Three-particle decays on the lattice

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REGIONAL DOCTORAL PROGRAM IN THEORETICAL AND EXPERIMENTAL PARTICLE PHYSICS

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Plan

- Introduction: scale separation and the choice of the EFT
- The role of the relativistic invariance
- Lellouch-Lüscher formula for $K \to 3\pi$ decays
- Renormalization
- The role of the three-particle force
- Conclusions

Why three particles on the lattice?

- Three-pion decays of K, η, ω
- $a_1(1260) \to \rho\pi \to 3\pi$
- $a_1(1420) \to f_0(980)\pi \to 3\pi$
- Exotica: $T_{cc}(3875)^+ \to DD^* \to DD\pi$, ...
- Roper resonance: πN and $\pi\pi N$ final states
- Few-body physics: reactions with the light nuclei

Lattice vs. infinite volume: observables

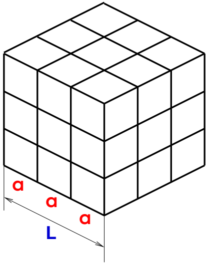
- Infinite volume:
  Three-particle bound states; Elastic scattering; Rearrangement reactions; Breakup; Three-particle resonances; Decay matrix elements (complex): e.g., $\langle \pi\pi\pi | H_W | K \rangle$
- Finite volume:
  Two- and three-particle energy levels; Matrix elements between eigenstates (real)

How does one connect these two sets? EFT serves as a bridge!

"Scattering" in a finite volume

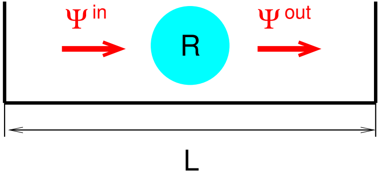
The scattering observables cannot be directly extracted from the amplitudes calculated on the lattice!

- (Periodic) boundary conditions imposed
- The spatial size of the box, L, is finite
- Assume the temporal size $L_t \gg L$, $L_t \to \infty$
- Three-momenta are quantized $\mathbf{p} = \frac{2\pi}{L}\mathbf{n}\,, \quad \mathbf{n} \in \mathbb{Z}^3$
- Discrete energy levels: $E_{n+1} - E_n = O(L^{-2})$



How does one extract the scattering observables:
phase shifts, cross sections, ... from the measured quantities
on the lattice?

EFT meets lattice



- When $R < L$, well-separated hadrons can be formed, QCD $\to$ ChPT
- When $R \ll L$, wave function has asymptotic form $\to$ scattering
- Since $p \sim 1/L$ and $R \sim 1/m$, then from $R \ll L$: follows $p \ll m$: non-relativistic EFT (NREFT)
  - Polarization effects, caused by creation/annihilation of the particles, are exponentially small and can be neglected

Scale separation: QCD (in a finite volume) $\Rightarrow$ EFT (in a finite volume)

Three-particle quantization condition

- Is the three-particle spectrum determined solely in terms of the S-matrix?

  K. Polejaeva and AR, 2012: Yes!

- Three different but equivalent formulations of the three-particle quantization condition are available
  - RFT (Relativistic Field Theory): Hansen & Sharpe, 2014
  - NREFT (Non-Relativistic Effective Field Theory): Hammer, Pang & AR, 2017
  - FVU (Finite-Volume Unitarity): Mai & Döring, 2017

- Enables one to extract scattering observables in the three-body sector from the measured finite-volume spectrum

Non-relativistic EFT: essentials

- Propagator:

$$\frac{1}{m^2 - p^2} = \underbrace{\frac{1}{2w(\mathbf{p})(w(\mathbf{p}) - p^0 - i\varepsilon)}}_{particle} + \underbrace{\frac{1}{2w(\mathbf{p})(w(\mathbf{p}) + p^0 - i\varepsilon)}}_{anti-particle}$$

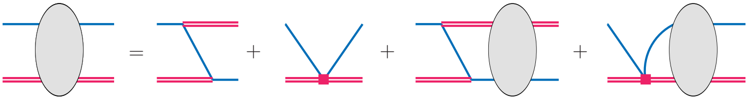
- The vertices in the Lagrangian conserve particle number:

$$\mathscr{L} = \phi^\dagger(i\partial_t - w)(2w)\phi + \underbrace{\frac{C_0}{4}\phi^\dagger\phi^\dagger\phi\phi + \frac{D_0}{36}\phi^\dagger\phi^\dagger\phi^\dagger\phi\phi\phi + \cdots}_{C_0, D_0 \text{ encode short-range physics}}$$

- Dimer: an alternative description of an infinite bubble sum; dummy field in the path integral

dimer :

The quantization condition (CM frame) (Hammer, Pang & AR, 2017)



Bethe-Salpeter equation

$$\mathcal{M}(\mathbf{p},\mathbf{q};E) = Z(\mathbf{p},\mathbf{q};E) + 8\pi \int^{\Lambda} \frac{d^3\mathbf{k}}{(2\pi)^3 2w(\mathbf{k})} Z(\mathbf{p},\mathbf{k};E)\tau(\mathbf{k};E)\mathcal{M}(\mathbf{k},\mathbf{q};E)$$

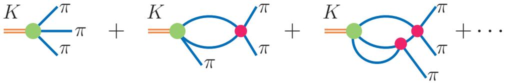
$$Z(\mathbf{p},\mathbf{q};E) = \frac{1}{2w(\mathbf{p}+\mathbf{q})(w(\mathbf{p})+w(\mathbf{q})+w(\mathbf{p}+\mathbf{q})-E)} + \tilde{H}_0 + \cdots$$

2-body amplitude: $4w(k^*)\tau^{-1}(\mathbf{k};E) = k^*\cot\delta(k^*) + \underbrace{\sqrt{\frac{s_2}{4} - m^2}}_{=k^*}$

Finite volume: $\mathbf{p} = \frac{2\pi}{L}\mathbf{n}$, $\mathbf{n} \in \mathbb{Z}^3$, poles of the scattering matrix $\Rightarrow$ spectrum

The 3-particle analog of the Lellouch-Lüscher formula (F. Müller & AR, 2020)

- Final-state interactions lead to an irregular L-dependence of the matrix element



- The non-relativistic Lagrangian

$$\begin{aligned}\mathscr{L} &= \phi^\dagger(i\partial_t - w)(2w)\phi + \frac{C_0}{4}\phi^\dagger\phi^\dagger\phi\phi + \cdots + K^\dagger(i\partial_t - w_K)(2w_K)K \\ &+ g(K^\dagger\phi\phi\phi + \text{h.c.}) + \cdots\end{aligned}$$

- Calculate the decay matrix element in a finite and in the infinite volume, extract g
- Matrix elements are related through

$$|\langle n|H_W|K\rangle_L| = \underbrace{\mathbb{L}_3(L)}_{\textit{depends on pion interactions}} |\langle\pi\pi\pi; out|H_W|K\rangle_\infty|$$

Relativistic invariance in the three-particle sector

- In the infinite volume, one resorts to the formalism that involves on-mass shell matrix elements only, in order to avoid ambiguities. The observables in a finite volume should be directly related to the infinite-volume S-matrix

  $\hookrightarrow$ Three-dimensional formalism, manifest Lorentz invariance is lost

  $\hookrightarrow$ Only Lorentz-invariant operators in the Lagrangian?

  $\hookrightarrow$ Proliferation of the independent couplings that should be extracted from lattice data in different moving frames

- In two-particle sector, the problem is solved by dim.reg.+threshold expansion. Sectors with more particles?

How does one interpret the relativistic invariance of the QC?

- A finite box breaks Lorentz/rotational invariance $\rightarrow$ only infinite volume
- Scalar particles, S-wave:

$$\text{Particle-dimer amplitude} \quad : \quad \mathscr{M}(P,p;Q,q) = \mathscr{M}(P',p';Q',q')$$

$$\text{Three-particle amplitude} \quad : \quad T(p_1,p_2,p_3;p_1,q_2,q_3) = T(p_1',p_2',p_3';p_1',q_2',q_3')$$

- Including spins and higher partial waves in the three-dimensional formalism accomplished albeit technically challenging
- Enables to describe the data taken in different moving frames by using the relativistic-invariant three-body force $\rightarrow$ less independent fitting parameters
- Methodology: write down the scattering equations in the manifestly Lorentz-invariant form

Explicitly Lorentz-invariant three-particle formalism

(F. Müller, J.-Y. Pang, AR and J.-J. Wu, JHEP 02 (2022) 158)

- Choose "quantization axis" in direction of an arbitrary unit vector v^μ, $v^2=1$
- The Lagrangian:

$$\mathscr{L}=\phi^\dagger(i(v\partial)-w_v)(2w_v)\phi+\sum_\ell \sigma_\ell T^\dagger_{\mu_1\cdots\mu_\ell}T^{\mu_1\cdots\mu_\ell}+\frac{1}{2}\left(\sum_\ell T^\dagger_{\mu_1\cdots\mu_\ell}O^{\mu_1\cdots\mu_\ell}+\text{h.c.}\right)+\cdots$$

- Here, $w_v=\sqrt{m^2+\partial^2-(v\partial)^2}$ and $O^{\mu_1\cdots\mu_\ell}$ denote the covariant operators, constructed out of two ϕ fields
- The propagator:

$$\langle 0|T\phi(x)\phi^\dagger(x)|0\rangle=\int\frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik(x-y)}}{2w_v(k)(w_v(k)-(vk)-i\varepsilon)}$$

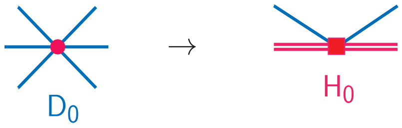
Particle-dimer picture

- Particle-dimer Lagrangian:

$$\mathscr{L} = \phi^\dagger(i(v\cdot\partial) - w_v)(2w_v)\phi + \sigma T^\dagger T + \Big(T^\dagger\big[\frac{f_0}{2}\phi\phi + \cdots\big] + \text{h.c.}\Big)$$

- Matching: $f_0, \ldots \leftrightarrow C_0, \ldots \leftrightarrow a, r, \ldots, \quad \sigma = \pm 1.$
- v^μ is a unit vector in the direction of the total four-momentum of the three-particle system

Particle-dimer picture in the three-particle sector

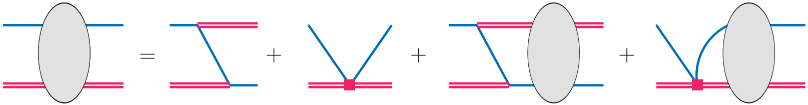


- The particle-dimer Lagrangian in the three-particle sector

$$\mathscr{L}_3 = h_0 T^\dagger T \phi^\dagger \phi + \cdots$$

- Matching: $h_0, \ldots \leftrightarrow D_0, \ldots$
- Terms with higher derivatives, higher dimer spin and orbital momentum should be added

The scattering equation in the infinite volume



Bethe-Salpeter equation

$$\mathcal{M}(p,q) = Z(p,q) + 8\pi \int \frac{d^3 k_\perp}{(2\pi)^3 2w_v(k)} \theta(\Lambda^2 + m^2 - (vk)^2) Z(p,k)\tau(K-k)\mathcal{M}(k,q)$$

$$\tau(P) = \frac{2\sqrt{P^2}}{k^* \cot\delta(k^*) - ik^*} \qquad k^* = \sqrt{\frac{P^2}{4} - m^2}$$

$$Z(p,q) = \frac{1}{2w_v(K-p-q)(w_v(p)+w_v(q)+w_v(K-p-q)-(vK)-i\varepsilon)} + \tilde{H}_0 + \cdots$$

Relativistic invariant QC in the three-body sector

(F. Müller, J.-Y. Pang, AR and J.-J. Wu, JHEP 02 (2022) 158)

$$\mathcal{M}_L(p,q) = Z(p,q) + \frac{8\pi}{L^3}\sum_{\mathbf{k}} \theta(\Lambda^2 + m^2 - (vk)^2) Z(p,k) \frac{\tau_L(K-k)}{2w(\mathbf{k})} \mathcal{M}_L(k,q)$$

$$\tau_L(P) = \frac{2\sqrt{K^2}}{k^* \cot\delta(k^*) - \frac{2}{\sqrt{\pi}L\gamma} Z_{00}^{\mathbf{P}}(1;q_0^2)}\,, \qquad q_0 = \frac{k^* L}{2\pi}$$

$$Z(p,q) = \frac{1}{2w_v(K-p-q)(w_v(p)+w_v(q)+w_v(K-p-q)-(vK)-i\varepsilon} + \tilde{H}_0 + \cdots$$

- Quantization condition:

$$\det \mathscr{A} = 0\,, \qquad \mathscr{A}_{pq} = L^3 2w(\mathbf{p})\delta^3_{\mathbf{pq}}(8\pi\tau_L(K-p))^{-1} - Z(p,q)$$

Quantization condition: essentials

- The solution for the scattering amplitude in the infinite volume is manifestly relativistic invariant:

$$\mathscr{M}(p,q;K;v)=\mathscr{M}(p',q';K';v')$$

- Relativistic invariance is achieved by expressing v^μ in terms of the external momenta

$$\text{a natural choice:}\quad v^\mu=K^\mu/\sqrt{K^2}$$

- Three-body amplitude expressed through particle-dimer amplitude
  $\to$ relativistic-invariant
- Analysis of lattice data: two-body interactions as an input: $k^*\cot\delta(k^*)$ fitted in the two-particle sector
- Extracting short-range quantities encoded in the three-body couplings $\tilde{H}_0,\ldots$
  – should be fitted to the three-particle energies

Relativistic invariant formalism for the three-particle decays

(F. Müller, J.-Y. Pang, AR and J.-J.Wu, JHEP 02 (2023) 214)

Applicable to: decays through the weak or electromagnetic interactions; isospin-breaking decays: pole on the real axis, example: $K \to 3\pi$

- The Lagrangian is manifestly relativistic invariant

$$
\begin{aligned}
\mathscr{L}_K &= K^\dagger (i(v\partial) - w_v^K)(2w_v^K) K \\
&+ \sqrt{4\pi} \sum_{\ell m} \frac{(-1)^\ell}{\sqrt{2\ell+1}} \left(K^\dagger G_\ell(\Delta_T) \left((\mathscr{Y}_{\ell,-m}(\underline{w}))^* \phi\right) T_{\ell m} + \text{h.c.}\right)
\end{aligned}
$$

- The effective couplings

$$
G_\ell(\Delta_T) = G_\ell^{(0)} + G_\ell^{(1)} \Delta_T + \cdots
$$

The 3-particle LL factor

The LL formula relates matrix elements in a finite and infinite volume

$$
\begin{aligned}
L_\alpha^{3/2}\langle n|J_K^\dagger(0)|0\rangle &= \sum_{\ell,i} A_\ell^{(i)}(K_\alpha, L_\alpha)\,{\color{red}G_\ell^{(i)}} \\
\langle \pi(k_1)\pi(k_2)\pi(k_3)|J_K^\dagger(0)|0\rangle &= \sum_{\ell,i} X_\ell^{(i)}(K)\,{\color{red}G_\ell^{(i)}}
\end{aligned}
$$

$$
{\color{red}\hookrightarrow} \quad \langle \pi(k_1)\pi(k_2)\pi(k_3); out|H_W|K\rangle_\infty = \mathbb{L}_3(\{k\})L^{3/2}\langle n|H_W|K\rangle_L
$$

- The factor $\mathbb{L}_3(\{k\})$ depends on the $\pi\pi$, $\pi\pi\pi$ interactions and on L, but not on the couplings that describe the short-range part of the $K \to 3\pi$ amplitude!
- The derivative couplings emerge at higher orders; decay amplitudes into different final states mix. The three-particle LL factor becomes a matrix

LL factor in the $K \to 3\pi$ decays

(J.-Y. Pang, R. Bubna, F. Müller, AR and J.-J. Wu, arXiv:2312.04391)

Objectives:

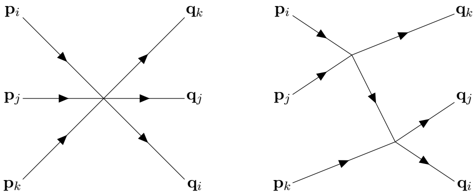
- Provide a numerical implementation of the analytic expressions (lowest order in the EFT expansion, only S-wave). Include isospin channels explicitly
- Discuss renormalization and cutoff-independence of the LL factor
- Apart from the two-body scattering parameters, the LL factor depends on the three-body force, which should be determined prior to calculating the matrix element. Is this dependence significant numerically?

Matching of the parameters of the Lagrangian

- Two-body sector: S-wave scattering lengths a_0, a_2
- Two independent three-body couplings without derivatives:

$$\mathscr{L}_3 = D_1(\pi^\dagger\pi)^3 + D_2(\pi^\dagger\pi^\dagger)(\pi^\dagger\pi)(\pi\pi)$$

- Tree-level matching to ChPT:



- Two independent $K \to 3\pi$ couplings

$$\mathscr{L}_K = G_1\big(K_+^\dagger \pi_0\pi_0\pi_+ + \text{h.c.}\big) + G_2\big(K_+^\dagger \pi_+\pi_+\pi_- + \text{h.c.}\big)$$

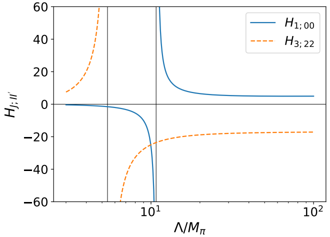
- At lowest order in EFT expansion, $G_2 = -2G_1 + O(G_F^2)$

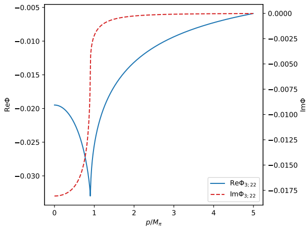
Renormalization and cutoff independence

- 5 independent particle-dimer couplings but only 2 independent three-particle couplings at lowest order. Are all observables cutoff-independent after matching three-particle amplitudes at threshold?

$\rightarrow$ Matching renders all observables cutoff-independent. Unphysical quantities, in general, are cutoff-dependent; The running of the particle-dimer coupling constants is irregular (reminiscent of the log-periodic behavior)

- The Faddeev eq. for the particle-dimer w.f. is solved, using contour deformation





LL factor for the $K \to 3\pi$ decays

$$\begin{pmatrix} \langle \pi^+(p_1)\pi^+(p_2)\pi^-(p_3)|K^+\rangle \\ \langle \pi^0(p_1)\pi^0(p_2)\pi^+(p_3)|K^+\rangle \end{pmatrix} = \begin{pmatrix} X_{c0} & X_{c2} \\ X_{n0} & X_{n2} \end{pmatrix} \begin{pmatrix} g^{(1,0)} \\ g^{(3,2)} \end{pmatrix}$$

Infinite-volume particle-dimer scattering wave functions $\to X_{\alpha\beta}$

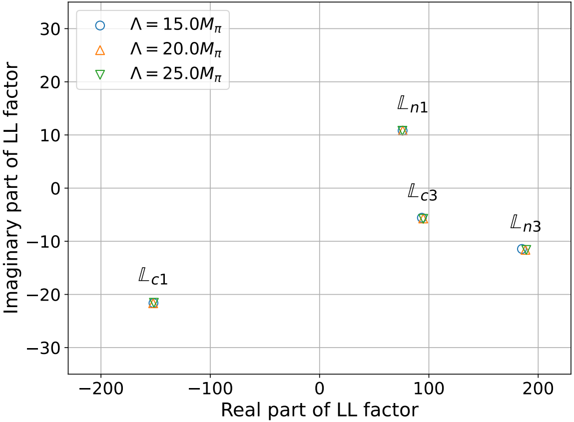
$$\begin{pmatrix} L^{3/2}\langle \Gamma n, 1|K^+\rangle \\ L^{3/2}\langle \Gamma n, 3|K^+\rangle \end{pmatrix} = \begin{pmatrix} A_{10} & A_{12} \\ A_{30} & A_{32} \end{pmatrix} \begin{pmatrix} g^{(1,0)} \\ g^{(3,2)} \end{pmatrix}$$

Finite-volume particle-dimer wave functions $\to A_{ij}$

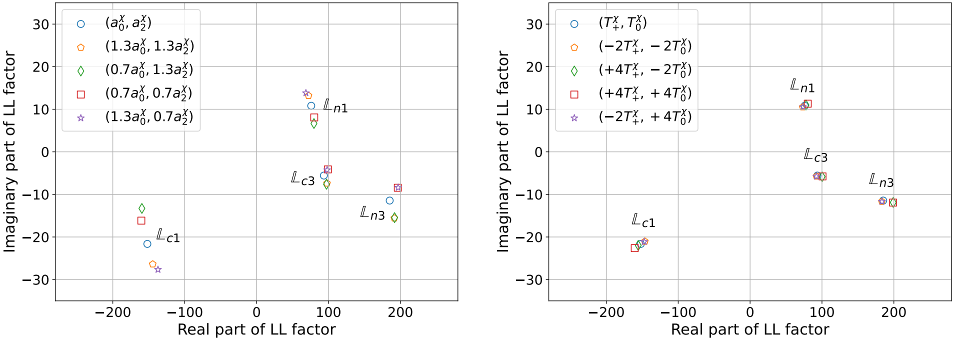
$$\begin{pmatrix} \langle \pi^+\pi^+\pi^-|K^+\rangle \\ \langle \pi^0\pi^0\pi^+|K^+\rangle \end{pmatrix} = \begin{pmatrix} \mathbb{L}_{c1} & \mathbb{L}_{c3} \\ \mathbb{L}_{n1} & \mathbb{L}_{n3} \end{pmatrix} \begin{pmatrix} L^{3/2}\langle \Gamma n, 1|K^+\rangle \\ L^{3/2}\langle \Gamma n, 3|K^+\rangle \end{pmatrix},$$

$$\begin{pmatrix} \mathbb{L}_{c1} & \mathbb{L}_{c3} \\ \mathbb{L}_{n1} & \mathbb{L}_{n3} \end{pmatrix} = \begin{pmatrix} X_{c0} & X_{c2} \\ X_{n0} & X_{n2} \end{pmatrix} \begin{pmatrix} A_{10} & A_{12} \\ A_{30} & A_{32} \end{pmatrix}^{-1}$$

Cutoff independence of the LL factor



Sensitivity towards scattering lengths vs three-body force



- 2×2 LL factor, corresponding to $K^+ \to \pi^+\pi^+\pi^-$ and $K^+ \to \pi^0\pi^0\pi^+$ decays
- Sensitive to the values of a_0, a_2, very little dependence on the three-body force!

Conclusions

- In the analysis of lattice data, EFT can be used to systematically relate the finite- and infinite-volume observables. This facilitates the extraction of scattering observables from lattice data
- The crucial point: decoupling of short- and long-range physics
- The quantization condition and an analog of the LL formula in the three-particle sector is derived in a manifestly Lorentz-invariant form
- Decays of K^+ into three pions: everything ready!
  - LL factor worked out explicitly to lowest order; all isospin channels are included
  - Renormalization is addressed in detail. Number of independent particle-dimer three-body couplings exceeds the number of independent three-pion amplitudes. However, matching renders all observables cutoff-independent
  - Very little dependence on the three-body input! At the first stage, matching to the ChPT amplitude suffices (power counting, away from singular values of couplings)

Spares

Higher partial waves, derivative couplings

- The dimer field with an arbitrary (integer) spin

$$T_{\ell m}=\sum_{\mu_i,\nu_i}(c^{-1})^{\ell m}_{\mu_1\cdots\mu_\ell}\underline{\Lambda}^{\mu_1}_{\nu_1}\cdots\underline{\Lambda}^{\mu_\ell}_{\nu_\ell}T^{\nu_1\cdots\nu_\ell}\,,\qquad \underline{\Lambda}^{\mu}_{\nu}v^{\nu}=v^{\mu}_{0}=(1,\mathbf{0})$$

- (Symmetric) dimer field obeys the constraints

$$v_{\mu_i}T^{\mu_1\cdots\mu_\ell}=0,\qquad T^{\mu_1\cdots\mu_i\cdots\mu_\ell}_{\mu_i}=0$$

- Interaction of a dimer with two particles

$$\mathscr{L}_2=\sum_{\ell m}\sigma_\ell T^\dagger_{\ell m}T_{\ell m}+\sum_{\ell m}(T^\dagger_{\ell m}O_{\ell m}+\text{h.c.})$$

Two-particle vertices

- Generalization of the on-shell three-momentum to moving frames:

$$\bar{w}_\perp^\mu = \bar{w}^\mu - v^\mu(v\bar{w})\,, \qquad \bar{w}^\mu = \Lambda^\mu_\nu w^\nu\,, \qquad w^\mu = v^\mu w_v + i(\partial^\mu - v^\mu(v\partial))$$

- The boost Λ renders the total momentum of the pair P^μ parallel to v^μ

$$\Lambda^\mu_\nu(v,u)u^\mu = v^\mu\,, \qquad u^\mu = \frac{P^\mu}{\sqrt{P^2}}\,, \quad P^\mu = \tilde{p}_1^\mu + \tilde{p}_2^\mu \text{ (on-shell)}$$

- The vertices:

$$\begin{aligned} O &= \frac{f_0^{(0)}}{2}\phi^2 + \frac{f_0^{(2)}}{4}(\phi\bar{w}_\perp^\mu\bar{w}_{\perp\mu}\phi - \bar{w}_\perp^\mu\phi\bar{w}_{\perp\mu}\phi) + \cdots \\ O^{\mu\nu} &= \frac{f_2^0}{2}\Big(3(\phi\bar{w}_\perp^\mu\bar{w}_\perp^\mu\phi - \bar{w}_\perp^\mu\phi\bar{w}_\perp^\nu\phi) - (g^{\mu\nu} - v^\mu v^\nu)(\phi\bar{w}_\perp^\lambda\bar{w}_{\perp\lambda}\phi - \bar{w}_\perp^\lambda\phi\bar{w}_{\perp\lambda}\phi)\Big) + \cdots \end{aligned}$$

...and so on

Three-particle force

$$\mathscr{L}_3 = \sum_{\ell m, \ell' m'} \sum_{LL'JM} T^{\dagger}_{\ell' m'} \left(\mathscr{Y}^{JM}_{L'\ell'}(\underline{w}, m')\phi^{\dagger}\right) T^{\ell'\ell}_{JL'L}(\Delta, \vec{\Delta}_T, \overleftarrow{\Delta}_T) \left(\left(\mathscr{Y}^{JM}_{L\ell}(\underline{w}, m)\right)^{*}\phi\right) T_{\ell m}$$

$$\mathscr{Y}^{JM}_{L\ell}(\mathbf{k}, m) = \langle L(M-m), \ell m | JM\rangle \mathscr{Y}_{L(M-m)}(\mathbf{k}), \quad \underline{w}^{\mu} = \underline{\Lambda}^{\mu}_{\nu} w^{\nu}$$

- The three-body force is parameterized by effective couplings

$$T^{\ell'\ell}_{JL'L}(\Delta, \vec{\Delta}_T, \overleftarrow{\Delta}_T) = h_0 + h_1\Delta + h_2(\vec{\Delta}_T + \overleftarrow{\Delta}_T) + \cdots$$

$$\underbrace{\Delta = K^2 - (3m)^2}_{\text{3-body system}}, \qquad \underbrace{\Delta_T = P^2 - (2m)^2}_{\text{2-body subsystem}}$$

- Number of independent couplings depends on the detailed dynamics of the system!

The Bethe-Salpeter equation

$$\mathscr{M}_{\ell' m', \ell m} = Z_{\ell' m', \ell m} + \sum_{\ell''} Z_{\ell' m', \ell'' m''} S_{\ell''} \mathscr{M}_{\ell'' m'', \ell m}$$

- The two-body propagator

$$S_\ell(s) = -\frac{1}{\sigma_\ell - f_\ell^2(s)\frac{1}{2}\, p^{2\ell}(s) I(s)}$$

- The driving term

$$Z_{\ell' m', \ell'' m''}(p, q) = \frac{4\pi \, (\mathscr{Y}_{\ell' m'}(\tilde{\mathbf{p}}))^* \, f_{\ell'}(s_p)\, f_\ell(s_q) \mathscr{Y}_{\ell m}(\tilde{\mathbf{q}})}{2 w_v (K - p - q)(w_v(p) + w_v(q) + w_v(K - p - q) - vK - i\varepsilon)}$$

$$+ \; 4\pi \sum_{LL'} \sum_{JM} \mathscr{Y}_{JM}^{L'\ell'}(\underline{\mathbf{p}}, m')\, T_{JL'L}^{\ell'\ell}(\Delta, \Delta_p, \Delta_q) \left(\mathscr{Y}_{JM}^{L\ell}(\underline{\mathbf{q}}, m)\right)^*$$

Relativistic invariance of the framework

- Two types of momenta: $\tilde{p} = \underline{\Lambda}(v)\Lambda(v,u)p$ and $\underline{p} = \underline{\Lambda}(v)p$
- Wigner-Thomas rotation

$$\underline{\Lambda}(v_\Omega) = R\underline{\Lambda}(v)\Omega^{-1}\,, \qquad R = R(\Omega, v)$$

- Lorentz-transformation of the momenta

$$\underline{p} \to \underline{\Lambda}(v_\Omega)p_\Omega = R\underline{\Lambda}(v)\Omega^{-1}\Omega p = R\underline{p}\,, \qquad \tilde{p} \to R\tilde{p}$$

- Lorentz-transformation of the kernel

$$Z_{\ell' m',\ell m}(\Omega p, \Omega q, \Omega K) = \sum_{m''' m''} \mathscr{D}^{(\ell')}_{m' m'''}(R) Z_{\ell' m''',\ell m''}(p,q,K) \left(\mathscr{D}^{(\ell)}_{m'' m}(R)\right)^*$$

$$\to \quad \mathscr{M}_{\ell' m',\ell m}(\Omega p, \Omega q, \Omega K) = \sum_{m''' m''} \mathscr{D}^{(\ell')}_{m' m'''}(R) \mathscr{M}_{\ell' m''',\ell m''}(p,q,K) \left(\mathscr{D}^{(\ell)}_{m'' m}(R)\right)^* \quad \checkmark$$

Relativistic invariant three-body QC

$$\det\left(\mathscr{A}\right)=0$$

$$\mathscr{A}_{\ell' m',\ell m}(p,q)=2w(\mathbf{p})\delta_{\mathbf{pq}}\left(S^L_{\ell' m',\ell m}(K-p)\right)^{-1}-\frac{1}{L^3}Z_{\ell' m',\ell m}(p,q)$$

- Even in a finite volume, dimer propagator S^L does not depend on v^μ
- Projection on the irreps of the cubic group and its subgroups can be done in a standard manner
- Meaning of the relativistic invariance in a finite volume: Parameterizing the three-body force in a Lorentz-invariant manner and fitting it to data in different frames, the finite-volume corrections to the extracted effective couplings will be exponentially suppressed.