

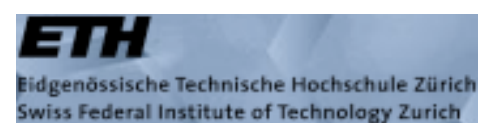
Measurements of the proton charge and magnetic radii with muonic hydrogen: the mystery deepens!

Paul Indelicato



CREMA: Muonic Hydrogen Collaboration

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 P.E. Knowles, F. Kottmann, J.A.M. Lopes, E. Le Bigot, Y.-W. Liu, L. Ludhova,
 C.M.B. Monteiro, F. Mulhauser, T. Nebel, F. Nez, R. Pohl, P. Rabinowitz,
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EMMI days 2012

<http://muhy.web.psi.ch>

<http://www.lkb.ens.fr/-Metrology-of-simple-systems-and->

Form factor

A rapid definition

2 quarks **up** ($2/3 e$) + 1 quark **down** ($-1/3 e$) + strong interaction (**gluons**)

Vertex EM interaction: Dirac and Pauli Form factors

(S, P : spin and 4-momentum of nucleon, f : quark flavor)

$$\langle P', S' | V_{(f)}^\mu | P, S \rangle = \bar{U}(P', S') \left[\gamma^\mu F_1^{(f)}(Q^2) + i\sigma^{\mu\nu} \frac{q_\nu}{2M_N} F_2^{(f)}(Q^2) \right] U(P, S),$$

$$V_{(f)}^\mu = \bar{\psi}_{(f)} \gamma^\mu \psi_{(f)},$$

Physical charge density are derived from the Sachs Form factors

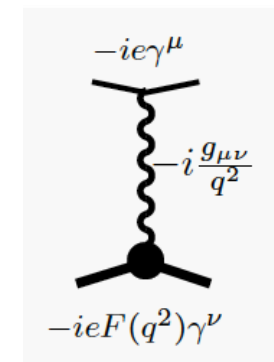
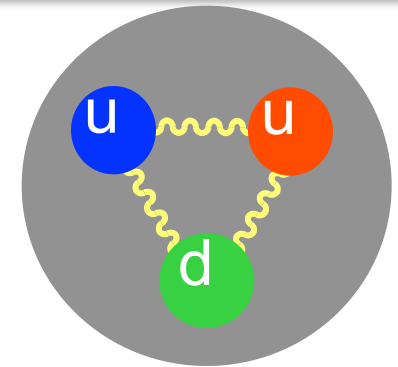
$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{(2M_N)^2} F_2(Q^2),$$

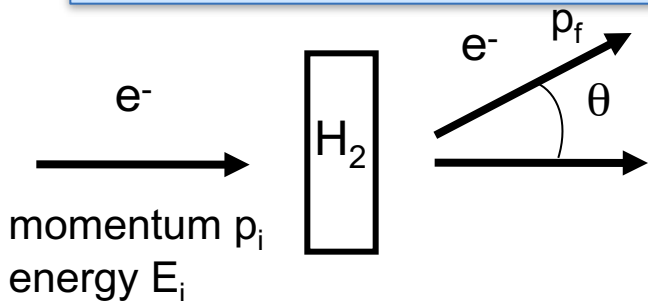
$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2).$$

Measure the moments of the charge distribution:

$$G_N(q^2) = \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \frac{\rho_N(\mathbf{r})}{4\pi},$$

$$\langle r^n \rangle = \int_0^\infty r^{2+n} \rho(r) dr,$$





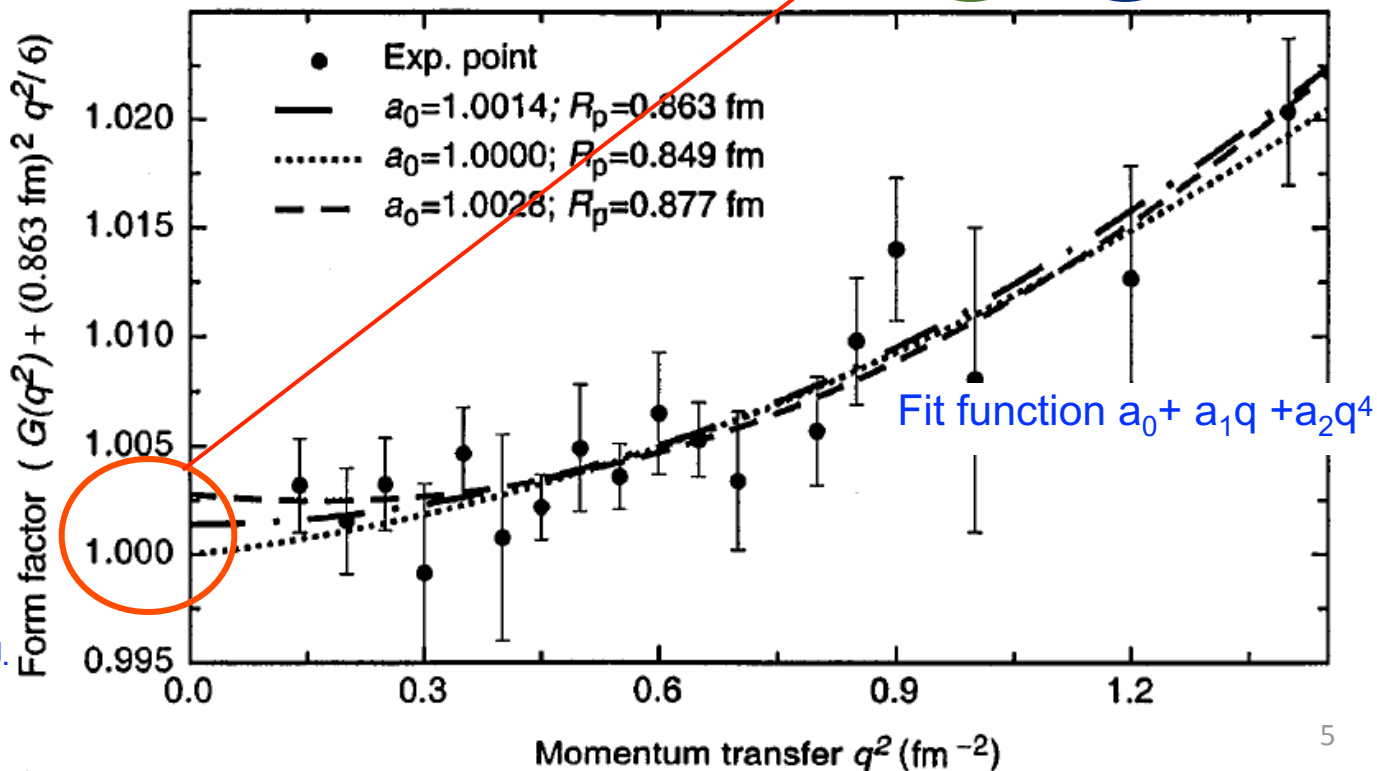
$$q = 2p_f \sin\left(\frac{\theta}{2}\right)$$

$$\vec{q} = \vec{p}_f - \vec{p}_i$$

$$G_N(q^2) = \frac{1}{\left(1 + \frac{R^2 q^2}{12}\right)^2} \approx 1 - \frac{R^2}{6} q^2 + \frac{R^4}{48} q^4 + \dots$$

$$\frac{d\sigma(E_i, \theta)}{d\omega} = \frac{d\sigma_{\text{Rut.}}(E_i, \theta)}{d\omega} G_E(q^2)$$

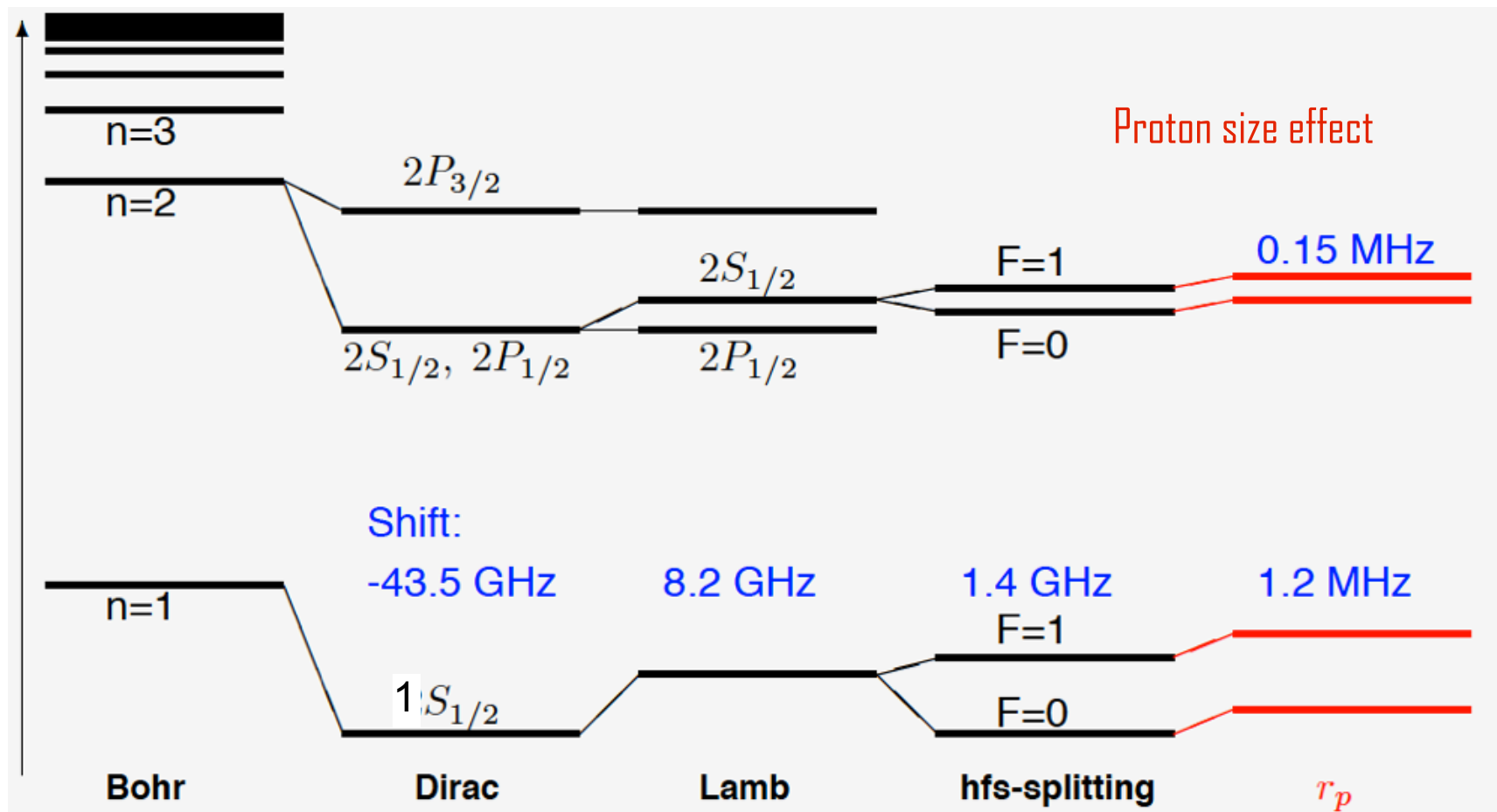
$$G_N(q^2) = e^{-\frac{1}{6} R^2 q^2} \approx 1 - \frac{R^2}{6} q^2 + \frac{R^4}{72} q^4 + \dots$$

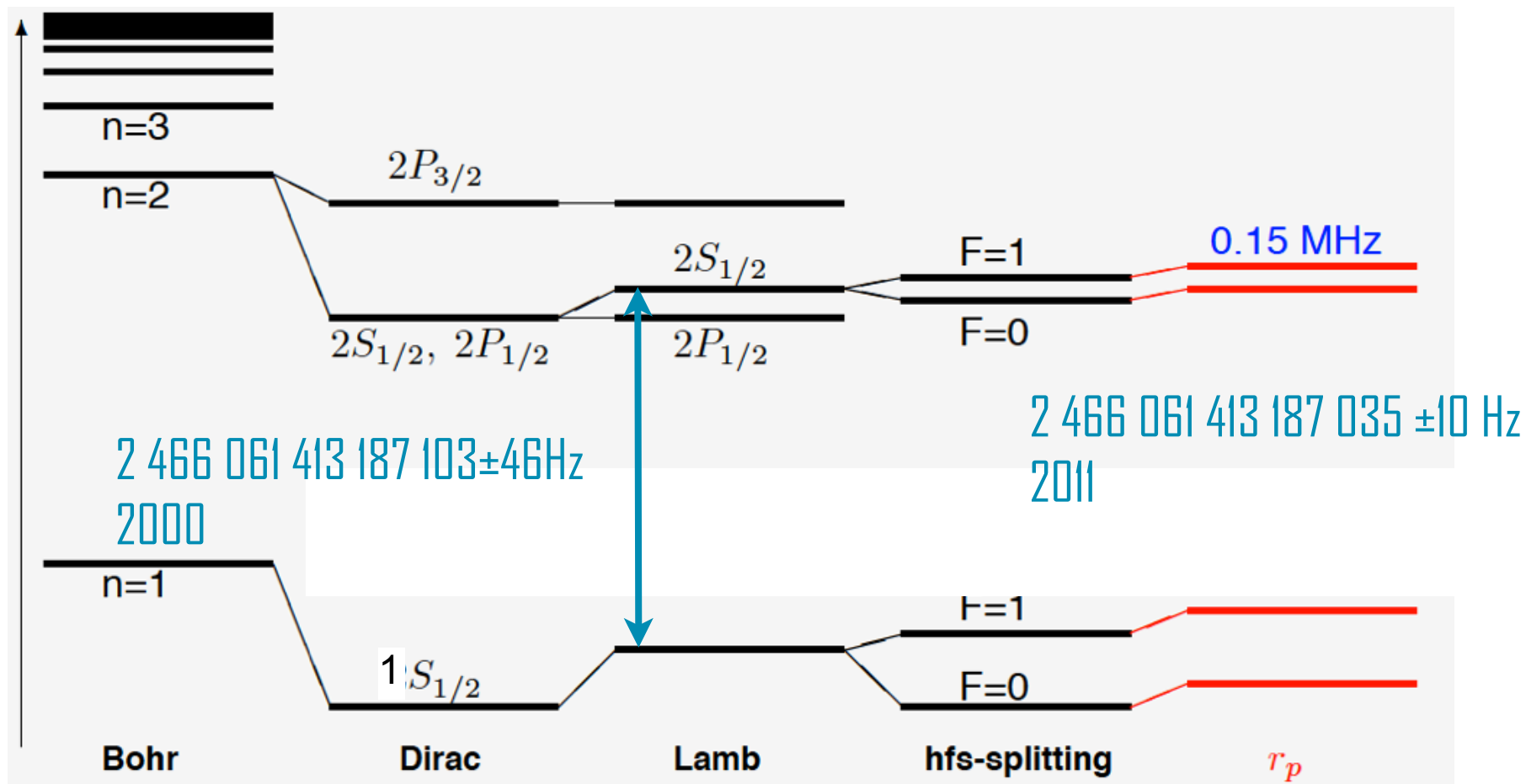


see S. Karshenboim in Can. J. Phys. 77, 241-266 (1999) and refs therein

Metrology in hydrogen

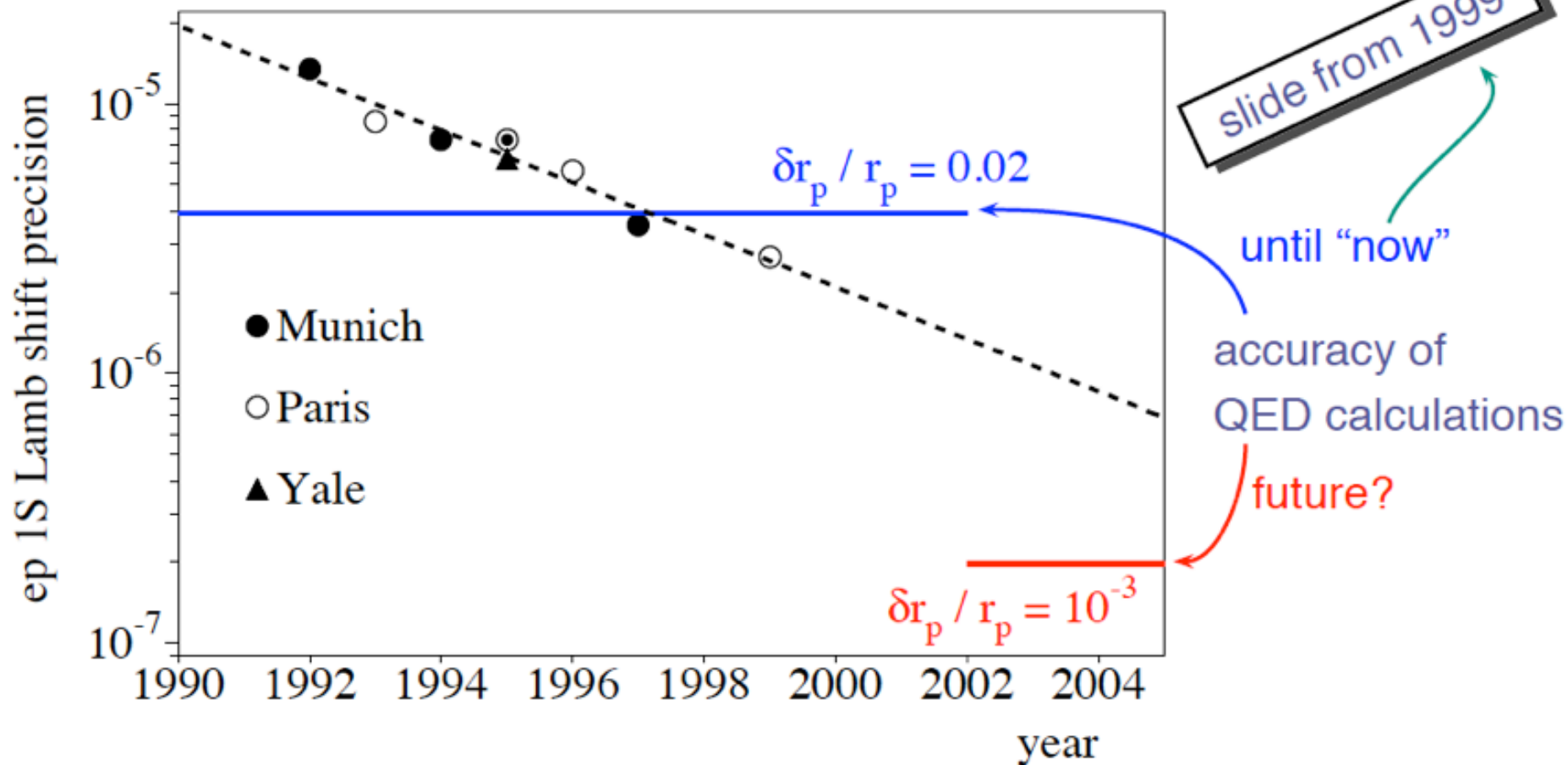
Highest precision experiments





Why re-measure the proton charge radius?

1S Lamb shift in hydrogen: $L_{1S}(r_p) = 8171.636(4) + 1.5645 \langle r_p^2 \rangle$ MHz

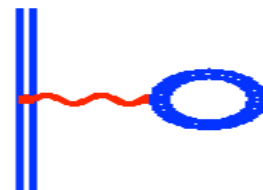


QED-test is limited by the uncertainty of the **proton rms charge radius**.

Hydrogen

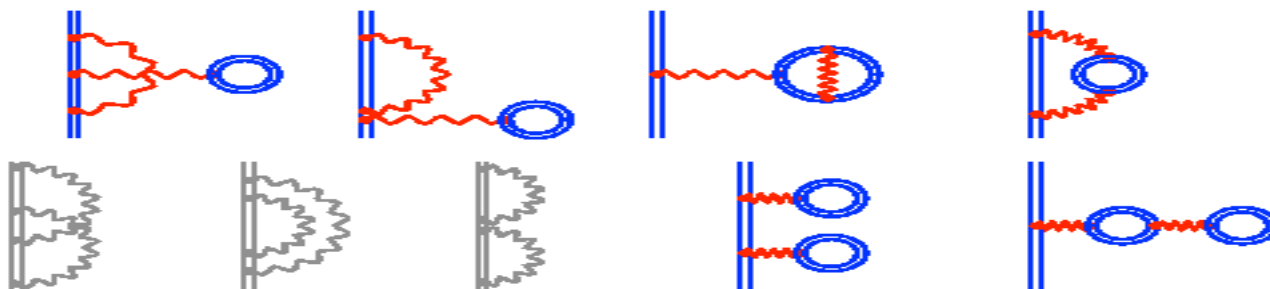
QED corrections

Self Energy



Vacuum Polarization

H-like "One Photon" order α

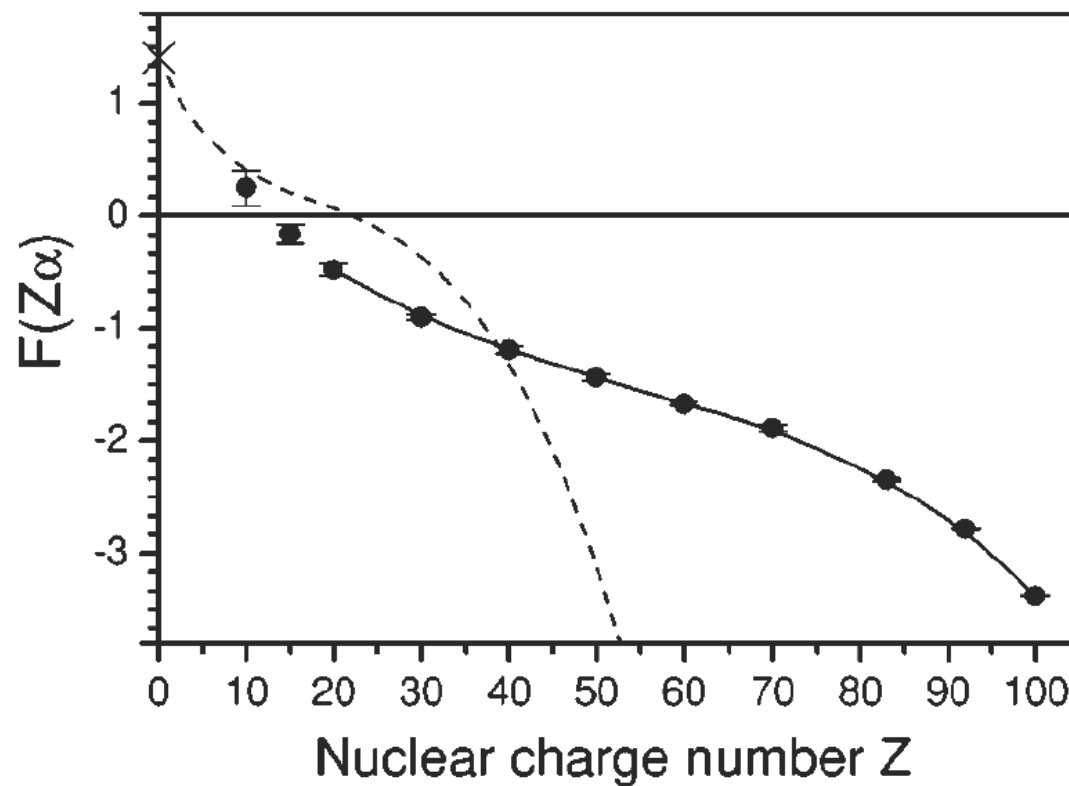


H-like "Two Photon" order α^2



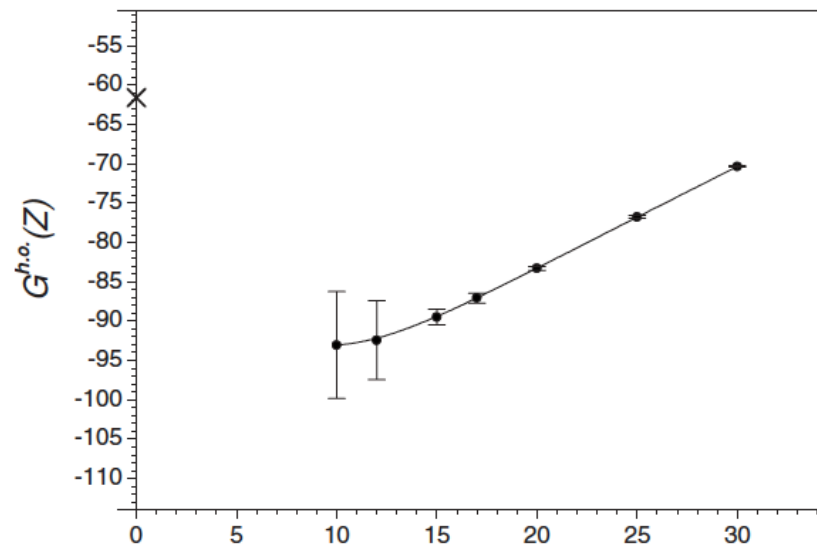
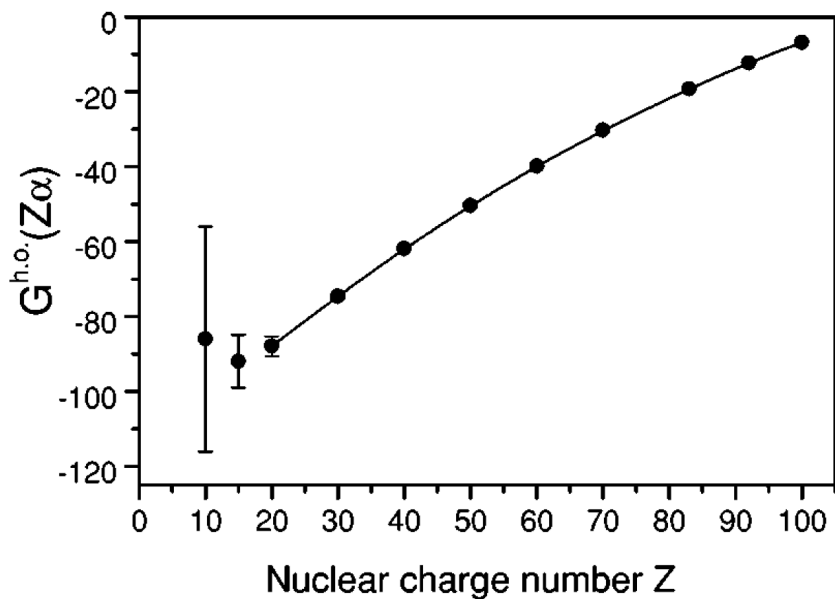
$Z\alpha$ expansion; replace exact Coulomb propagator by expansion in number of interactions with the nucleus

$$\Delta E = m \left(\frac{\alpha}{\pi} \right)^2 \frac{(Z\alpha)^4}{n^3} F(Z\alpha)$$



V. A. Yerokhin, P. Indelicato, and V. M. Shabaev, Phys. rev. A 71, 040101(R) (2005).

$$\Delta E_{\text{SESE}} = m \left(\frac{\alpha}{\pi} \right)^2 (Z\alpha)^4 \{ B_{40} + (Z\alpha) B_{50} + (Z\alpha)^2 \\ \times [L^3 B_{63} + L^2 B_{62} + L B_{61} + G_{\text{SESE}}^{\text{h.o.}}(Z)] \},$$



V. A. Yerokhin, Physical Review A 80, 040501 (2009)

Using muonic hydrogen

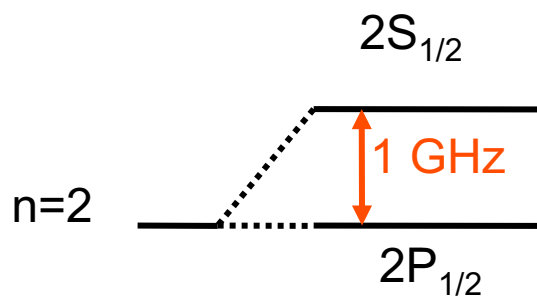
The exotic way...

Lamb shift



Self-energy:

The heavier the particle, the smaller (in relative term) it is

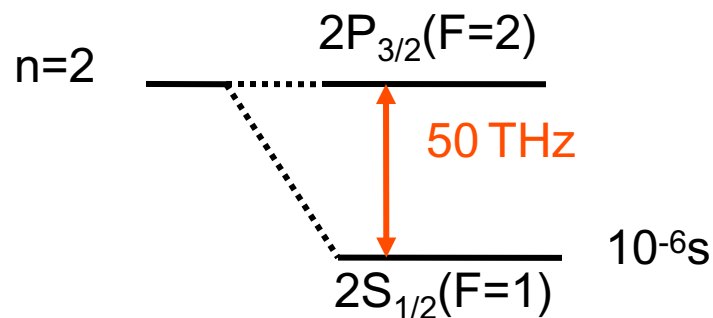


Hydrogen (electron)
Effect of R: 6×10^{-11}



Vacuum Polarization:

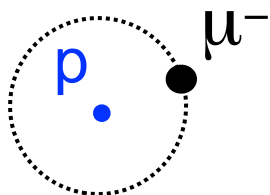
The closer the particle is, the stronger it is



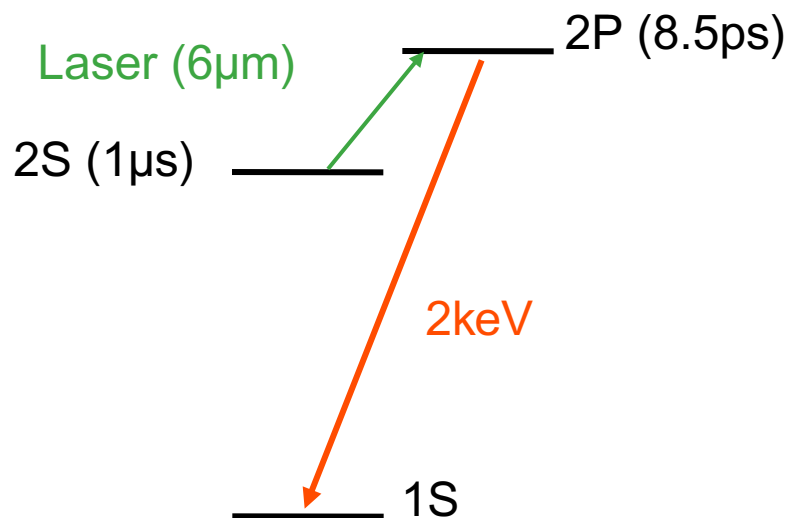
Muonic Hydrogen
(muon 207 times heavier than the electron)
Effect of R: 1.7%

muonic hydrogen 2S Lamb shift determination of the "proton radius"

Exotic atom



Experiment



Challenges

- production of muonic hydrogen in 2S
- powerful triggerable 6 μ m laser
- small signal analysis

Aim : better determination of proton radius r_p

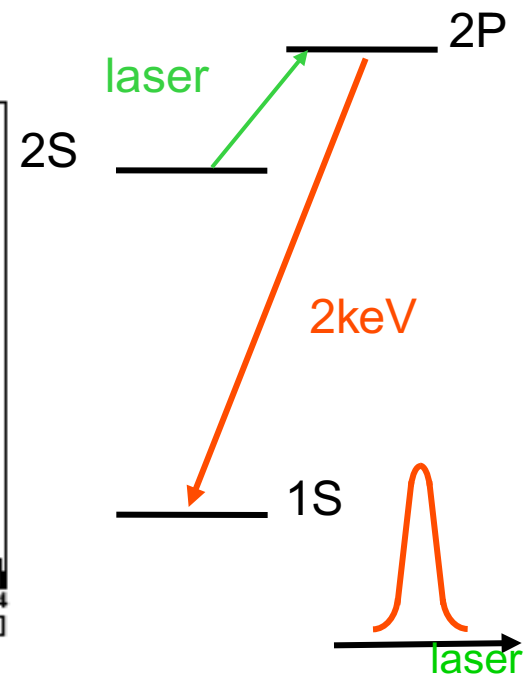
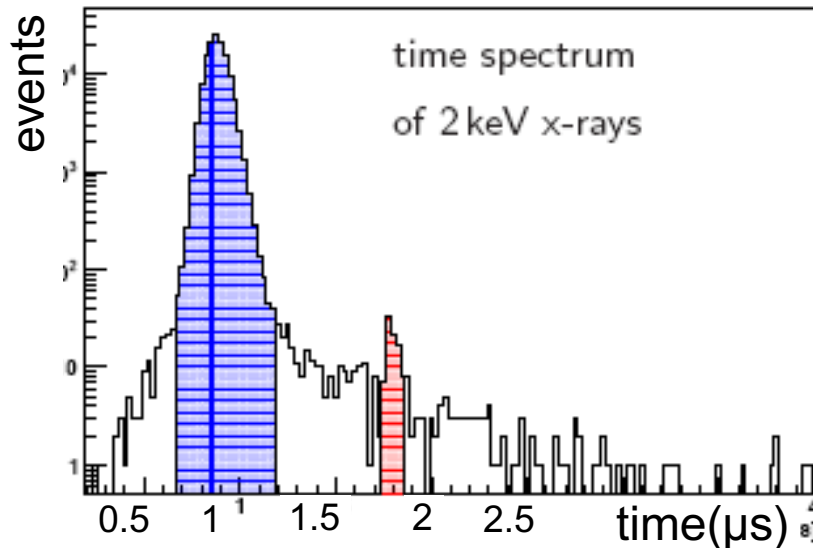
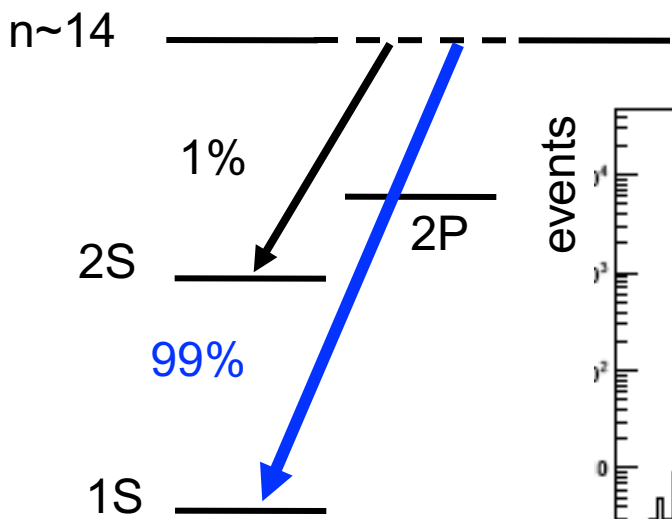
The muonic hydrogen experiment

Getting up close and personal with the proton!

“prompt” ($t \sim 0$)



“delayed” ($t \sim 1\mu\text{s}$)



μ^- stop in H_2 gas
 $\Rightarrow \mu p^*$ atoms formed ($n \sim 14$)

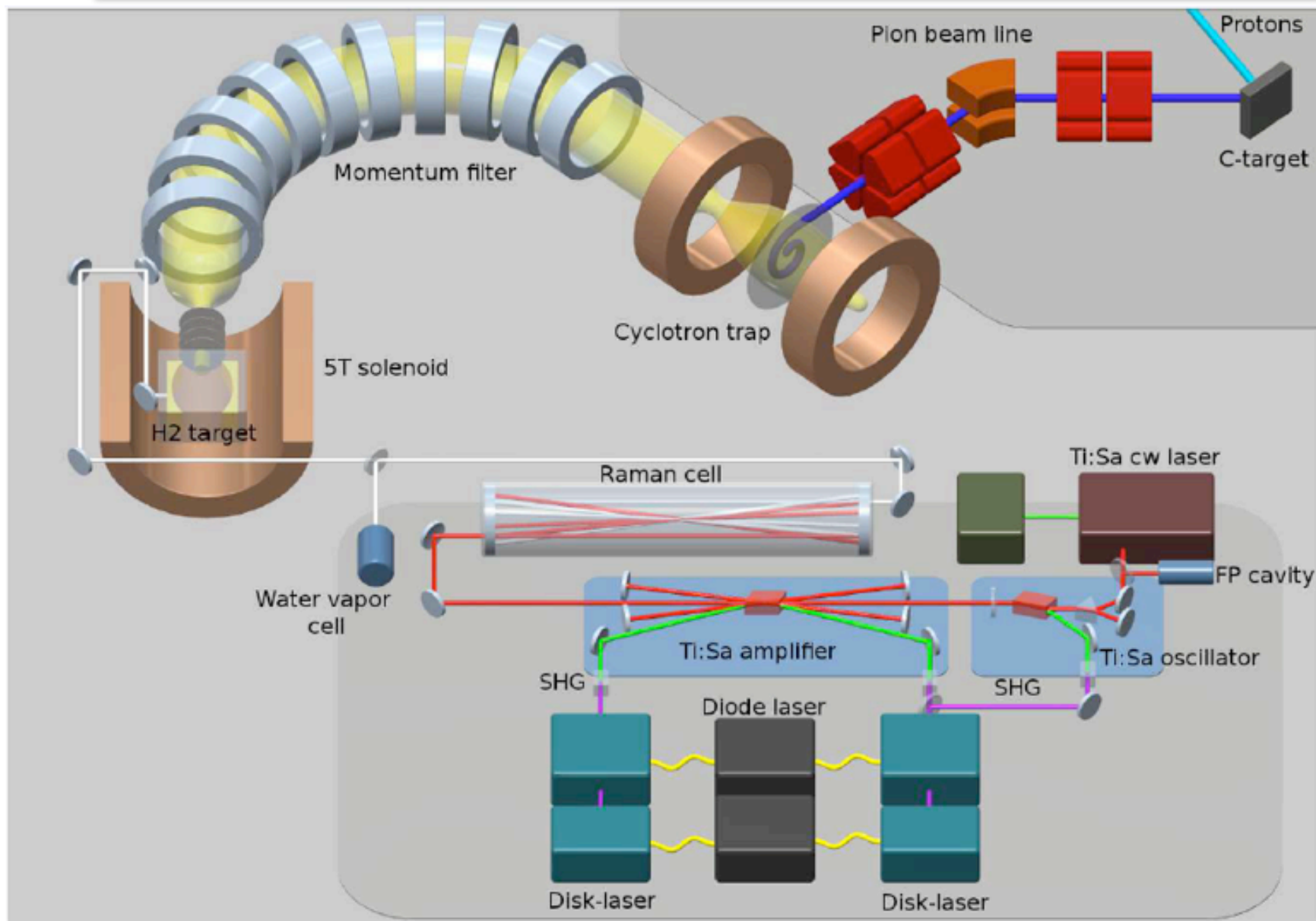
99%: cascade to $1S$ emitting prompt $K\alpha, K\beta, \dots$

1%: long lived $2S$ state ($\tau \sim 1\mu\text{s}$ at 1mbar)

Fire laser ($\lambda \sim 6\mu\text{m}$, $\Delta E \sim 0.2\text{eV}$)
 \Rightarrow induce $\mu p(2S-2P)$

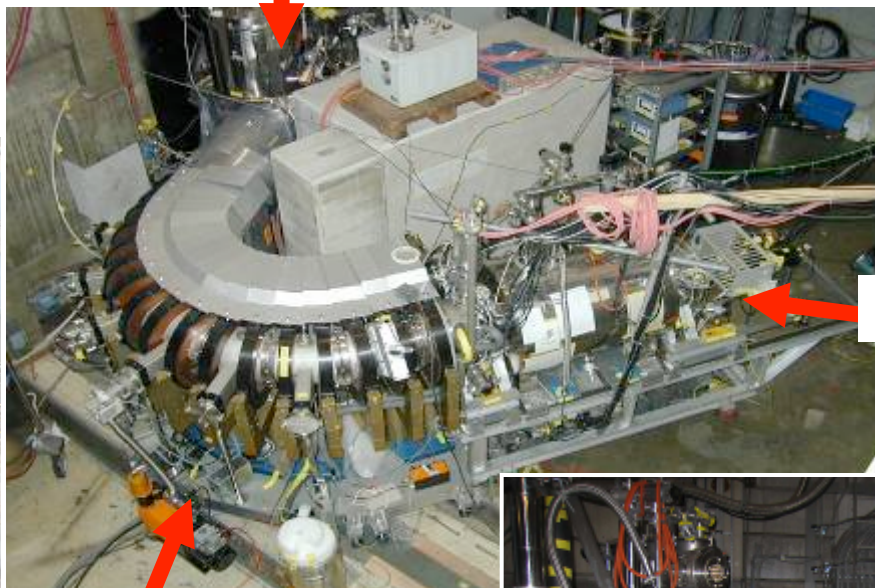
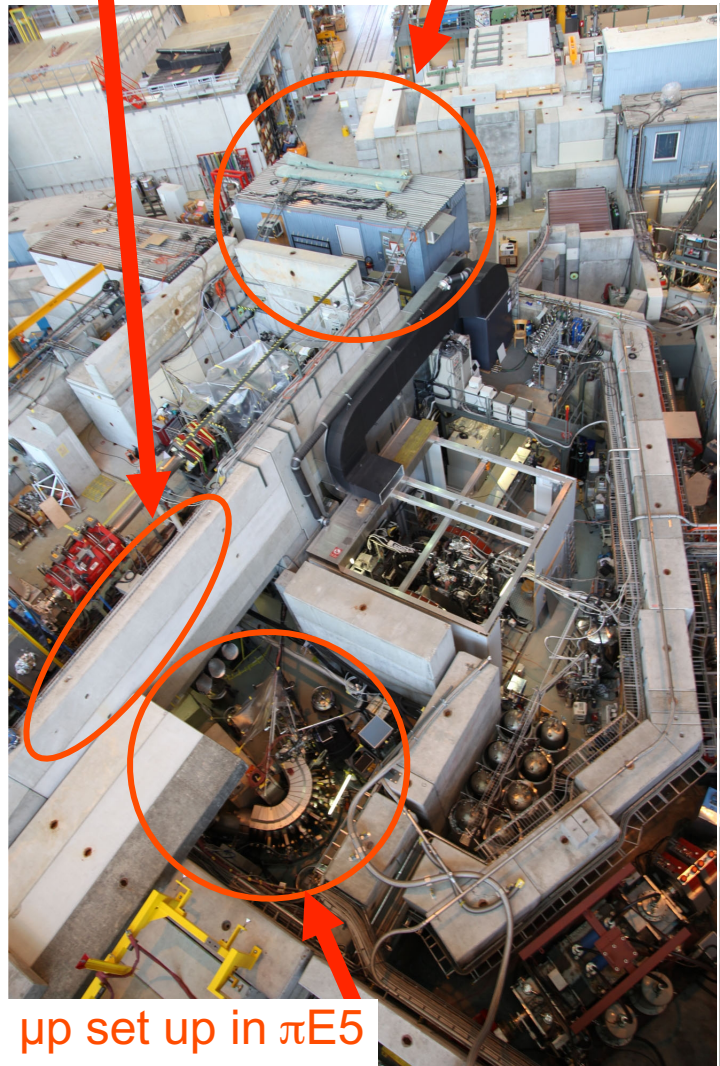
\Rightarrow observe delayed $K\alpha$ x-rays

\Rightarrow normalize $\frac{\text{delayed } K\alpha}{\text{prompt } K\alpha}$ x-rays



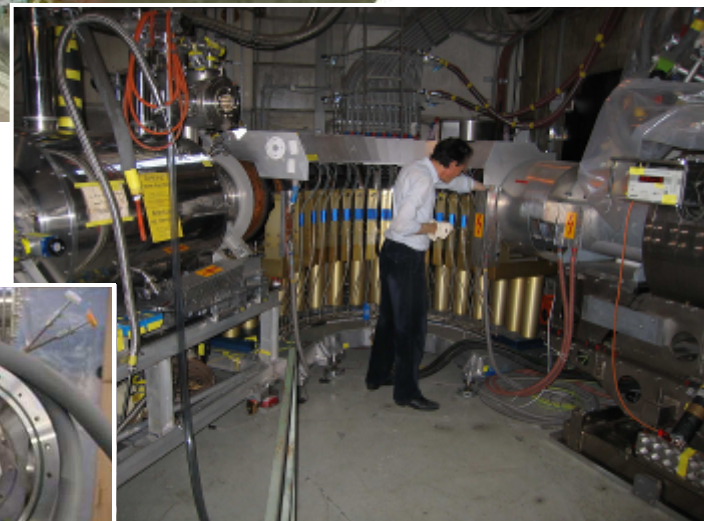
laser hut
below
concrete blocks

counting room

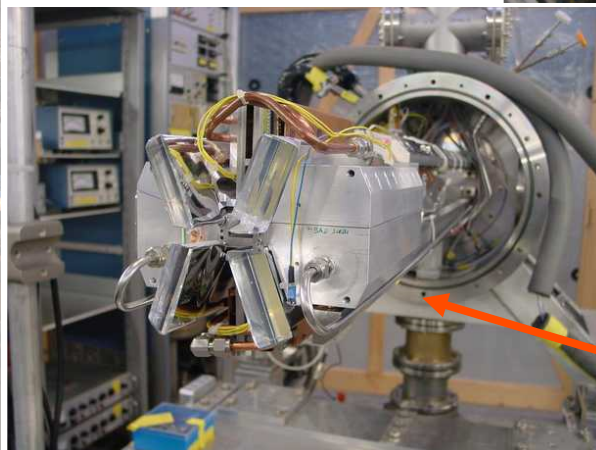


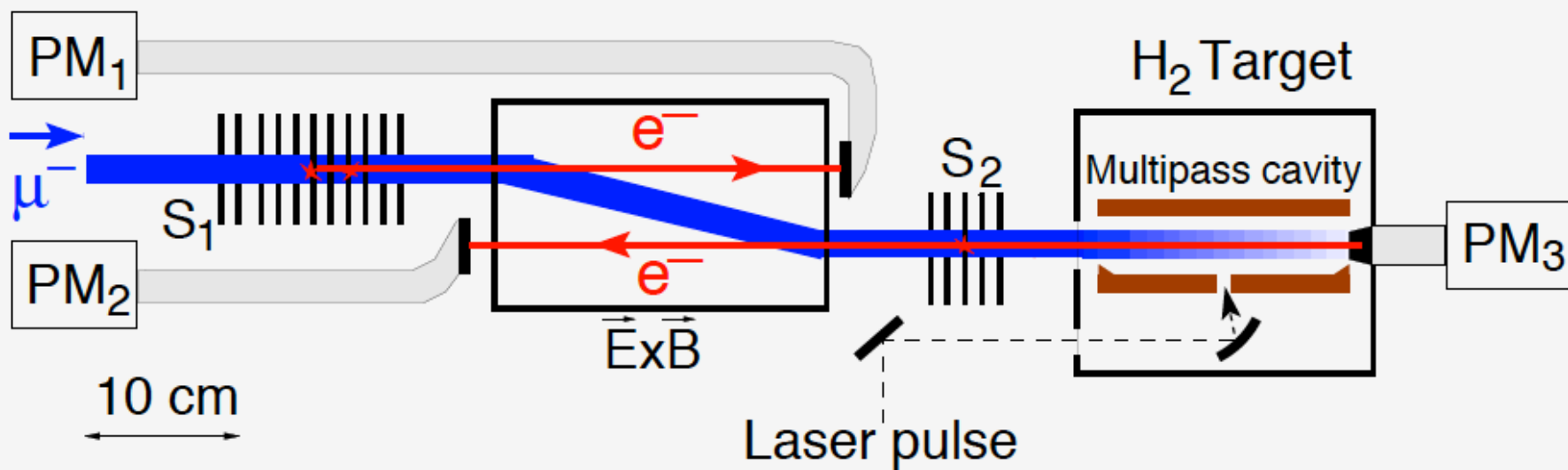
PSC solenoid,

Muon extraction
channel

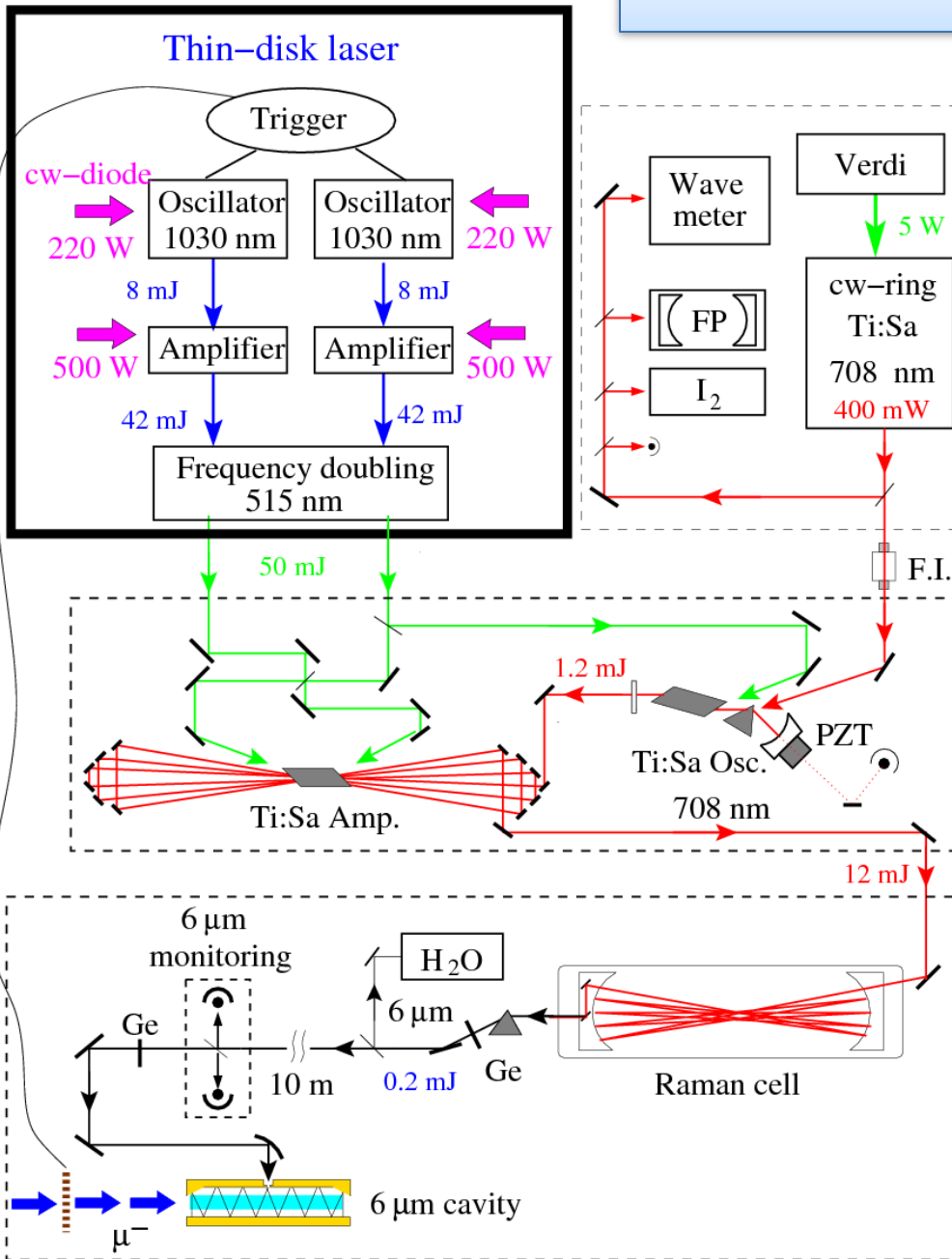


H2 target, laser cavity,
detectors

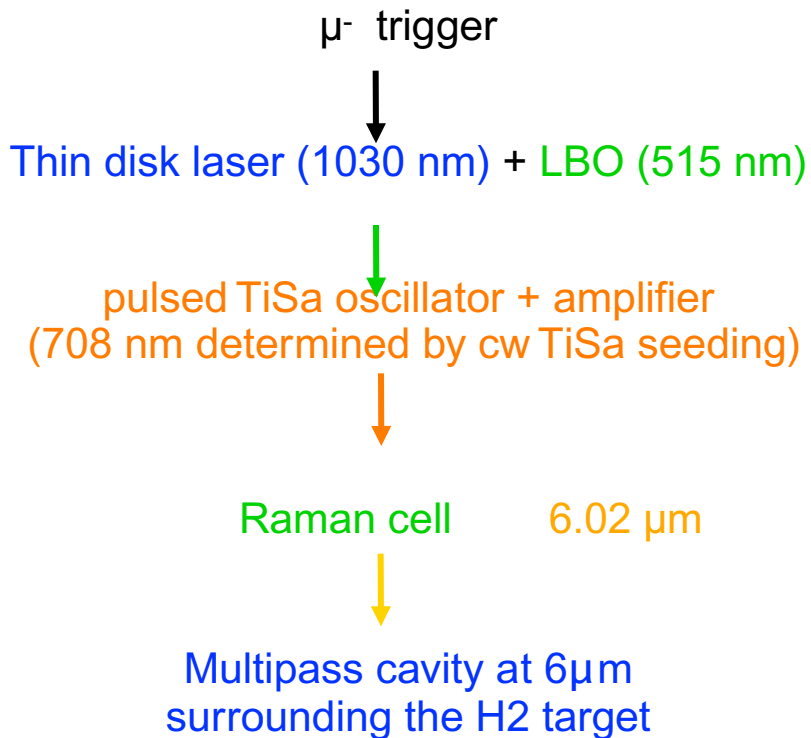




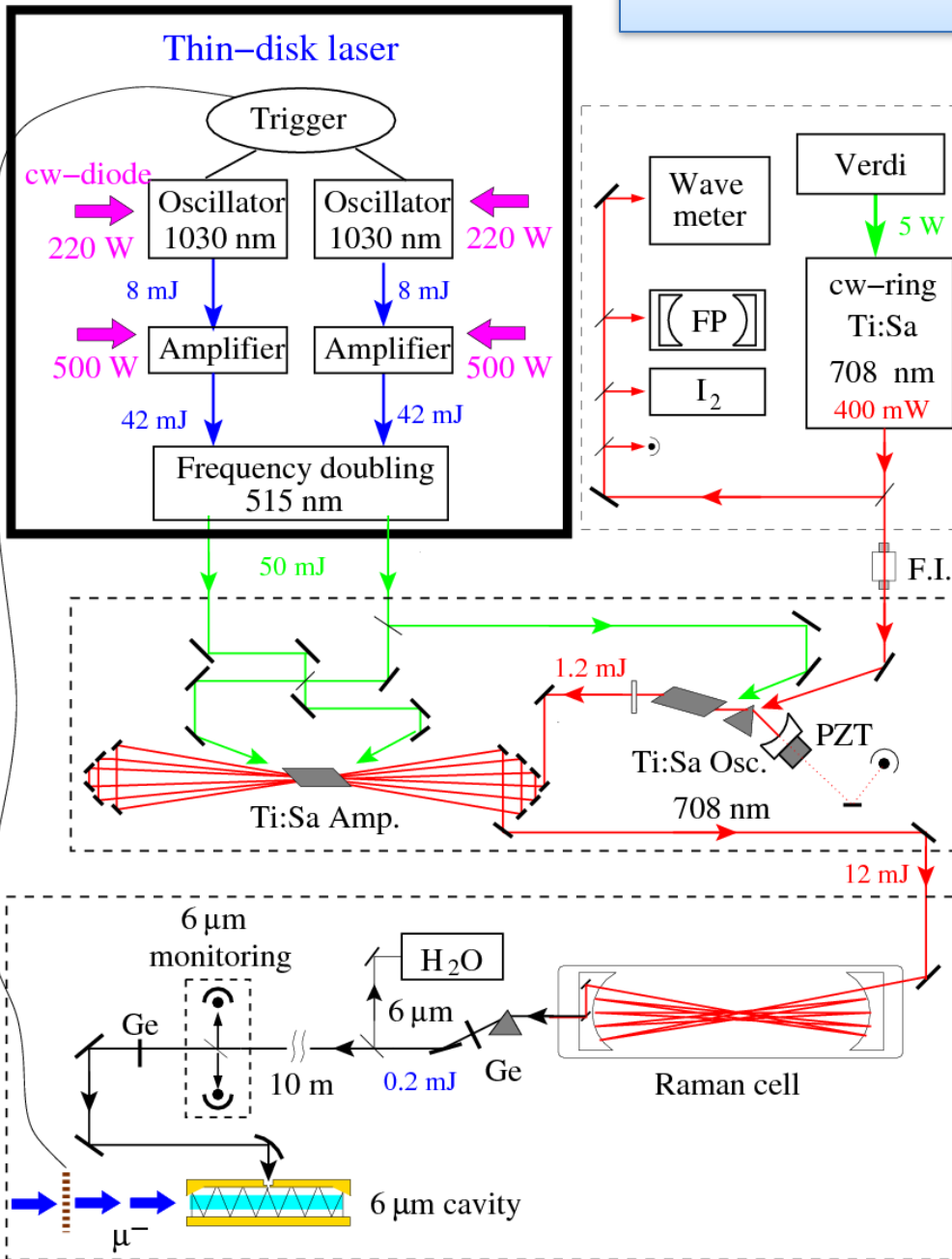
Laser chain



- Each single muon triggers the laser system (random trigger)
- 2S lifetime $\sim 1\mu\text{s}$ \rightarrow short laser delay (disk laser)
- 6 μm tunable laser pulse (0.2mJ)



Laser chain

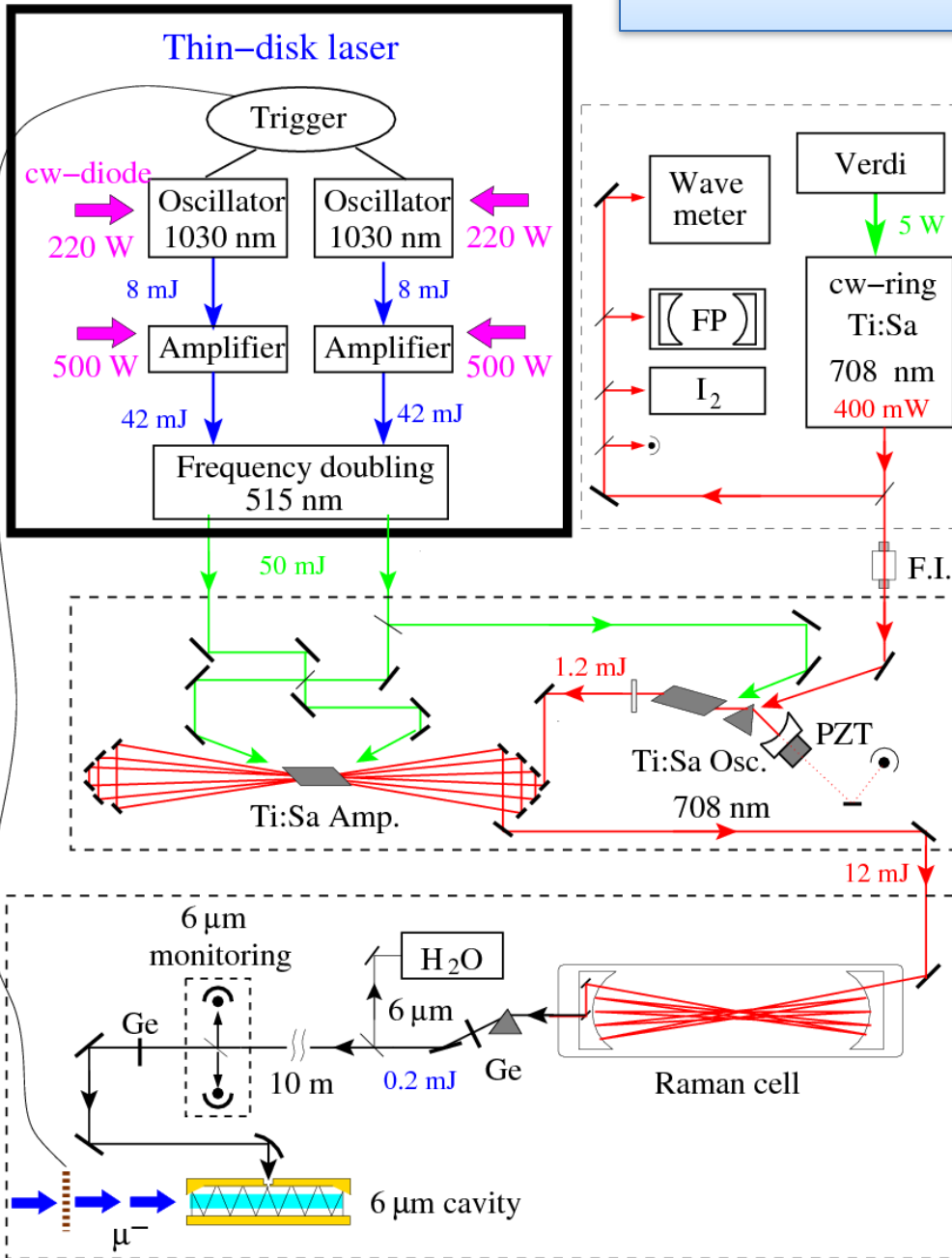


Thin disk laser

- Large pulse energy: 85 (160) mJ
- Short trigger-to-pulse delay: ≤ 400 ns
- Random trigger
- Pulse-to-pulse delays down to 2 ms (rep. rate ≥ 500 Hz)

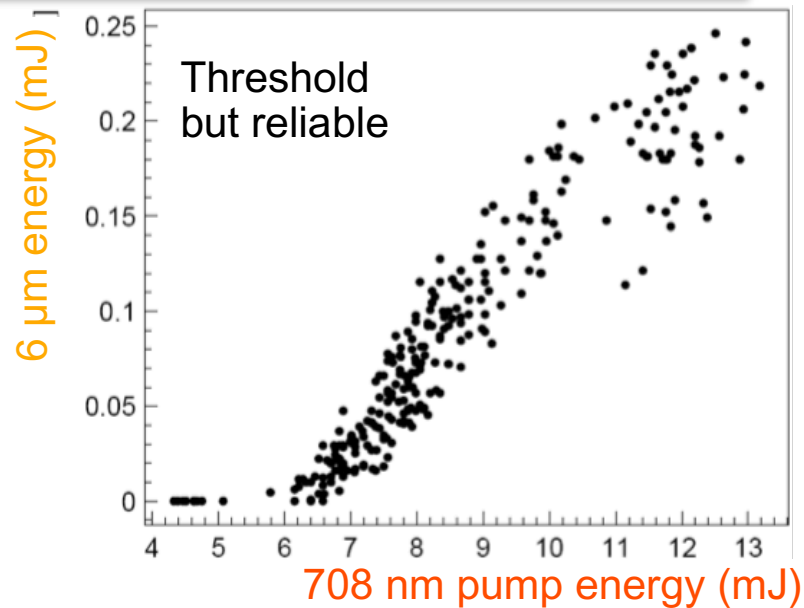
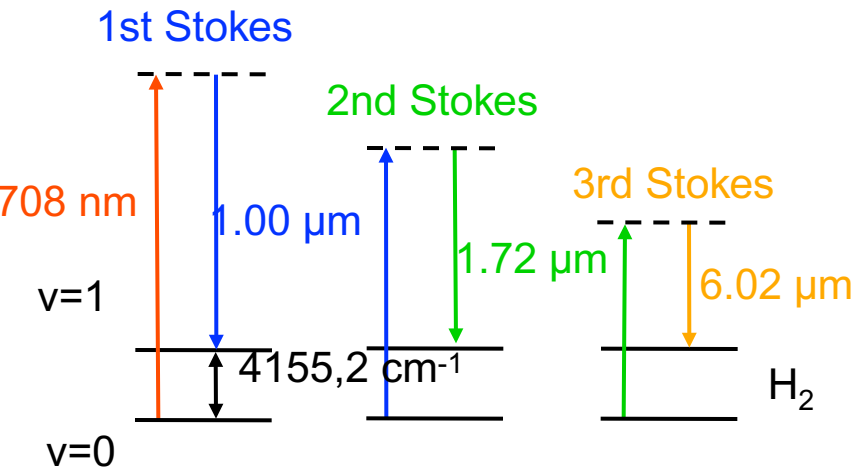
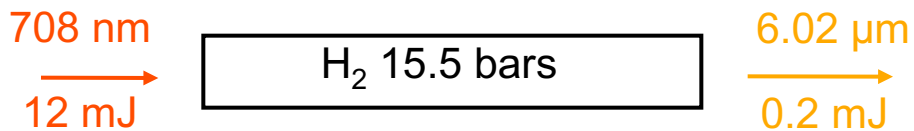
Laser chain

MOPA TiSa laser

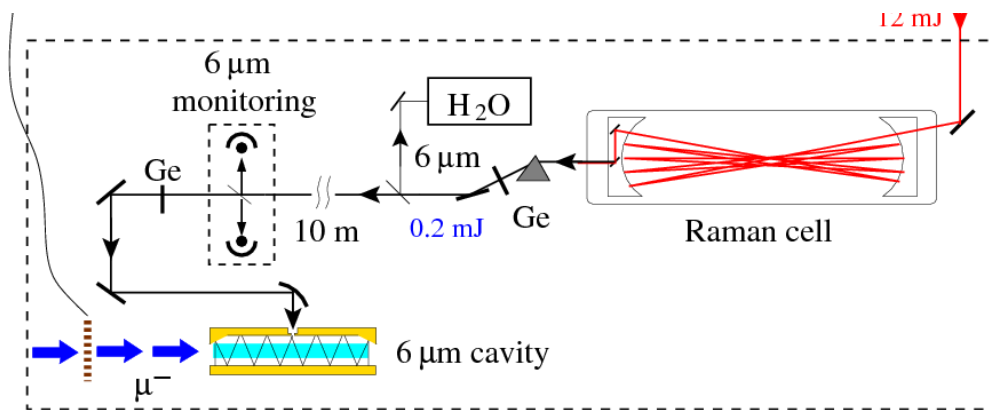
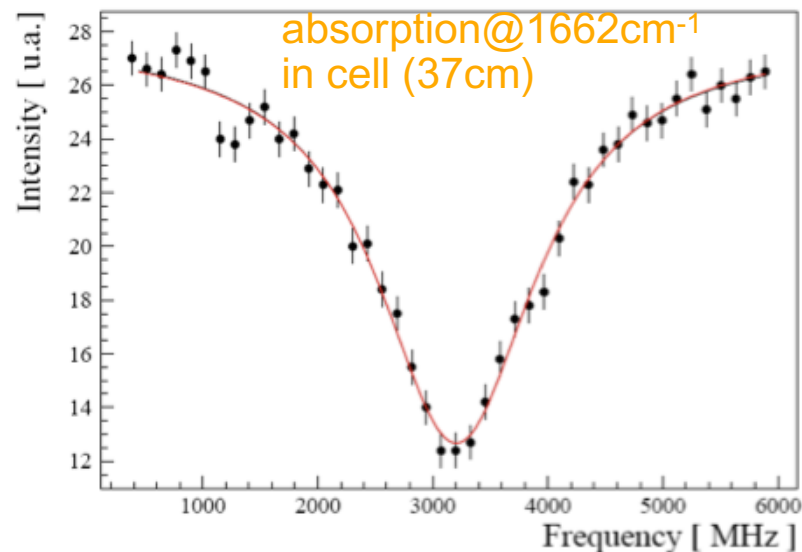


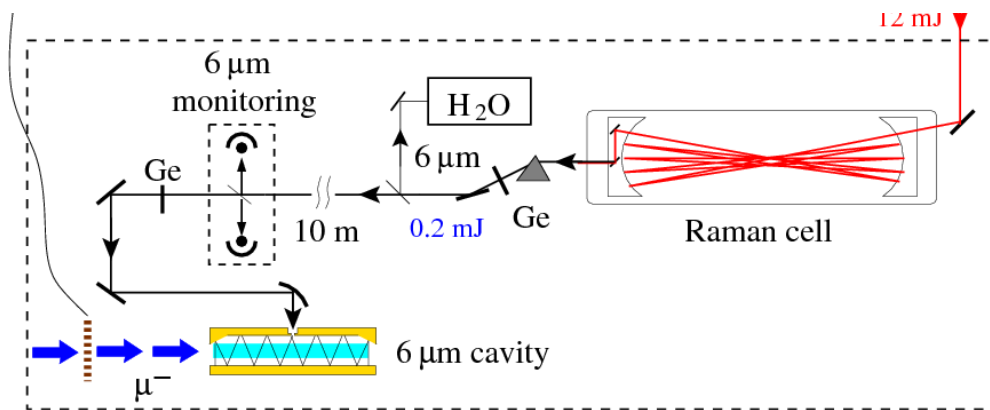
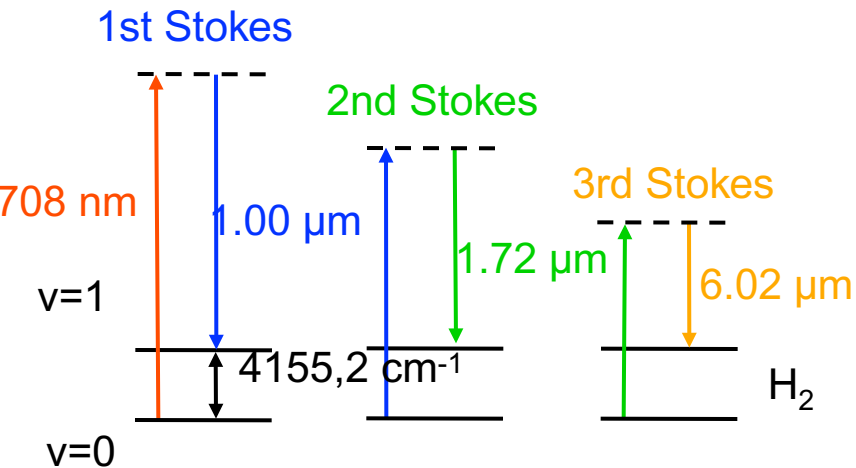
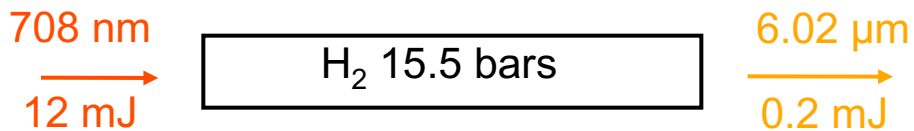
- cw laser, frequency stabilized
 - referenced to a stable FP cavity
 - FP cavity calibrated with I₂, Rb, Cs lines
 - FP = N · FSR (free spectral range)
 - FSR = 1497.344(6) MHz
- cw TiSa frequency absolutely known to 30 MHz
- $\Gamma_{2P-2S} = 18.6$ GHz
- Seeded oscillator
- TiSa = cw → pulsed TiSa (frequency chirp ≤ 100 MHz)
- Multipass amplifier (2f- configuration)
 - gain=10

Laser chain : Raman cell

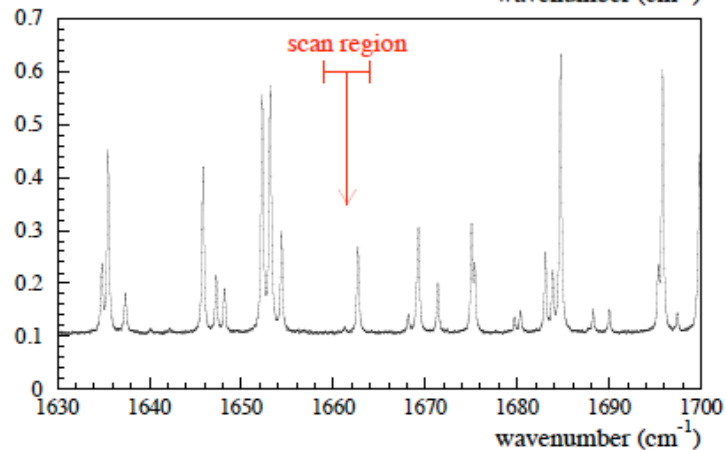
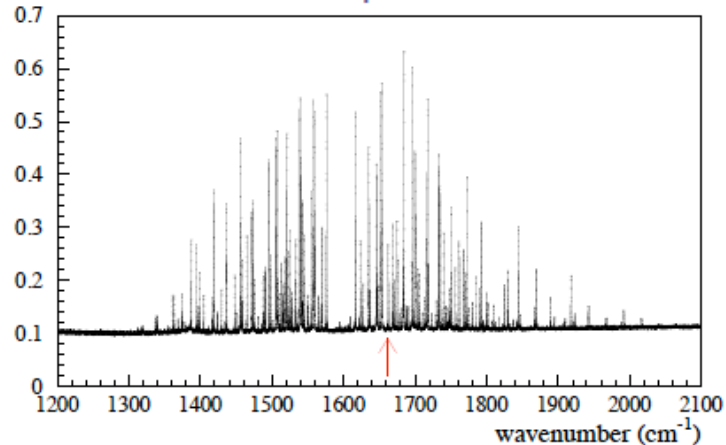


6 μm frequency calibration : H₂O lines



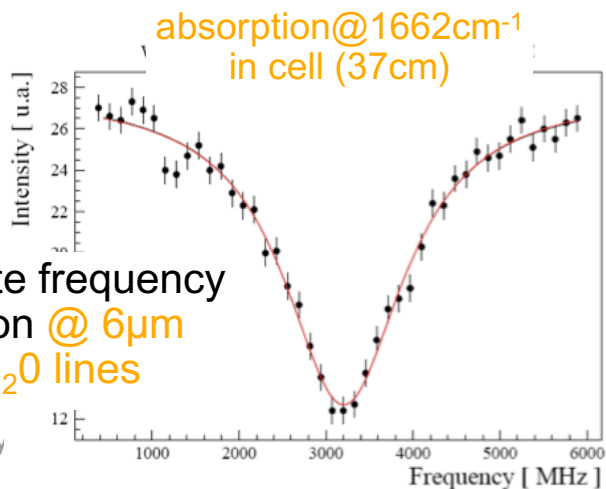
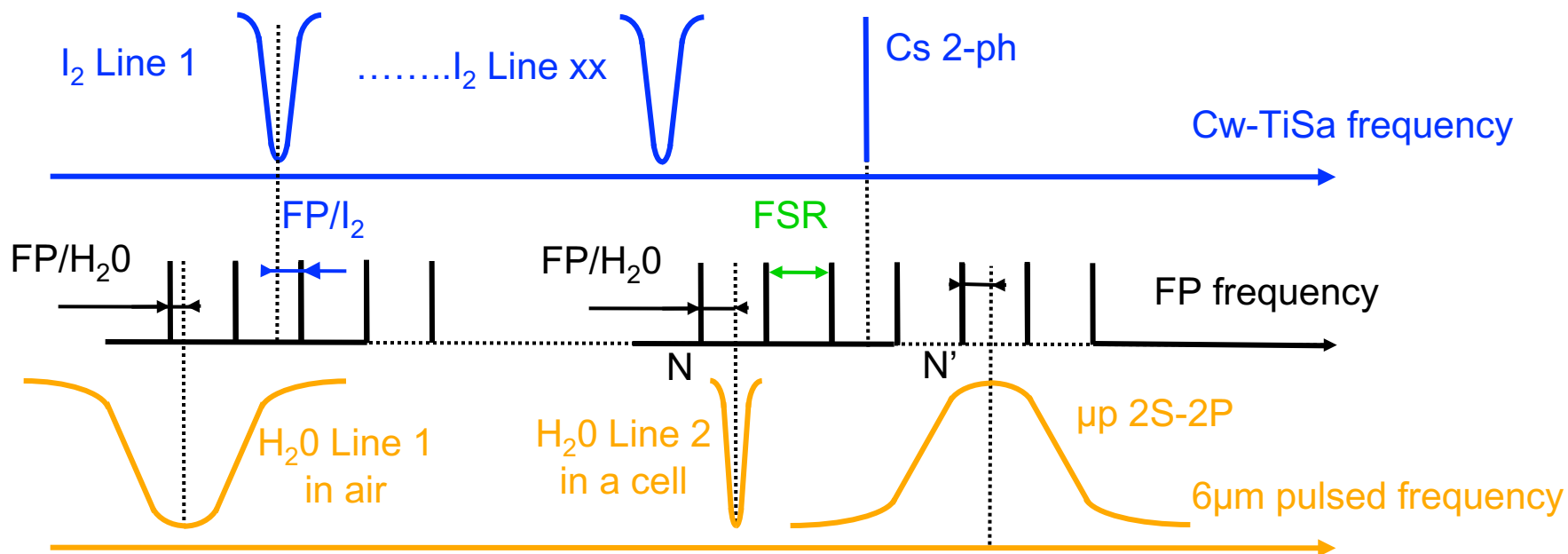


Water absorption



- Vacuum tube for 6μm laser beam transport
- Direct frequency calibration at 6μm
- Well known lines

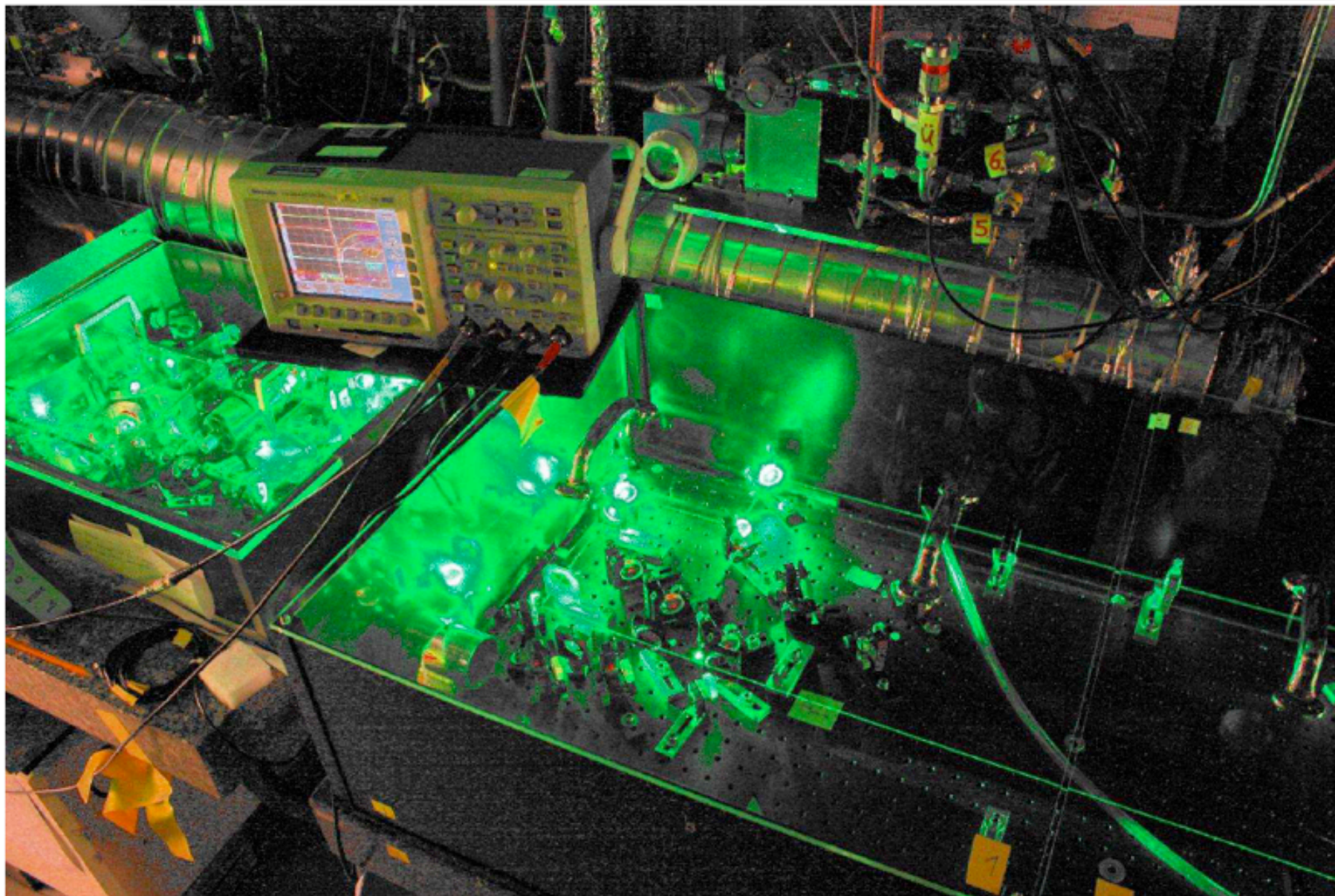
FSR measured/controlled in cw with I₂ (1 ph abs), Cs (2 ph fluo), Rb (2 ph fluo), lines



$$\nu(\mu\text{p}:2\text{S}-2\text{P}) = \nu(\text{H}_2\text{O Line 2}) + (\text{N}-\text{N}') \text{ FSR}$$

FP absolute frequency calibration @ 6µm with H₂O lines



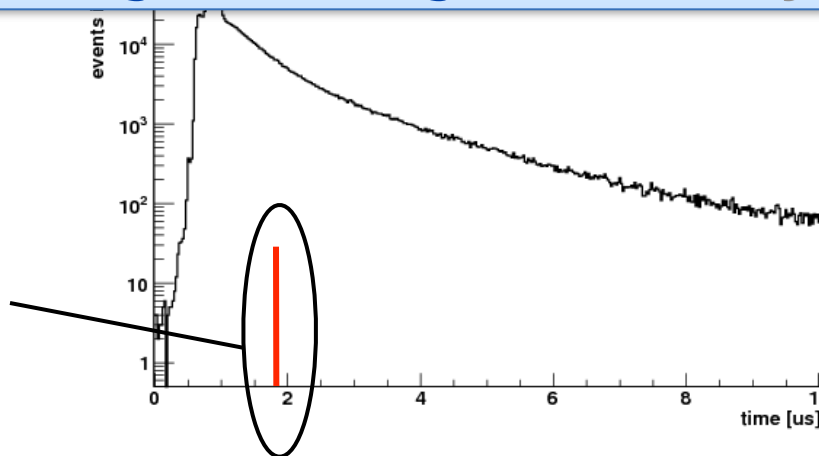


X-rays analysis → event gate sorting → noise rejection

Example : FP 900 - 11 hrs meas.

1.56 million detector events

expected 2-3 laser induced events/hour !

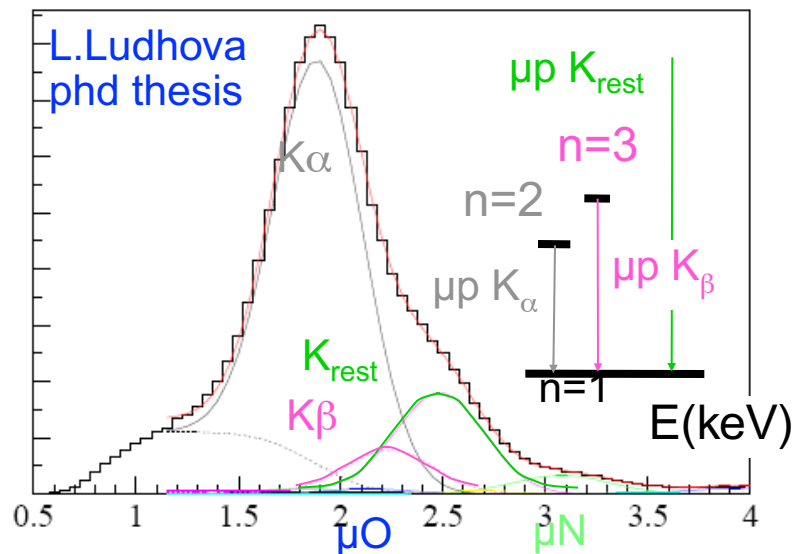
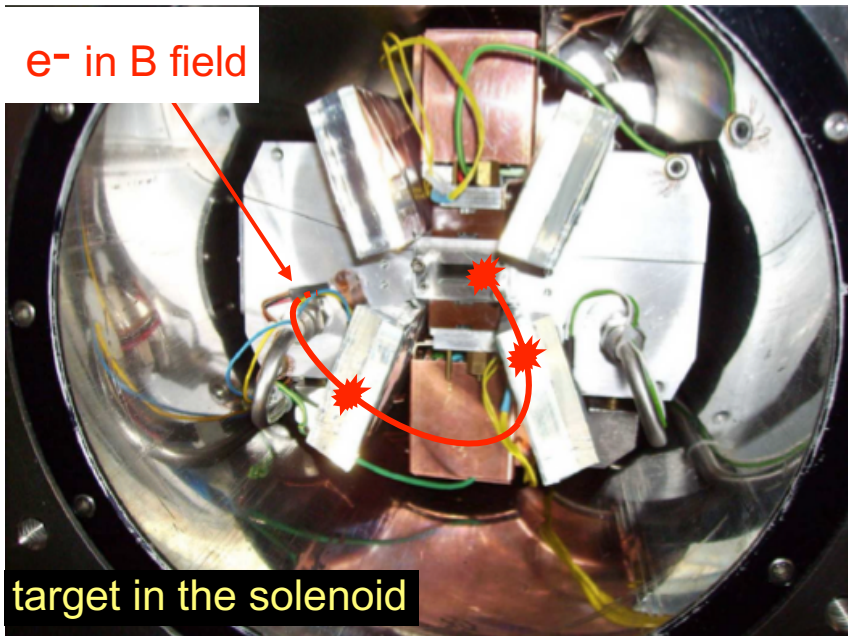


time signature in LAAPD

- photon < 10keV → 1 shot in the LAAPD
- e⁻ in B = 5T → many counts in detectors

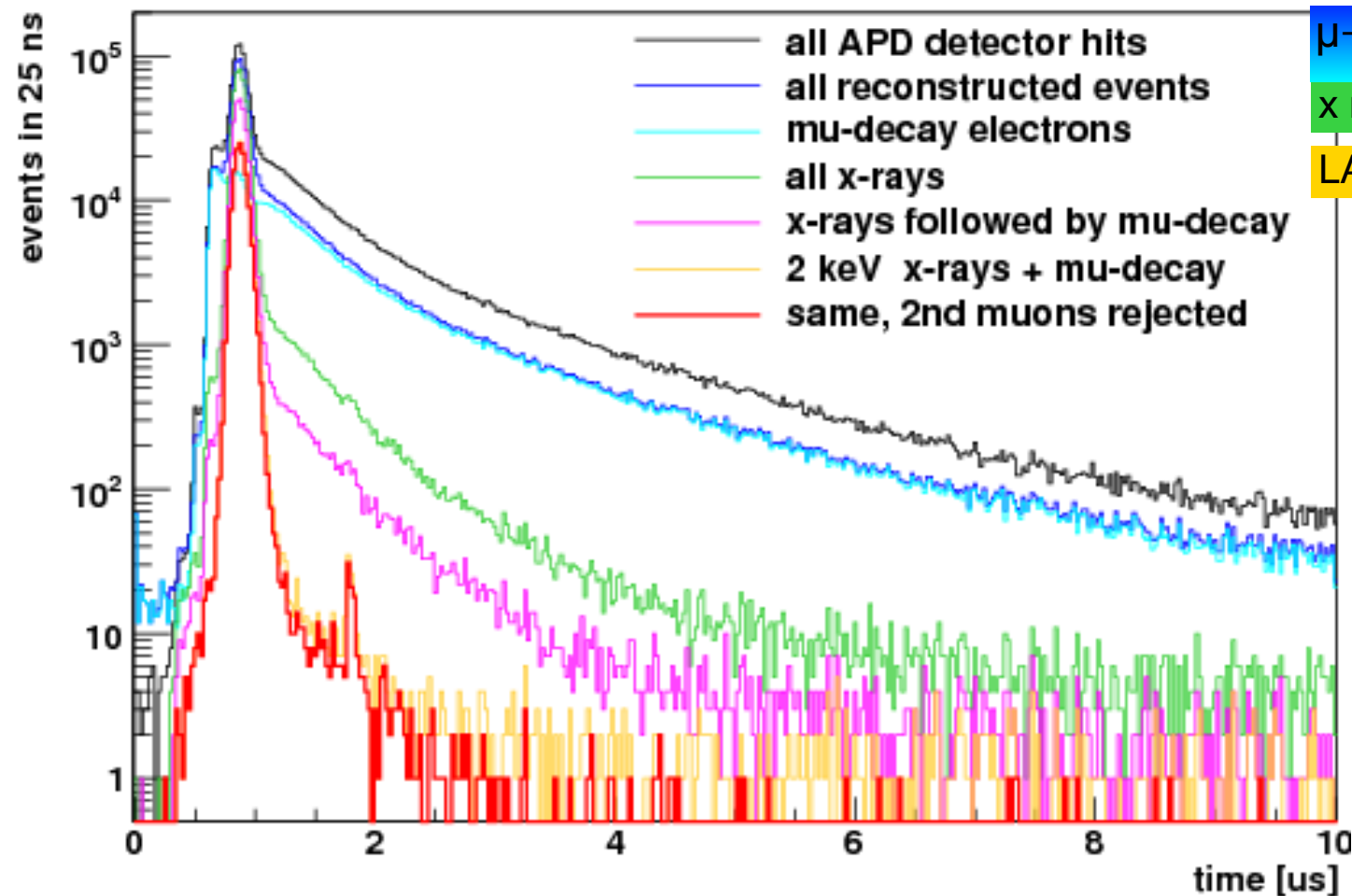
energy signature in LAAPD

- E > 8keV ⇔ electron
- 1keV < E < 8keV ⇔ X ray
- E < 1keV ⇔ neutron



Example : FP 900 - 11 hrs meas.

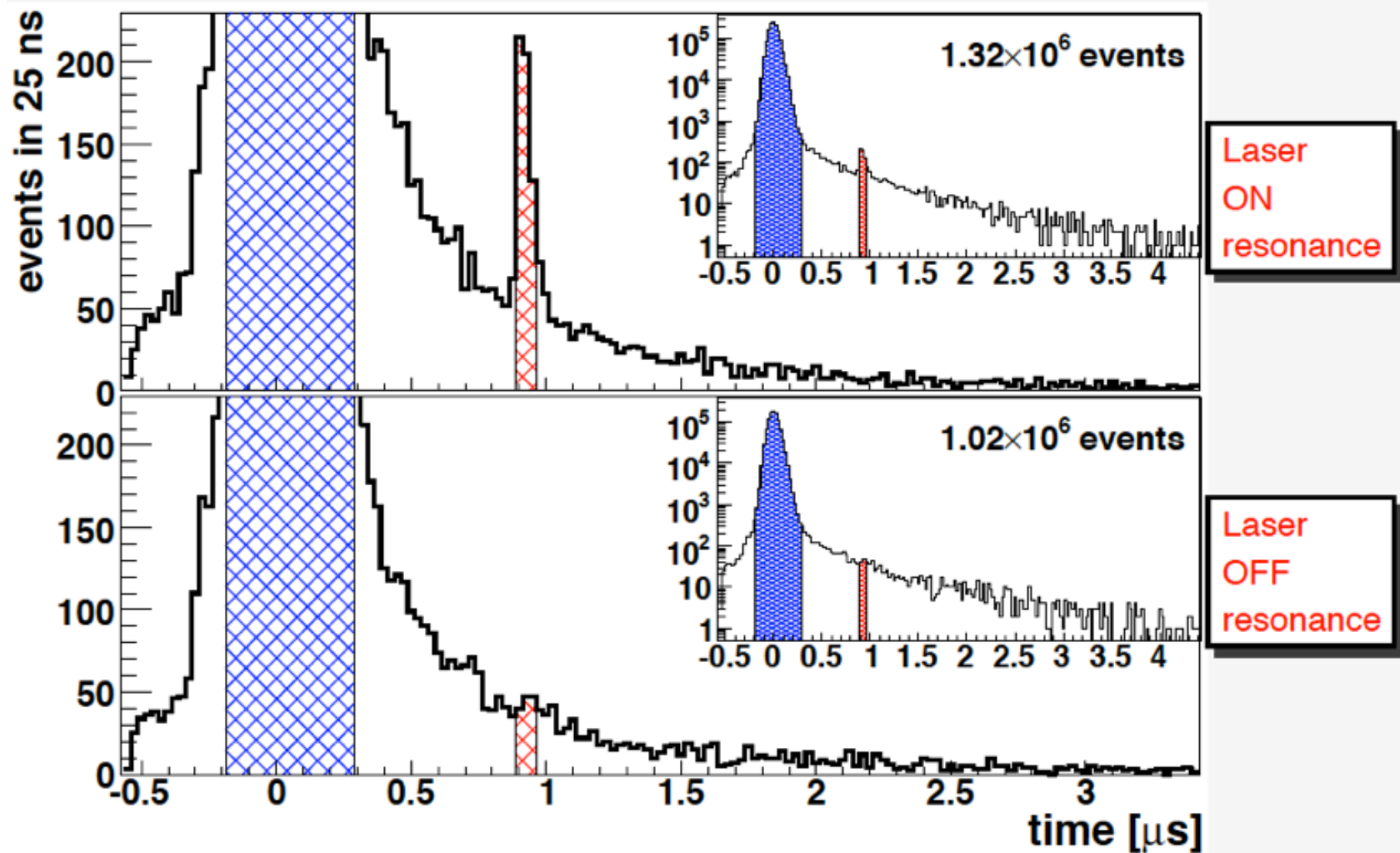
- 400 μ /s
- 240 laser shot/s
- 860 000 laser shot/hour
- 1.56 million detector clicks
- 19600 clicks in the laser region
- expected 2-3 laser induced events/hour !

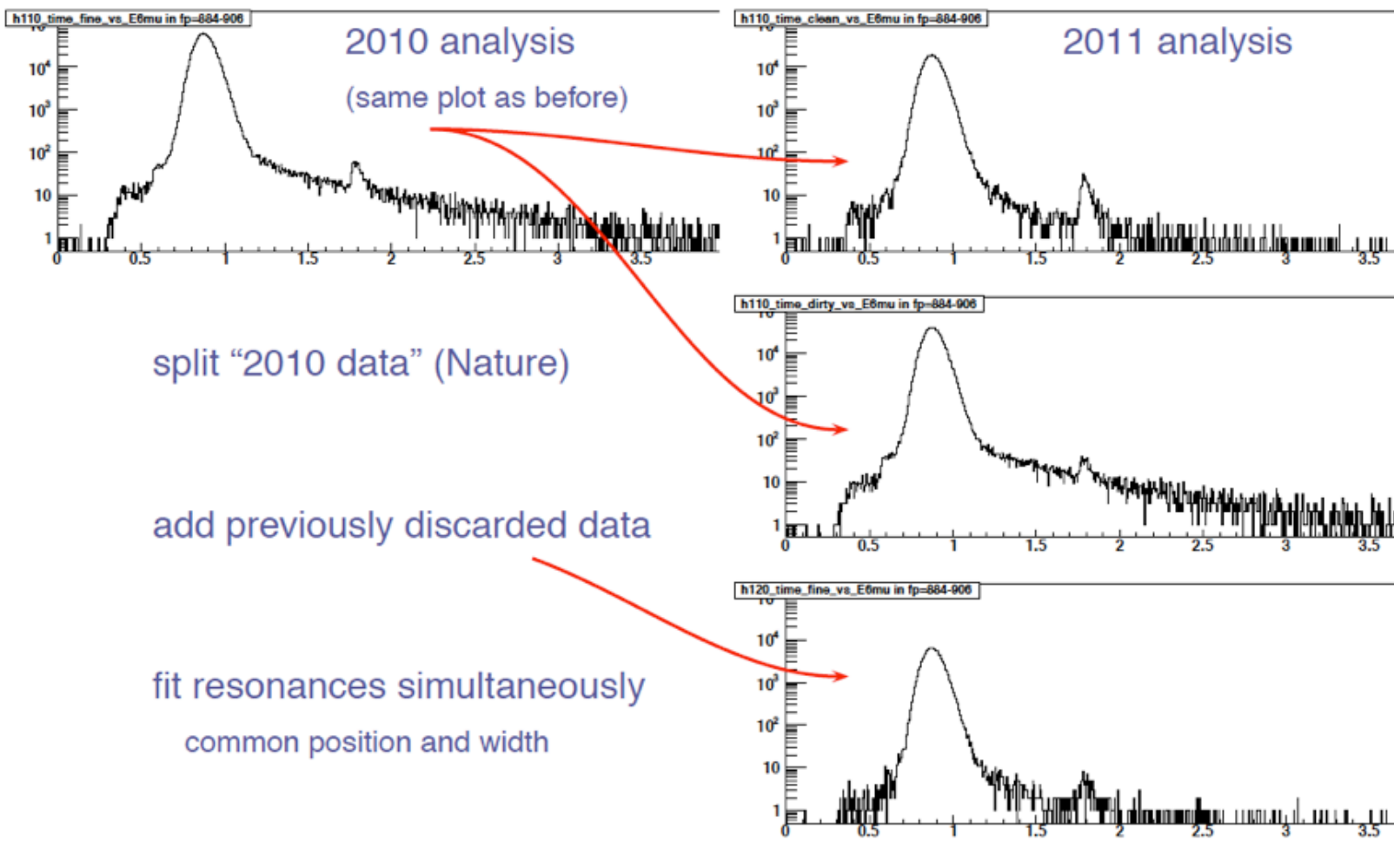


$\mu \rightarrow e \nu_{\mu} \bar{\nu}_e$

x rays multiplicity 1

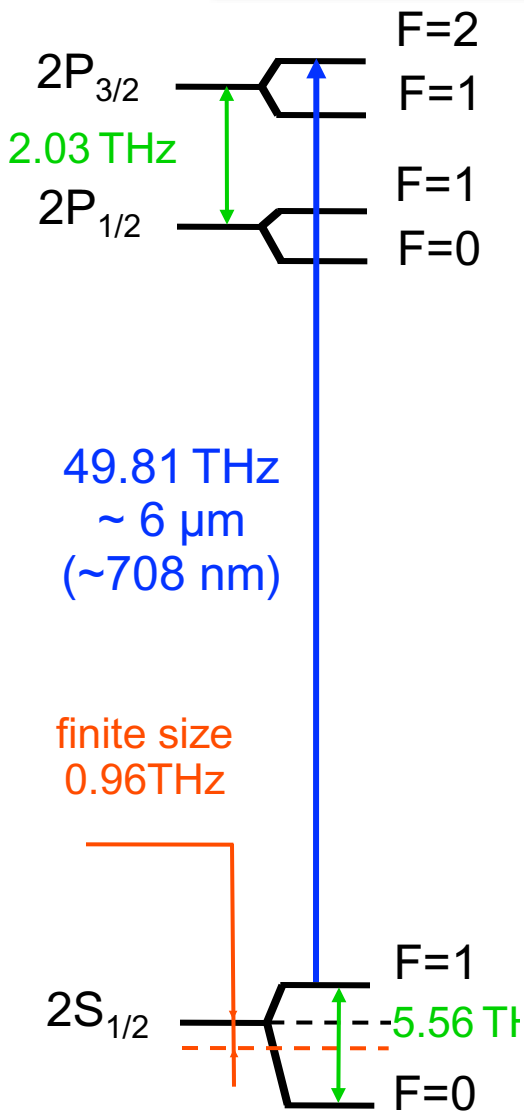
LAAPD energy resolution



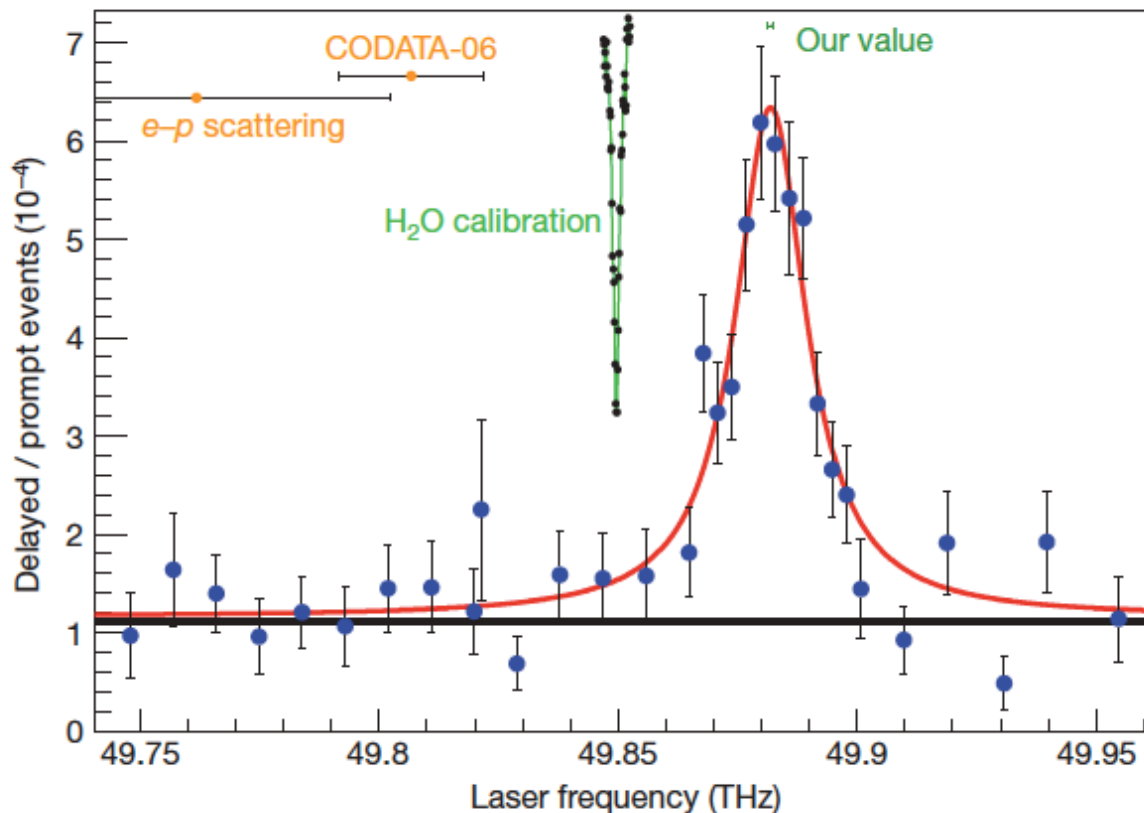


1.9 keV Ka x-ray must followed by the detection of an MeV-energy electron, but there are several detectors

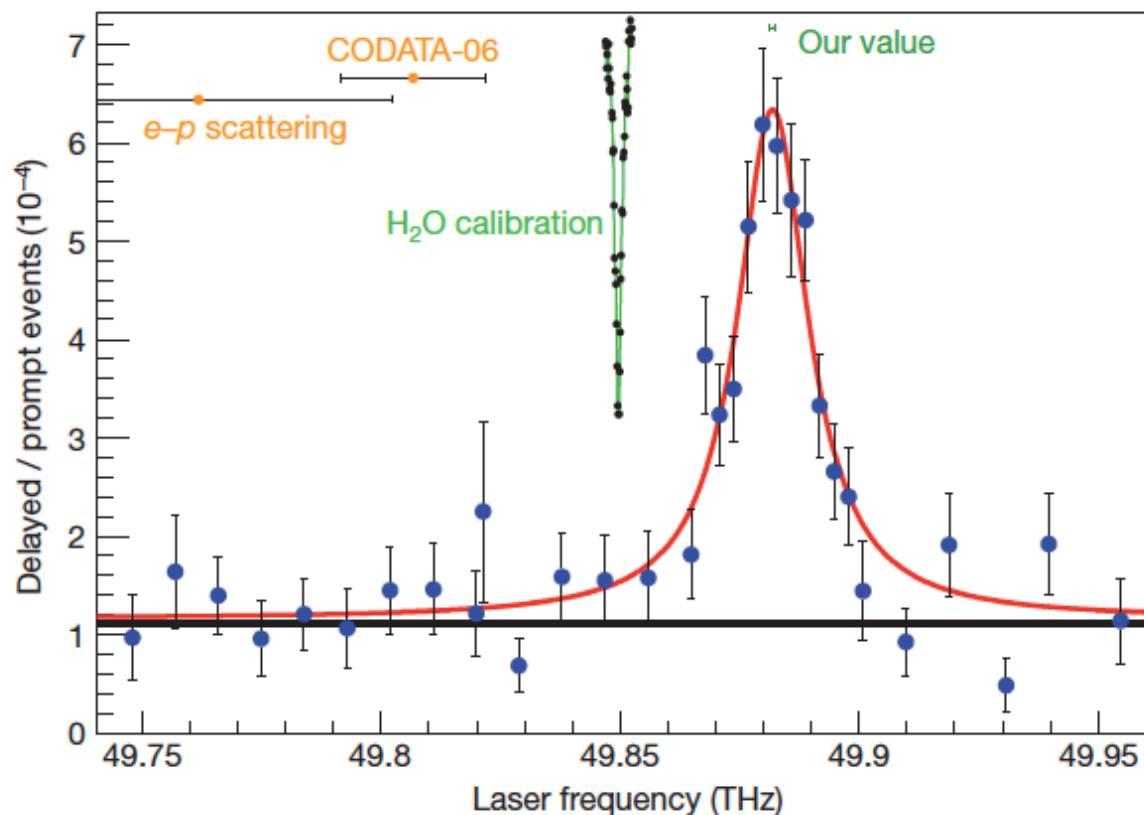
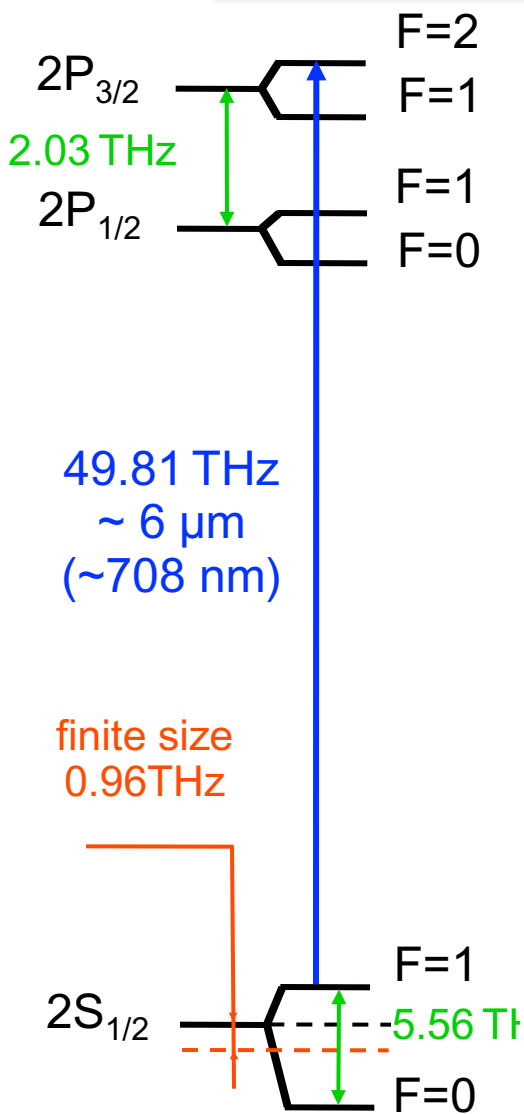
muonic hydrogen : $^2S_{1/2}(F=1) - ^2P_{3/2}(F=2)$



- 550 events measured
- 155 backgrounds
- 31 FP fringes
- 250 hours



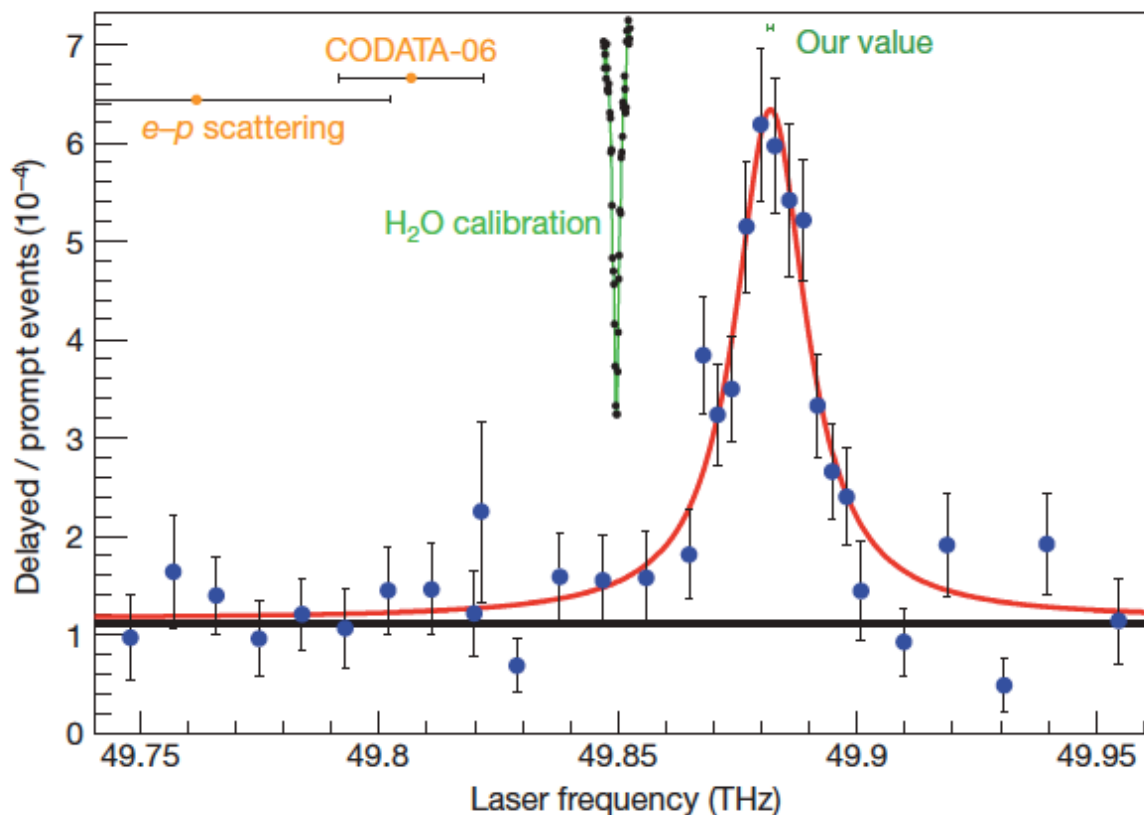
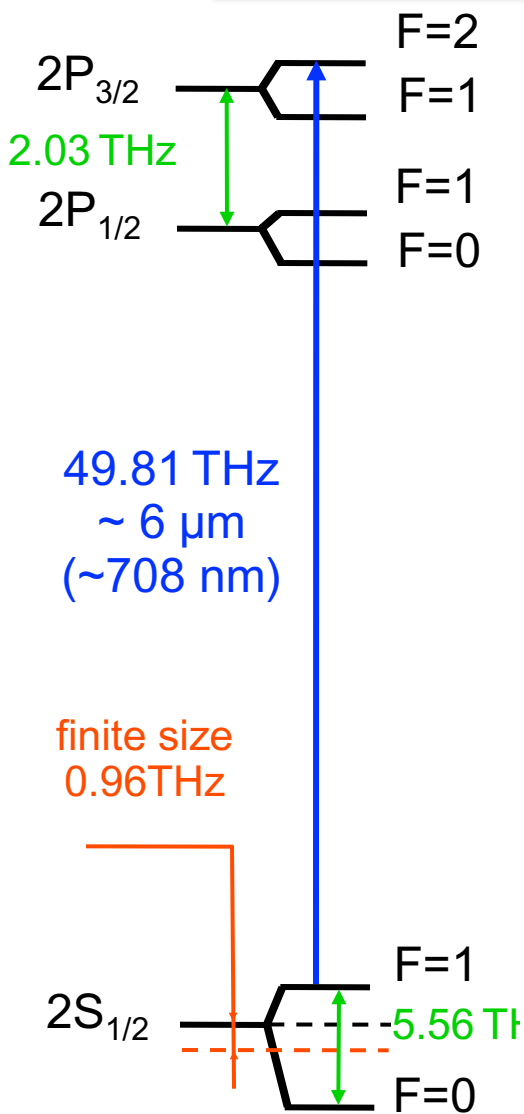
R. Pohl, A. Antognini, F. Nez, et al., Nature 466, 213 (2010).



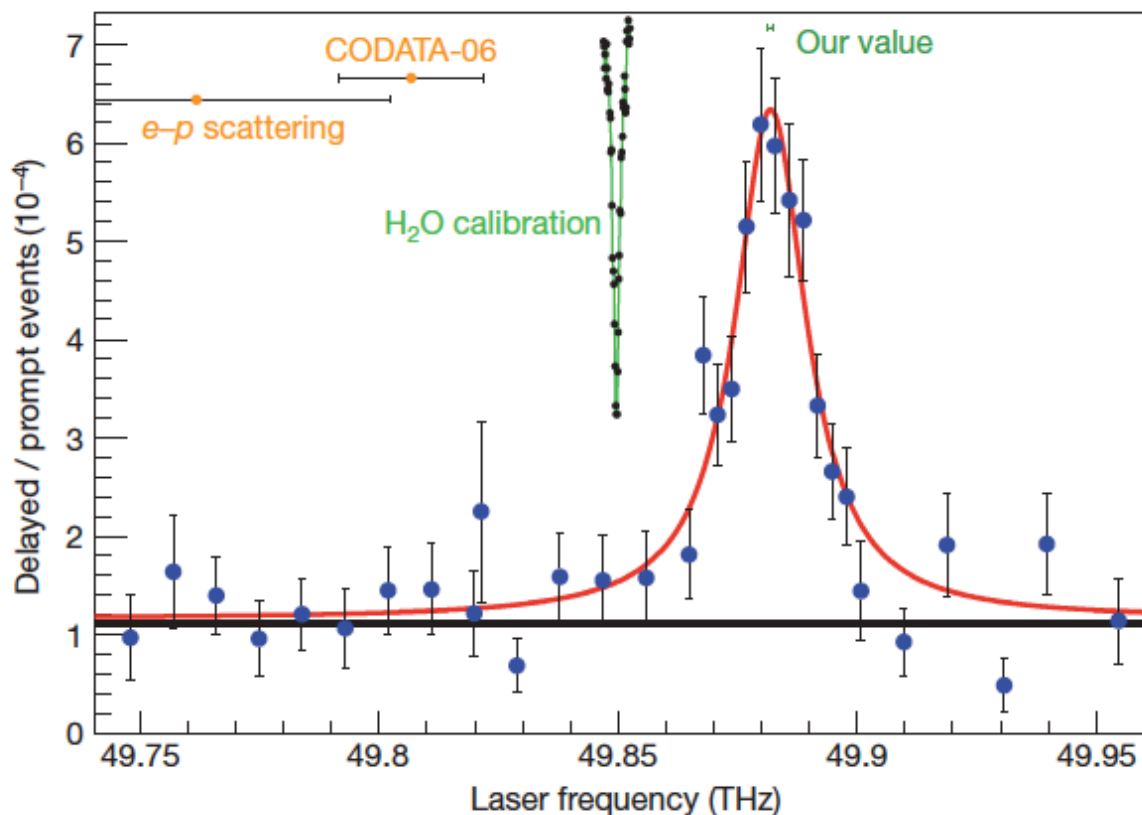
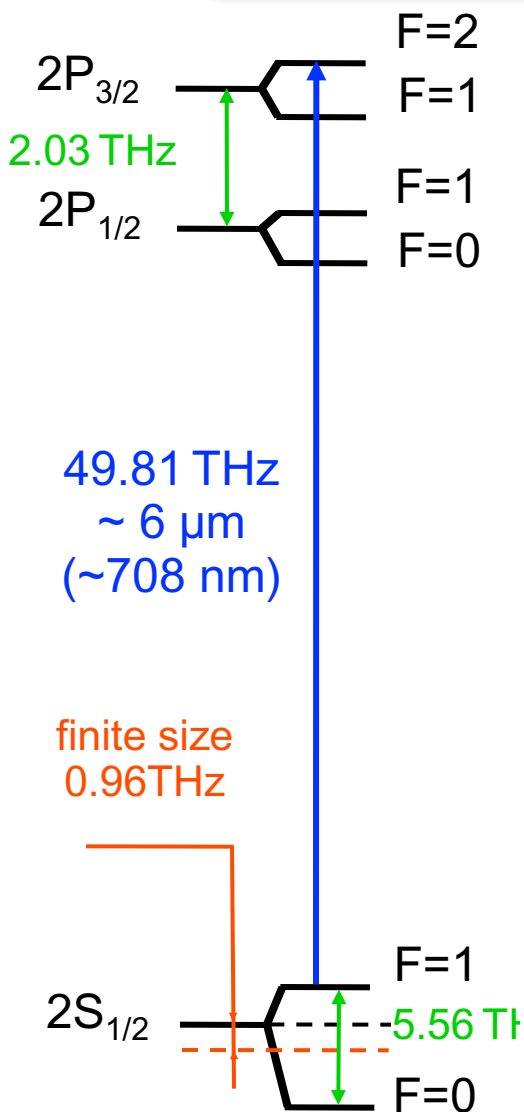
R. Pohl, A. Antognini, F. Nez, et al., Nature 466, 213 (2010).

muonic hydrogen : $^2S_{1/2}(F=1) - ^2P_{3/2}(F=2)$

Discrepancy CODATA
2010: 7σ (75 GHz)

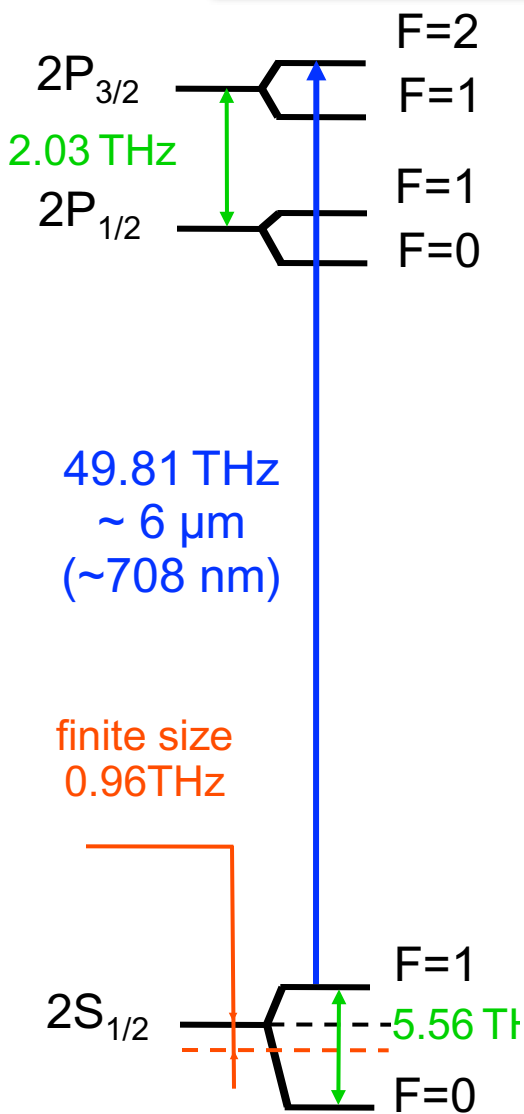


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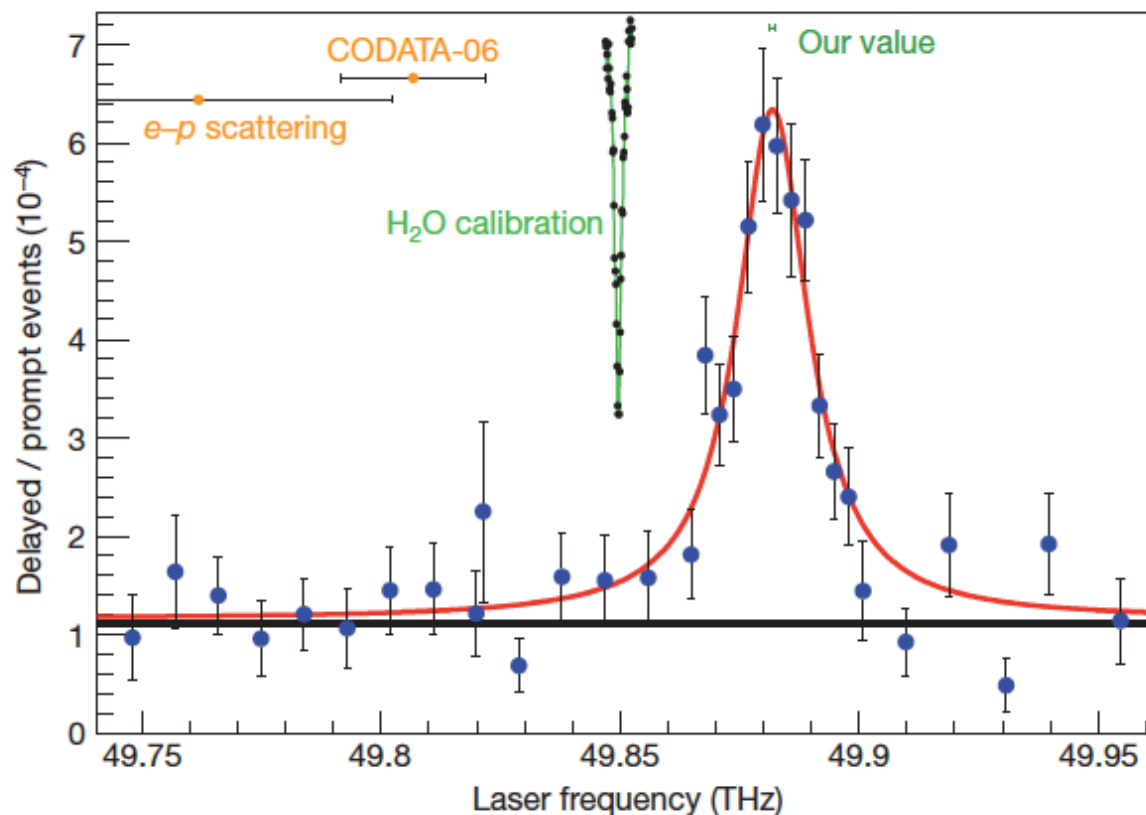
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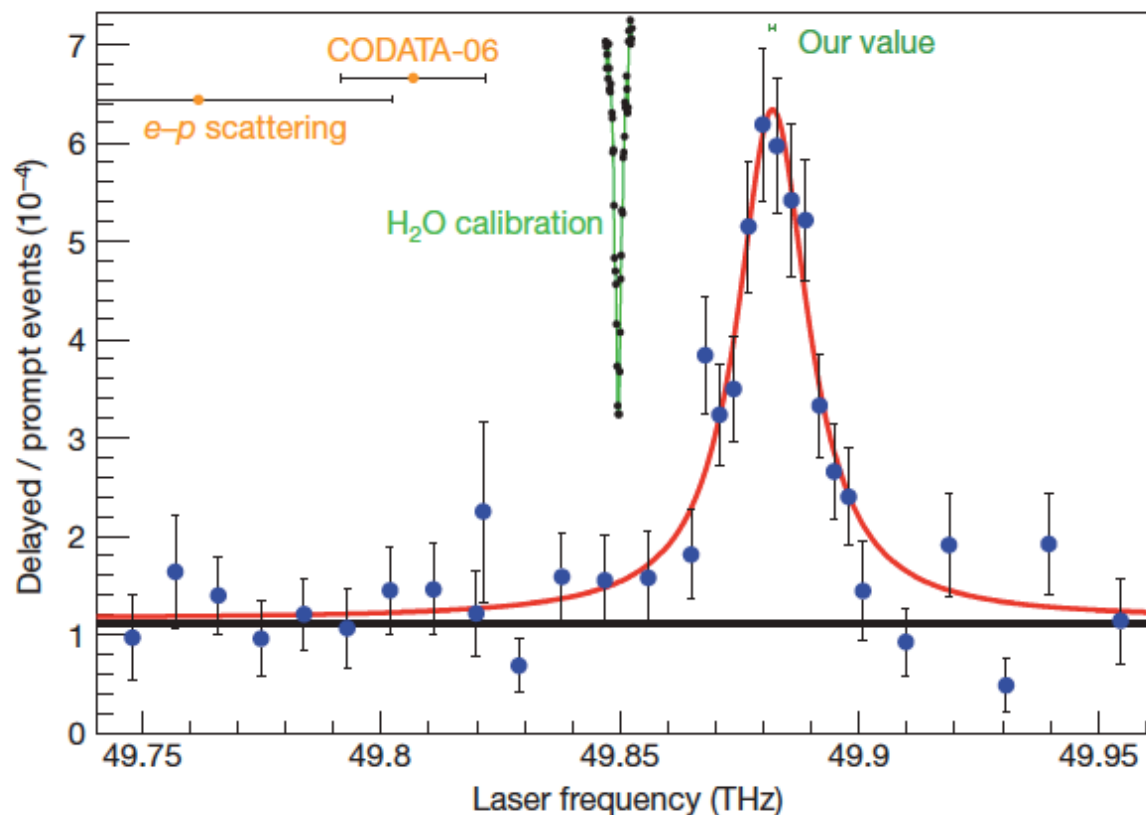
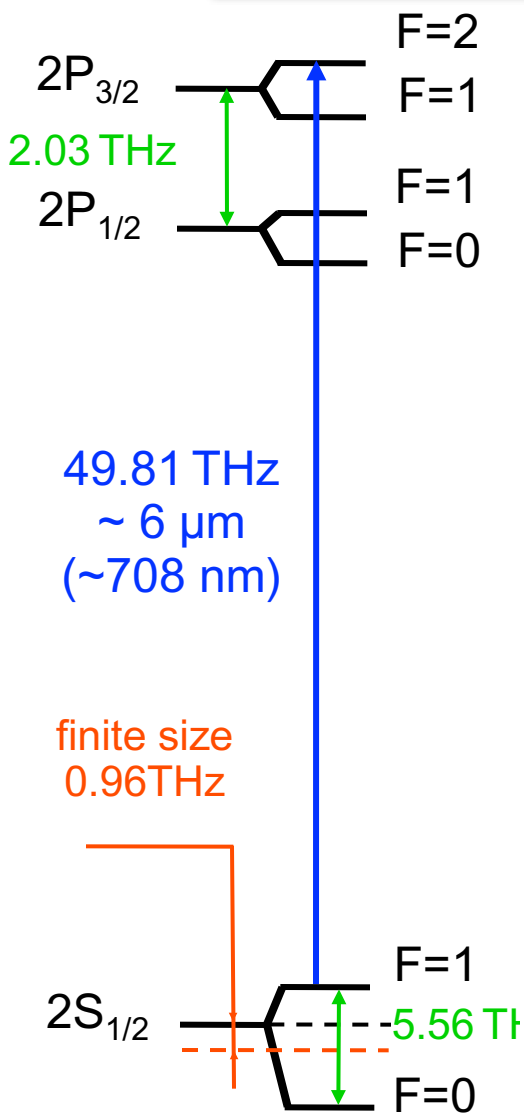


Water-line/laser
wavelength:
300 MHz uncertainty



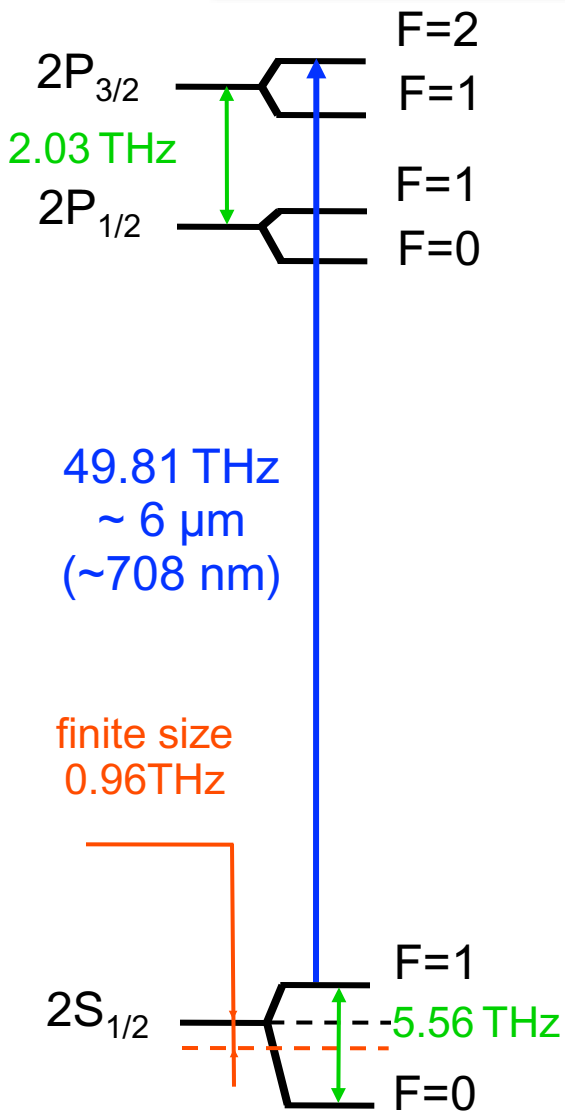
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muonic hydrogen : $^2S_{1/2}(F=1) - ^2P_{3/2}(F=2)$



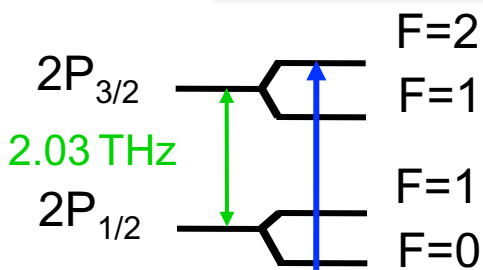
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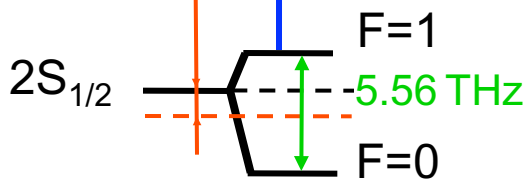


49.81 THz
 ~ 6 μm
 (~708 nm)

550 events measured on resonance
 where 155 bgr events are expected
 fit Lorentz + flat bgr $\Rightarrow \chi^2/\text{dof} = 28.1/28$

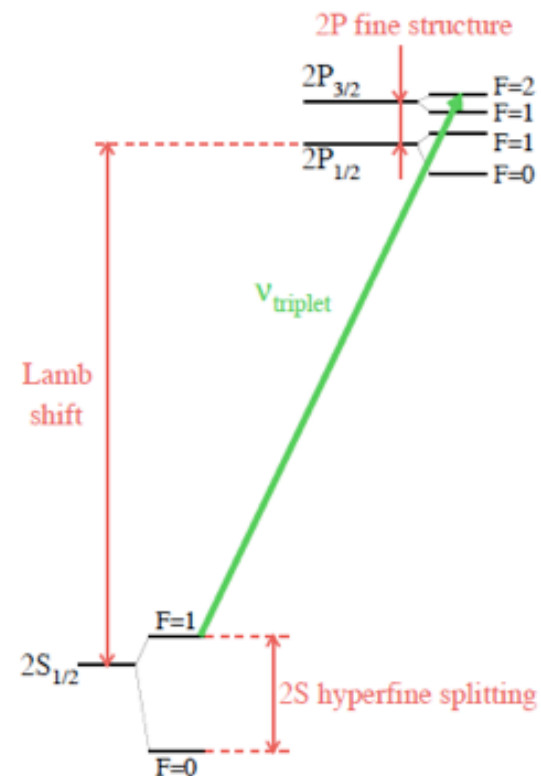
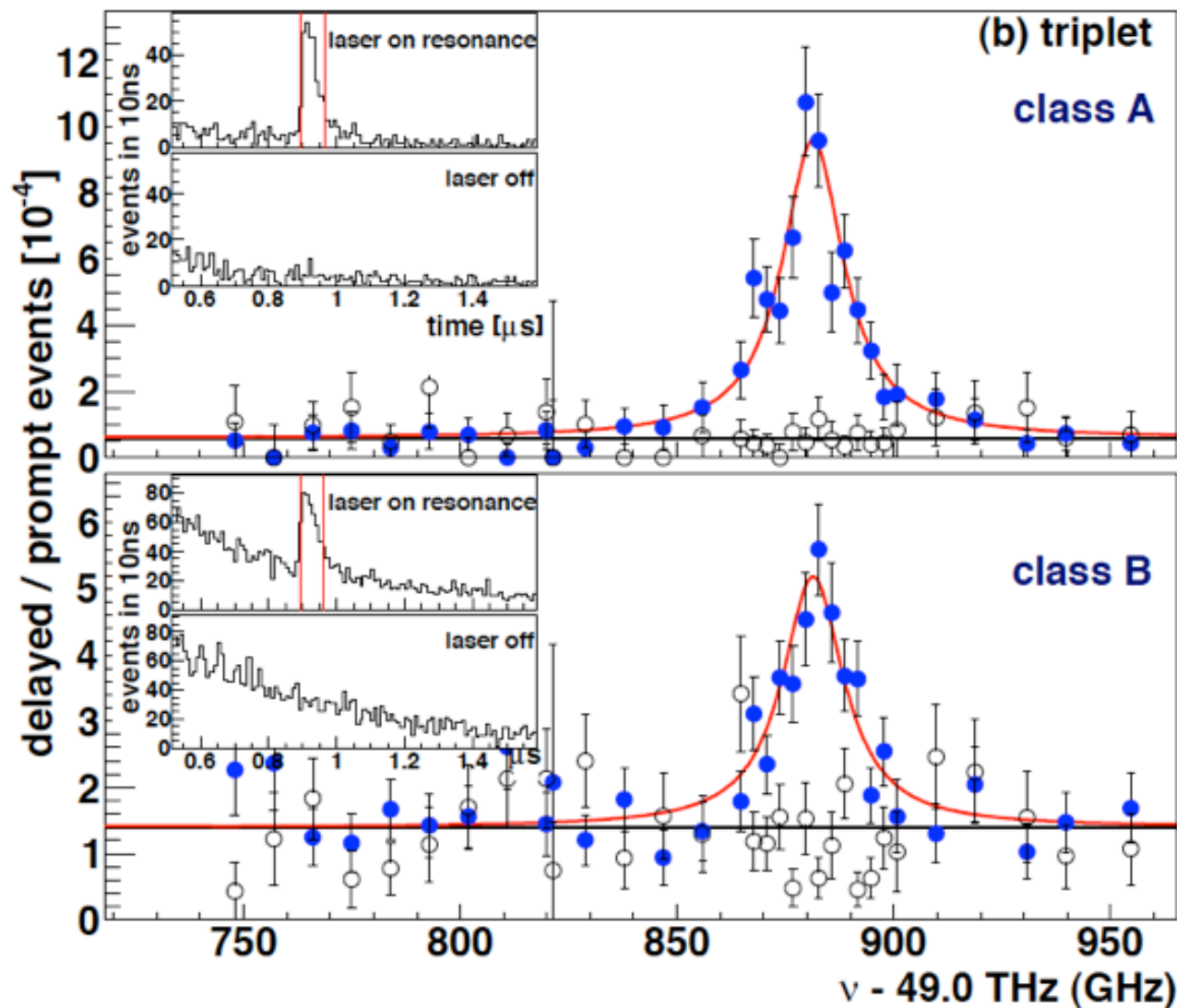
width agrees with expectation
 bgr agrees with laser OFF data
 $\chi^2/\text{dof} = 283/31$ for flat line $\rightarrow 16\sigma$

finite size
 0.96 THz



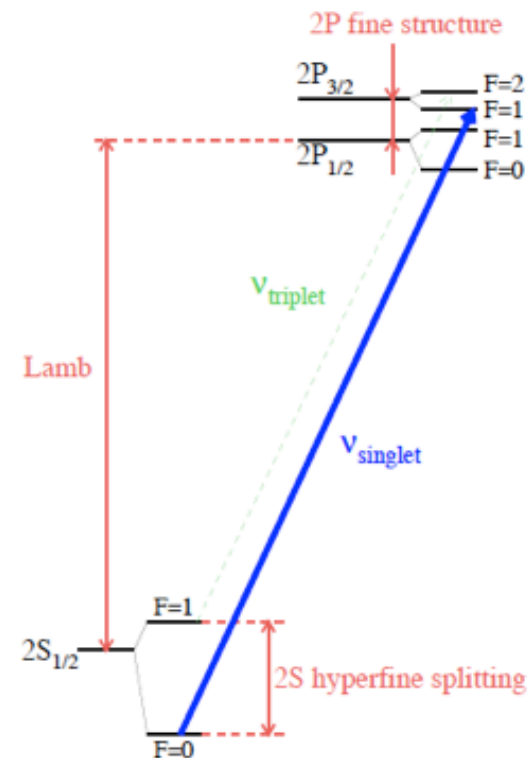
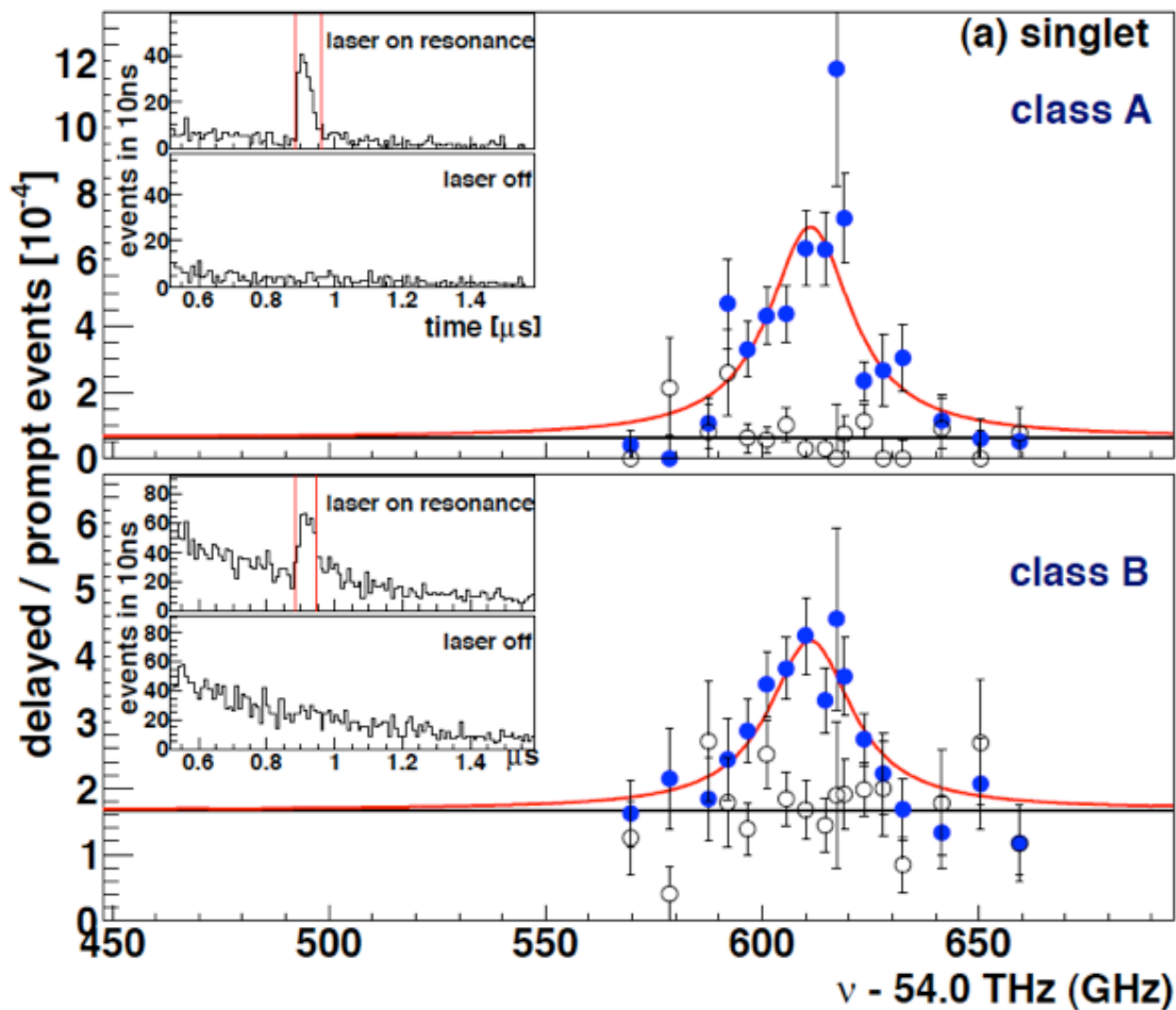
R. Pohl, A. Antognini, F. Nez, et al., Nature 466, 213 (2010).

$\mu p (2S_{1/2}(F=1) \rightarrow 2P_{3/2}(F=2))$ at $\lambda = 6.0 \mu\text{m}$



A. Antognini *et al.*, submitted (2012)

$\mu\text{P} (2S_{1/2}(F=0) \rightarrow 2P_{3/2}(F=1))$ at $\lambda = 5.5 \mu\text{m}$



A. Antognini *et al.*, submitted (2012)

Statistics

- uncertainty on position (fit)

541 MHz ($\sim 3\%$ of Γ_{nat})

$$\Delta\nu_{\text{experimental}} = 20 (1) \text{ GHz} \quad (\Gamma_{\text{nat}} = 18.6 \text{ GHz})$$

Sources :

- Laser frequency (H_2O calibration, lines known to ~ 1 MHz) 300 MHz
- AC and DC stark shift < 1 MHz
- Zeeman shift (5 Telsa) < 30 MHz
- Doppler shift < 1 MHz
- Collisional shift 2 MHz

TOTAL UNCERTAINTY ON FREQUENCY

618 MHz

Broadening :

- 6 μm laser line width ~ 2 GHz
- Doppler Broadening < 1 GHz
- Collisional broadening 2.4 MHz

Updated: $\nu (\mu\text{p} : {}^2\text{S}_{1/2}(F=1) - {}^2\text{P}_{3/2}(F=2)) < 1\sigma$ (12.5 ppm)

Nature: $\nu (\mu\text{p} : {}^2\text{S}_{1/2}(F=1) - {}^2\text{P}_{3/2}(F=2)) = 49\,881.88 (76) \text{ GHz}$ (16 ppm)

Statistics

- uncertainty on position (fit)

960 MHz

Sources :

- Laser frequency (H_2O calibration)
- AC and DC stark shift
- Zeeman shift (5 Telsa)
- Doppler shift
- Collisional shift

300 MHz

< 1 MHz

< 30 MHz

< 1 MHz

2 MHz

TOTAL UNCERTAINTY ON FREQUENCY

1006 MHz

Broadening :

- 6 μm laser line width
- Doppler Broadening
- Collisional broadening

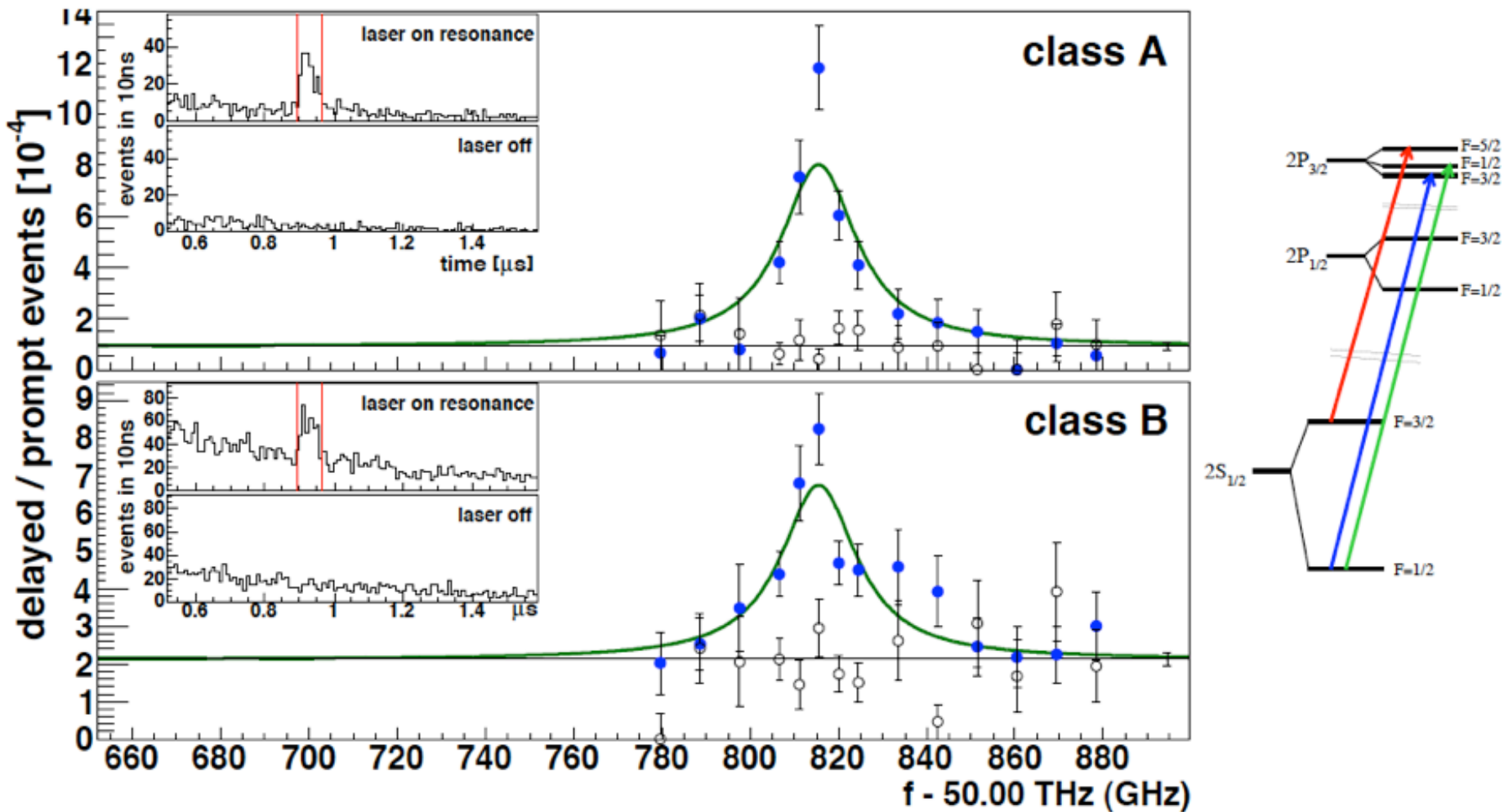
~ 2 GHz

< 1 GHz

2.4 MHz

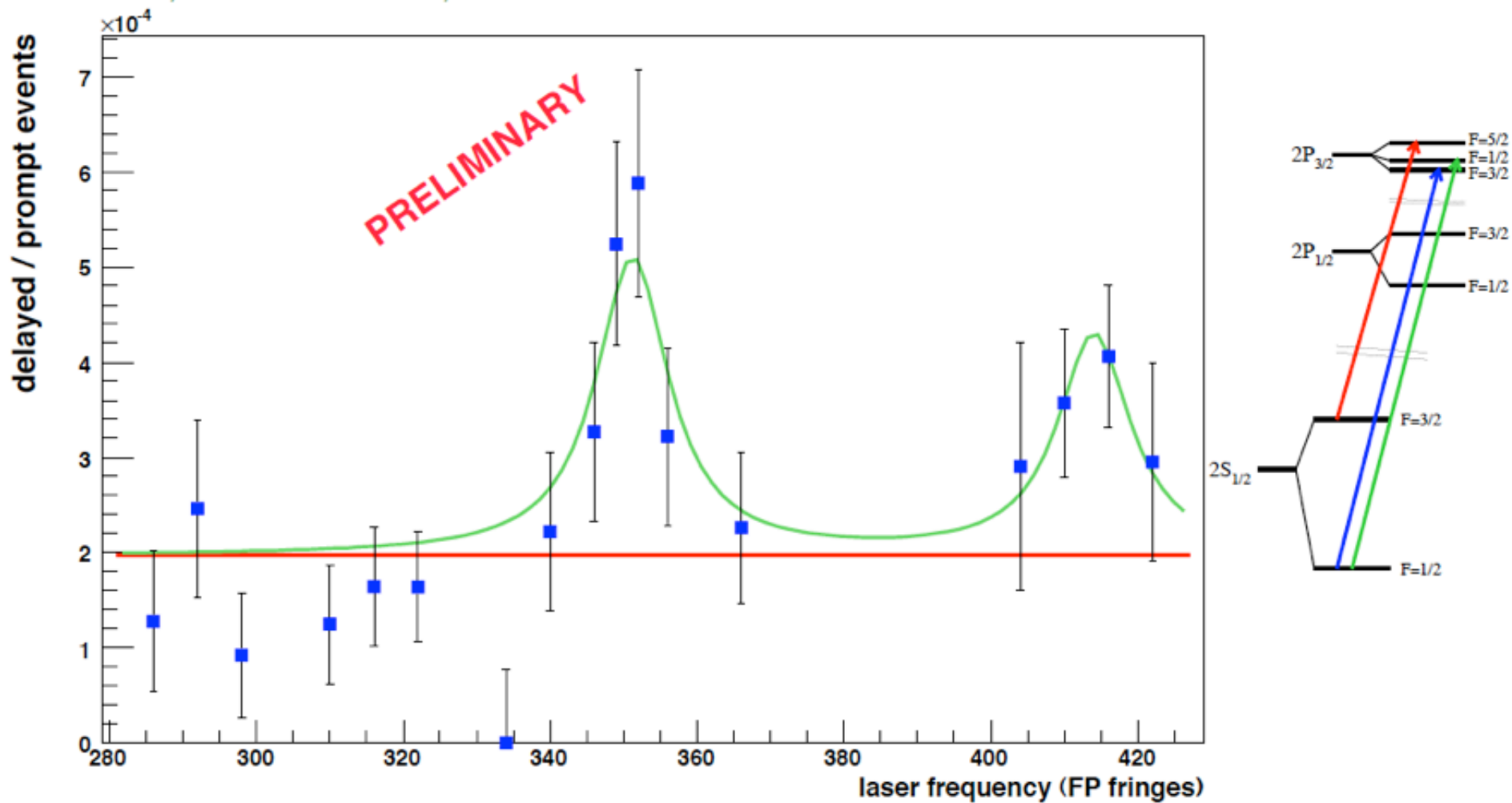
ν ($\mu\text{p} : {}^2\text{S}_{1/2}(F=0) - {}^2\text{P}_{3/2}(F=1)$) good agreement with the other
(18.5ppm)

$\mu d ({}^2S_{1/2}(F=3/2) \rightarrow {}^2P_{3/2}(F=5/2)) : 50\,815.5(8)(3) \text{ GHz}$ still preliminary



μd ($2S_{1/2}(F=1/2) \rightarrow 2P_{3/2}(F=3/2)$) : $52\,061 \pm 2.3_{\text{stat}}$ GHz VERY preliminary

μd ($2S_{1/2}(F=1/2) \rightarrow 2P_{3/2}(F=1/2)$) : $52\,156 \pm 3.5_{\text{stat}}$ GHz VERY preliminary







Extraction of the radii

Charge, magnetic and Zemach's radii

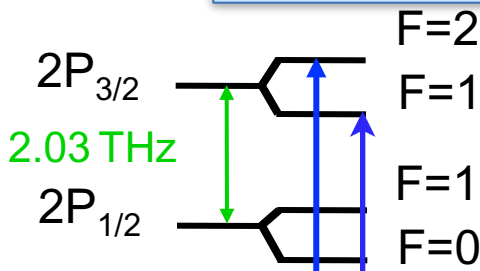
- 2010 CODATA value uses improved theory for hydrogen and Mainz electron-proton scattering is now at 6.9σ mostly by a reduction of σ :
 - 0.8775 (59) fm 2010
 - 0.8768 (69) fm 2006
- We have analyzed in details the second transition that was observed, using an improved algorithm that correct for the variation of the laser pulse energy from shot to shot
- We take into account more events
- We have reanalyzed the first observed line using the improved method
- This lead to a slightly reduced error bar for the first transition, an accurate value of a second transition which allows to
 - Get a measurement of the magnetic moment distribution mean radius
 - An improved charge radius

Discrepancy=0.31 meV
 Th. uncertainty=0.005 meV
 $\implies 60\delta(\text{theory})$ deviation

Main contributions to the μp Lamb shift

Discrepancy	
Polarisability	How well is it calculated?
Finite size	How dependent on nuclear model?
Recoil	QED
Muon self-energy + muon VP	 QED
Källen Sabry	 QED
One-loop VP	 QED

■ The two **measured** lines obey to:



$$E_{5P_{3/2}} - E_{3S_{1/2}} = \Delta E_{LS} + \Delta E_{FS} + \frac{3}{8} \Delta E_{HFS}(2p_{3/2}) - \frac{1}{4} \Delta E_{HFS}(2s)$$

$$E_{3P_{3/2}} - E_{1S_{1/2}} = \Delta E_{LS} + \Delta E_{FS} - \frac{5}{8} \Delta E_{HFS}(2p_{3/2}) + \frac{3}{4} \Delta E_{HFS}(2s)$$

49.81 THz
~ 6 μm
(~708 nm)

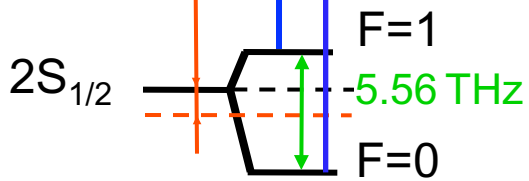
finite size
0.96 THz

Lamb shift

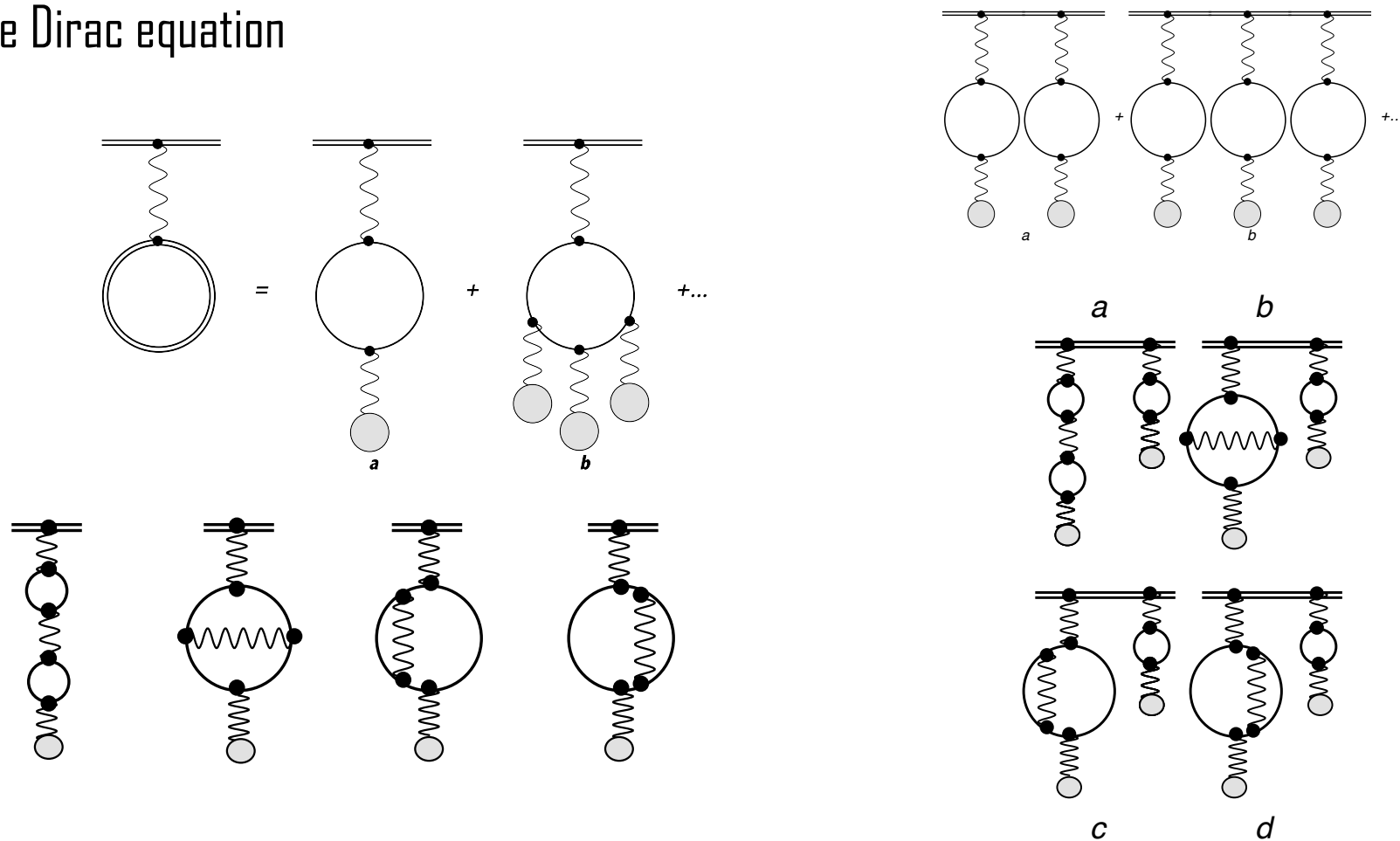
Fine structure

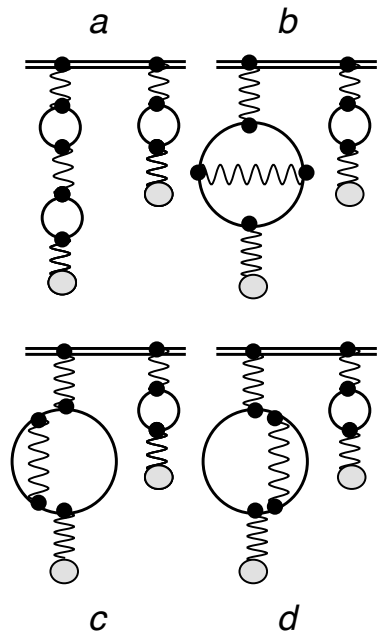
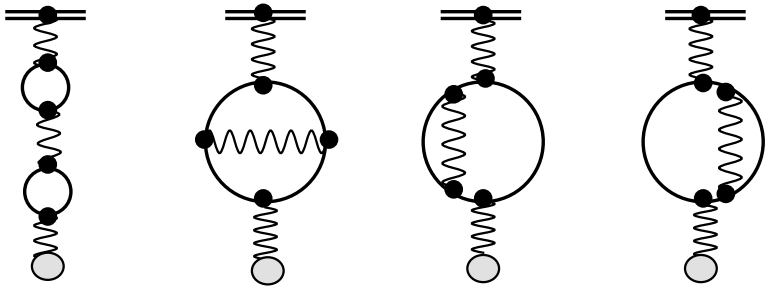
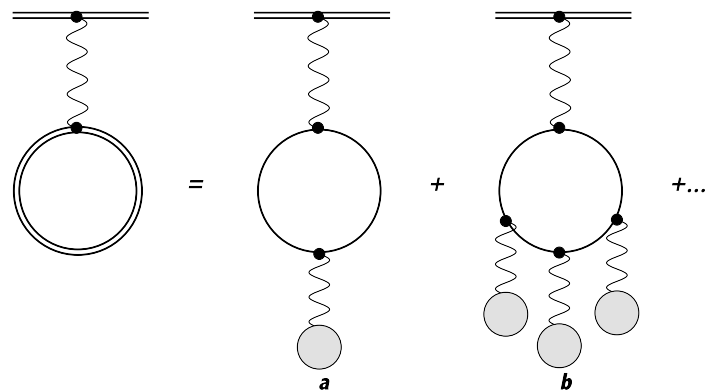
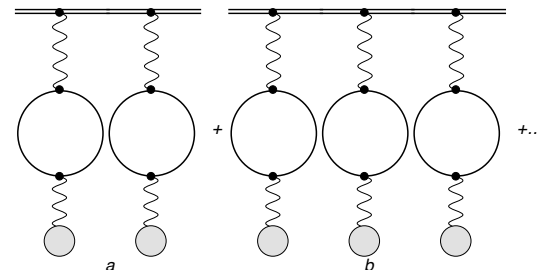
2p Hyperfine structure

2s hyperfine structure



All-order: the charge distribution is included exactly in the wavefunction and in the operator, when relevant. Higher order Vacuum Polarization contribution included by numerical solution of the Dirac equation





$$(\mathbf{c}\boldsymbol{\alpha} \cdot \mathbf{p} + \beta\mu_r c^2 + V_{\text{Nuc}}(\mathbf{r})) \Phi_{n\kappa\mu}(\mathbf{r}) = \mathcal{E}_{n\kappa\mu} \Phi_{n\kappa\mu}(\mathbf{r}),$$

Bohr radius/particle mass:

$$\begin{aligned} V_{11}^{pn}(r) &= -\frac{\alpha(Z\alpha)}{3\pi} \int_1^\infty dz \sqrt{z^2 - 1} \left(\frac{2}{z^2} + \frac{1}{z^4} \right) \frac{e^{-2m_e r z}}{r} \\ &= -\frac{2\alpha(Z\alpha)}{3\pi} \frac{1}{r} \chi_1 \left(\frac{2}{\lambda_e} r \right) \end{aligned}$$

Electron Compton wavelength/ $2\pi=440$ R

$n=1$ in hydrogen: $a_0=137\lambda_e=60340$ R

$n=2$ in muonic H: $a=2.65\lambda_e$

$n=1$ in h-like ($Z=52$): $a=2.65\lambda_e$

- Fit to the Coulomb+Vacuum polarization contribution to $2s$ - $2p_{1/2}$ separation, plus higher order corrections using Friar functional form

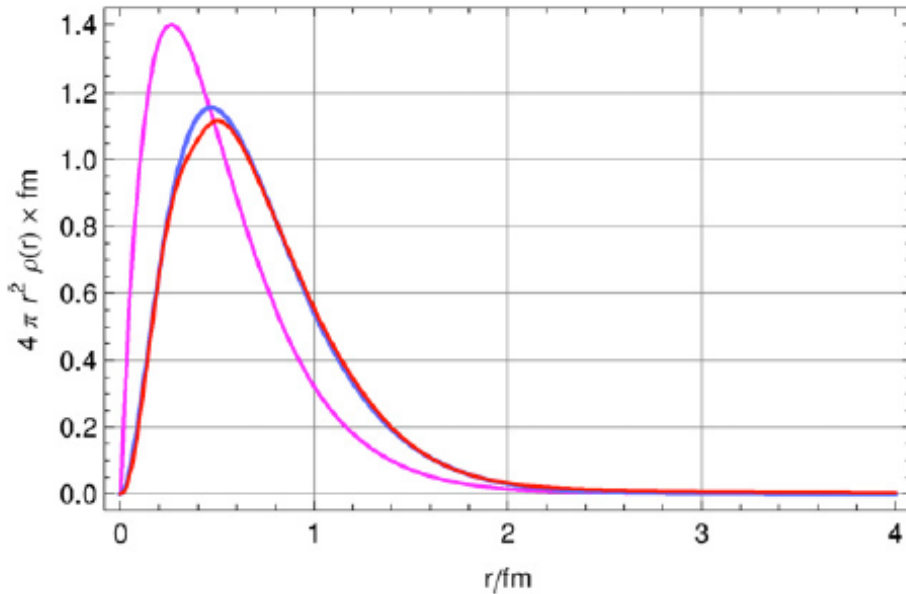
$$\begin{aligned} E_{2p_{1/2}}^{\text{Tot,fs}} - E_{2s_{1/2}}^{\text{Tot,fs}}(R) = & 206.046613695 - 5.226988678R^2 \\ & + 0.03532068001R^3 \\ & + 0.00006692700063R^4 \\ & + 0.0002962967640R^2 \log(R) \\ & - 0.00004751147090R^4 \log(R) \text{ meV.} \end{aligned}$$

#	Contribution	Reference	Value	Unc.
1	NR three-loop electron VP (Eq. (11), (15), (18) and (23))	[73]	0.00529	
2	Virtual Delbrück scattering (2:2)	[75, 78]	0.00115	0.00001
3	Light by light electron loop contribution (3:1)	[75, 78]	-0.00102	0.00001
4	Mixed self-energy vacuum polarization	[35, 84, 107]	-0.00254	
5	Hadronic vacuum polarization	[108–110]	0.01121	0.00044
6	Recoil contribution Eqs. (82) and (83)	[11, 36, 62, 85]	0.05747063	
7	Relativistic recoil of order $(Z\alpha)^5$ Eq. (84)	[11, 37–39, 41]	-0.04497053	
8	Relativistic Recoil of order $(Z\alpha)^6$ Eq. (86)	[11, 37]	0.0002475	
9	Recoil correction to VP of order m/M and $(m/M)^2$ in Eq. (4)	[72]	-0.001987	
10	Proton Self-energy	[35, 37, 41, 111]	-0.0108	0.0010
11	Proton polarization	[18, 37, 109, 112, 113]	0.0129	0.0040
12	Electron loop in the radiative photon of order $\alpha^2(Z\alpha)^4$	[98, 114–116]	-0.00171	
13	Mixed electron and muon loops	[117]	0.00007	
14	Rad. Recoil corr. $\alpha(Z\alpha)^5$	[61]	0.000136	
15	Hadronic polarization $\alpha(Z\alpha)^5 m_r$	[109, 110]	0.000047	
16	Hadronic polarization in the radiative photon $\alpha^2(Z\alpha)^4 m_r$	[109, 110]	-0.000015	
17	Polarization operator induced correction to nuclear polarizability $\alpha(Z\alpha)^5 m_r$	[110]	0.00019	
18	Radiative photon induced correction to nuclear polarizability $\alpha(Z\alpha)^5 m_r$	[110]	-0.00001	
	Total		0.0256	0.0041

The role of the nuclear model

Dependence on the charge distribution





- $\langle r^3 \rangle_{(2)} = 3.789 \langle r^2 \rangle^{3/2}$ Dipole
- $\langle r^3 \rangle_{(2)} = 1.960 \langle r^2 \rangle^{3/2}$ Gauss
- $\langle r^3 \rangle_{(2)} = 3.91 \langle r^2 \rangle^{3/2}$ Arrington (2007)
- $\langle r^3 \rangle_{(2)} = 3.78(13) \langle r^2 \rangle^{3/2}$ Friar & Sick (2005)
- $\langle r^3 \rangle_{(2)} = 4.18(13) \langle r^2 \rangle^{3/2}$ Distler, Bernauer, Walcher (2011)
- $\langle r^3 \rangle_{(2)} = 36.6 \pm 7.3 = 51 \langle r^2 \rangle^{3/2}$ De Rújula (2010), retracted (?) in 2011...

[1] QED is not endangered by the proton's size, A. De Rújula. Physics Letters B 693, 555-558 (2010)

[2] QED confronts the radius of the proton, A. De Rújula. Physics Letters B 697, 26-31 (2011)

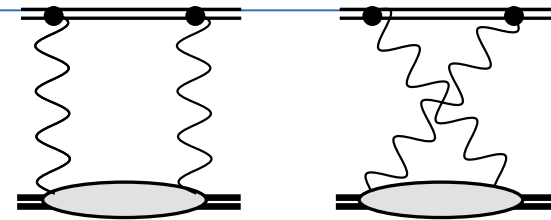
[3] The RMS charge radius of the proton and Zemach moments, M.O. Distler, J.C. Bernauer et T. Walcher. Physics Letters B 696, 343-347 (2011).

- Using the electronic density from Arrington et al. I get
 - $R_p = 0.85035$ fm $R_m = 0.831$ fm $R_z = 1.0466$ fm
- Dirac + Uehling vacuum polarization with this density:
 - $E = 201.2789$ meV
- Dirac + Uehling vacuum polarization with same radius and other models
 - Gauss: $E = 201.2680$ (-0.0109) meV
 - Dipole: $E = 201.2700$ (-0.0089) meV
 - Uniform: $E = 201.2669$ (-0.0120) meV
 - Fermi: $E = 201.2686$ (-0.0102)
- Solving $E_{\text{Dipole}}(R) = 201.2789$ meV gives:
 - $R = 0.84934$ (-0.00101) fm

- Several calculations

- Rosenfelder (1999)

$$\Delta E_{2s}^{\text{p.pol}} = -\frac{136 \pm 30}{n^3} \mu\text{eV} = -0.017 \pm 0.004 \text{ meV.}$$



- Pachucki (1999) $\Delta E_{2s}^{\text{p.pol}} = -0.012 \pm 0.002 \text{ meV,}$

- Martynenko (2006) $\Delta E_{2s}^{\text{p.pol}} = \Delta E^{\text{subt.}} + \Delta E^{\text{inel.}}$
 $= 0.0023 - 0.01613 \text{ meV}$
 $= -0.0138(29) \text{ meV,}$

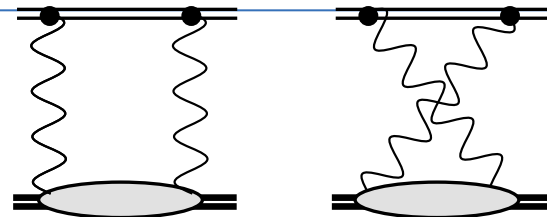
- Carlson and Vanderhaeghen (2011) $\Delta E_{2s}^{\text{p.pol}} = \Delta E^{\text{subt.}} + \Delta E^{\text{inel.}} + \Delta E^{\text{el.}}$
 $= 0.0053(19) - 0.0127(5) - 0.0295(13) \text{ meV}$
 $= -0.0074(20) - 0.0295(13) \text{ meV.}$

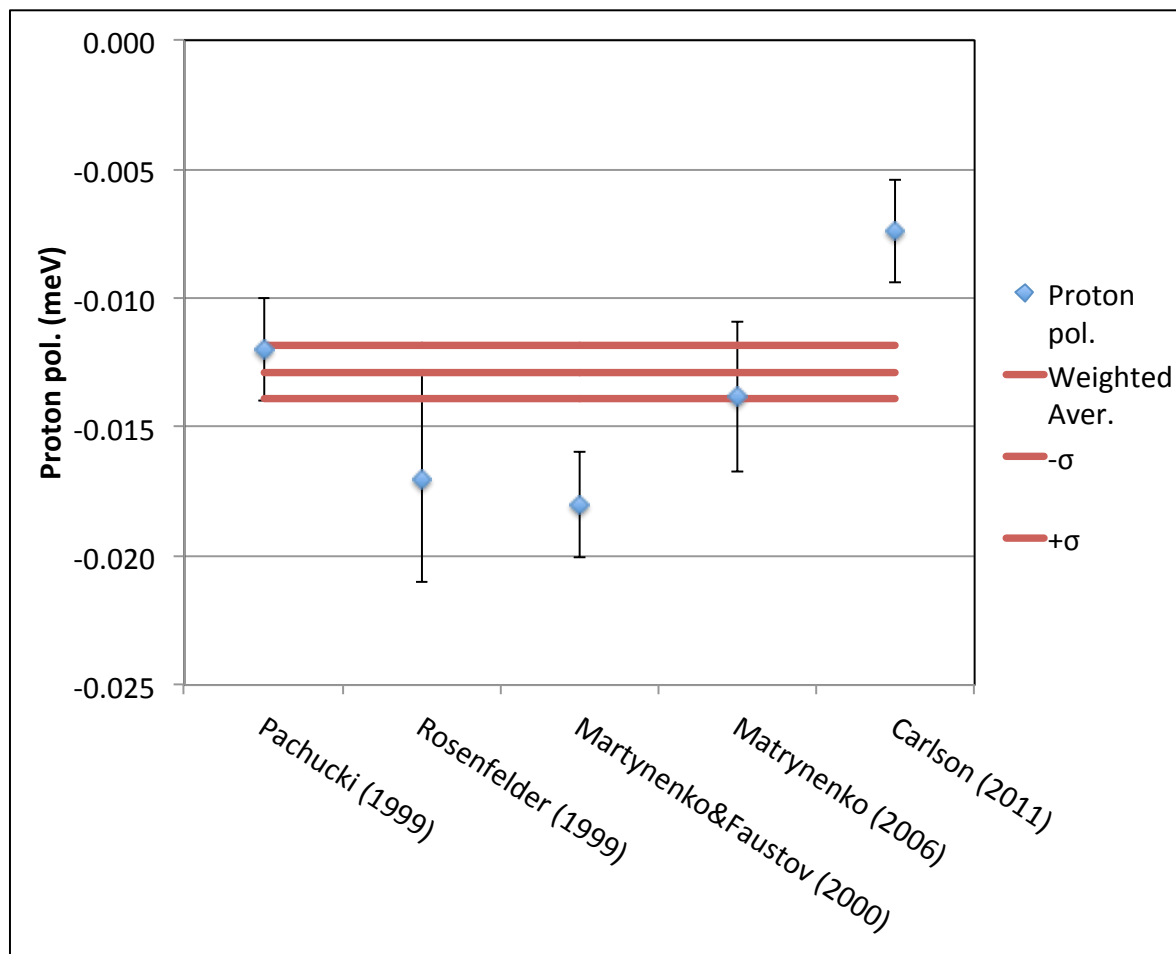
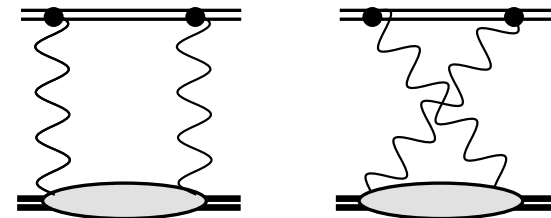
- Hill and Paz (2011+DPF 2011) $\Delta E_{2s}^{\text{p.pol}} = \Delta E^{\text{subt.}} + \Delta E^{\text{inel.}}$
 $= [\delta E^{W_1(0,Q^2)} + \delta E^{\text{proton pole}}] + \delta E^{\text{continuum}}$

Could be wrong by 0.04 meV \longrightarrow $[\delta E^{W_1(0,Q^2)} + 0016] - 0.0127(5) \text{ meV,}$

- Several calculations

- McGovern and Birse (2012): chiral perturbation theory
 - OK with previous calculations, Hill and Paz hypothesis does not hold





Hyperfine structure

Beyond Zemach

$$\Delta E_{M1}^{HFS} = A \frac{g\alpha}{2M_p} \int_0^\infty dr \frac{P_1(r)Q_2(r) + P_2(r)Q_1(r)}{r^2},$$

$$\begin{aligned} \Delta E^{BW} &= -A \frac{g\alpha}{2M_p} \int_0^\infty dr_n r_n^2 \mu(r_n) \\ &\quad \times \int_0^{r_n} dr \frac{P_1(r)Q_2(r) + P_2(r)Q_1(r)}{r^2}, \end{aligned}$$

$$\begin{aligned} \Delta E_{HFS}^Z &= -\frac{2}{3} \langle \mathbf{S}_p \cdot \mathbf{S}_\mu \rangle |\phi_C(0)|^2 \\ &\quad \times \left(1 - 2\alpha m_\mu \int \rho(\mathbf{u}) |\mathbf{u} - \mathbf{r}| \mu(r) d\mathbf{u} dr \right), \\ &= E_F \left(1 - 2\alpha m_\mu \int \rho(\mathbf{u}) |\mathbf{u} - \mathbf{r}| \mu(r) d\mathbf{u} dr \right), \end{aligned}$$

$$\Delta E_{HFS}^Z = E_F \left(1 - 2\alpha m_\mu \langle r_Z \rangle \right),$$

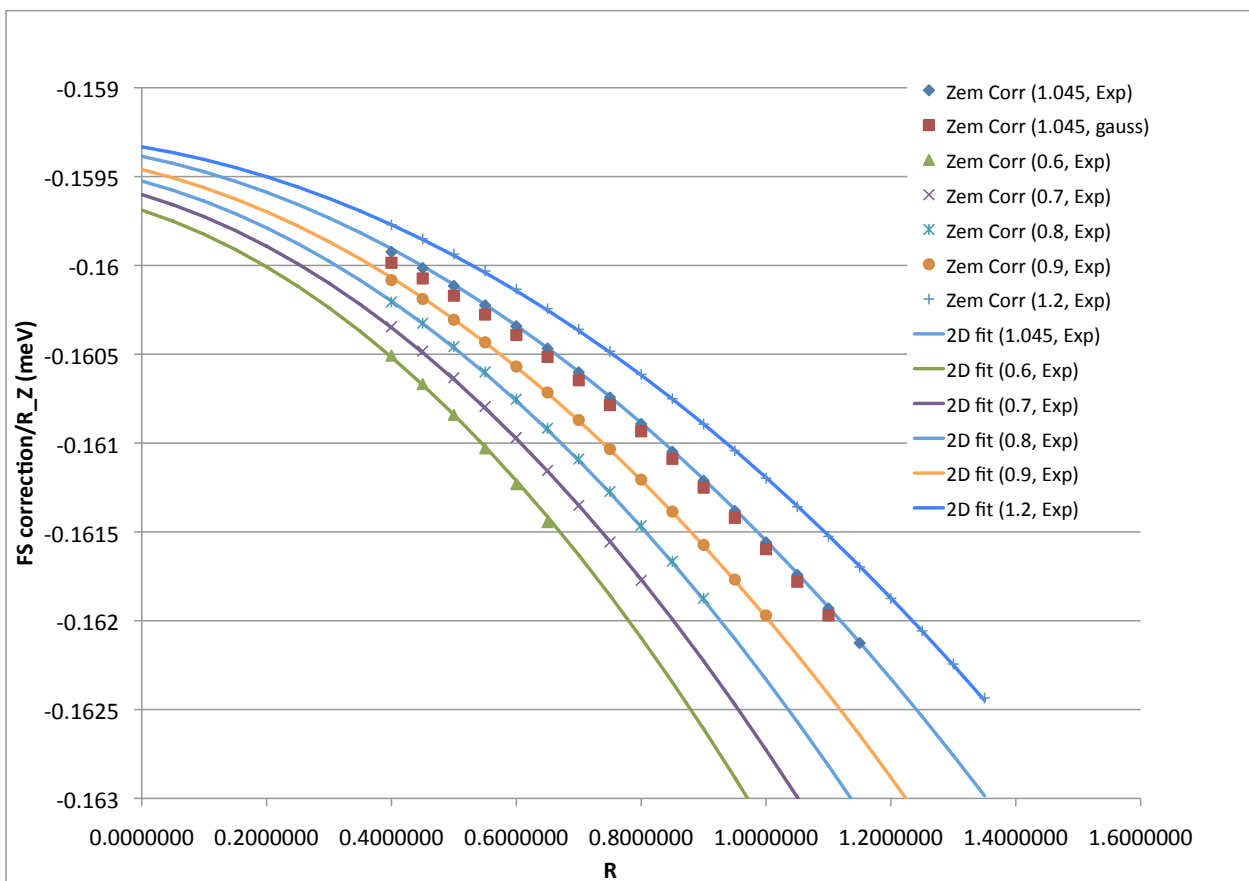
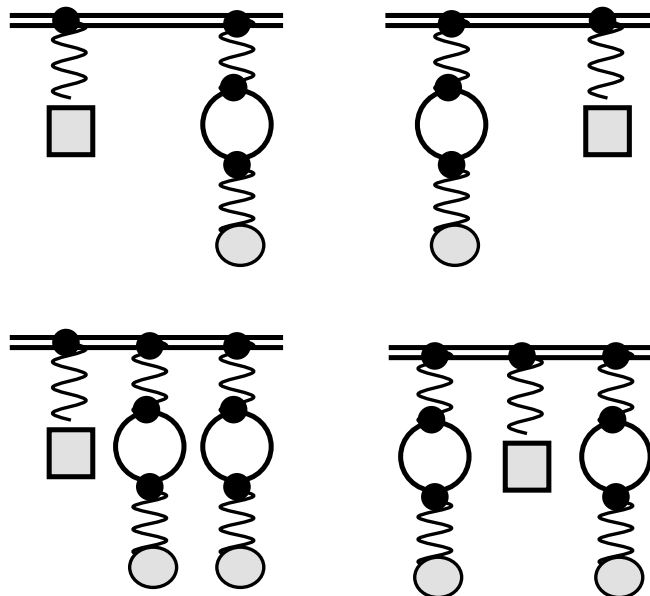


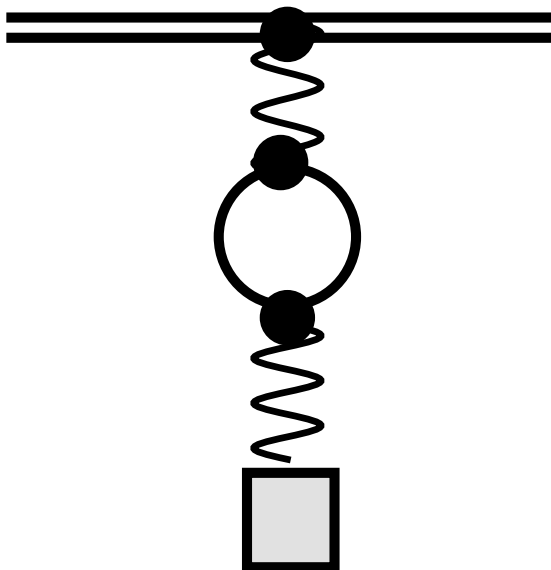
Fig. 10. Finite charge and magnetic moment distribution energy shift for the $2s$ state as a function of the charge R and Zemach radius R_Z for the Gaussian ($R_Z = 1.045\text{fm}$) and exponential model, divided by R_Z . The lines correspond to the function in Eq. (90).

$$\begin{aligned}
 E_{\text{HFS}}^{2s}(R_Z, R) = & 22.807995 \\
 & - 0.0022324349R^2 + 0.00072910794R^3 \\
 & - 0.000065912957R^4 - 0.16034434R_Z \\
 & - 0.00057179529RR_Z \\
 & - 0.00069518048R^2R_Z \\
 & - 0.00018463878R^3R_Z \\
 & + 0.0010566454R_Z^2 \\
 & + 0.00096830453RR_Z^2 \\
 & + 0.00037883473R^2R_Z^2 \\
 & - 0.00048210961R_Z^3 \\
 & - 0.00041573690RR_Z^3 \\
 & + 0.00018238754R_Z^4 \text{ meV.}
 \end{aligned}$$

$$\begin{aligned}
 E_{\text{HFS}}^{2s,VP}(R_Z, R) = & 0.074369030 + 0.000074236132R^2 \\
 & + 0.00013277334R^3 - 8.0987285 \times 10^{-6}R^4 \\
 & - 0.0017880269R_Z - 0.00017204505RR_Z \\
 & - 0.00037499458R^2R_Z \\
 & - 0.000070355379R^3R_Z \\
 & - 0.00022093411R_Z^2 + 0.00035038656RR_Z^2 \\
 & + 0.00020554316R^2R_Z^2 + 0.00025100642R_Z^3 \\
 & - 0.00017200435RR_Z^3 \\
 & - 0.000061266973R_Z^4 \text{ meV.}
 \end{aligned}$$



Full calculation beyond Zemach



Would need to be evaluated as well

	#	Ref. [40]	Ref. [70]	This work
Fermi energy	1	22.8054	22.8054	
Dirac Energy (includes Breit corr.)	2			22.807995
Vacuum polarization corrections of orders α^5, α^6 in 2nd-order perturbation theory ϵ_{VP1}	3	0.0746	0.07443	
All-order VP contribution to HFS, with finite magnetisation distribution	4			0.07244
finite extent of magnetisation density correction to the above	5		-0.00114	
Proton structure corr. of order α^5	6	-0.1518	-0.17108	-0.17173
Proton structure corrections of order α^6	7	-0.0017		
Electron vacuum polarization contribution+ proton structure corrections of order α^6	8	-0.0026		
contribution of 1γ interaction of order α^6	9	0.0003	0.00037	0.00037
$\epsilon_{VP}2E_F$ (neglected in Ref. [40])	10		0.00056	0.00056
muon loop VP (part corresponding to ϵ_{VP2} neglected in Ref. [40])	11		0.00091	0.00091
Hadronic Vac. Pol.	12	0.0005	0.0006	0.0006
Vertex (order α^5)	13		-0.00311	-0.00311
Vertex (order α^6) (only part with powers of $\ln(\alpha)$ - see Ref. [103])	14		-0.00017	-0.00017
Breit	15	0.0026	0.00258	
Muon anomalous magnetic moment correction of order α^5, α^6	16	0.0266	0.02659	0.02659
Relativistic and radiative recoil corrections with proton anomalous magnetic moment of order α^6	17	0.0018		
One-loop electron vacuum polarization contribution of 1γ interaction of orders α^5, α^6 (ϵ_{VP2})	18	0.0482	0.04818	0.04818
finite extent of magnetisation density correction to the above	19		-0.00114	-0.00114
One-loop muon vacuum polarization contribution of 1γ interaction of order α^6	20	0.0004	0.00037	0.00037
Muon self energy+proton structure correction of order α^6	21	0.001		0.001
Vertex corrections+proton structure corrections of order α^6	22	-0.0018		-0.0018
"Jellyfish" diagram correction+ proton structure corrections of order α^6	23	0.0005		0.0005
Recoil correction Ref. [104]	24		0.02123	0.02123
Proton polarizability contribution of order α^5	25	0.0105		
Proton polarizability Ref. [104]	26		0.00801	0.00801
Weak interaction contribution	27	0.0003	0.00027	0.00027
Total		22.8148	22.8129	22.8111

- Using the second line measured during the experiment we can improve the charge radius and get a value for the Zemach's radius.

$$\begin{aligned}
 E_{2p_{3/2}}^{F=2} - E_{2s_{1/2}}^{F=1}(R_Z, R) = & 209.92451 - 5.2265012R^2 \\
 & + 0.035105381R^3 \\
 & + 0.000085386880R^4 \\
 & + 1.5472388 \times 10^{-8}R^5 \\
 & - 2.1359270 \times 10^{-9}R^6 \\
 & + 0.040533092Rz \\
 & + 0.00018596008RRz \\
 & + 0.00026754376R^2Rz \\
 & + 0.000063748539R^3Rz \\
 & - 0.00020892783Rz^2 \\
 & - 0.00032967277RRz^2 \\
 & - 0.00014609447R^2Rz^2 \\
 & + 0.000057775798Rz^3 \\
 & + 0.00014693531RRz^3 \\
 & - 0.000030280142Rz^4 \\
 & + 0.00029629676R^2 \log(R) \\
 & - 0.000047511471R^4 \log(R) \\
 & \text{meV.}
 \end{aligned}$$

$$\begin{aligned}
 E_{2p_{3/2}}^{F=1} - E_{2s_{1/2}}^{F=0}(R_Z, R) = & 229.66172 - 5.2286594R^2 \\
 & + 0.035967212R^3 \\
 & + 0.000011416693R^4 \\
 & - 0.12159928R_Z - 0.00055788025RR_Z \\
 & - 0.00080263129R^2R_Z \\
 & - 0.00019124562R^3R_Z \\
 & + 0.00062678350R_Z^2 \\
 & + 0.00098901832RR_Z^2 \\
 & + 0.00043828342R^2R_Z^2 \\
 & - 0.00017332740R_Z^3 \\
 & - 0.00044080593RR_Z^3 \\
 & + 0.000090840426R_Z^4 \\
 & + 0.00029629676R^2 \log(R) \\
 & - 0.000047511471R^4 \log(R) \\
 & \text{meV.}
 \end{aligned}$$

Simultaneous solution of these two equations with the two line energies

$R_c = 0.84100(63)$ fm and $R_z = 1.086(40)$ fm

Assuming a dipole model, this gives R_m : $0.879(50)$ fm

Mainz results: $R_c = 0.879$ fm, $R_m = 0.777$ fm and $R_z = 1.047$ fm

- We can extract the magnetic radius by using the dipole model to be consistent with the calculations.

$$R_Z^{\text{Exp.}} = \frac{3R^4 + 9R^3R_M + 11R^2R_M^2 + 9RR_M^3 + 3R_M^4}{2\sqrt{3}(R + R_M)^3}$$

Simultaneous solution of the two equations with the two line energies

$R_c = 0.84100(63)$ fm and $R_z = 1.086(40)$ fm

Assuming a dipole model, this gives R_m : $0.879(50)$ fm

Mainz results: $R_c = 0.879$ fm, $R_m = 0.777$ fm and $R_z = 1.047$ fm

A magnetic radius larger than the charge radius leads to large discrepancies when applied to electron proton scattering data

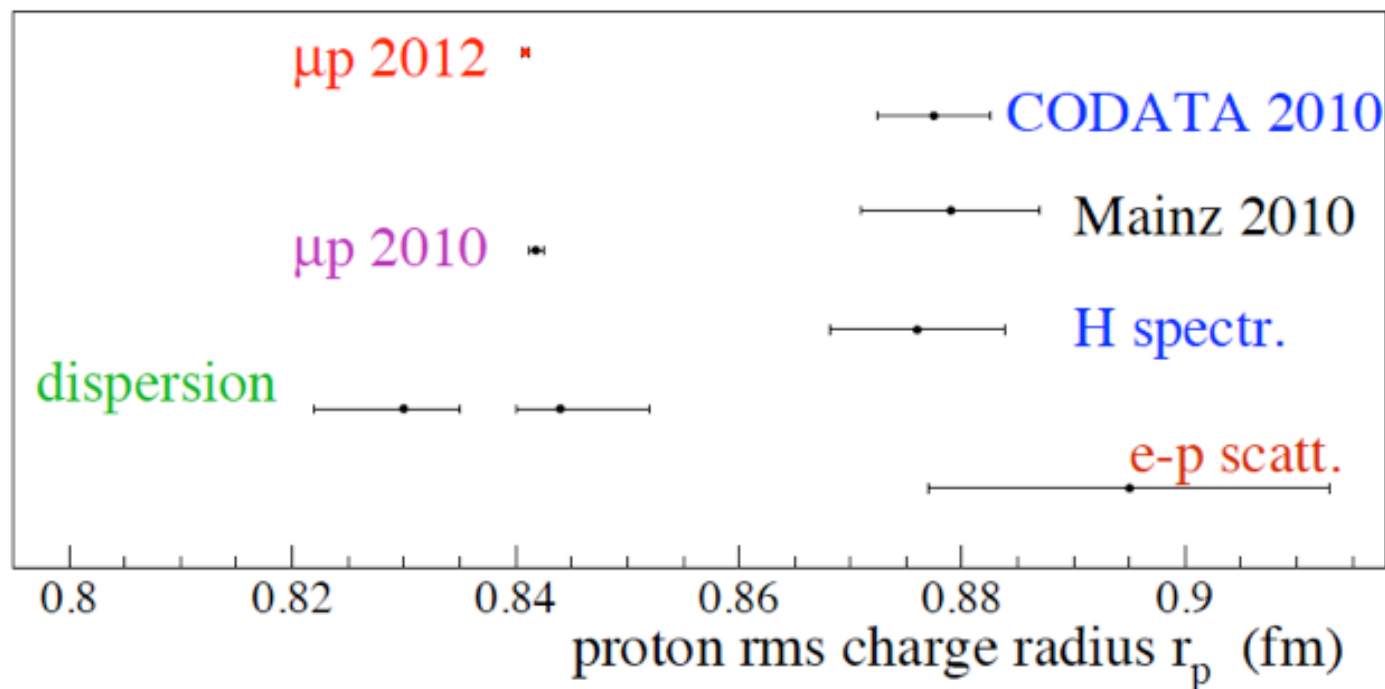
$$\nu(2S_{1/2}^{F=1} \rightarrow 2P_{3/2}^{F=2}) = 49881.88(76) \text{ GHz} \quad \text{R. Pohl et al., Nature 466, 213 (2010)}$$

$$49881.35(64) \text{ GHz}$$

$$\nu(2S_{1/2}^{F=0} \rightarrow 2P_{3/2}^{F=1}) = 54611.16(1.04) \text{ GHz} \quad \text{A. Antognini et al., submitted (2012)}$$

Proton charge radius: $r_p = 0.84089 (26)_{\text{exp}} (29)_{\text{th}} = 0.84089 (39) \text{ fm}$

μp theory: A. Antognini et al., arXiv :1208.2637 (atom-ph)



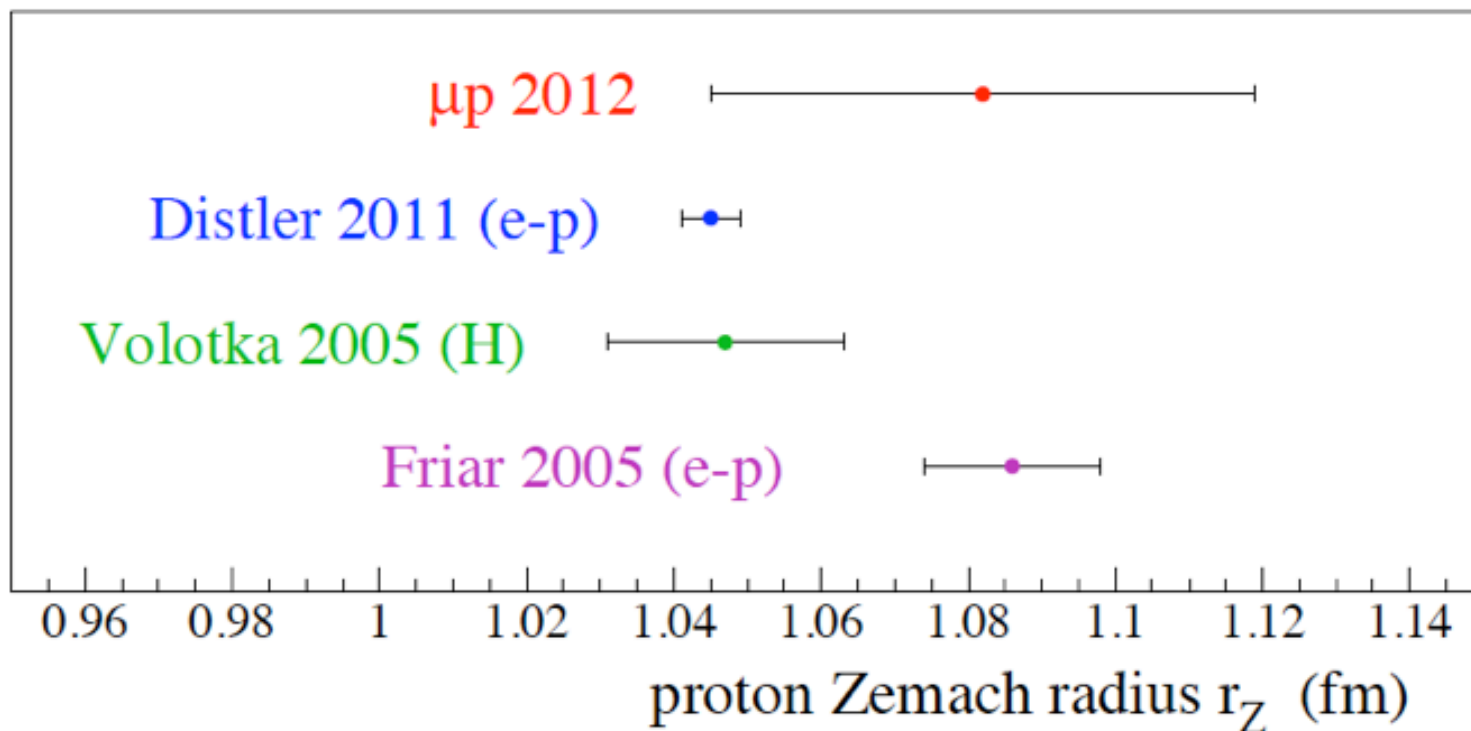
Summary of results: Zemach radius

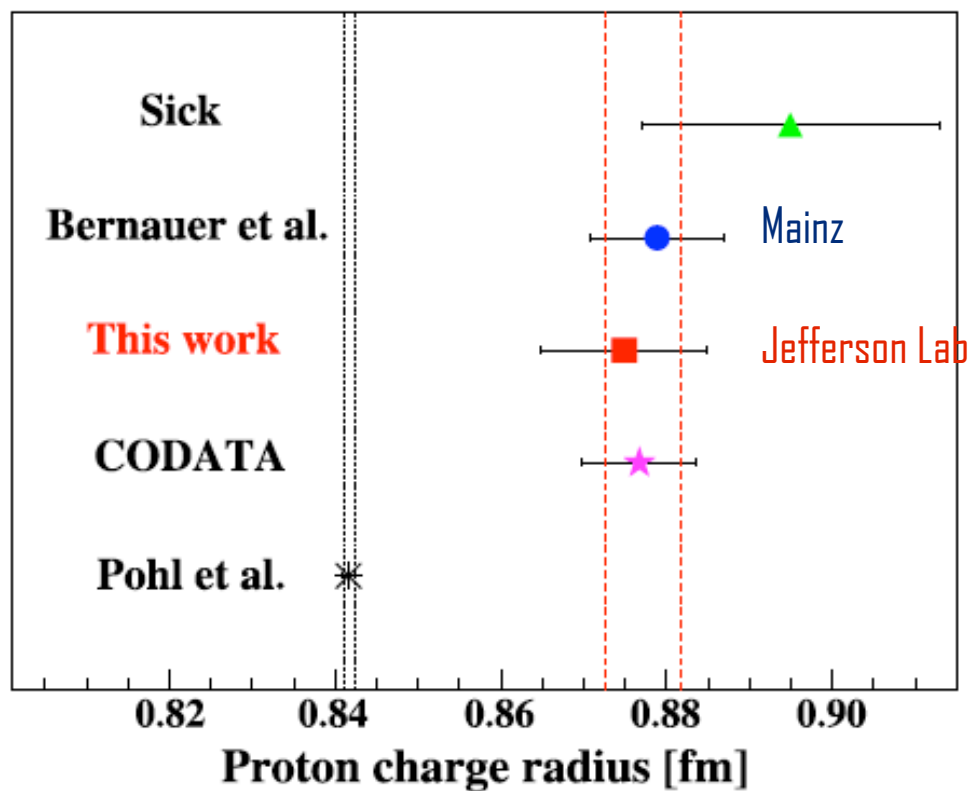
2S hyperfine splitting in μp is: $\Delta E_{\text{HFS}} = 22.8089(51) \text{ meV}$

gives a proton Zemach radius $r_Z = \int d^3r \int d^3r' r \rho_E(r) \rho_M(r - r')$

$$r_Z = 1.082(31)_{\text{exp}}(20)_{\text{th}} = 1.082(37) \text{ fm}$$

μp theory: A. Antognini *et al.*, arXiv :1208.2637 (atom-ph)

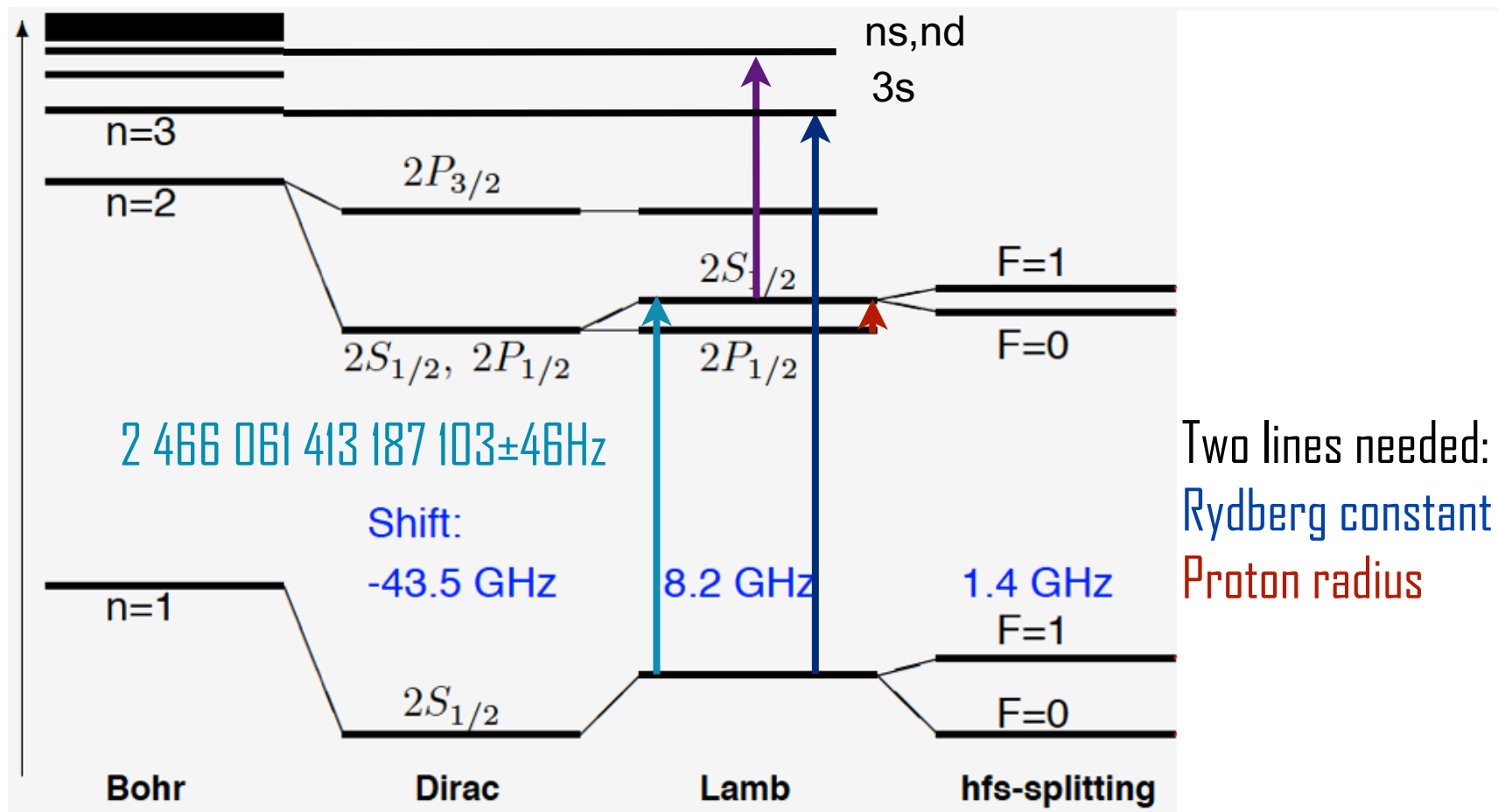




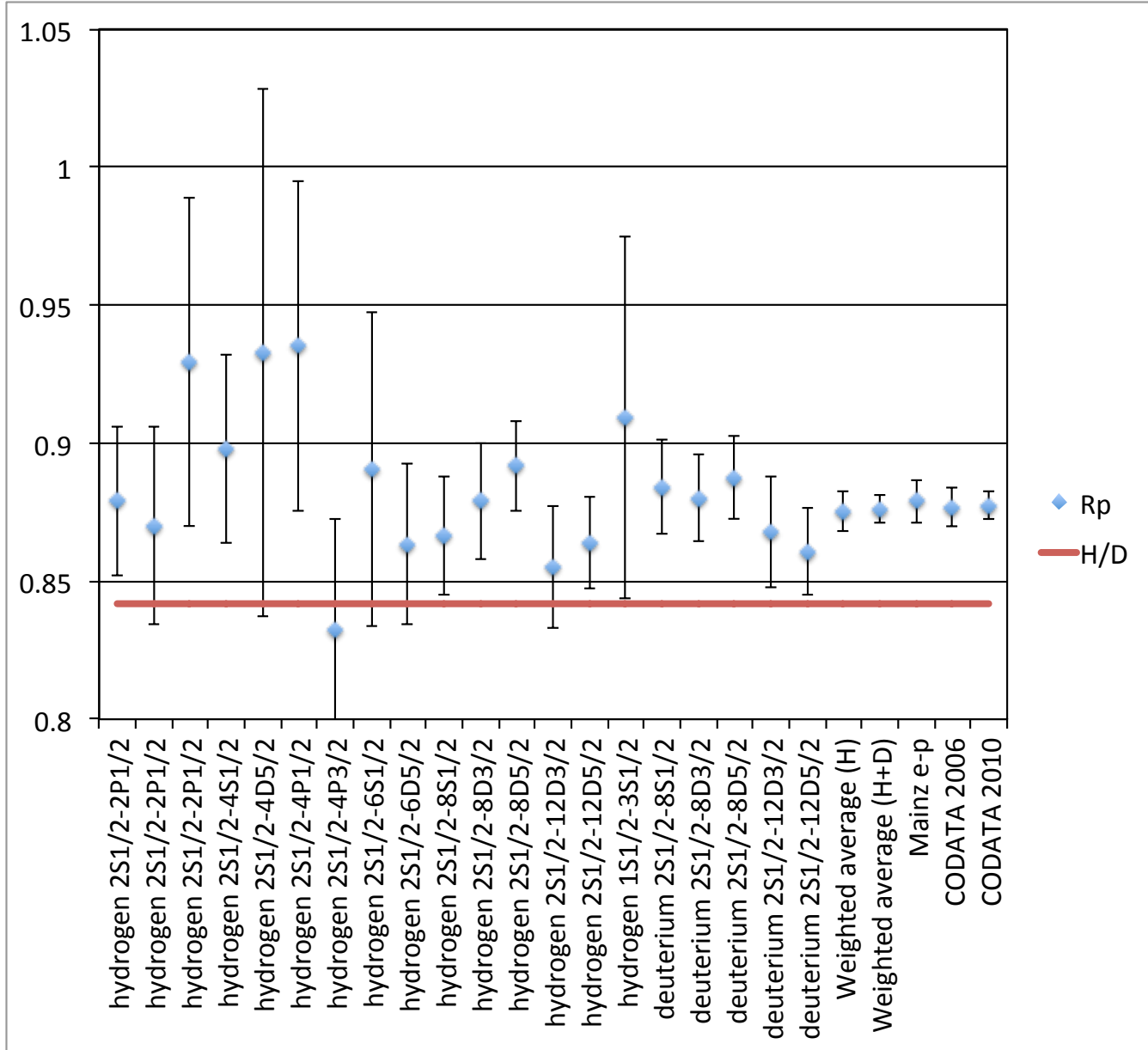
High-precision measurement of the proton elastic form factor ratio at low, X. Zhan, et al Physics Letters B 705, 59-64 (2011).

Other measurements

How to get the radius from hydrogen and electron-proton scattering



Analysis by F. Biraben



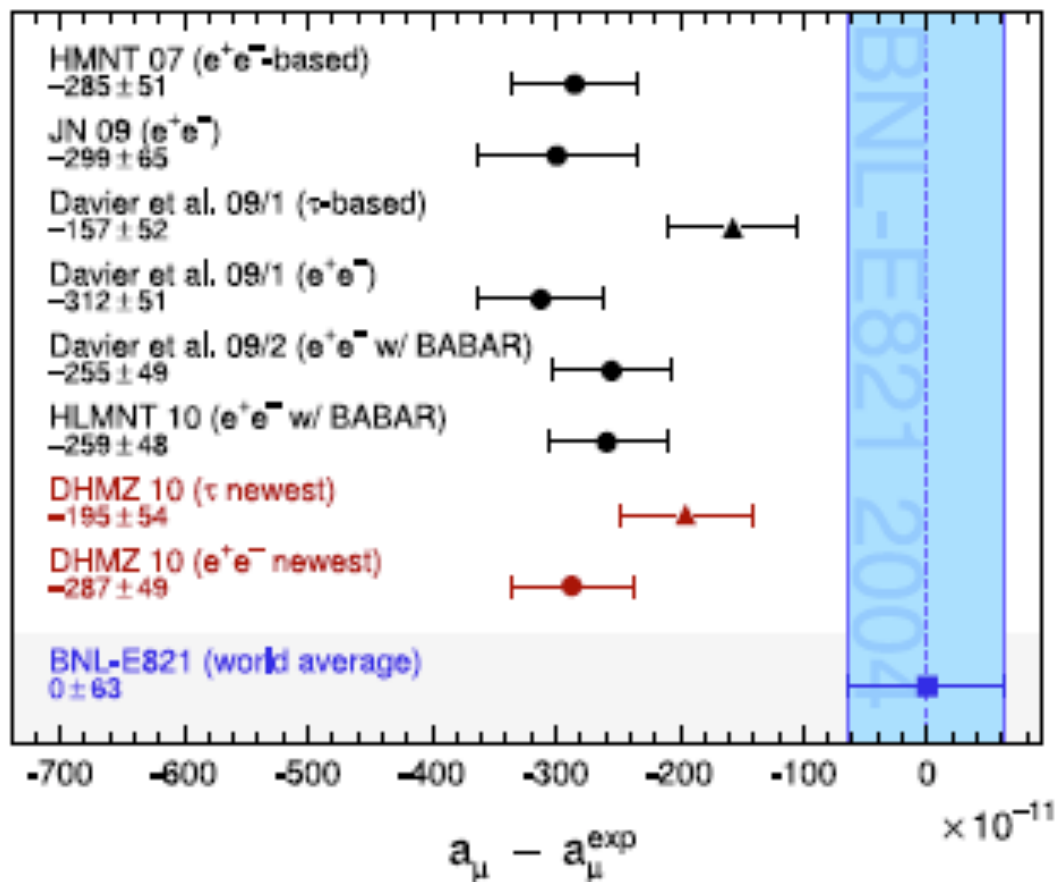
- Needed: new measurements with independent systematic errors and get an independent Rydberg constant value:
 - 2S-4P in H (Garching)
 - 2S-nS,D in H (J. Flowers, NPL)
 - 1S-3S (Garching, Paris)
 - transitions between Rydberg states of heavy H-like ion (NIST)
 - 1S-2S and 1S hfs in e⁻ (A. Antonini, PSI)

Possible origin of the discrepancy

Systematic errors or new physics?

- Frequency shift: unlikely - several redundant measurements at 708 nm (Fabry-Perot, two-photon transition in Rb) and $6\mu\text{m}$ (water lines)
- $\mu e^- p$ molecules or $p p \mu$ molecules. Not possible - Why Three-Body Physics Does Not Solve the Proton-Radius Puzzle, J.-P. Karr and L. Hilico. Phys. Rev. Lett. 109, 103401 (2012).
- Experimental problems, e.g., a small air leak in the hydrogen target: we see characteristic μN and μO x-rays
 - Less than 1% of all created μP atoms see any N_2 molecules
 - Less than 0.1% of all μP in 2S state see any N_2 molecule during laser time
- μP theory: many checks, no effect seems large enough to explain a 0.3 meV energy shift, probably not even proton polarization (30 times too small)

- Electron-proton elastic scattering data analysis
- Under-estimated systematic errors in some hydrogen measurements
 - possible, but many different kind of experiments (microwave, $1s-3s$, $2s-n_s$ and $2s-n_d$)
- Proton structure
- New physics
 - Constraints:
 - $g-2$ of the muon (3σ),
 - $g-2$ of the electron (Harvard)+fine structure constant from atomic recoil (LKB)
 - Hydrogen
 - Precision highly charged ions experiments at GSI (if long range interaction)
 - ...



Example: muon $g-2$, discrepancy not solved after improved QED calculation

For electrons:

$$\alpha^{-1}(a_e) = 137.0359991727(68)(46)(19)(331)$$

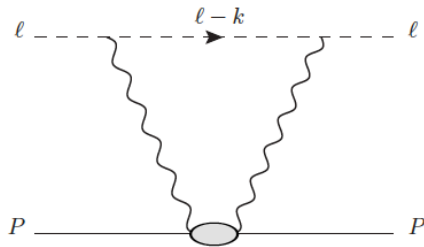
$$\alpha^{-1} = 137.035999037(91)$$

Reevaluation of the hadronic contributions to the muon $g-2$ and to $\alpha(M^2_Z)$, M. Davier, A. Hoecker, B. Malaescu and Z. Zhang. The European Physical Journal C - Particles and Fields **71**, 1-13 (2011).

Tenth-Order QED Contribution to the Electron $g-2$ and an Improved Value of the Fine Structure Constant, T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio. Phys. Rev. Lett. **109**, 111807 (2012).

Complete Tenth-Order QED Contribution to the Muon $g-2$, T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio. Phys. Rev. Lett. **109**, 111808 (2012).

Toward a resolution of the proton size puzzle, G.A. Miller, A.W. Thomas, J.D. Carroll and J. Rafelski. Phys. Rev. A 84, 020101 (2011).



0.31 meV for μH and 9Hz for hydrogen
(with model dependent parametrization)

FIG. 1: Direct two-photon exchange graph corresponding to the hitherto neglected term. The dashed line denotes the lepton; the solid line, the nucleon and the ellipse the off-shell nucleon.

We next seek values of the model parameters λ, b, ξ of $F(-q^2)$, chosen to reproduce the value of the energy shift, 0.31 meV, to resolve the puzzle. With $\xi = 0$, $\tilde{\Lambda} = \Lambda$, $\lambda/b^2 = 2.35/(79 \text{ MeV})^2$ is required. With this value, the corresponding change in the electronic H Lamb shift for the 2S state is about 9 Hz, significantly below the current uncertainty in both theory and experiment [3]. If ξ is changed substantially from 0 to 1 our value of λ would be increased by about 10%. Other tests of this effect could show sensitivity to the value of ξ or $\tilde{\Lambda}$.

No other work supports such a large effect

- A muon edm? If $d_\mu = 2 \times 10^{-19}$ e·cm would shift the energy level < 200 MHz
- Charge equality between e^- and μ^- generation? Checked to $u_r = 10^{-8}$ (from μ^+e^-)

- Deviation from Coulomb's law: probe of hidden sector

- Test of Coulomb law via spectroscopy is very clean and model independent probe of new particles
It is independent on stability and decay channel.

$$V(r) = -\frac{Z\alpha}{r}(1 + \alpha'e^{-mr}) \quad \text{or} \quad V(r) = -\frac{Z\alpha}{r}(1 + \alpha''(\mathbf{s}_1 \cdot \mathbf{s}_2)e^{-mr})$$

- From simple atoms there are constraints on light bosons with ultra-weak coupling:

$$m \in [1\text{eV}, \text{MeV}] \text{ and } \alpha' < 10^{-13}, \alpha'' < 10^{-17} \quad [\text{PRL } 104,220406 \text{ (2010), arXiv:1008.3536v2}]$$

- Minicharge particles? [Jaeckel and Roy (2010), Jentschura(2010)]

Vacuum polarization with pair production of light fermions with $q = \varepsilon e$ and masses $m_\varepsilon < m_e$

No parameter found explaining r_p puzzle without contradicting, simple atoms spectroscopy, $g_e/\mu - 2$, $\alpha...$

- New bound-state QED theory?

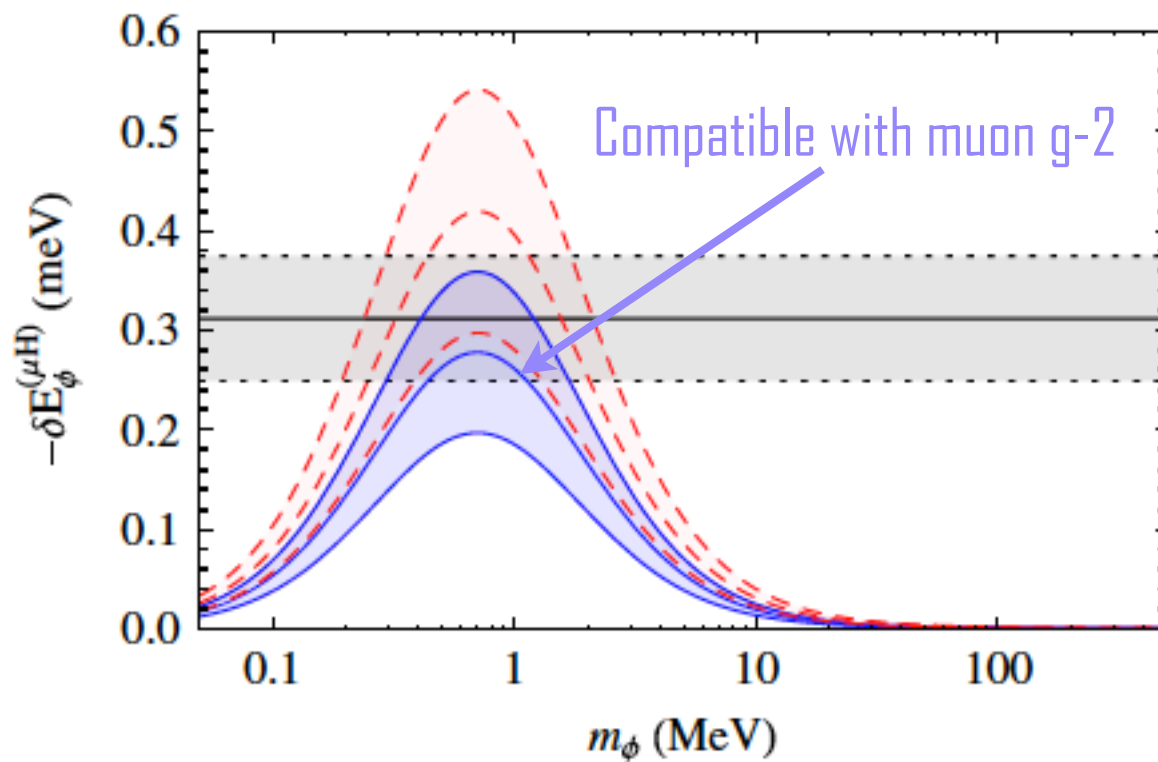
- Non-local in time interaction...(R. K. Gainutdinov)?

- New physics and the proton radius problem, C.E. Carlson and B.C. Rislow. *Physical Review D* 86, 035013 (2012):
 - Particles that couple to muons and hadrons but not electrons
 - For the scalar-pseudoscalar model, masses between 100 to 200 MeV are not allowed.
 - For the vector model, masses below about 200 MeV are not allowed. The strength of the couplings for both models approach that of electrodynamics for particle masses of about 2 GeV.
 - New physics with fine-tuned couplings may be entertained as a possible explanation for the Lamb shift discrepancy.
- New Parity-Violating Muonic Forces and the Proton Charge Radius, B. Batell, D. McKeen and M. Pospelov. *Phys. Rev. Lett.* 107, 011803 (2011).
 - We identify a class of models with gauged right-handed muon number, which contains new vector and scalar force carriers at the 100 MeV scale or lighter, that is consistent with observations.
 - Such forces would lead to an enhancement by several orders-of-magnitude of the parity-violating asymmetries in the scattering of low-energy muons on nuclei.

Muonic hydrogen and MeV forces, D. Tucker-Smith et I. Yavin. Physical Review D 83, 101702 (2011).

We explore the possibility that a new interaction between muons and protons is responsible for the discrepancy between the CODATA value of the proton-radius and the value deduced from the measurement of the Lamb shift in muonic hydrogen. We show that a new force carrier with roughly MeV-mass can account for the observed energy-shift as well as the discrepancy in the muon anomalous magnetic moment. However, measurements in other systems constrain the couplings to electrons and neutrons to be suppressed relative to the couplings to muons and protons, which seems challenging from a theoretical point of view. One can nevertheless make predictions for energy shifts in muonic deuterium, muonic helium, and true muonium under the assumption that the new particle couples dominantly to muons and protons.

Muonic hydrogen and MeV forces, D. Tucker-Smith et I. Yavin. Physical Review D 83, 101702 (2011).

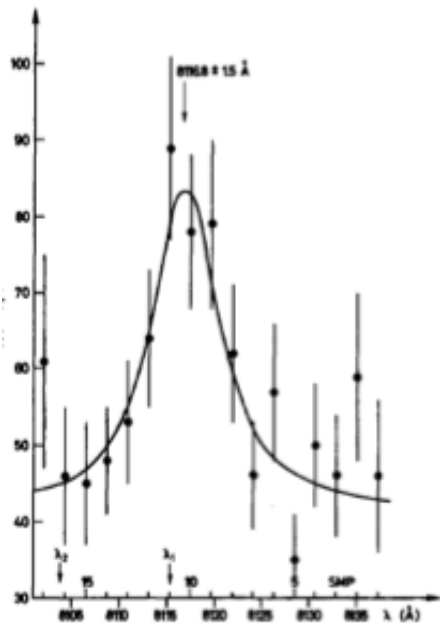


What's next

Deuterium: deuteron polarization very large
Helium?

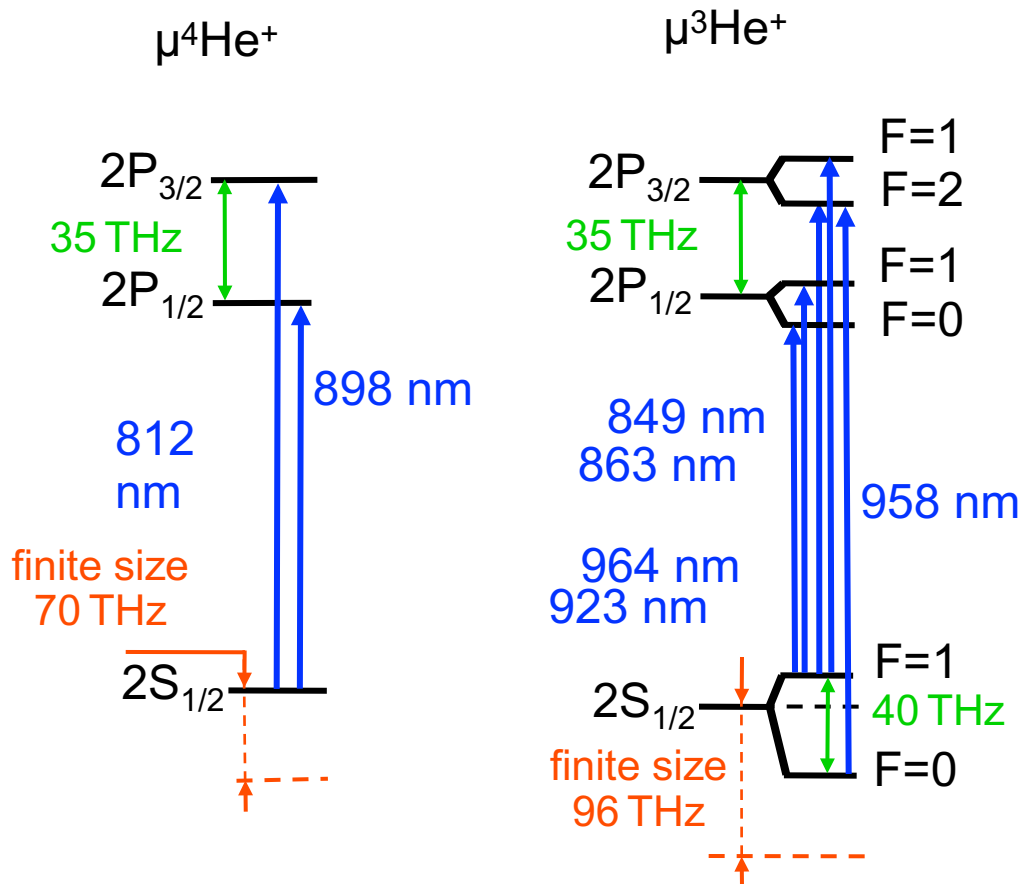
812 nm
 $\rho_{\text{He}} = 40 \text{ bars}$

2011-2013 → muonic helium spectroscopy (4 mbar)

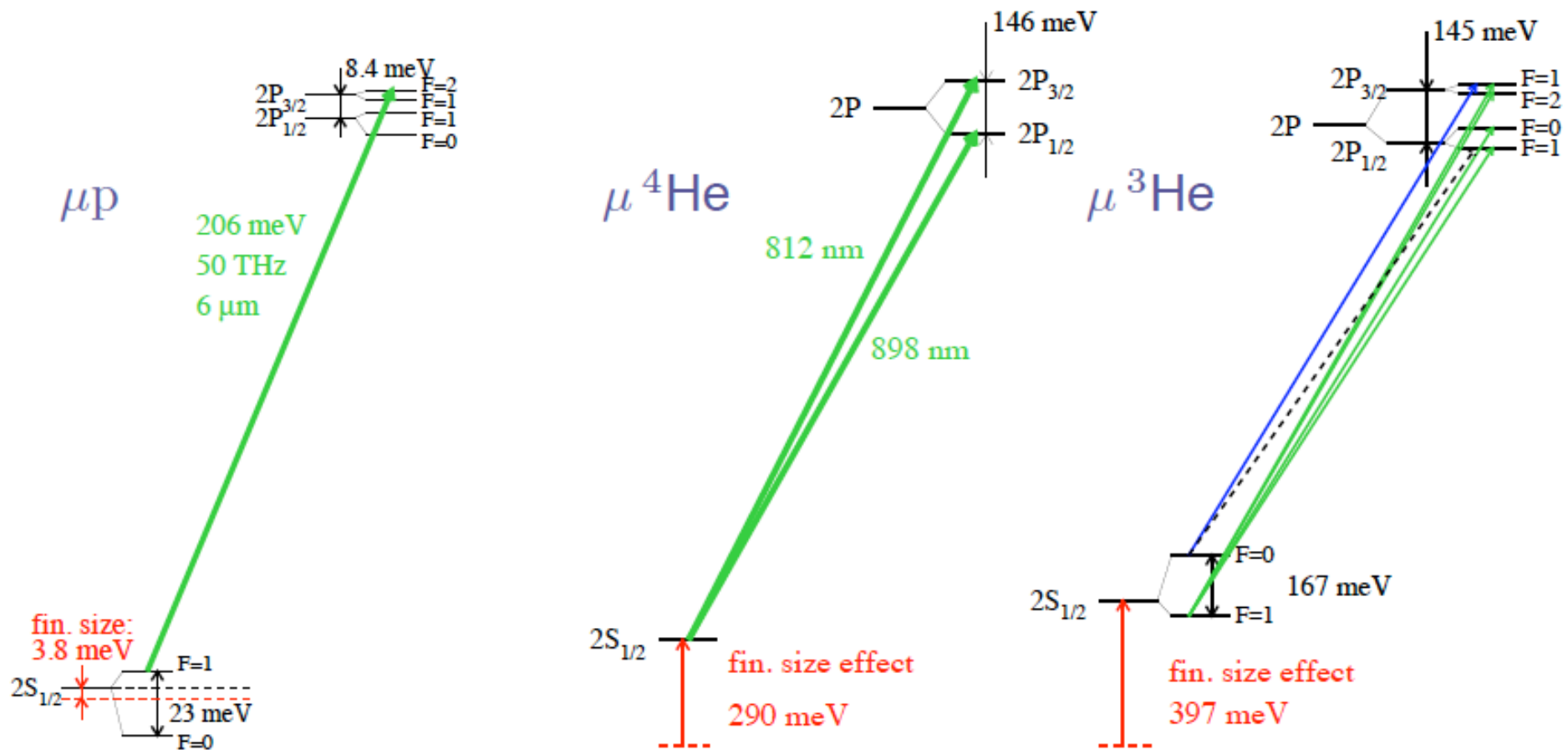


Nuclear Physics A278 (1977) p. 381

but signal never reproduced
 (10 bars, 40 bars)

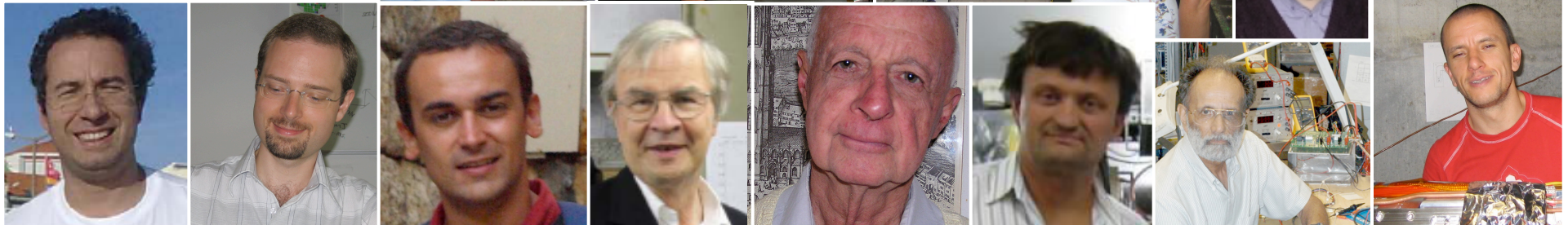


- μHe^+ spectroscopy + He^+ spectroscopy → QED test ($Z\alpha$)
- improve He spectroscopy



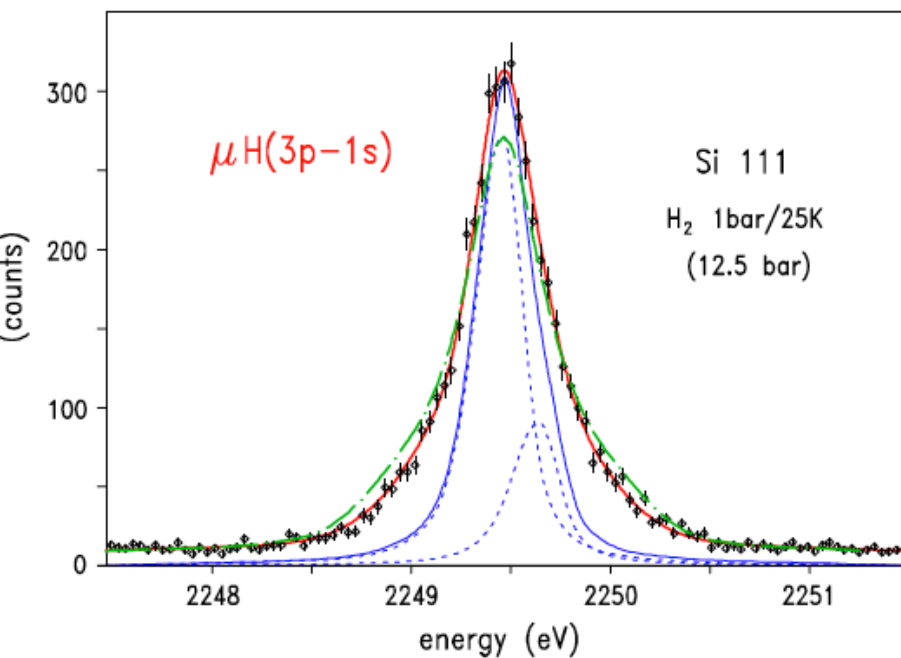
- We have performed a 12.5 ppm measurement of the Lamb-shift in muonic hydrogen
- The deduced proton radius using a Dipole model is 6.9 standard deviations away from the hydrogen and electron-proton elastic scattering data
- Better modeling of the proton form-factor and polarization required to confirm or reduce the disagreement
- Experiment confirmed with 2nd μH line
- 3 μD lines observed and being analyzed
- Muonic He in 2013 (check of theory, different laser wavelength-in the red) predictions of measurable effects from new physics!!



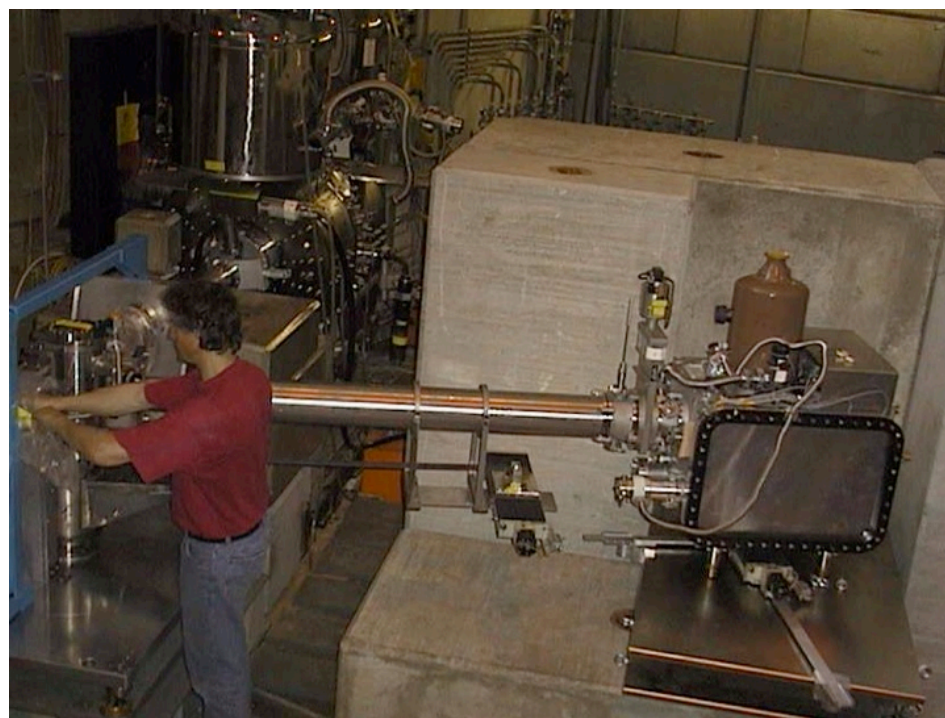


Proton Size Investigators thank you for your attention

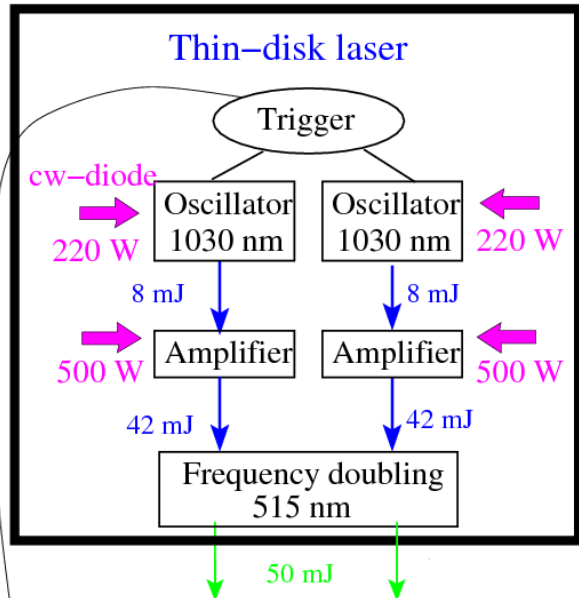




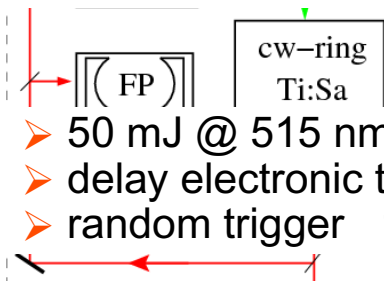
Line Shape of the mu H(3p-1s) Hyperfine Transitions,
D.S. Covita, D.F. Anagnostopoulos, H. Gorke, D. Gotta,
A. Gruber, A. Hirtl, T. Ishiwatari, [P. Indelicato](#), [E.-O.L. Bigot](#),
M. Nekipelov, J.M.F.d. Santos, P. Schmid, L.M. Simons,
M. Trassinelli, J.F.C.A. Veloso and J. Zmeskal.
Phys. Rev. Lett. 102, 023401 (2009).



Laser chain : thin-disk Yb:YAG laser

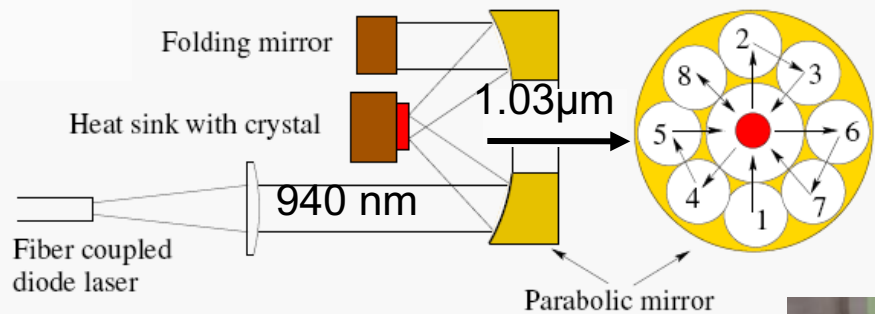


- Q-switch oscillator (cw prelasing mode → short delay between electronic trigger and opt. output (8mJ, xxx ns, delay 250ns)
- 12 pass amplifier (20m propagation) (42mJ)
- LBO frequency doubling 25mJ @ 515nm

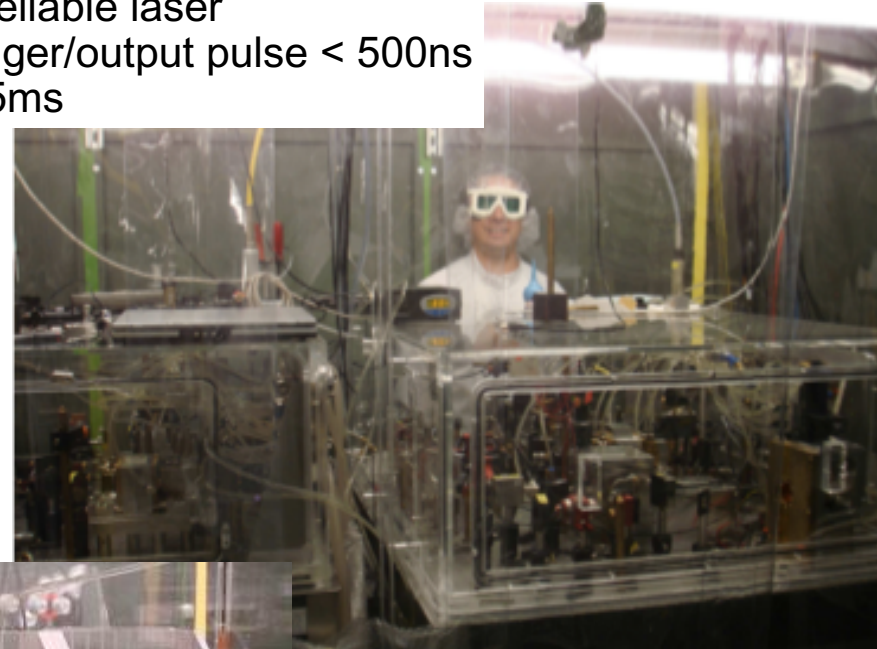
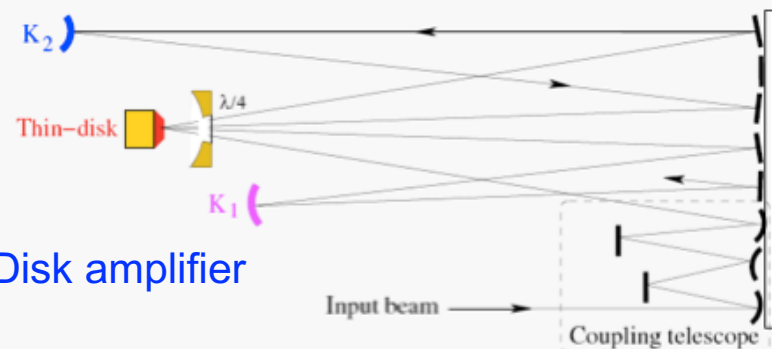


- 50 mJ @ 515 nm, reliable laser
- delay electronic trigger/output pulse < 500ns
- random trigger 1.5ms

Disk oscillator

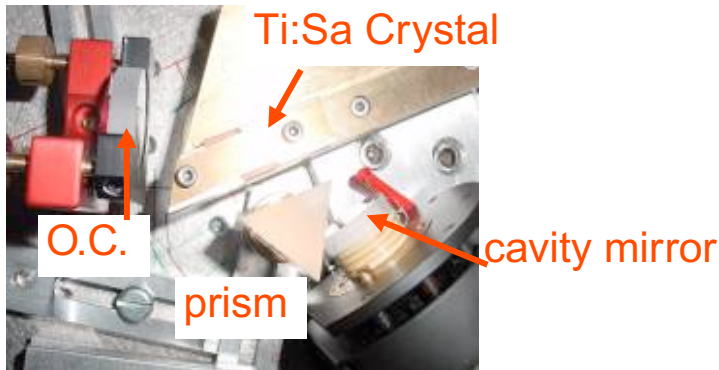
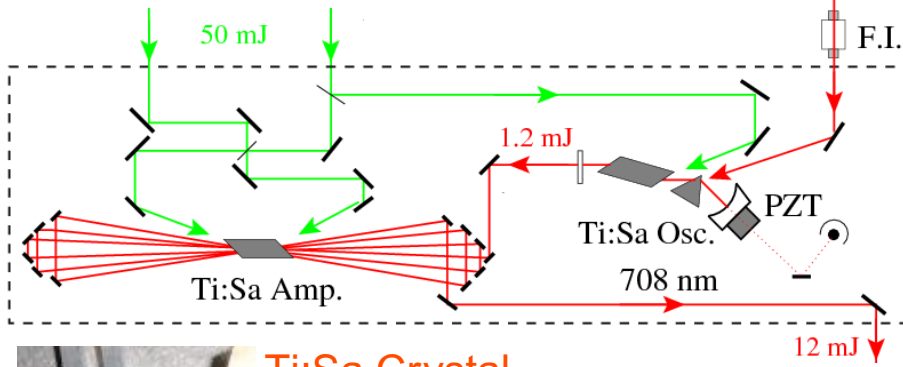
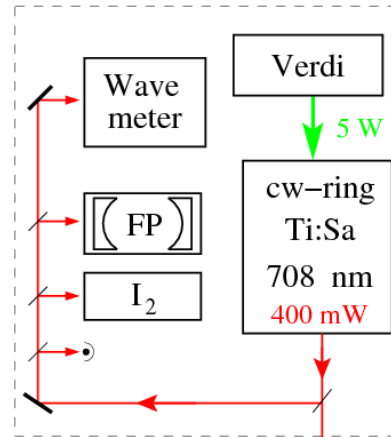
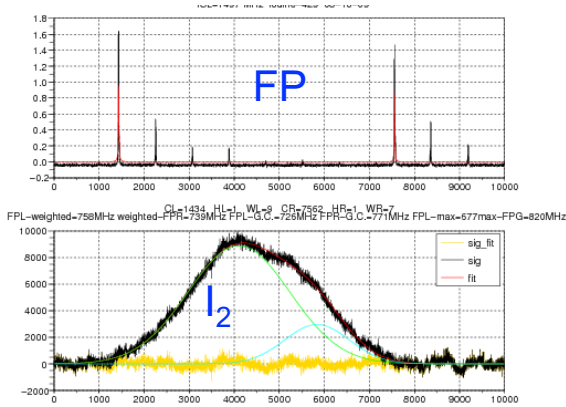


Disk amplifier

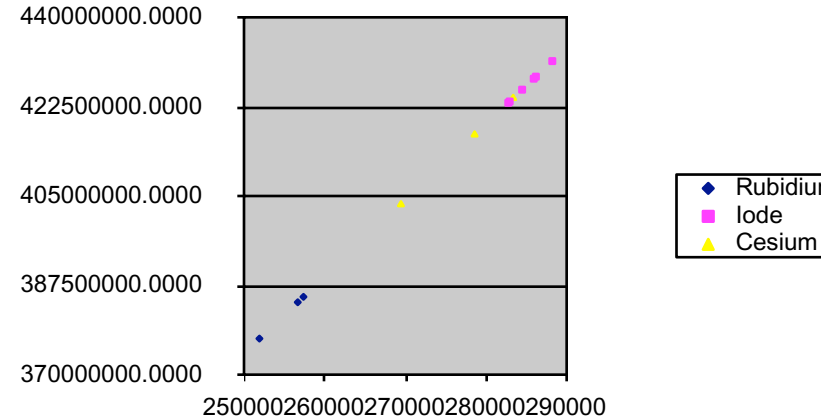


Laser chain : Ti:Sa

Calibration FP/ I₂, Cs, Rb (700-794nm)



Frequencies of the FP peaks



- home made Ti:Sa lasers
- cw-Ti:Sa frequency controlled with FP, atom, molecule (abs. freq. <50MHz)
- short length pulsed oscillator seeded with Cw-Ti:Sa (1.2 mJ, 6 ns, delay 50 ns, $\Delta\nu = 200\text{MHz}$)
- multipass amplifier $\times 6$ (12 mJ, 5 ns)

New physics

Predictions are always difficult, in particular about the Future
[N. Bohr]

Proton Size Anomaly, V. Barger, C.-W. Chiang, W.-Y. Keung et al. Phys. Rev. Lett. 106, 153001 (2011):

We explore the possibility that new scalar, pseudoscalar, vector, and tensor flavor-conserving nonuniversal interactions may be responsible for the discrepancy. We consider exotic particles that, among leptons, couple preferentially to muons and mediate an attractive nucleon-muon interaction. We find that the many constraints from low energy data disfavor new spin-0, spin-1, and spin-2 particles as an explanation.

New Parity-Violating Muonic Forces and the Proton Charge Radius, B. Batell, D. McKeen et M. Pospelov. Phys. Rev. Lett. 107, 011803 (2011).

The recent discrepancy between proton charge radius measurements extracted from electron-proton versus muon-proton systems is suggestive of a new force that differentiates between lepton species. We identify a class of models with gauged right-handed muon number, which contains new vector and scalar force carriers at the 100 MeV scale or lighter, that is consistent with observations. Such forces would lead to an enhancement by several orders-of-magnitude of the parity-violating asymmetries in the scattering of low-energy muons on nuclei. The relatively large size of such asymmetries, $O(10^{-4})$, opens up the possibility for new tests of parity violation in neutral currents with existing low-energy muon beams.

B. Batell, D. McKeen, and M. Pospelov, Phys. Rev. Lett. 107, 011803 (2011).

$$\Delta r_p^2|_{e\text{-H}} = -\frac{6\kappa^2}{m_V^2}; \quad \Delta r_p^2|_{\mu\text{-H}} = -\frac{6(\kappa^2 + \eta)}{m_V^2} f(am_V) \quad (11)$$

where $a = (\alpha m_\mu m_p)^{-1}(m_\mu + m_p)$ is the μ -H Bohr radius, $f(\hat{x}) \equiv \hat{x}^4(1 + \hat{x})^{-4}$, and $\eta \equiv \kappa g_R/(2e)$. The difference $\Delta r_p^2|_{e\text{-H}} - \Delta r_p^2|_{\mu\text{-H}}$ must be consistent with the observed pattern (5) and requires η to be positive. In the scaling regime of $am_V \gg 1$ one has

$$\frac{\eta}{m_V^2} \simeq \frac{\Delta r^2}{6} \simeq 0.01 \text{ fm}^2 \simeq \frac{2.5 \times 10^{-5}}{(10 \text{ MeV})^2}. \quad (12)$$

In the same regime, the model predicts that future experiments with μ -He would detect the effective charge radius of the helium nucleus shifted down by $\Delta r_{\text{He}}^2 = -0.06 \text{ fm}^2$.

Direct numerical solution of Dirac equation, numerical grid, 10000 points, ~4000 inside the proton

$$(c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta\mu_r c^2 + V_{\text{Nuc}}(\mathbf{r})) \Phi_{n\kappa\mu}(\mathbf{r}) = \mathcal{E}_{n\kappa\mu} \Phi_{n\kappa\mu}(\mathbf{r}),$$

$$\begin{aligned} V_{11}^{pn}(r) &= -\frac{\alpha(Z\alpha)}{3\pi} \int_1^\infty dz \sqrt{z^2 - 1} \left(\frac{2}{z^2} + \frac{1}{z^4} \right) \frac{e^{-2m_e r z}}{r} \\ &= -\frac{2\alpha(Z\alpha)}{3\pi} \frac{1}{r} \chi_1 \left(\frac{2}{\lambda_e} r \right) \end{aligned}$$

$$\begin{aligned} V_{11}(r) &= -\frac{2\alpha(Z\alpha)}{12\pi} \frac{1}{r} \int_0^\infty dr' r' \rho(r') \\ &\quad \times \left[\chi_2 \left(\frac{2}{\lambda_e} |r - r'| \right) - \chi_2 \left(\frac{2}{\lambda_e} |r + r'| \right) \right]. \end{aligned}$$



$$\chi_n(x) = \int_1^\infty dz e^{-xz} \frac{1}{z^n} \left(\frac{1}{z} + \frac{1}{2z^3} \right) \sqrt{z^2 - 1}.$$

Analytical expression for the evaluation of vacuum-polarization potentials in muonic atoms, S. Klarsfeld. Physics Letters 66B, 86-88 (1977).