (1993)].

Low energy contribution. Relativistic Bethe logarithm

1. Relativistic corrections to the electron wave function

$$E_{L1} = \frac{2\alpha^5}{3\pi m^2}\int_0^\Lambda k\,dk\;\delta_{H_B}\left\langle \mathbf{p}\left(\frac{1}{E_0-H-k}\right)\mathbf{p}\right\rangle = \frac{2\alpha^5}{3\pi m^2}\int_0^\Lambda k\,dk\;P_{rc}^{(1)}(k)$$

$$P_{rc}^{(1)}(k) = 2\left\langle H_B Q(E_0-H)^{-1}Q\mathbf{p}\,(E_0-H-k)^{-1}\,\mathbf{p}\right\rangle + \left\langle \mathbf{p}\,(E_0-H-k)^{-1}\left(H_B+\langle H_B\rangle\right)(E_0-H-k)^{-1}\,\mathbf{p}\right\rangle$$

2. Retardation

$$E_{L2} = \frac{2\alpha^5}{3\pi m^2}\int_0^\Lambda k\,dk\;P_{nq}(k)$$

$$P_{nq}(k) = \frac{3k^2}{8\pi}\int_S dS_{\mathbf{n}}\left(\delta^{ij}-n^in^j\right)\left\{\left\langle p^i(\mathbf{n}\cdot\mathbf{r})\,(E_0-H-k)^{-1}\,(\mathbf{n}\cdot\mathbf{r})p^i\right\rangle - \left\langle p^i(\mathbf{n}\cdot\mathbf{r})^2\,(E_0-H-k)^{-1}\,p^i\right\rangle\right\}$$

Low energy contribution. Relativistic Bethe logarithm

3. Modification of vertex interactions in the SE diagram

$$E_{L3} = \frac{4\alpha^5}{3\pi m^2}\int_0^\Lambda k\,dk \left\langle \delta\mathbf{j}\left(\frac{1}{E_0-H-k}\right)\mathbf{p}\right\rangle = \frac{4\alpha^5}{3\pi m^2}\int_0^\Lambda k\,dk\; P_{rc}^{(2)}(k)$$

$$P_{rc}^{(2)}(k) = \left\langle\left(-p^2p^i - \frac{1}{2}\sigma^{ij}\nabla^j V\right)(E_0-H-k)^{-1}\,p^i\right\rangle$$

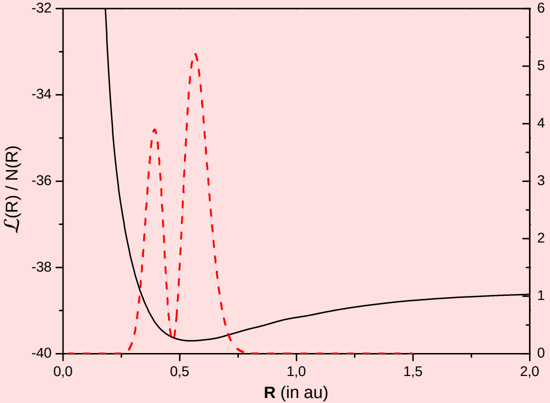
Then the relativistic Bethe logarithm is defined

$$\mathcal{L} = \beta_1+\beta_2+\beta_3$$

$$\beta_1 = \frac{2}{3}\int_0^{E_h} k\,dk\left[P_{rc}^{(1)}(k) - \frac{A_{rc}^{(1)}}{k} - \frac{B_{rc}^{(1)}}{k^{3/2}}\right] + \frac{2}{3}\int_{E_h}^{\infty} k\,dk\left[P_{rc}^{(1)}(k) - \frac{A_{rc}^{(1)}}{k} - \frac{B_{rc}^{(1)}}{k^{3/2}} - \frac{C_{rc}^{(1)}\ln k}{k^2} - \frac{D_{rc}^{(1)}}{k^2}\right]$$

etc.

Relativistic corrections to the Bethe logarithm in the two-center problem



$$N(R) = \pi \left(Z_1^3 \delta(\mathbf{r}_1) + Z_2^3 \delta(\mathbf{r}_2) \right).$$

The two-photon $(36,34) \to (34,32)$ transition

ΔE_{nr}	$=$	$1\,522\,150\,208.13$
ΔE_{α^2}	$=$	$-50\,320.63$
ΔE_{α^3}	$=$	$7\,070.28$
ΔE_{α^4}	$=$	113.11
ΔE_{α^5}	$=$	$-9.82(20)$
ΔE_{α^6}	$=$	$-0.06(6)$
ΔE_{total}	$=$	$1\,522\,107\,061.0(2)$

Transition $(36,34) \to (34,32)$ (in MHz).
CODATA10 recommended values of constants.

Conclusions

Conclusions:

1. The nonrelativistic energies as well as relativistic corrections both $m\alpha^4$ and $m\alpha^6$ orders are available with required precision;
2. The two major sources of numerical and theoretical uncertainty have been addressed and completed in the past year.
3. The theoretical precision of 10^{-10} for the transition frequencies is achieved.
4. Determination of the improved electron-to-(anti)proton mass ratio from $\mathrm{He}^+\bar{p}$ spectroscopy now become real from the theoretical point of view.

Thank You for Your Attention!