Single-boson exchange fRG and its application to the Hubbard model

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Functional Methods in Strongly Correlated Systems (FunSCS2023)

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In collaboration with:

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Alessandro Toschi (Technische Universität Wien)

Pietro M. Bonetti (Max-Planck-Institute Stuttgart)

Demetrio Vilardi (Max-Planck-Institute Stuttgart)

1. Single-boson exchange fRG
   - SBE formalism
   - SBE fRG formalism
   - Application to the Hubbard model

2. Extension to strong couplings using DMF^{2}RG
   - Merits of fRG with correlated starting point
   - Application to the Hubbard model

Single-boson exchange fRG

Section based on:

KF, S. Heinzelmann, P.M. Bonetti, A. Al-Eryani, D. Vilardi, A. Toschi, and S. Andergassen, SBE fRG application to the 2D Hubbard model at weak coupling, Eur. Phys. J. B 95, 202 (2022)

Consider a fermionic theory with a quartic interaction:

$$S\left[\overline{\psi},\psi\right] = -\sum_{\alpha_1,\alpha_2} \overline{\psi}_{\alpha_1} G^{-1}_{0,\alpha_1|\alpha_2} \psi_{\alpha_2} + \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} U_{\alpha_1\alpha_2|\alpha_3\alpha_4} \overline{\psi}_{\alpha_1} \overline{\psi}_{\alpha_2} \psi_{\alpha_3} \psi_{\alpha_4}$$

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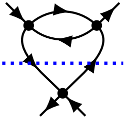
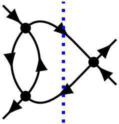
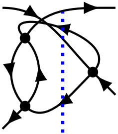
- 2P-reducible diagram = diagram that can be disconnected after cutting two **propagator lines** ($\Leftrightarrow G$):

ph : , $\overline{ph}$: , pp : 

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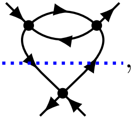
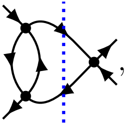
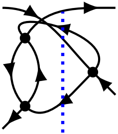
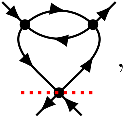
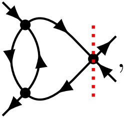
ph : 

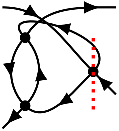
$\overline{ph}$: 

pp : 

- U-reducible diagram = diagram that can be disconnected after cutting one **bare interaction vertex** ($\Leftrightarrow U$):

ph : 

$\overline{ph}$: 

pp : 

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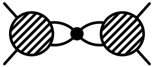

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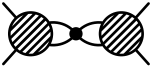
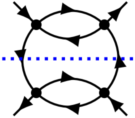
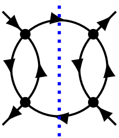
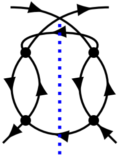


$$U\text{-reducibility} \Rightarrow \text{2P-reducibility}$$

- Consider the following 2P-reducible diagrams:

ph : , $\overline{ph}$: , pp : 

$$\text{2P-reducibility} \not\Rightarrow U\text{-reducibility}$$

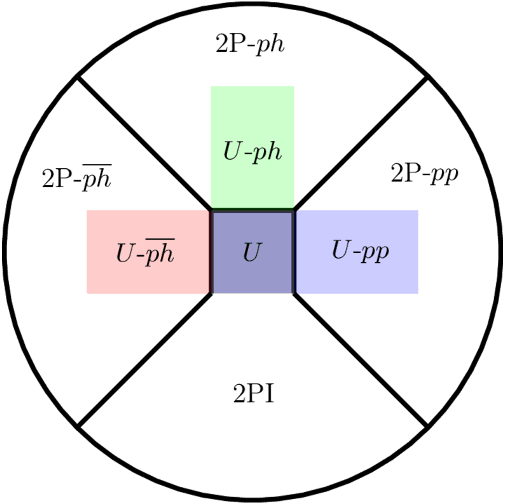
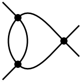


Figure adapted from Krien, Valli, Capone, PRB 100, 155149 (2019)

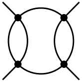
$\Rightarrow$ The class of 2P-reducible diagrams fully contains that of U-reducible diagrams

Single-boson exchange (SBE) decomposition (Krien, Valli, Capone, PRB 100, 155149 (2019)):

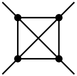
$$\Gamma^{(4)} = \left.\frac{\delta^4\Gamma\left[\overline{\psi},\psi\right]}{\delta\overline{\psi}\delta\overline{\psi}\delta\psi\delta\psi}\right|_{\psi=\overline{\psi}=0} = \sum_{\mathrm{x}=\{\mathrm{pp},\mathrm{ph},\overline{\mathrm{ph}}\}} \left(\nabla^{\mathrm{x}} + M^{\mathrm{x}} - U\right) + \mathcal{I}^{2\mathrm{PI}}$$

with $\nabla^{\mathrm{x}} =$  $+ \ldots$

only U-reducible graphs

$M^{\mathrm{x}} =$  $+ \ldots$

only U-irreducible and 2P-reducible graphs

$\mathcal{I}^{2\mathrm{PI}} =$  $+$  $+ \ldots$

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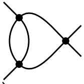
with $\nabla^{\mathrm{x}} =$  $+ \ldots$

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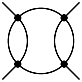
$M^{\mathrm{x}} =$  $+ \ldots$

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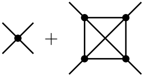
$\mathcal{I}^{\mathrm{2PI}} =$  $+ \ldots$

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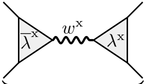
$\nabla^{\mathrm{x}} =$ 

1 boson exchanged

w : bosonic propagator
λ : Yukawa coupling
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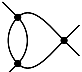
with $\nabla^{\mathrm{x}} =$  $+ \ldots$

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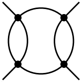
$M^{\mathrm{x}} =$  $+ \ldots$

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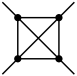
$\mathcal{I}^{\mathrm{2PI}} =$  $+$  $+ \ldots$

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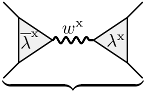
$\nabla^{\mathrm{x}} =$ 

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$\Rightarrow$ Focus on translationally invariant and $SU(2)$-spin-symmetric systems (see Gievers, Walter, Ge, von Delft, Kugler, EPJB 95, 108 (2022) for a more general formulation of the SBE decomposition):

$$\boxed{\nabla^{\mathrm{x}}_{kk'}(Q) = \overline{\lambda}^{\mathrm{x}}_{k}(Q)w^{\mathrm{x}}(Q)\lambda^{\mathrm{x}}_{k'}(Q)} \qquad k^{(\prime)} = \left(\mathbf{k}^{(\prime)},\nu^{(\prime)}\right) \quad Q = (\mathbf{Q},\Omega)$$

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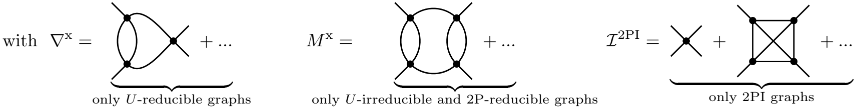

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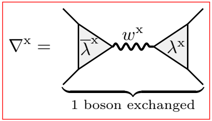

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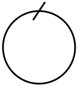
$\Rightarrow$ SBE decomposition introduces **bosonic dofs** and treats **all channels equitably**

Functional renormalization group (fRG) in a nutshell (Wetterich, PLB 301, 90 (1993)):

Introduce cutoff function: $G_0 \to G_0^\Lambda \;\Rightarrow\; \partial_\Lambda \Gamma_\Lambda =$  $\Rightarrow$

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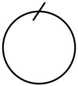
Introduce cutoff function: $G_0 \to G_0^\Lambda \;\Rightarrow\; \partial_\Lambda \Gamma_\Lambda =$  $\Rightarrow$

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Illustration of the fRG flow:

fRG solution $(\Lambda = \Lambda_{\text{fin}})$

Exact solution

Starting point $(\Lambda = \Lambda_{\text{ini}})$

$G_0^{\Lambda_{\text{ini}}} = 0 \;\Rightarrow\; \Gamma_{\Lambda_{\text{ini}}} = S$

Solve set of differential equations:

$$\partial_\Lambda \Sigma_\Lambda = \ldots$$
$$\partial_\Lambda \Gamma_\Lambda^{(4)} = \ldots$$
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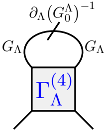
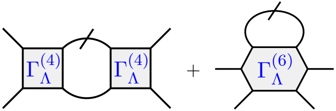
fRG for fermionic systems typically relies on a **hierarchy of equations for the 1PI vertices** derived from a vertex expansion of the Wetterich equation:

$\partial_\Lambda \Sigma_\Lambda =$ 

$\partial_\Lambda \Gamma_\Lambda^{(4)} =$ 

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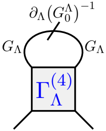
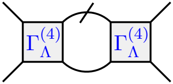
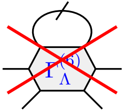
$\partial_\Lambda \Sigma_\Lambda =$ 

$\partial_\Lambda \Gamma^{(4)}_\Lambda =$ 

$+$ 

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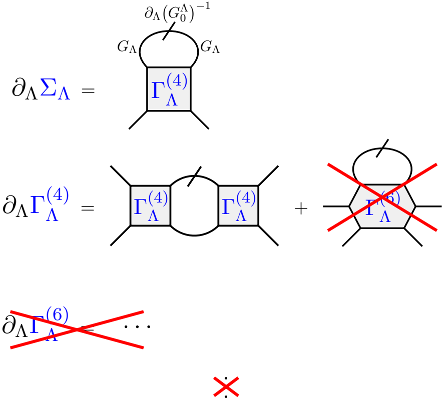

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⚠ fRG approach **restricted to weak couplings** for fermionic systems

Inserting the SBE decomposition of $\Gamma^{(4)}$ into the flow equation $\partial_\Lambda \Gamma^{(4)}_\Lambda \equiv \partial_\Lambda \Gamma^{(4)} = \ldots$ yields (Bonetti, Toschi, Hille, Andergassen, Vilardi, PRR 4, 013034 (2022)):

$$\partial_\Lambda w^{\mathrm{X}}(Q) = \left[w^{\mathrm{X}}(Q)\right]^2 \int_p \lambda_p^{\mathrm{X}}(Q) \left[\widetilde{\partial}_\Lambda \Pi_p^{\mathrm{X}}(Q)\right] \lambda_p^{\mathrm{X}}(Q)$$

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with $\mathcal{I}^{\mathrm{X}} \sim \Gamma^{(4)} - \lambda^{\mathrm{X}} w^{\mathrm{X}} \lambda^{\mathrm{X}}$, $\Pi^{\mathrm{X}} \sim GG$, $\widetilde{\partial}_\Lambda G = \partial_\Lambda G|_{\Sigma=\mathrm{const}}$ and X physical channels $(\mathrm{X} = \mathrm{M}, \mathrm{D}, \mathrm{SC})$

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⚠ Still solve the flow equation for the self-energy $\partial_\Lambda \Sigma_\Lambda = \ldots$ as in the conventional fermionic fRG:

$$\partial_\Lambda \Sigma(k) = \int_p \left[2\Gamma^{(4)}(k,k,p) - \Gamma^{(4)}(p,k,k)\right] \underbrace{\widetilde{\partial}_\Lambda G(p)}_{S(p)}$$

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$$\partial_\Lambda \Sigma(k) = \int_p \left[2\Gamma^{(4)}(k,k,p) - \Gamma^{(4)}(p,k,k)\right] \underbrace{\widetilde{\partial}_\Lambda G(p)}_{S(p)}$$

⚠ Still use the frequency-dependent cutoff:

$$G^\Lambda_0(\mathbf{k},\nu) = \frac{\nu^2}{\nu^2 + \Lambda^2} G_0(\mathbf{k},\nu) \qquad \text{with } G_0^{\Lambda_{\mathrm{ini}}=\infty} = 0$$

Illustration of the 1-loop SBE fRG flow:

Starting point $(\Lambda = \Lambda_{\mathrm{ini}})$

$$\Sigma_{\Lambda_{\mathrm{ini}}} = 0,\ w^{\mathrm{X}}_{\Lambda_{\mathrm{ini}}} = U,$$
$$\lambda^{\mathrm{X}}_{\Lambda_{\mathrm{ini}}} = 1,\ M^{\mathrm{X}}_{\Lambda_{\mathrm{ini}}} = 0$$

Solve set of differential equations:

$$\partial_\Lambda \Sigma_\Lambda = \partial_\Lambda \Sigma^{(1)}_\Lambda$$
$$\partial_\Lambda w^{\mathrm{X}}_\Lambda = \partial_\Lambda w^{\mathrm{X}(1)}_\Lambda$$
$$\partial_\Lambda \lambda^{\mathrm{X}}_\Lambda = \partial_\Lambda \lambda^{\mathrm{X}(1)}_\Lambda$$
$$\left(\partial_\Lambda M^{\mathrm{X}}_\Lambda = \partial_\Lambda M^{\mathrm{X}(1)}_\Lambda\right)$$

- fRG solution $(\Lambda = \Lambda_{\mathrm{fin}})$
- Exact solution

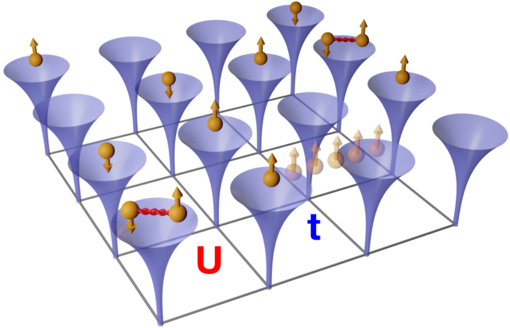
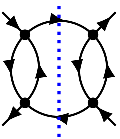


Figure taken from Qin, Schäfer, Andergassen, Corboz, Gull, ARCMP 13, 275 (2022)

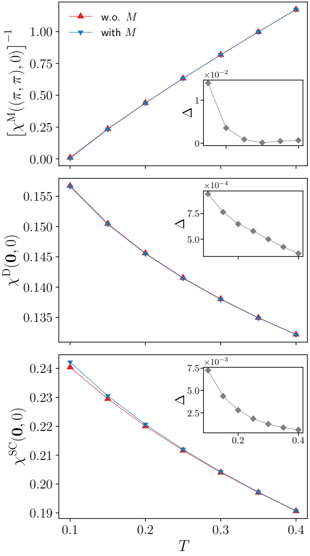
Chosen playground = 2D Hubbard model:

$$\mathcal{H} = \sum_{i \neq j, \sigma} t_{ij} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_i c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow} - \mu \sum_{i,\sigma} c^\dagger_{i\sigma} c_{i\sigma}$$

- $t_{ij} = -t = -1$ for i, j nearest-neighbor sites (0 otherwise)
- U = **on-site** Coulomb interaction

$M =$  $+ \ldots$

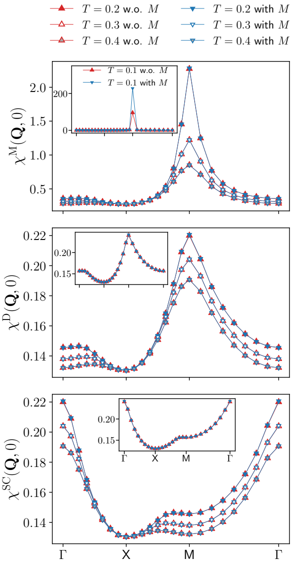
M has **negligible** effects on physical observables in **most** studied parameter regimes



Parameters: $U = 2$, half filling

⚠ Violation of **Mermin-Wagner theorem** observed with and without M

⇒ **Pseudo-critical transition** observed at half filling



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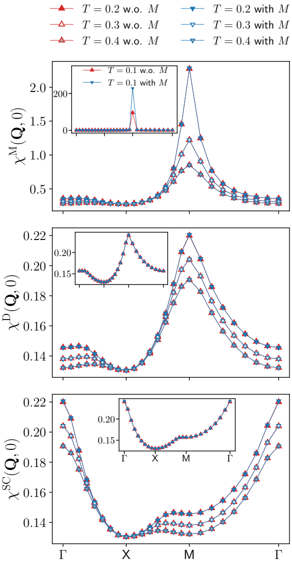
↪ Feature inherent to our truncation of the fRG hierarchy but not to the SBE scheme



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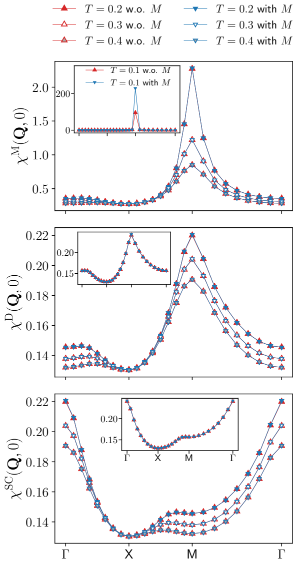
What happens to the SBE quantities (w, λ, M) near the pseudo-critical transition?



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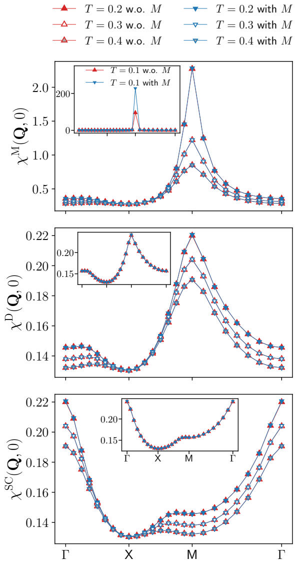
$\hookrightarrow$ Feature inherent to our truncation of the fRG hierarchy but not to the SBE scheme

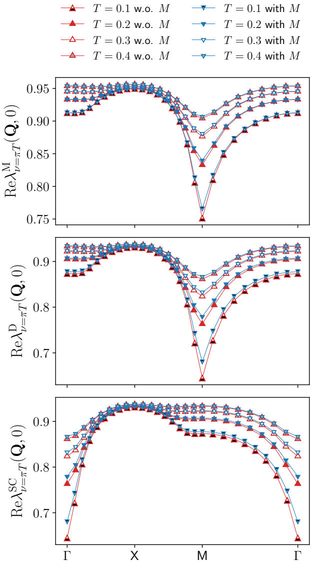
What happens to the SBE quantities (w, λ, M) near the pseudo-critical transition?

$\Rightarrow w^{\mathrm{M}}$ diverges $(w^{\mathrm{X}}(Q) = U \pm U^2\chi^{\mathrm{X}}(Q))$



Parameters: $U = 2$, half filling

$\Rightarrow \lambda^{\mathrm{M}}$ remains finite

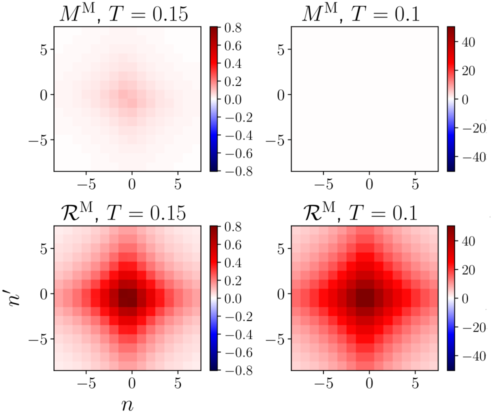


Parameters: $U = 2$, half filling

$$\mathcal{R}^{\mathrm{X}}_{kk'}(Q) = \phi^{\mathrm{X}}_{kk'}(Q) - \mathcal{K}^{(1)\mathrm{X}}(Q) - \mathcal{K}^{(2)\mathrm{X}}_{k}(Q) - \mathcal{K}^{(2)\mathrm{X}}_{k'}(Q)$$

$$\mathcal{K}^{(1)\mathrm{X}}(Q) = \lim_{\nu,\nu'\to\infty} \phi^{\mathrm{X}}_{(\mathbf{k},\nu),(\mathbf{k}',\nu')}(Q)$$

$$\mathcal{K}^{(2)\mathrm{X}}_{k}(Q) = \lim_{\nu'\to\infty} \phi^{\mathrm{X}}_{k,(\mathbf{k}',\nu')}(Q) - \mathcal{K}^{(1)\mathrm{X}}(Q)$$



Results for $M^{\mathrm{M}}_{\nu\nu'}((\pi,\pi),0)$ and $\mathcal{R}^{\mathrm{M}}_{\nu\nu'}((\pi,\pi),0)$
Parameters: $U = 2$, half filling

$\Rightarrow M^{\mathrm{M}}$ remains finite

$$\mathcal{R}^{\mathrm{X}}_{kk'}(Q) = \phi^{\mathrm{X}}_{kk'}(Q) - \mathcal{K}^{(1)\mathrm{X}}(Q) - \mathcal{K}^{(2)\mathrm{X}}_{k}(Q) - \mathcal{K}^{(2)\mathrm{X}}_{k'}(Q)$$

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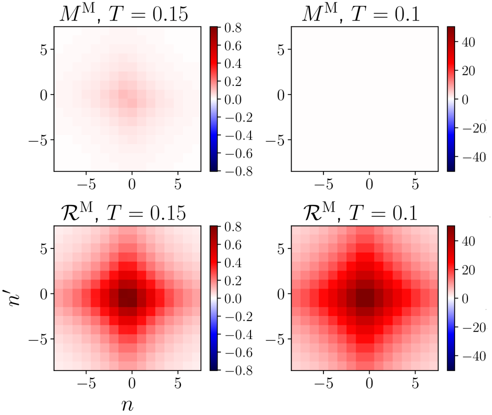


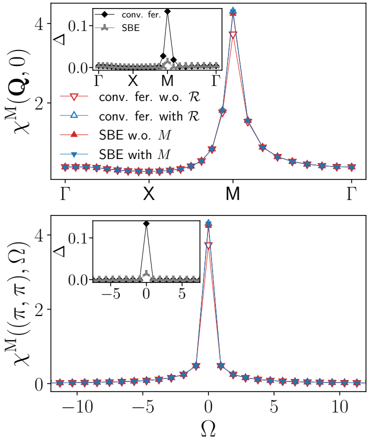
Results for $M^{\mathrm{M}}_{\nu\nu'}((\pi,\pi),0)$ and $\mathcal{R}^{\mathrm{M}}_{\nu\nu'}((\pi,\pi),0)$
Parameters: $U = 2$, half filling

$\Rightarrow M^{\mathrm{M}}$ remains finite

$\hookrightarrow$ Divergence in $\mathcal{R}^{\mathrm{M}}$ canceled out by w^{M}:

$$M^{\mathrm{X}}_{kk'}(Q) = \mathcal{R}^{\mathrm{X}}_{kk'}(Q) - [\lambda^{\mathrm{X}}_{k}(Q) - 1]w^{\mathrm{X}}(Q)[\lambda^{\mathrm{X}}_{k'}(Q) - 1]$$

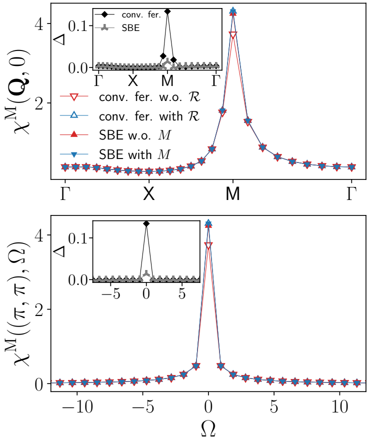
Neglecting M in SBE fRG clearly better approximation than $\mathcal{R}$ in conv. fer. fRG at 1 loop



Parameters: $U = 2$, $T = 0.15$, half filling

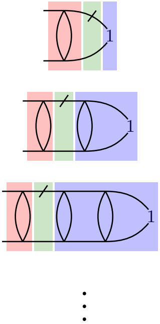
Neglecting M in SBE fRG clearly better approximation than $\mathcal{R}$ in conv. fer. fRG at 1 loop

$\hookrightarrow$ Why does the SBE approximation induce so little loss?



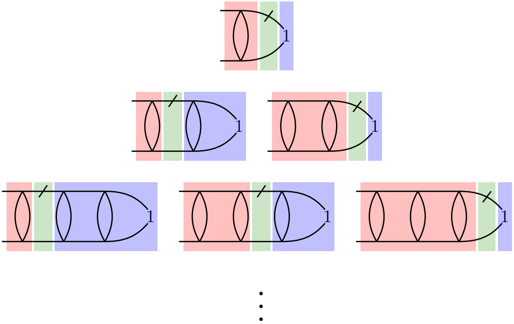
Parameters: $U = 2$, $T = 0.15$, half filling

$\partial_\Lambda \lambda^{\mathrm{X}}$ in SBE



$$\partial_\Lambda \lambda^{\mathrm{X}}_k(Q) = \int_p \mathcal{I}^{\mathrm{X}}_{kp}(Q) \left[\widetilde{\partial}_\Lambda \Pi^{\mathrm{X}}_p(Q)\right] \lambda^{\mathrm{X}}_p(Q)$$

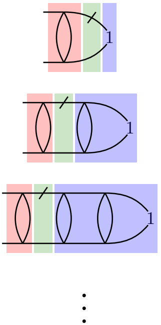
$\partial_\Lambda \lambda^{\mathrm{X}}$ in SBE with M



$$\partial_\Lambda \lambda^{\mathrm{X}}_k(Q) = \int_p \mathcal{I}^{\mathrm{X}}_{kp}(Q) \left[\widetilde{\partial}_\Lambda \Pi^{\mathrm{X}}_p(Q)\right] \lambda^{\mathrm{X}}_p(Q)$$
$$\left(+\partial_\Lambda M^{\mathrm{X}}_{kk'}(Q) = \int_p \mathcal{I}^{\mathrm{X}}_{kp}(Q) \left[\widetilde{\partial}_\Lambda \Pi^{\mathrm{X}}_p(Q)\right] \mathcal{I}^{\mathrm{X}}_{pk'}(Q)\right)$$

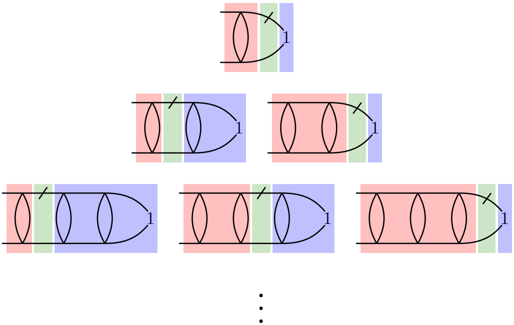
Recall: $\mathcal{I}^{\mathrm{X}} \sim \Gamma^{(4)} - \lambda^{\mathrm{X}} w^{\mathrm{X}} \lambda^{\mathrm{X}}$ and $\Pi^{\mathrm{X}} \sim GG$

$\partial_\Lambda \lambda^{\mathrm{X}}$ in SBEa



$$\partial_\Lambda \lambda^{\mathrm{X}}_k(Q) = \int_p \mathcal{I}^{\mathrm{X}}_{kp}(Q)\left[\widetilde{\partial}_\Lambda \Pi^{\mathrm{X}}_p(Q)\right]\lambda^{\mathrm{X}}_p(Q)$$

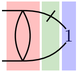
$\partial_\Lambda \lambda^{\mathrm{X}}$ in SBE with M



$$\partial_\Lambda \lambda^{\mathrm{X}}_k(Q) = \int_p \mathcal{I}^{\mathrm{X}}_{kp}(Q)\left[\widetilde{\partial}_\Lambda \Pi^{\mathrm{X}}_p(Q)\right]\lambda^{\mathrm{X}}_p(Q)$$
$$\left(+\partial_\Lambda M^{\mathrm{X}}_{kk'}(Q) = \int_p \mathcal{I}^{\mathrm{X}}_{kp}(Q)\left[\widetilde{\partial}_\Lambda \Pi^{\mathrm{X}}_p(Q)\right]\mathcal{I}^{\mathrm{X}}_{pk'}(Q)\right)$$

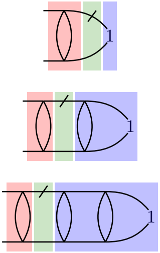
Recall: $\mathcal{I}^{\mathrm{X}} \sim \Gamma^{(4)} - \lambda^{\mathrm{X}} w^{\mathrm{X}} \lambda^{\mathrm{X}}$ and $\Pi^{\mathrm{X}} \sim GG$

$\partial_\Lambda \lambda^{\mathrm{X}}$ in SBEb



$$\partial_\Lambda \lambda^{\mathrm{X}}_k(Q) = \int_p \mathcal{I}^{\mathrm{X}}_{kp}(Q) \left[\widetilde{\partial}_\Lambda \Pi^{\mathrm{X}}_p(Q)\right] 1$$

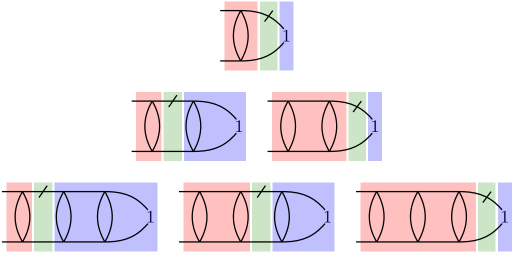
$\partial_\Lambda \lambda^{\mathrm{X}}$ in SBEa



$\vdots$

$$\partial_\Lambda \lambda^{\mathrm{X}}_k(Q) = \int_p \mathcal{I}^{\mathrm{X}}_{kp}(Q) \left[\widetilde{\partial}_\Lambda \Pi^{\mathrm{X}}_p(Q)\right] \lambda^{\mathrm{X}}_p(Q)$$

$\partial_\Lambda \lambda^{\mathrm{X}}$ in SBE with M



$\vdots$

$$\partial_\Lambda \lambda^{\mathrm{X}}_k(Q) = \int_p \mathcal{I}^{\mathrm{X}}_{kp}(Q) \left[\widetilde{\partial}_\Lambda \Pi^{\mathrm{X}}_p(Q)\right] \lambda^{\mathrm{X}}_p(Q)$$
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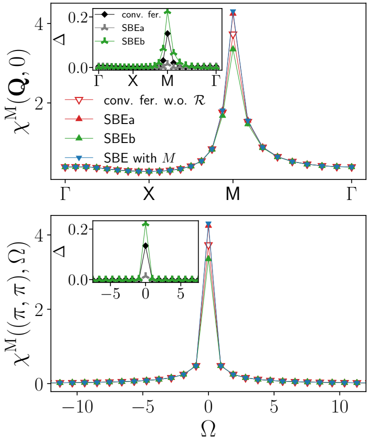
Recall: $\mathcal{I}^{\mathrm{X}} \sim \Gamma^{(4)} - \lambda^{\mathrm{X}} w^{\mathrm{X}} \lambda^{\mathrm{X}}$ and $\Pi^{\mathrm{X}} \sim GG$

Important loss of accuracy in SBEb as compared to SBEa but SBEb restores cutoff independence...

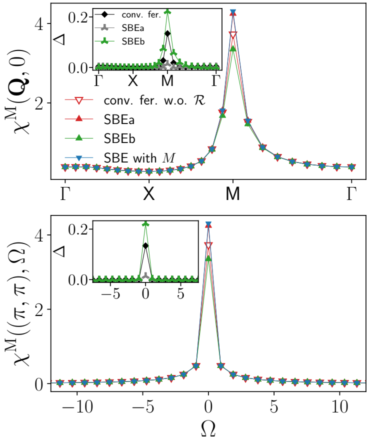


Parameters: $U = 2$, $T = 0.15$, half filling

Important loss of accuracy in SBEb as compared to SBEa but SBEb restores cutoff independence...

⚠ SBEb is a self-consistent solution of parquet equations without M but SBEa inherent to fRG framework

$\Rightarrow$ Advantage of fRG formulation of SBE scheme



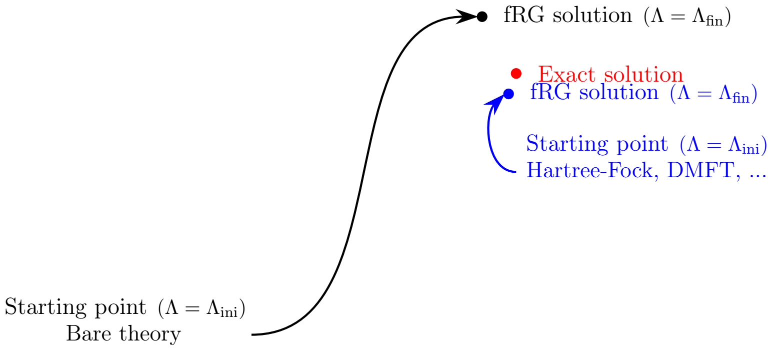
Parameters: $U = 2$, $T = 0.15$, half filling

Extension to strong couplings using DMF^{2}RG

Section based on:

P.M. Bonetti, A. Toschi, C. Hille, S. Andergassen, and D. Vilardi, SBE representation of the fRG for strongly interacting many-electron systems, Phys. Rev. Research 4, 013034 (2022)

Illustration of the fRG flow with/without **correlated starting point**:

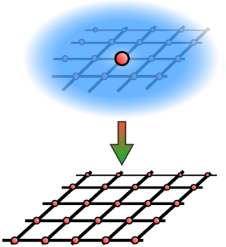


Taranto, Andergassen, Bauer, Held, Katanin, Metzner, Rohringer, Toschi, PRL 112, 196402 (2014)
Dupuis, PRB 89, 035113 (2014)
Wentzell, Taranto, Katanin, Toschi, Andergassen, PRB 91, 045120 (2015)
Katanin, PRB 99, 115112 (2019)
Vilardi, PRB 99, 104501 (2019)

General principle of DMF^{2}RG:

$$\mathcal{S} \overset{\Lambda=\Lambda_{\mathrm{ini}}}{=} -\int_0^\beta d\tau d\tau' \sum_\sigma c_\sigma^\dagger(\tau)\mathcal{G}_{\mathrm{AIM}}^{-1}(\mathbf{k},\tau-\tau')^{-1} c_\sigma(\tau') + \mathcal{S}_{\mathrm{int}}$$

$$\mathcal{G}_0^{\Lambda_{\mathrm{ini}}}(\mathbf{k},i\omega) = \mathcal{G}_{\mathrm{AIM}}(i\omega)$$



$$\mathcal{S} \overset{\Lambda=\Lambda_{\mathrm{fin}}}{=} -\int_0^\beta d\tau d\tau' \sum_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}^\dagger(\tau)\mathcal{G}_{\mathrm{latt}}^{-1}(\mathbf{k},\tau-\tau')^{-1} c_{\mathbf{k}\sigma}(\tau') + \mathcal{S}_{\mathrm{int}}$$

$$\mathcal{G}_0^{\Lambda_{\mathrm{fin}}}(\mathbf{k},i\omega) = \mathcal{G}_{\mathrm{latt}}(\mathbf{k},i\omega)$$

- Use the **DMFT solution as starting point** of the flow (DMF^{2}RG = DMFT+fRG)
- Incorporate **nonlocal correlations** on top of the local ones throughout the flow

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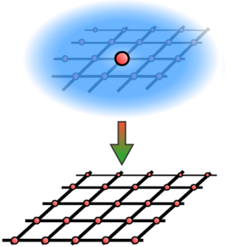

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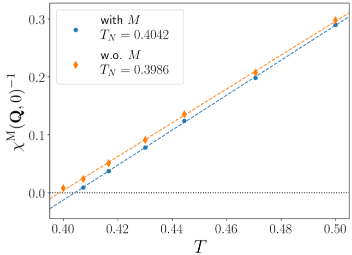
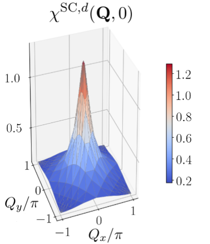
DMF^{2}RG formulated also in the SBE scheme:



Parameters: $U = 8$, half filling



Parameters: $U = 8$, $T = 0.044$, $\Lambda \neq \Lambda_{\text{fin}}$, finite doping

Conclusion

Main conclusions of our SBE fRG study:

- **Rest function M negligible** in most studied cases at 1 loop, at weak and strong couplings (with DMF^{2}RG)
- **Advantages** of the SBE fRG **over the conventional fermionic fRG** especially in the vicinity of the pseudo-critical transition at 1 loop

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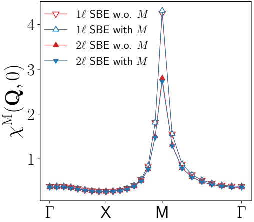
Outlooks for the SBE fRG:

- Design a quantitative fRG approach at strong couplings
  $\hookrightarrow$ Calculate multiloop corrections for DMF^{2}RG in the SBE scheme
  Kugler, von Delft, PRL 120, 057403 (2018), PRB 97, 035162 (2018)
  Gievers, Walter, Ge, von Delft, Kugler, EPJB 95, 108 (2022)



Parameters: $U = 2$, $T = 0.15$, half filling

- Comparison with other fRG schemes based on Hubbard-Stratonovich transf.
  $\hookrightarrow$ Connection with Denz, Mitter, Pawlowski, Wetterich, Yamada, PRB 101, 155115 (2020)

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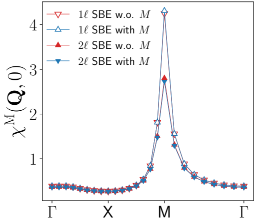

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See A. Al-Eryani's talk for applications to non-local interactions

Backup slides

Conventional decomposition of the 2-particle vertex $\Gamma^{(4)}$ within the fermionic fRG (Wentzell, Li, Tagliavini, Taranto, Rohringer, Held, Toschi, Andergassen, PRB 102, 085106 (2020)):

$$\phi^{\mathrm{X}}_{kk'}(Q) \equiv \left(V - \mathcal{I}^{\mathrm{2PI}}\right)^{\mathrm{X}}_{kk'}(Q) = \mathcal{K}^{(1)\mathrm{X}}(Q) + \mathcal{K}^{(2)\mathrm{X}}_{k}(Q) + \mathcal{K}^{(2)\mathrm{X}}_{k'}(Q) + \mathcal{R}^{\mathrm{X}}_{kk'}(Q)$$

with the high-frequency asymptotics:

$$\mathcal{K}^{(1)\mathrm{X}}(Q) = \lim_{\nu,\nu'\to\infty} \phi^{\mathrm{X}}_{(\mathbf{k},\nu),(\mathbf{k}',\nu')}(Q)$$

$$\mathcal{K}^{(2)\mathrm{X}}_{k}(Q) = \lim_{\nu'\to\infty} \phi^{\mathrm{X}}_{k,(\mathbf{k}',\nu')}(Q) - \mathcal{K}^{(1)\mathrm{X}}(Q)$$

and $\mathcal{R}$ the rest function

⚠ $\mathcal{R}$ = counterpart of the multiboson exchange contributions M for the conv. fer. fRG

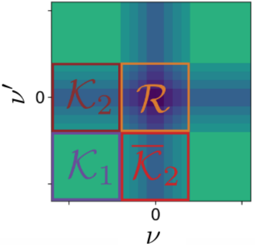

Figure adapted from C. Hille, PhD thesis (2020)

Possibility to express SBE quantities in terms of those inherent to the conventional fermionic fRG:

$$\text{Consider}\quad \begin{cases} \phi^{\mathrm{X}}_{kk'}(Q) = \lambda^{\mathrm{X}}_{k}(Q)\, w^{\mathrm{X}}(Q)\, \lambda^{\mathrm{X}}_{k'}(Q) + M^{\mathrm{X}}_{kk'}(Q) - U \\ \phi^{\mathrm{X}}_{kk'}(Q) = \mathcal{K}^{(1)\mathrm{X}}(Q) + \mathcal{K}^{(2)\mathrm{X}}_{k}(Q) + \mathcal{K}^{(2)\mathrm{X}}_{k'}(Q) + \mathcal{R}^{\mathrm{X}}_{kk'}(Q) \end{cases}$$

$\Rightarrow$ w and λ vs $\mathcal{K}$ functions:

$$w^{\mathrm{X}}(Q) = U + \mathcal{K}^{(1)\mathrm{X}}(Q)$$

$$\lambda^{\mathrm{X}}_{k}(Q) = 1 + \frac{\mathcal{K}^{(2)\mathrm{X}}_{k}(Q)}{w^{\mathrm{X}}(Q)}$$

$\Rightarrow$ SBE rest function M vs conv. fer. rest function $\mathcal{R}$:

$$\boxed{M^{\mathrm{X}}_{kk'}(Q) = \mathcal{R}^{\mathrm{X}}_{kk'}(Q) - [\lambda^{\mathrm{X}}_{k}(Q) - 1]w^{\mathrm{X}}(Q)[\lambda^{\mathrm{X}}_{k'}(Q) - 1]}$$