Gaussian approximation within linear sigma models

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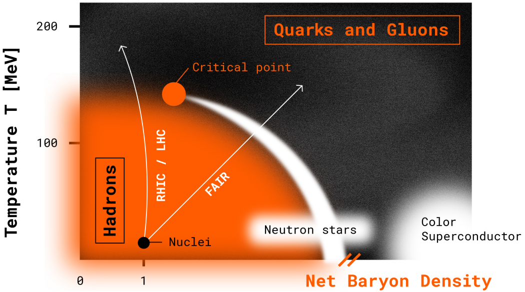
FAIRness, 2024. 09. 23-27. Croatia

QCD phase diagram

Large density:
Effective models



Mean-field approximation

pros:

- Clear and simple
- It works well! (mostly)

cons:

- It can miss something important
- It can generate nonphysical effects

How to go beyond?

- Functional methods
- Add further correction, e.g. in a gaussian approximation

What will be shown is one way to go beyond mean-field

Start with a simple Yukawa model (1 meson + 1 fermion):

$$\mathcal{L}_Y = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V_{cl}(\phi) + \bar{\psi}(i\partial\!\!\!/ - g\phi)\psi, \qquad V_{cl}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

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To get the thermodynamics:

partition function $Z \Rightarrow$ effective potential Ω from $Z = e^{-i\mathcal{V}_4\Omega}$

Partition function

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\phi e^{i\int_x \mathcal{L}_Y(x)}$$

Integrate out fermions

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\int_x \bar{\psi}(i\mathcal{S}^{-1})\psi} = \mathrm{Det}(i\mathcal{S}^{-1}) \qquad \Rightarrow \qquad Z = \int \mathcal{D}\phi e^{iS}, \qquad (S = S_m + S_f)$$

What to do with the meson fields?

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$$\boxed{\phi \to \Phi + \varphi}$$

Φ: homogeneous background, φ: fluctuating field

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Mean-field approximation: Set $\varphi \rightarrow 0 \quad \Rightarrow \quad Z = e^{iS(\Phi)}$

$$S(\Phi) = \int_x (-\frac{1}{2}m^2\Phi^2 - \frac{1}{4!}\Phi^2) + S_f(\Phi)$$

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To go beyond: Expansion in ϕ

$$S(\Phi+\varphi) = S(\Phi) + \varphi\,\frac{\delta S(\phi)}{\delta\phi}\Big|_{\phi=\Phi} + \varphi\,\frac{\delta S(\phi)}{\delta\phi\delta\phi}\Big|_{\phi=\Phi}\varphi + \dots$$

Gaussian integral for mesons

Gaussian approximation

$$\int \mathcal{D}\varphi\, e^{\frac{i}{2}\int_x \varphi(i\mathcal{G}^{-1})\varphi} = Det(i\mathcal{G}^{-1})^{-\frac{1}{2}}$$

$$\Omega = \frac{1}{2}m^2\Phi^2 + \frac{1}{4!}\Phi^2 + i\mathrm{tr}\int_k \log(i\mathcal{S}^{-1}) - \frac{i}{2}\mathrm{tr}\int_k \log(i\mathcal{G}^{-1})$$

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Physics
all loop and
everything

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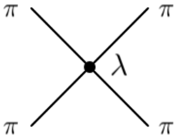

SOME CLARIFICATION

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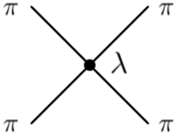

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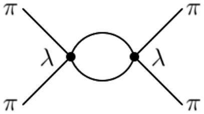

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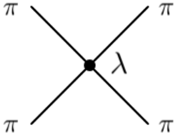

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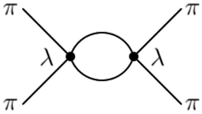

Even in the simplest case: **Physics contained** in the parameters
Keeps the **structure of the underlying symmetry**

Going beyond: **Extra structure** can be important

Simple models with:

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- **Chiral symmetry** as basic structure
- **Constituent quarks** in a Yukawa type term

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Lagrangian:

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}\left(\partial_\mu \phi^\dagger \partial^\mu \phi\right) - \frac{1}{2} m^2 \mathrm{Tr}\left(\phi^\dagger \phi\right) - \lambda_1 \left(\mathrm{Tr}(\phi^\dagger \phi)\right)^2 - \lambda_2 \mathrm{Tr}\left((\phi^\dagger \phi)^2\right)$$
$$+ \mathrm{Tr}(H\phi) + c_1 \left(\mathrm{Det}(\phi) + \mathrm{Det}(\phi^\dagger)\right) + \bar{\psi}\left(i\partial\!\!\!/ - g\mathcal{M})\right)\psi$$

with $\phi = S + i\,P$, containing pion, (kaon,) eta, sigma, etc..., while $\mathcal{M} = S + i\gamma_5 P$

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+ Vector and axial-vector mesons (ELSM)
+ Polyakov loop to mimic the confinement
+ Further hadronic fields to describe more dofs

Thermodynamics: **Mean-field level** effective potential

- Classical potential.
- Fermionic one-loop correction (vacuum part properly renormalized)

$$\Omega_{MF}(T,\mu_q) = V_{cl} + i\mathrm{tr}\int_K \log\left(iS_0^{-1}(T,\mu_q)\right)$$

Field equations (FE):

$$\frac{\partial \Omega_{MF}}{\partial \phi_N} = \frac{\partial \Omega_{MF}}{\partial \phi_S} = \frac{\partial \Omega_{MF}}{\partial \bar{\Phi}} = \frac{\partial \Omega_{MF}}{\partial \Phi} = 0$$

Curvature meson masses:

$$M^2_{(MF)ab} = \left.\frac{\partial^2 \Omega_{MF}}{\partial \varphi_a \partial \varphi_b}\right|_{\{\varphi_i\}=0}$$

Thermodynamics: **Gaussian** effective potential

- Classical potential.
- Fermionic one-loop correction (vacuum part properly renormalized)
- Meson fluctuations are included (only the pion for now and thermal fluctuations)

$$\Omega_G(T,\mu_q) = \Omega_{MF}(T,\mu_q) - \frac{i}{2}\mathrm{tr}\int_K \log\left(iG^{-1}(T,\mu_q,M^2_{(MF)})\right)$$

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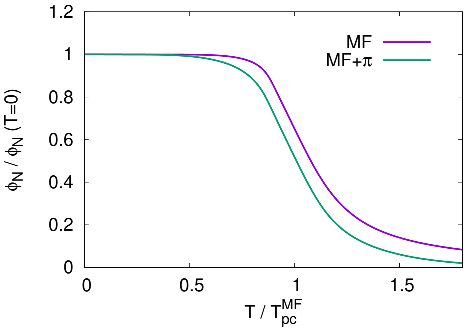
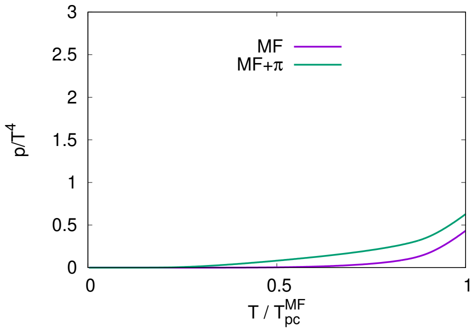
The transition temperature decreases (good news for ELSM)

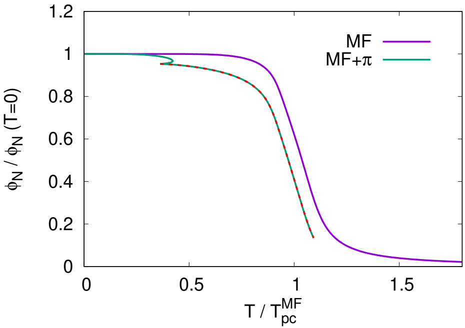
The mesonic pressure corrects the $T < T_{pc}$ behavior

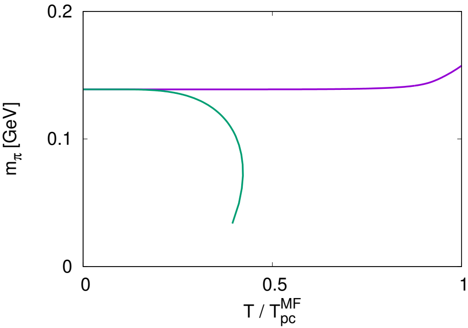




What can go wrong?
The pion mass!

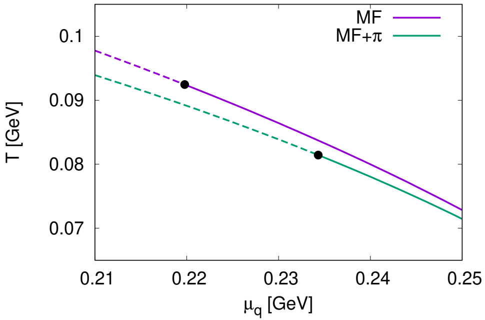
This is non-physical

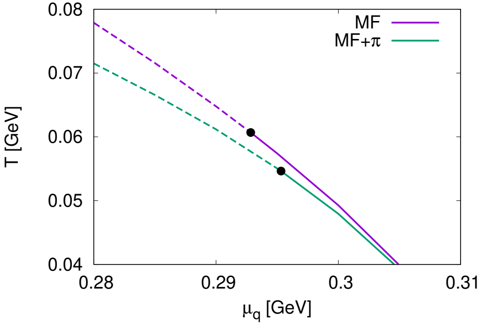




A slight shift in the critical endpoint

Note: the meson fluctuations have no μ_B dependence ($\mu_B = 3\mu_q$ in this case)





Summary

- Effective models are simple and useful tools to investigate the phase diagram.
- The usual mean-field approximation works surprisingly well.
- **We implemented a beyond mean-field, Gaussian approximation.**
- The Gaussian approximation slightly modifies the phase diagram but also qualitatively justifies the mean-field results.

- The problem with the pion mass shows that something might be incomplete.
- How to handle properly the mesonic vacuum fluctuations?
- What are the beyond mean-field effects? E.g. what can be seen for the critical scaling?
- Meson fluctuations in the meson masses?

Thank you!