JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

Studienstiftung des deutschen Volkes

An overview on

Critical dynamics from the real-time functional renormalization group

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FUNSCS2023, Hirschegg, 10-15 September 2023

Based on

JR, L. von Smekal, arXiv:2303.11817

JR, D. Schweitzer, L. J. Sieke, L. von Smekal, Phys. Rev. D **105**, 116017 (2022)

JR, S. Schlichting, L. von Smekal, Y. Ye, in preparation

DFG

CRC-TR 211 Strong-interaction matter under extreme conditions

HFHF Helmholtz Forschungsakademie Hessen für FAIR

HGS-HIRe for FAIR Helmholtz Graduate School for Hadron and Ion Research

Motivation: Why real time?

Study **QCD phase diagram** through heavy-ion collisions:

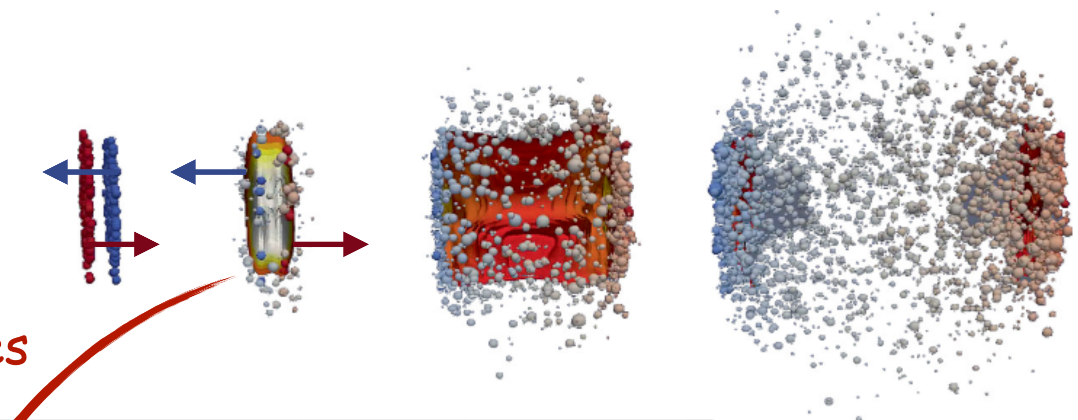


Figure from MADAI collaboration

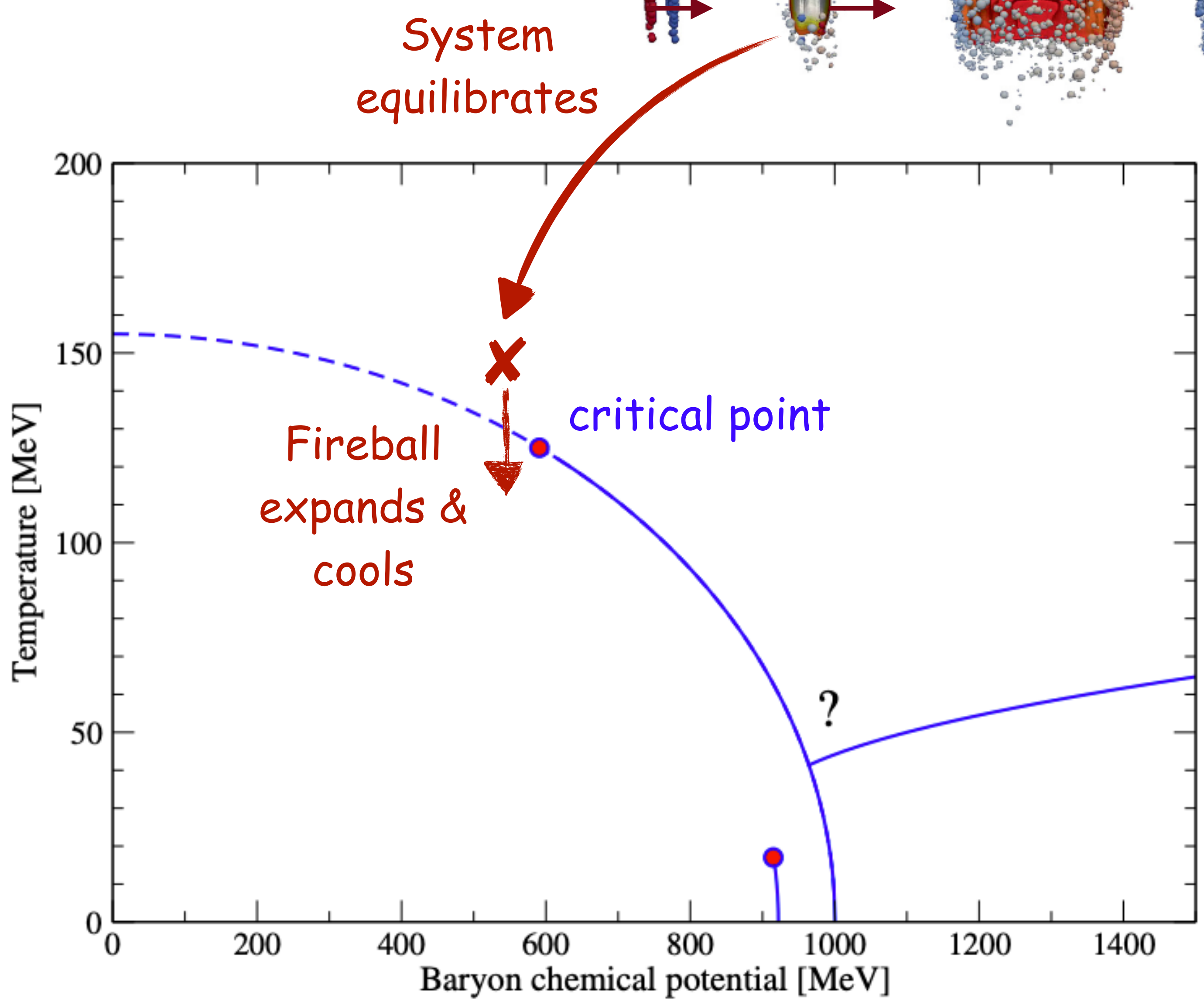


Figure adapted from C. S. Fischer, Prog. Part. Nucl. Phys. **105**, 1 (2019)

Fireball passes critical point:

Critical slowing down ⇝ **System falls out of equilibrium**

- non-perturbative
- off-equilibrium

⇝ proper description needs genuine **real-time methods**

Outline

1. Real-time functional renormalization group
2. Field-theory application: critical dynamics

Brief summary of Schwinger-Keldysh formalism

Goal: compute **non-equilibrium** correlation functions

⇝ Path integral requires **doubling number of fields:** L.V. Keldysh, Sov. Phys. JETP **20** (1965) 1018

$$\langle O(t)\rangle = \mathrm{tr}\left(O(t)\rho_0\right)$$ (Heisenberg picture)

$$= \mathrm{tr}\left(U(-\infty,t)OU(t,-\infty)\rho_0\right)$$

(extend evolution to t = +∞)

$$= \int_{\rho_0} \mathcal{D}\phi^+\,\mathcal{D}\phi^-\, e^{i(S[\phi^+]-S[\phi^-])}\, O\left(\phi^+(t),\phi^-(t)\right)$$

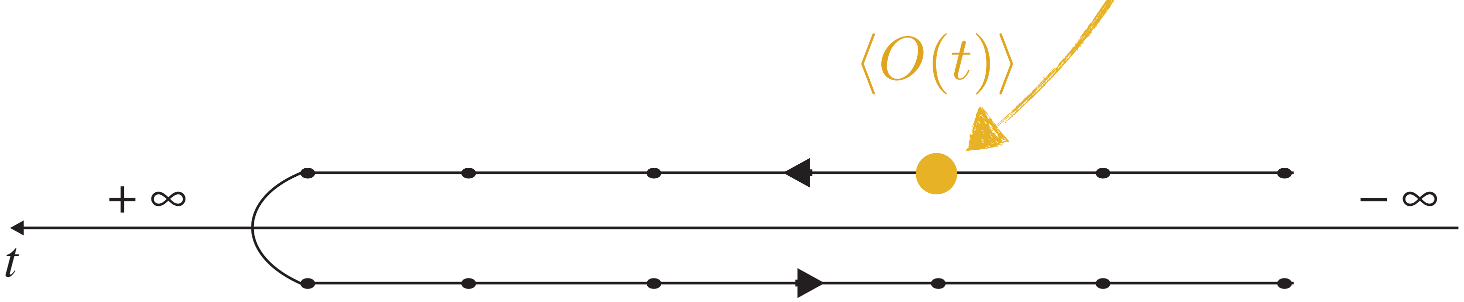


Figure adapted from
Kamenev, *Field Theory of Non-Equilibrium Systems*
(Cambridge University Press, 2011)

closed-time path

⇝ in particular: **direct access** to **real-time** Green functions

$$G^K(t,t') = i\langle\{\phi(t),\phi(t')\}\rangle$$

$$G^R(t,t') = i\theta(t-t')\langle[\phi(t),\phi(t')]\rangle$$

$$G^A(t,t') = i\theta(t'-t)\langle[\phi(t'),\phi(t)]\rangle$$

$$G^{\tilde{K}}(t,t') = 0$$

⇝ **Causal structure** built into the formalism!

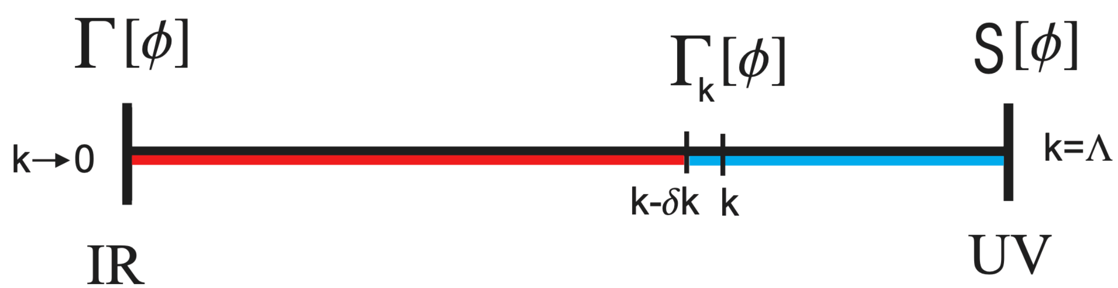
Functional RG (flow) equations

Wilson: introduce **infrared cutoff** to suppress fluctuations with $p \lesssim k$

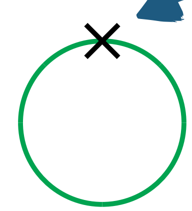
$$\Delta S_k[\phi] = \frac{1}{2}\int_{xx'} \phi^T(x) R_k(x,x')\phi(x') \qquad \phi = (\phi^c, \phi^q)^T$$

(scalar field theory)

Integrate fluctuations ‘momentum shell by momentum shell’

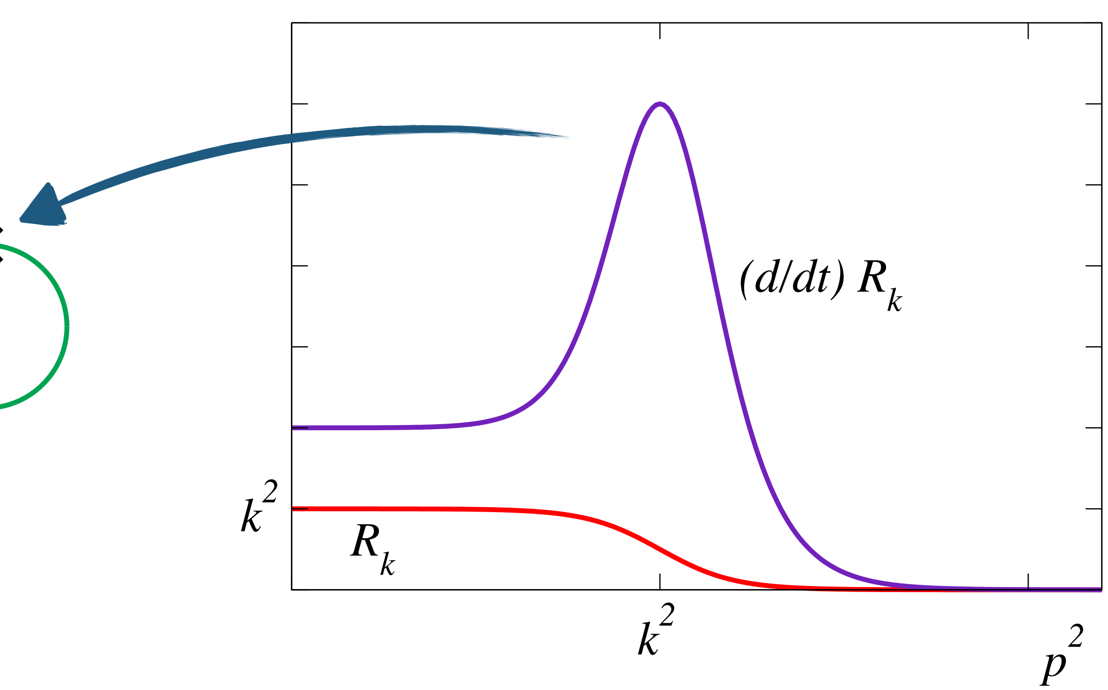


$$\partial_k \Gamma_k = \frac{i}{2}\mathrm{tr}\left\{\partial_k R_k \circ \left(R_k + \Gamma_k^{(2)}\right)^{-1}\right\} = -\frac{i}{2}$$



(exact ‘flow’ equation)

Typical form:

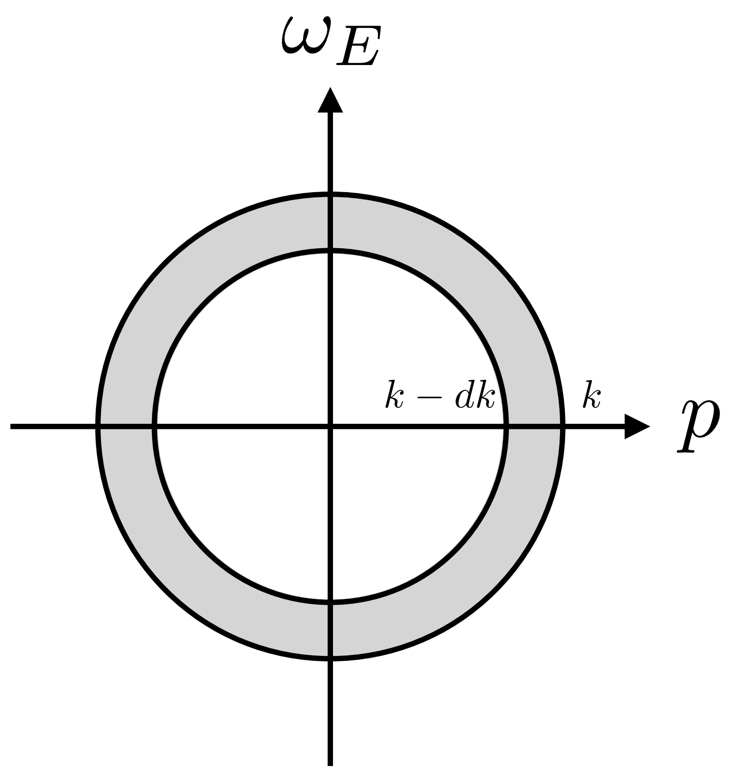


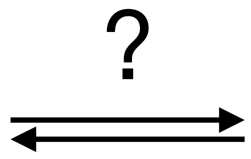
First derived by
C. Wetterich, Phys. Lett. B **301** (1993) 90-94

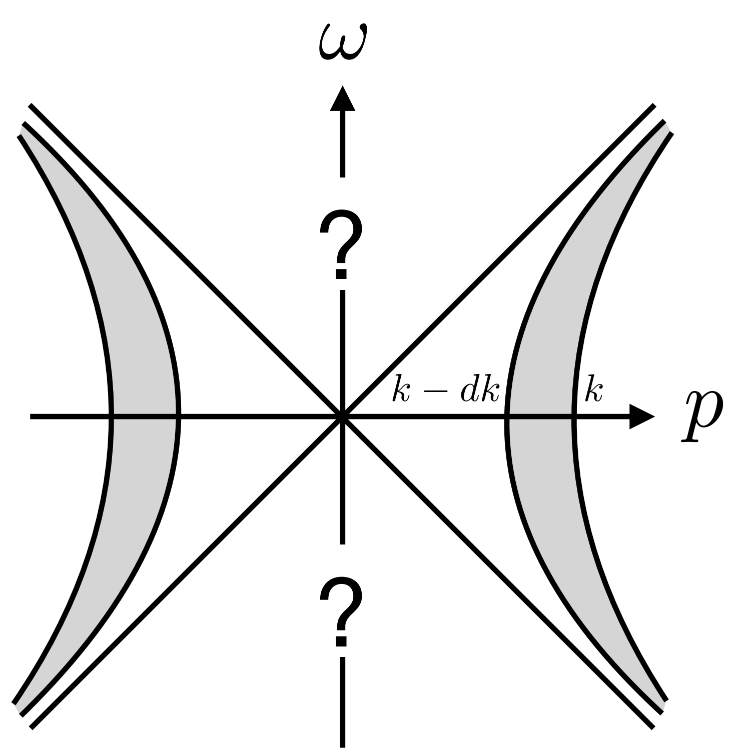
Real time:
J. Berges, D. Mesterházy, Nucl. Phys. B Proc. Suppl. **228** (2012) 37-60

Figure taken from H. Gies, Lect. Notes Phys. **852** (2012) 287-348

Wilsonian renormalization in Minkowski spacetime?







Wilsonian renormalization in Euclidean spacetime

vs.

Wilsonian renormalization in Minkowski spacetime

Conceptually easy:
integrate out (hyper-)spheres
no need to worry about causality

Conceptually intricate:
integrate hyperboloids?
timelike momenta?
causal structure of propagators?
...

Find: Frequency-dependent regulators usually violate **causal structure**

⇝ General construction scheme that **guarantees** causality?

Causal regulators

Solution: Observe that regulator is a self-energy

- Self-energies generally inherit **causal structure**

  ⤳ **Spectral representation** from (subtracted) Kramers-Kronig relations

mass-like part
(trivially causal)

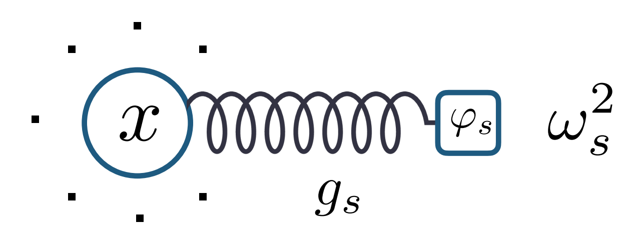
'spectral density'

$$J_k(\omega, \boldsymbol{p}) = 2\, \mathrm{Im}\, R_k^R(\omega, \boldsymbol{p})$$

$$R_k^{R/A}(\omega, \boldsymbol{p}) = R_k^{R/A}(0, \boldsymbol{p}) - \int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega^2 J_k(\omega', \boldsymbol{p})}{\omega'((\omega \pm i\varepsilon)^2 - \omega'^2)}$$

JR, von Smekal, arXiv:2303.11817
JR, Schweitzer, Sieke, von Smekal, Phys. Rev. D **105**, 116017 (2022)

- Interpret as coupling to **fictitious heat bath:**
  (Hubbard-Stratonovich transformation)



here: $J_k(\omega) = \pi \sum_s \frac{g_s^2(k)}{\omega_s(k)} \left(\delta(\omega - \omega_s(k)) - \delta(\omega + \omega_s(k))\right)$

⤳ **Physical** only for **positive-semidefinite** spectral densities $J_k(\omega, \boldsymbol{p}) \geq 0 \quad (\omega > 0)$

⤳ Spectral density encodes **spectrum of bath oscillators**

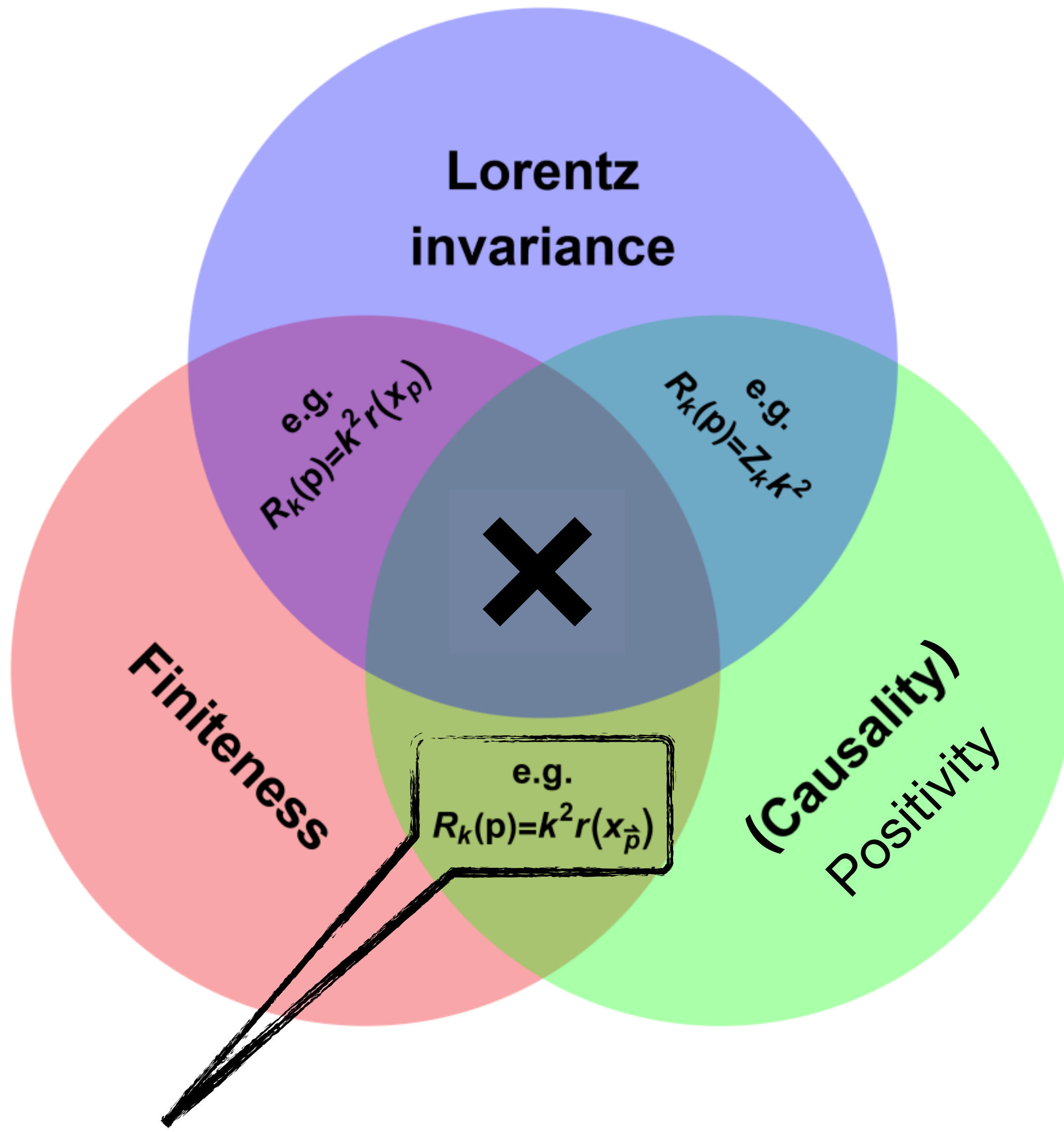
Regulator trinity in real-time FRG flows

requires invariant spectral distribution & momentum-independent mass shift

$$J_k(\omega,\boldsymbol{p}) = 2\pi\,\mathrm{sgn}(\omega)\,\theta(p^2)\,\widetilde{J}_k(p^2)\,, \qquad \Delta M_k^2(\boldsymbol{p}) = \Delta M_k^2$$

requires vanishing spectral density and mass shift in the UV

$$J_k(\omega,\boldsymbol{p}) \to 0 \text{ for } \omega \to \infty$$
$$\Delta M_k^2(\boldsymbol{p}) \to 0 \text{ for } \boldsymbol{p} \to \infty$$



requires positive-semidefinite spectral density

$$J_k(\omega,\boldsymbol{p}) \geq 0 \quad \text{for} \quad \omega > 0$$

use for our field-theory applications (next)

Figure adapted from fQCD collaboration, SciPost Phys. Core 6, 061 (2023)

Field-theory application: critical dynamics

Universality at (thermodynamic) critical points

Example: Ising spin model $\dots \downarrow \; \uparrow \; \downarrow \; \uparrow \dots$

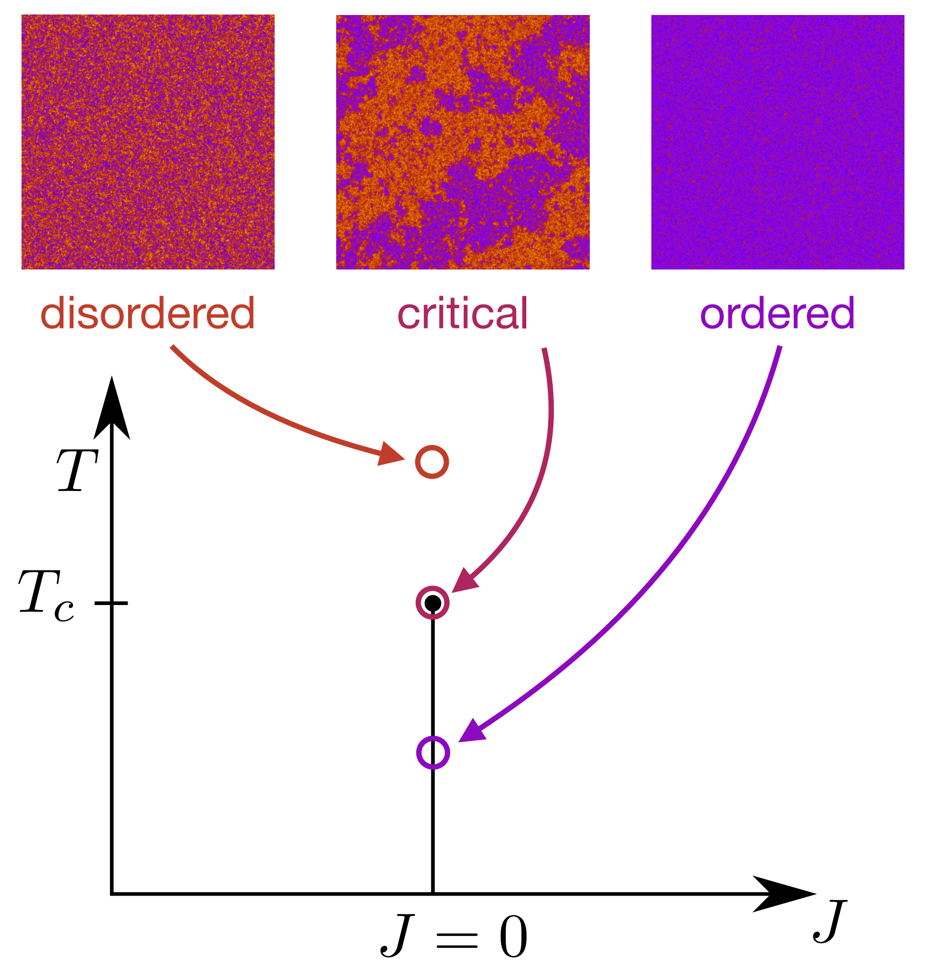
Critical point: Correlation length diverges

- Microscopic details irrelevant
- Different systems group into **universality classes**
- Observables described by power laws with **universal exponents:**

$$(\tau > 0) \qquad \xi = f^{+} \tau^{-\nu} + \text{less singular}$$

correlation length

reduced temperature $\tau = (T - T_c)/T_c$



Phase diagram of Ising model

(Figure adapted from D. Schweitzer)

Ising ferromagnets, liquid-gas transition in pure fluid, QCD's critical endpoint, ϕ^4 QFT, …

Z_2 symmetry ⇝ **'Ising' universality class** (has $\nu \approx 0.63$)

Universal (critical) dynamics

Universality also applies to (critical) **dynamics!**

Example: Spectral function

$$\rho(\omega) = \frac{1}{2\pi i} \int dt\, e^{i\omega t} \int d^d x\; i\langle[\phi(t,\boldsymbol{x}), \phi(0,\mathbf{0})]\rangle$$

typical non-critical spectral function (pion):

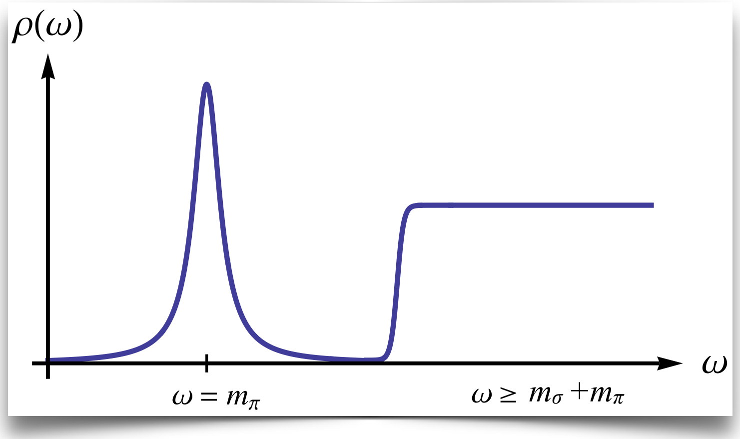


Figure adapted from A. Tripolt

also universal at criticality!

- typical critical form: $\rho(\omega) \sim \omega^{-\sigma}$
- scaling exponent: $\sigma = (2-\eta)/z$
- related to dynamic critical exponent z: $\xi_t \sim \xi^z$ **critical slowing down**

correlation time correlation length

- z determined by **dynamic** universality class

Dynamic universality classes

Static universality classes split up into **dynamic** universality classes:

classified into ‘Models A … J’ by Hohenberg and Halperin, Rev. Mod. Phys. **49**, 435 (1977)

Static Z_2 (Ising) universality class

Model A
No conserved quantities
No Poisson brackets

Model B
Order parameter conserved
No Poisson brackets

Model C
Conserved energy density
No Poisson brackets

Model H
Order parameter conserved
Poisson bracket $\{\phi, \vec{j}\}$ with shear modes

(work in progress)

Dynamic universality class of QCD’s critical endpoint!

Son and Stephanov, Phys. Rev. D **70**, 056001 (2004)

Dynamic universality classes

Similarly for *O*(4) universality class:

Static *O*(4) universality class

(Universality class of two-flavour QCD's chiral phase transition)

Rajagopal and Wilczek,
Nucl. Phys. B **399** (1993) 395-425

… … …

Model G

- *O*(4) order parameter ϕ not conserved
- Conserved (isovector and isoaxial-vector) charges n
- Poisson brackets $\{\phi, n\}$, $\{n, n\}$

see Yunxin Ye's talk

JR, Schlichting, von Smekal, Ye, in preparation

see also
Florio, Grossi, Soloviev, Teaney, PRD **105**, 054512 (2022)
Florio, Grossi, Teaney, arXiv:2306.06887

Dynamic universality classes

CRC-TR 211

Model A
$z = 2 + c\eta$

Statics: Landau-Ginzburg-Wilson functional

$$F = \int d^d x \,\left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) \right\}$$

equilibrium distribution:

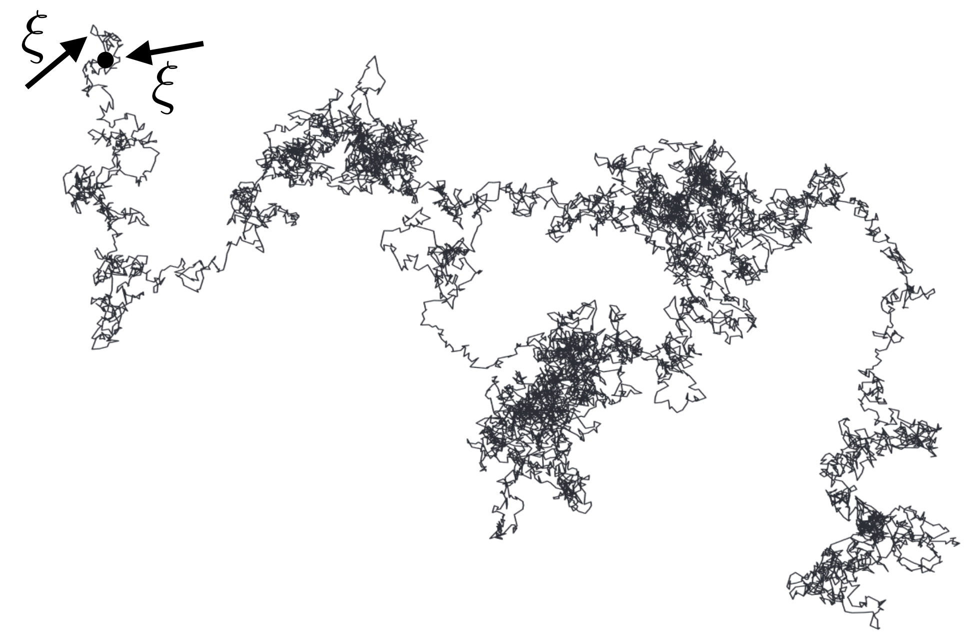
$$P[\varphi] \sim e^{-\beta F}$$

- Dynamics: Langevin equations of motion

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi$$

Gaussian white noise

describes particle
submerged in heat bath:



- No conservation laws here! ⇝ **Model A**
- **Slow modes** determine critical dynamics

(e.g. densities of conserved quantities) (generally true!)

Image adapted from P. Mörters, Y. Peres, *Brownian Motion*
(Cambridge University Press, 2010)

Dynamic universality classes

Model B
$z = 4 - \eta$

Statics: Landau-Ginzburg-Wilson functional

$$F = \int d^d x \, \left\{ \frac{1}{2} (\vec{\nabla}\varphi)^2 + V(\varphi) + B\varphi n + \frac{n^2}{2\chi_0} \right\}$$

equilibrium distribution:

$$P[\varphi, n] \sim e^{-\beta F}$$

- Dynamics: Langevin equations of motion

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi$$

Gaussian white noises

$$\partial_t n = \bar{\lambda} \vec{\nabla}^2 \frac{\delta F}{\delta n} + \vec{\nabla} \cdot \vec{\zeta}$$

diffusive!

- Critical dynamics dominated by diffusion ⤳ **Model B**
- Include hydrodynamic shear modes of energy-momentum tensor ⤳ **Model H**

Dynamic universality classes

Model C
$z = 2 + \alpha/\nu$

Statics: Landau-Ginzburg-Wilson functional

$$F = \int d^d x \, \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) \qquad + \frac{n^2}{2\chi_0} + \frac{g}{2}\varphi^2 n \right\}$$

equilibrium distribution:

$$P[\varphi, n] \sim e^{-\beta F}$$

- Dynamics: Langevin equations of motion

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi$$

Gaussian white noises

$$\partial_t n = \bar{\lambda} \vec{\nabla}^2 \frac{\delta F}{\delta n} + \vec{\nabla} \cdot \vec{\zeta}$$

diffusive!

- Order parameter not conserved but interacts non-linearly with conserved (energy) density ⇝ **Model C**

1PI vertex expansion around scale-dependent minimum $\phi_{0,k}$:

- effective average action:

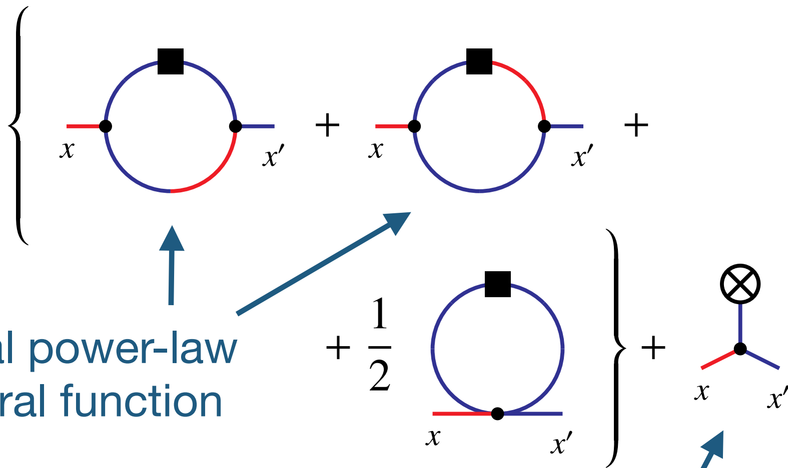
$$\Gamma_k = \frac{1}{2}\int_{xx'} \left(\phi^c - \phi^c_{0,k}, \phi^q\right)_x \begin{pmatrix} 0 & \Gamma^{cq}_k(x,x') \\ \Gamma^{qc}_k(x,x') & \Gamma^{qq}_k(x,x') \end{pmatrix} \begin{pmatrix} \phi^c - \phi^c_{0,k} \\ \phi^q \end{pmatrix}_{x'}$$
$$- \frac{\kappa_k}{\sqrt{8}} \int_x \left(\phi^c - \phi^c_{0,k}\right)^2 \phi^q - \frac{\lambda_k}{12}\int_x \left(\phi^c - \phi^c_{0,k}\right)^3 \phi^q$$

expand 2-point function in spatial gradients, but keep full frequency dependence:

$$\Gamma^{qc}_k(\omega, \boldsymbol{p}) = \Gamma^{qc}_{0,k}(\omega) - Z^{\perp}_k \boldsymbol{p}^2 + \cdots$$
$$\Gamma^{cq}_k(\omega, \boldsymbol{p}) = \Gamma^{cq}_{0,k}(\omega) - Z^{\perp}_k \boldsymbol{p}^2 + \cdots$$
$$\Gamma^{qq}_k(\omega, \boldsymbol{p}) = \frac{2T}{\omega}\left(\Gamma^{qc}_{0,k}(\omega) - \Gamma^{cq}_{0,k}(\omega)\right)$$

- flow of 2-point function:

$$\partial_k \Gamma^{qc}_k(x,x') = -i \Bigg\{$$

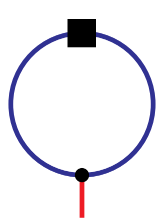


generate non-local power-law behavior in spectral function

'interaction' with scale-dependent minimum

- flow of effective potential:

$$\partial_k V'_k(\varphi) = -\frac{i}{\sqrt{8}}$$



use for squared mass and quartic coupling

- flow of couplings to density: (Model B)

vanish!
(coupling is linear ⤳ mixing)

for color coding and diagrammatic conventions, see
S. Huelsmann, S. Schlichting, P. Scior, Phys. Rev. D **102**, 096004 (2020)

1PI vertex expansion around $\phi = 0$:

- effective average action:

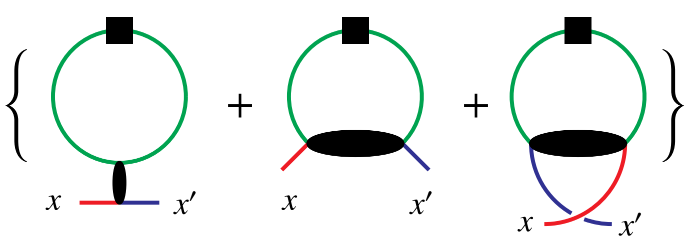
$$\Gamma_k = \frac{1}{2}\int_{xx'} (\phi^c, \phi^q)_x \begin{pmatrix} 0 & \Gamma_k^{cq}(x,x') \\ \Gamma_k^{qc}(x,x') & \Gamma_k^{qq}(x,x') \end{pmatrix} \begin{pmatrix} \phi^c \\ \phi^q \end{pmatrix}_{x'} + \frac{3\cdot 2^2}{4!}\int_{xx'} \phi^q(x)\phi^c(x) V_k^{an}(x,x')\phi^q(x')\phi^c(x') + \frac{3\cdot 2}{4!}\int_{xx'} \phi^q(x)\phi^c(x) V_k^{cl,R}(x,x')\phi^c(x')\phi^c(x') + \frac{3\cdot 2}{4!}\int_{xx'} \phi^c(x)\phi^c(x) V_k^{cl,A}(x,x')\phi^q(x')\phi^c(x')$$

expand 2- and 4-point functions in spatial gradients, but keep full frequency dependence:

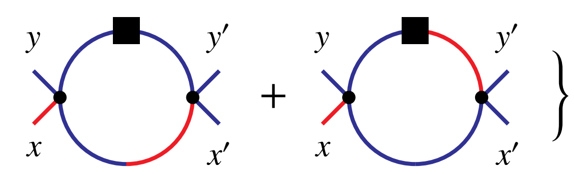
$$\Gamma_k^{qc}(\omega,\boldsymbol{p}) = \Gamma_{0,k}^{qc}(\omega) - Z_k^{\perp}\boldsymbol{p}^2 + \cdots$$
$$\Gamma_k^{cq}(\omega,\boldsymbol{p}) = \Gamma_{0,k}^{cq}(\omega) - Z_k^{\perp}\boldsymbol{p}^2 + \cdots$$
$$\Gamma_k^{qq}(\omega,\boldsymbol{p}) = \frac{2T}{\omega}\left(\Gamma_{0,k}^{qc}(\omega) - \Gamma_{0,k}^{cq}(\omega)\right)$$
$$V_k^{cl,A}(\omega,\boldsymbol{p}) = V_{0,k}^{cl,A}(\omega) + V_{1,k}^{cl,A}(0)\boldsymbol{p}^2 + \cdots$$
$$V_k^{cl,R}(\omega,\boldsymbol{p}) = V_{0,k}^{cl,R}(\omega) + V_{1,k}^{cl,R}(0)\boldsymbol{p}^2 + \cdots$$
$$V_k^{an}(\omega,\boldsymbol{p}) = \frac{2T}{\omega}\left(V_k^{cl,R}(\omega,\boldsymbol{p}) - V_k^{cl,A}(\omega,\boldsymbol{p})\right)$$

for the QM case, see
S. Huelsmann, S. Schlichting, P. Scior, Phys. Rev. D **102**, 096004 (2020)
JR, D. Schweitzer, L. J. Sieke, L. von Smekal, Phys. Rev. D **105**, 116017 (2022)

- flow of 2-point and 4-point functions:

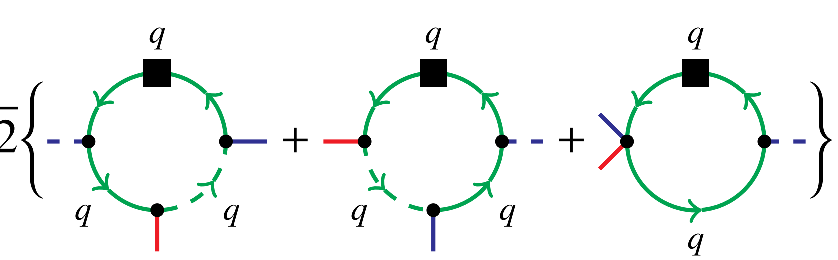
$$\partial_k \Gamma_k^{qc}(x,x') = -\frac{i}{2}\Big\{$$


$$\Big\}$$

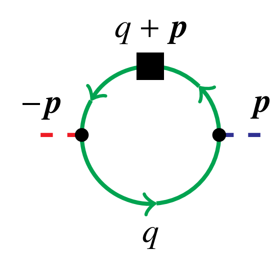
$$\partial_k V_k^{cl,R}(x,x') = -i\int_{\substack{x-y,\\ x'-y'}}\Big\{$$


$$\Big\}$$

- flow of couplings to density: (Model C)

$$\partial_k g_k = i\sqrt{2}\Big\{$$


$$\Big\}$$

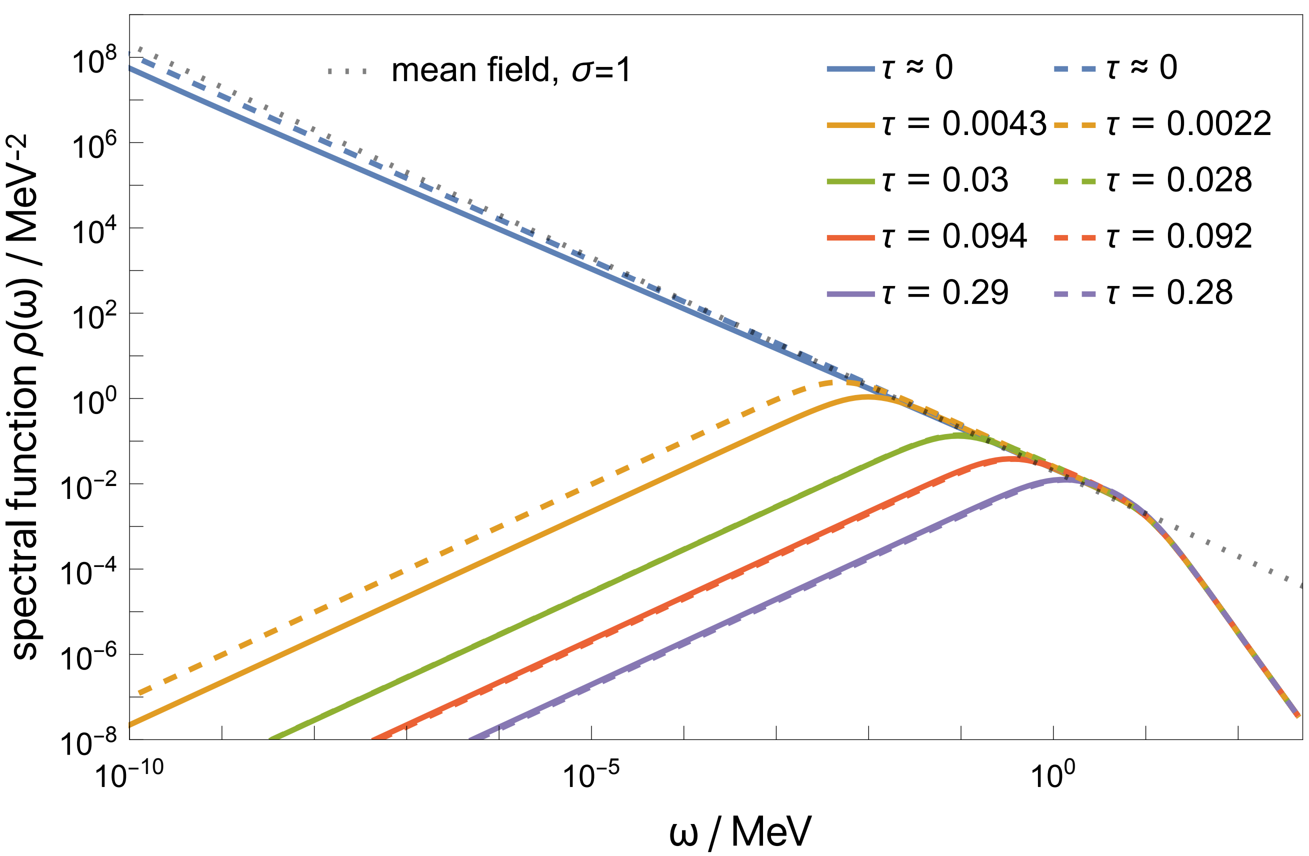
$$\partial_k \chi_{0,k}^{-1} = \frac{i}{\lambda}\lim_{\boldsymbol{p}\to 0}\frac{1}{\boldsymbol{p}^2}$$


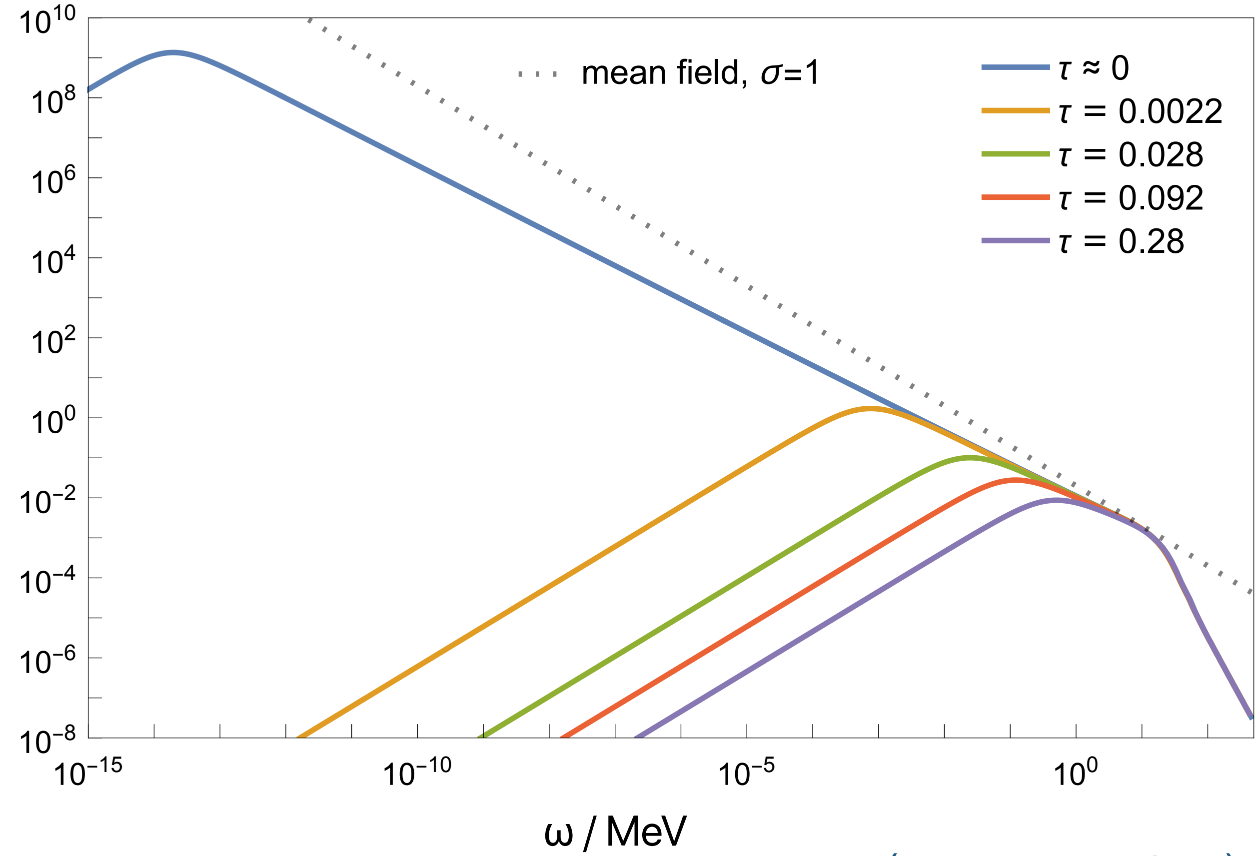
Spectral functions at criticality (Models A & C)

Model A
$z = 2 + c\eta$

$$\rho(\omega) \sim \omega^{-\sigma} \quad \text{with} \quad \sigma = \frac{2-\eta}{z}$$

Model C
$z = 2 + \alpha/\nu$





(zero momentum)

$d = 3$

$z \approx 2.042$ (dashed)

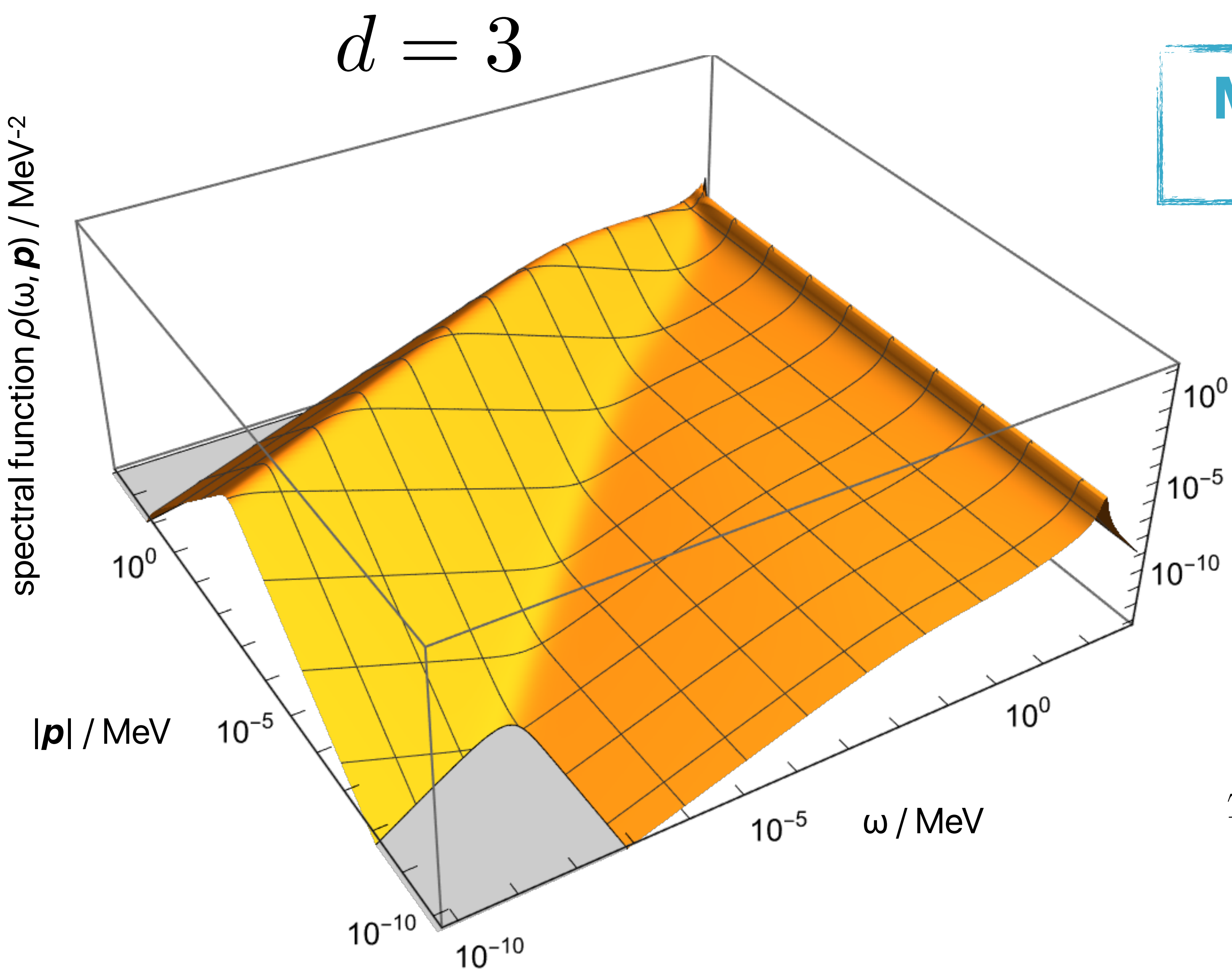
$z \approx 2.035$ (solid)

$z \approx 2.31$

[reduced temperature $\tau = (T - T_c)/T_c$]

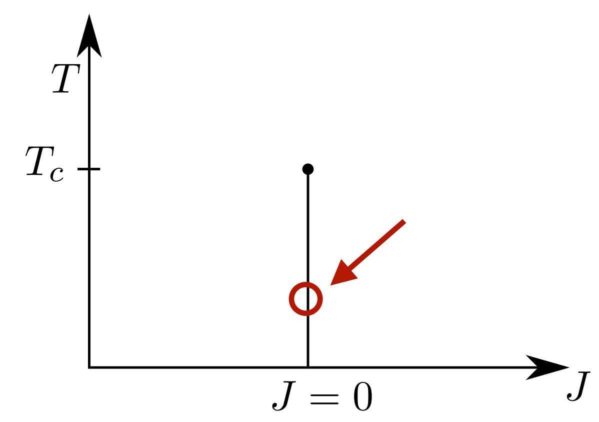
JR, L. von Smekal, arXiv:2303.11817

Spectral functions at criticality (Model B)



Model B
$z = 4 - \eta$

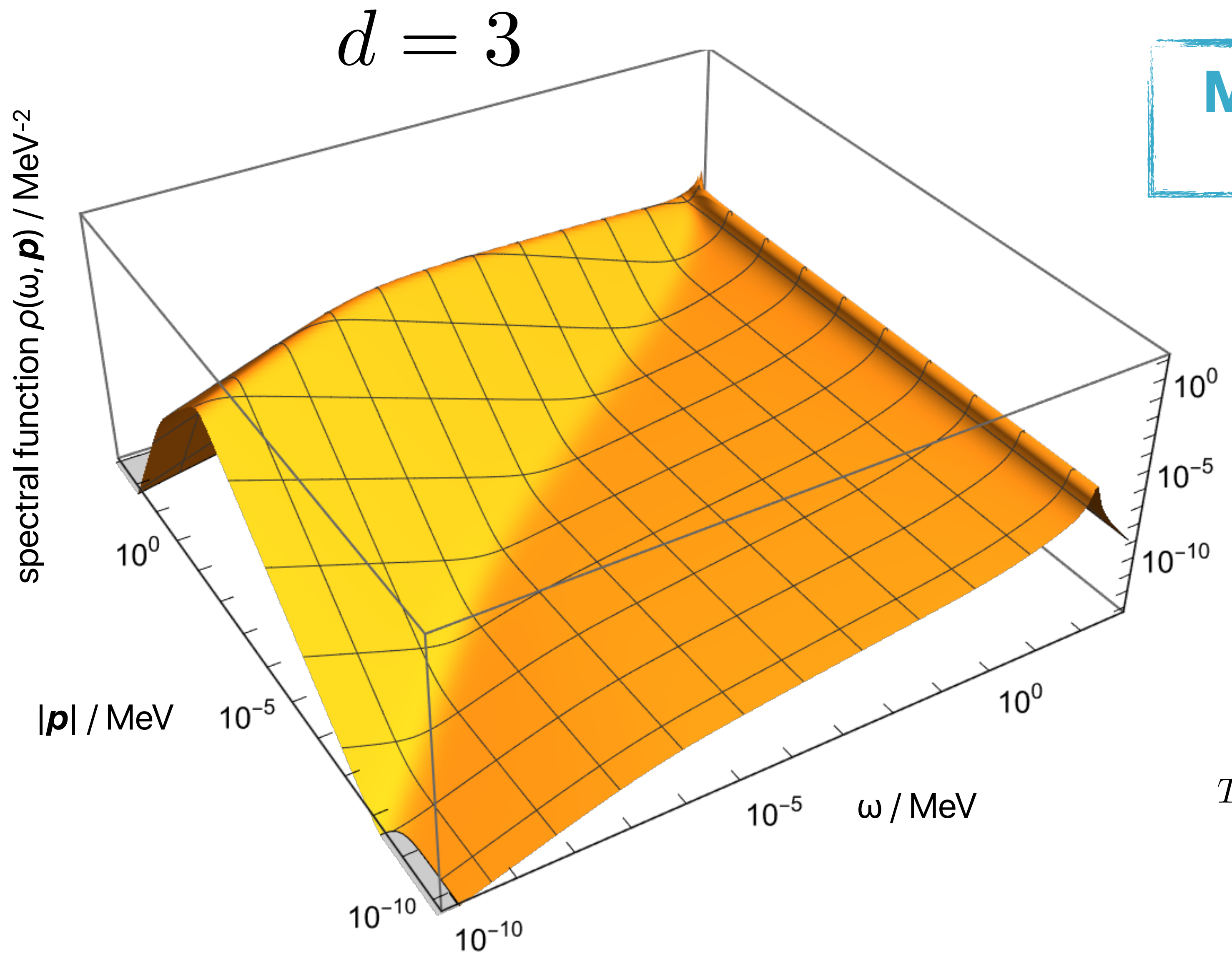
$\tau = -0.356$
(non-critical)



[reduced temperature $\tau = (T - T_c)/T_c$]

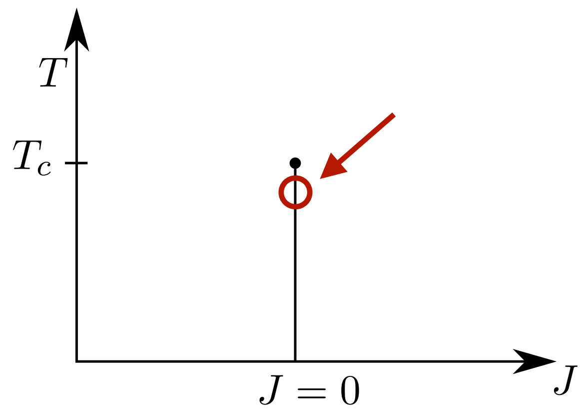
JR, L. von Smekal, arXiv:2303.11817

Spectral functions at criticality (Model B)



Model B
$z = 4 - \eta$

$\tau = -0.0021$
(near criticality)



[reduced temperature $\tau = (T - T_c)/T_c$]

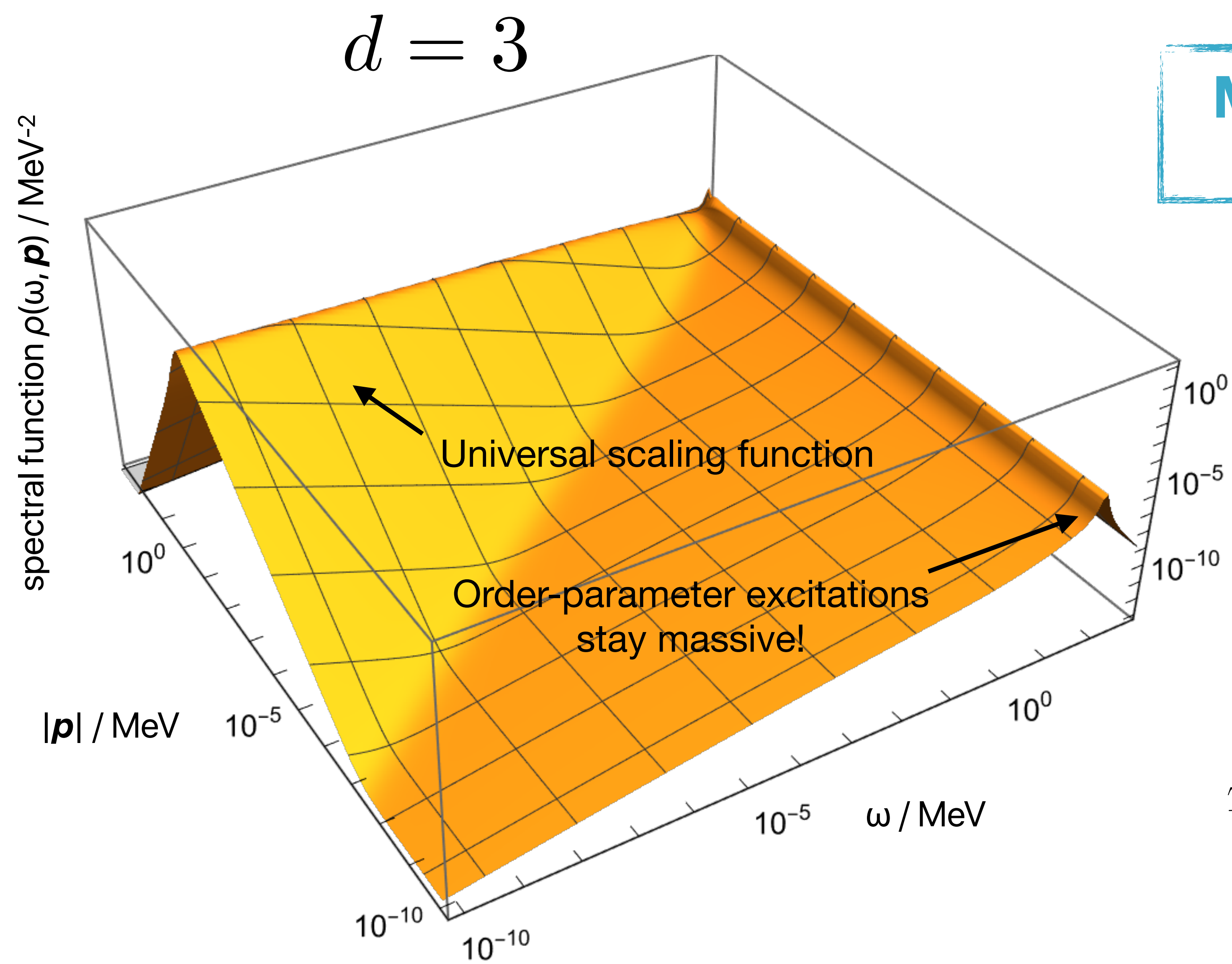
JR, L. von Smekal, arXiv:2303.11817

Spectral functions at criticality (Model B)

$$\omega \sim |\boldsymbol{p}|^z$$

$$z = 4$$

(here)

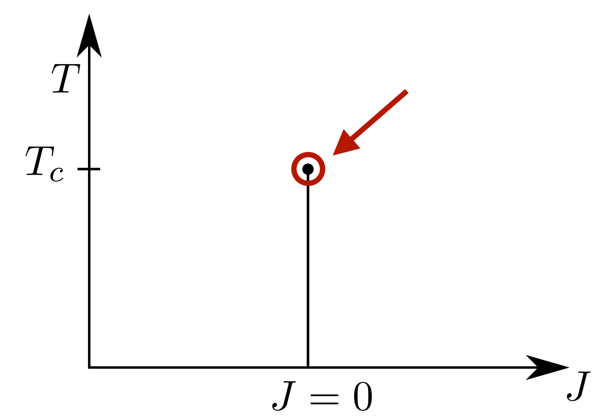


Model B

$z = 4 - \eta$

$\tau \approx 0$

(critical)



Universal scaling functions:
Model A, C: Schweitzer, Schlichting, von Smekal, Nucl. Phys. B **960**, 115165 (2020)
Model B, BC: Schweitzer, Schlichting, von Smekal, Nucl. Phys. B **984**, 115944 (2022)

[reduced temperature $\tau = (T - T_c)/T_c$]

JR, L. von Smekal, arXiv:2303.11817

Summary & Outlook

Summary:

- causal regulators for real-time FRG — JR, Schweitzer, Sieke, von Smekal, Phys. Rev. D **105**, 116017 (2022)
- critical spectral functions of **Models A, B, and C** — JR, von Smekal, arXiv:2303.11817

Outlook:

- dynamic critical exponent & scaling functions of **Model G** — JR, Schlichting, von Smekal, Ye, in preparation
- real-time dynamics of **Model H**
- new dynamic scaling functions
- non-equilibrium phase transitions (Kibble-Zurek scaling)

Thank you!

Backup

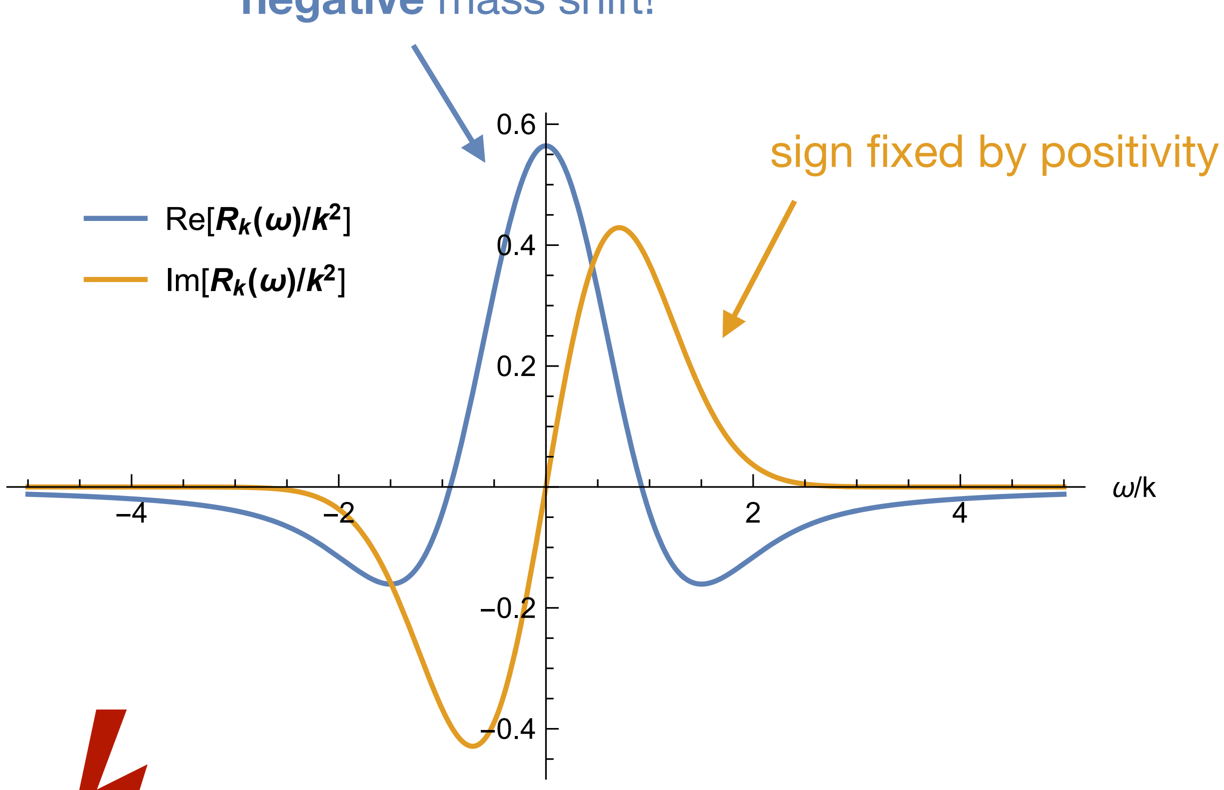
QM example for causal regulator

$$R_k^{R/A}(\omega) = R_k^{R/A}(0) - \int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega^2 J_k(\omega')}{\omega'((\omega \pm i\varepsilon)^2 - \omega'^2)} \quad \text{in} \quad {\Gamma_k^{(2)}}^R(\omega) = (\omega + i\varepsilon)^2 - m^2 + R_k^R(\omega)$$

- spectral density:

$$J_k(\omega) = 2k\omega\, e^{-\omega^2/k^2} = 2\,\mathrm{Im}\, R_k^R(\omega)$$

$\rightsquigarrow$ **Regulator** (retarded part):



- assume UV finiteness:

$$\Delta M_{UV}^2(k) = -R_k^{R/A}(0) + \underbrace{\int_0^\infty \frac{d\omega'}{\pi} \frac{J_k(\omega')}{\omega'}}_{\geq 0 \text{ (positivity)}} \stackrel{!}{=} 0$$

$\Rightarrow$ IR mass shift:

$$\Delta M_{IR}^2(k) = -R_k^{R/A}(0) < 0$$

is negative!

Solution: choose IR mass shift $\Delta M_{IR}^2(k) > 0$ positive (at cost of **UV finiteness**)