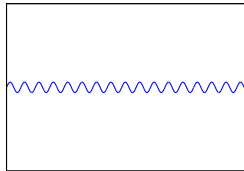
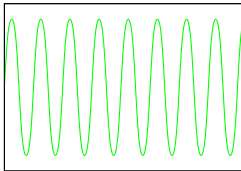
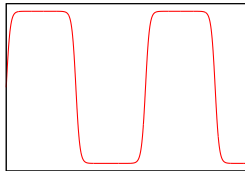


Inhomogeneous chiral symmetry breaking phases



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Stefano Carignano
Michael Buballa



Phys.Rev. D **82**, 054009 (2010)

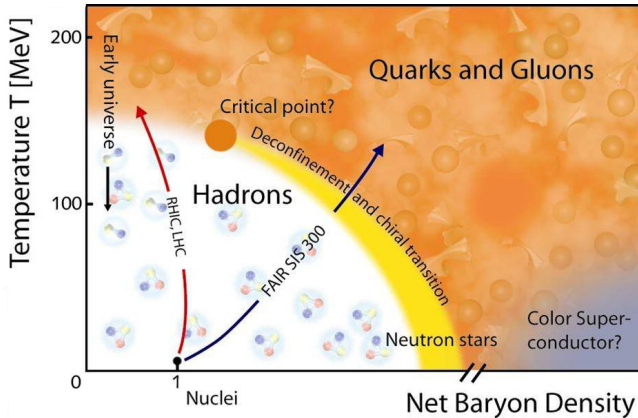
arXiv:1111.4400

arXiv:1203.5343

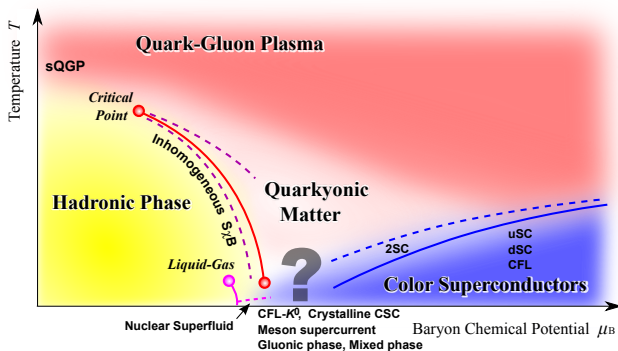
H-QM | Helmholtz Research School
Quark Matter Studies

HGS-HiRe *for FAIR*
Helmholtz Graduate School for Hadron and Ion Research

Motivation: the QCD phase diagram (so far ?)



Maybe it's not so simple...



Fukushima and Hatsuda, arXiv:1005.4814

- ▶ Choose an effective model
- ▶ Allow for spatial modulations of the chiral condensate
- ▶ Determine the favored ground state of the system
- ▶ Study the properties of chiral crystalline phases
- ▶ Extend the model and see how crystalline phases are affected

- ▶ Start from the two-flavor NJL Lagrangian

$$\mathcal{L}_{NJL} = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi + G_s \left((\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\tau^a\psi)^2 \right)$$

- ▶ Mean-field approximation
- ▶ Retain spatial dependence of the condensates

$$\langle \bar{\psi}\psi \rangle = S(\vec{x}), \quad \langle \bar{\psi}i\gamma^5\tau^a\psi \rangle = P_a(\vec{x})$$

- ▶ Mean-field Lagrangian:

$$\mathcal{L}_{MF} = \bar{\psi}(x)S^{-1}(x)\psi(x) - G_s (S(\vec{x})^2 + P(\vec{x})^2)$$

$$S^{-1} = i\gamma^\mu \partial_\mu - m + 2G_s (S(\vec{x}) + i\gamma^5\tau^a P_a(\vec{x})) \equiv \gamma^0(i\partial_0 - \mathcal{H}_{MF})$$

$$\begin{aligned}\Omega(T, \mu; S(\vec{x}), P(\vec{x})) &= -\frac{T}{V} \text{Log} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left(\int_{x \in [0, \frac{1}{T}] \times V} (\mathcal{L}_{MF} + \mu \bar{\psi} \gamma^0 \psi) \right) \\ &= -\frac{TN_c}{V} \sum_n \text{Tr}_{D,f,V} \text{Log} \left(\frac{1}{T} (i\omega_n + \mathcal{H}_{MF} - \mu) \right) + \frac{G_s}{V} \int_V (S(\vec{x})^2 + P(\vec{x})^2)\end{aligned}$$

- If we can calculate the eigenvalues $\{E_n\}$ of \mathcal{H}_{MF} , it's

$$\Omega(T, \mu; M(\vec{x})) = -\frac{TN_f N_c}{V} \sum_{E_n} \text{Log} \left(2 \cosh \left(\frac{E_n - \mu}{2T} \right) \right) + \frac{1}{V} \int_V \frac{|M(\vec{x}) - m|^2}{4G_s}$$

Having defined $M(\vec{x}) = m - 2G_s (S(\vec{x}) + iP(\vec{x}))$

- ▶ Restrict to lower-dimensional spatial modulations
- ▶ $[\mathcal{H}, P_{\perp}] = 0 \rightarrow$ Full spectrum from lower-dimensional eigenvalues λ
- ▶ Boost along the transverse directions:

$$(\lambda, \mathbf{0}) \rightarrow (\sqrt{p_{\perp}^2 + \lambda^2}, \mathbf{p}_{\perp})$$

$$\begin{aligned} \Omega(T, \mu; M(\vec{x})) &= -\frac{2TN_c}{V_{\parallel}} \sum_{\lambda} \int \frac{d\vec{p}_{\perp}}{(2\pi)^{d_{\perp}}} \ln \left(2 \cosh \left(\frac{\lambda \sqrt{1 + \vec{p}_{\perp}^2 / \lambda^2} - \mu}{2T} \right) \right) \\ &+ \frac{1}{V} \int_V \frac{|M(\vec{x}) - m|^2}{4G_s} + \text{const.} \end{aligned}$$

(D. Nickel, Phys.Rev.D80 074025, 2009)

One-dimensional modulations

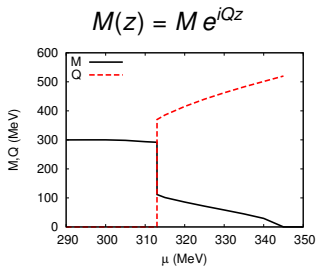
$$M(\vec{x}) \rightarrow M(z)$$

- ▶ Simplest case: one-dimensional modulations of the chiral condensate
- ▶ Different kinds of 1D modulations possible

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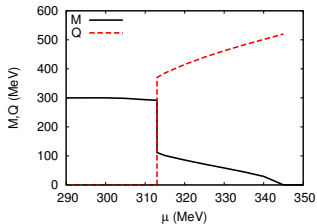


One-dimensional modulations

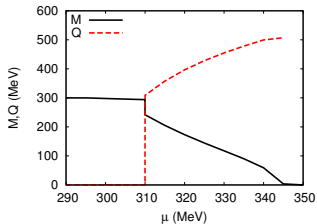
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- ▶ Simplest case: one-dimensional modulations of the chiral condensate
- ▶ Different kinds of 1D modulations possible

$$M(z) = M e^{iQz}$$



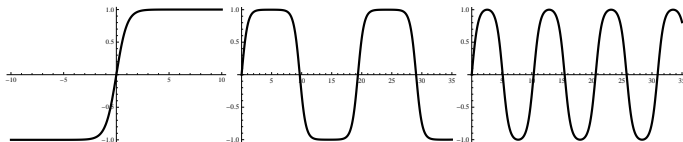
$$M(z) = M \cos(Qz)$$



A more general 1D modulation

- ▶ Self-consistent real solutions known from studies of 1+1D Gross-Neveu model
(M.Thies et al., Annals Phys. 314)

$$M(z) = \Delta \sqrt{\nu} \operatorname{sn}(\Delta z | \nu)$$

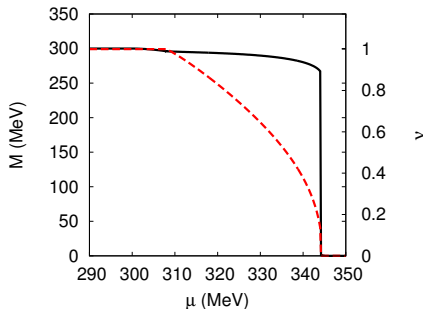


- ▶ Analytical expression for the eigenvalue spectrum of $\mathcal{H}_{MF} [M(z)]$
- ▶ Minimization of $\Omega[M(z)]$ w.r.t. two parameters (chiral limit): $\Omega(\Delta, \nu)$
- ▶ Away from chiral limit: add a third parameter δ

A more general 1D modulation

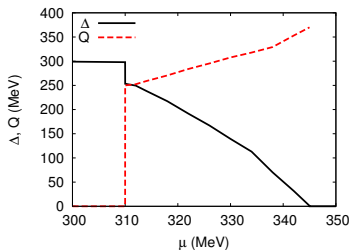
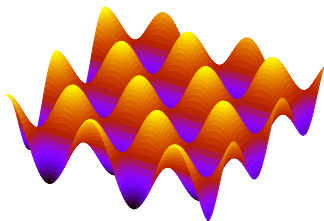
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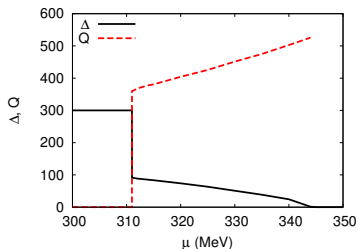
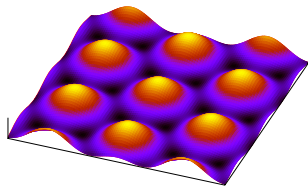
- ▶ Simple square crystal with LOFF-type modulation

$$M(x, y) = \Delta \cos(Qx) \cos(Qy)$$

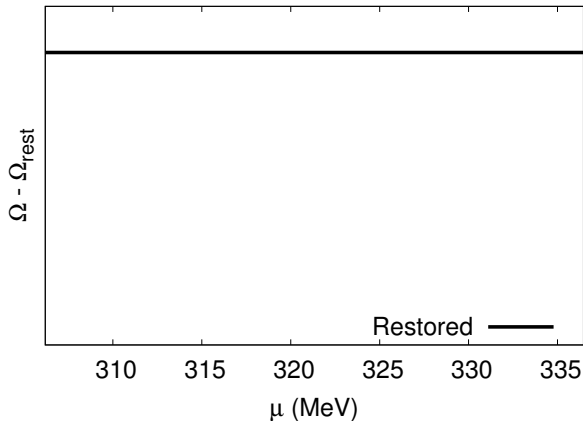


- ▶ Something more elaborate: hexagonal symmetry

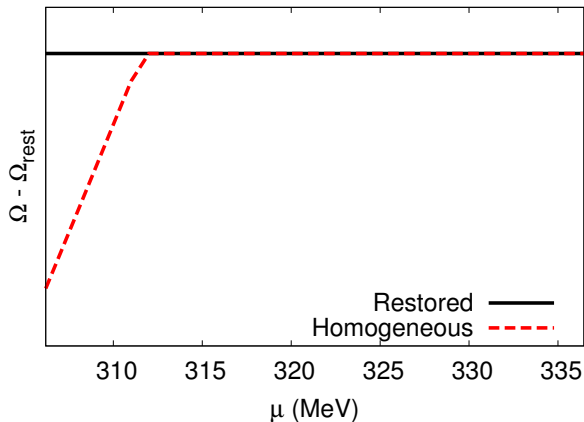
$$M(x, y) = \Delta \left[\cos(Qy) + 2 \cos\left(\frac{\sqrt{3}}{2} Qx\right) \cos\left(\frac{Q}{2} y\right) \right]$$



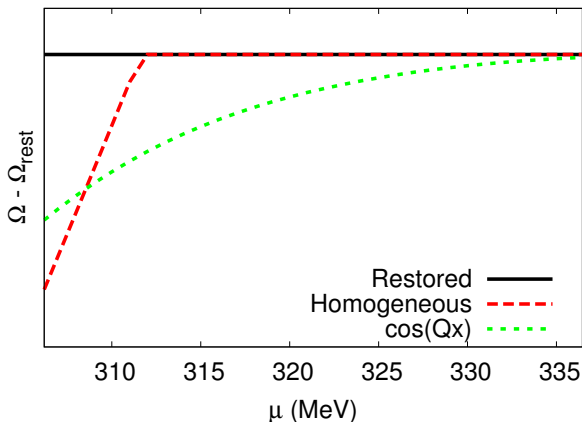
Comparison of free energies, $T = 0$



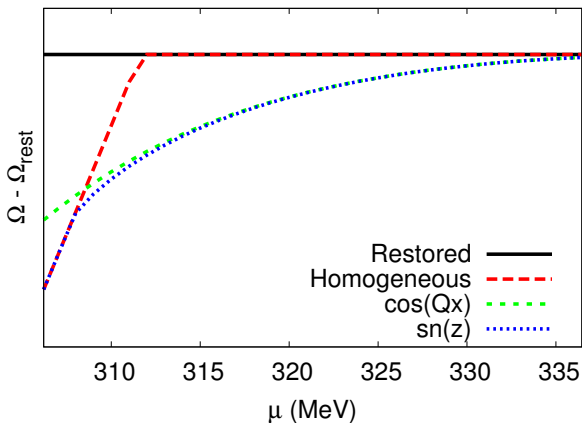
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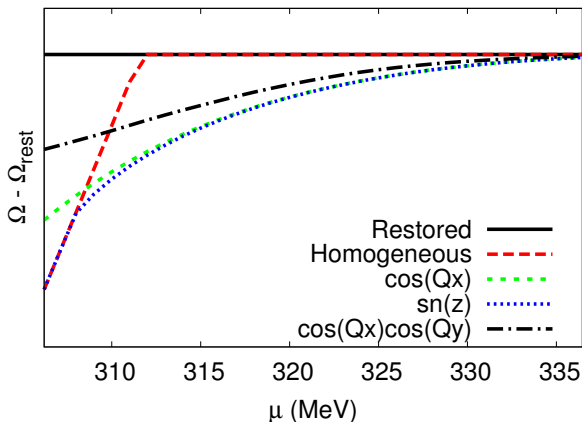
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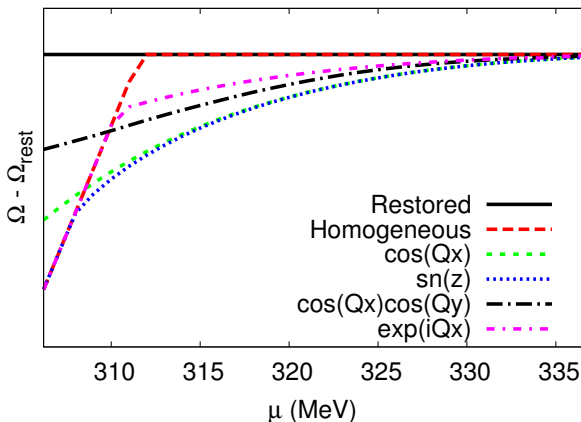
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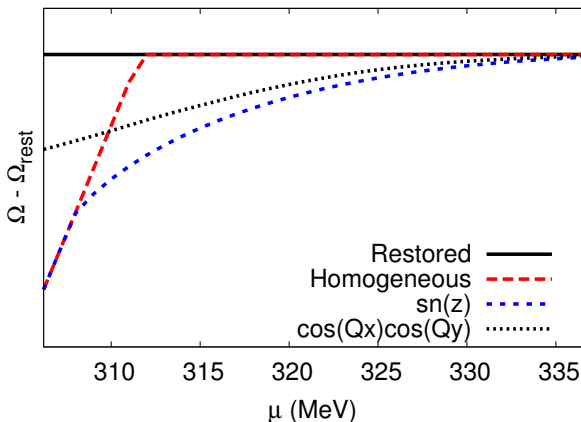
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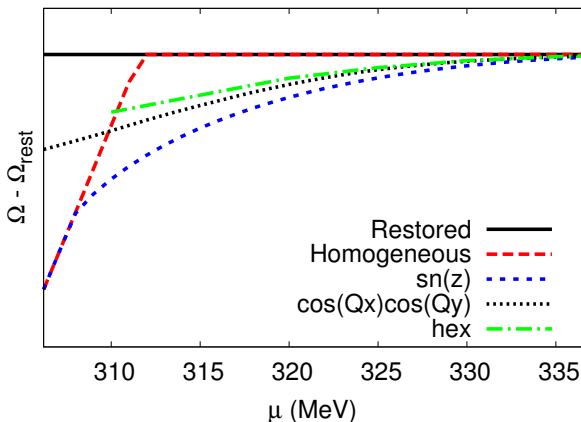
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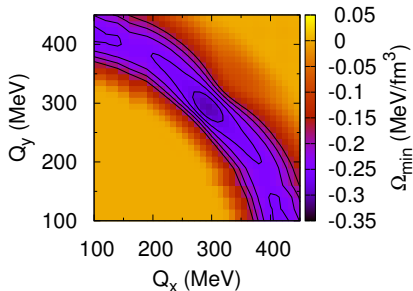


Comparison of free energies, $T = 0$

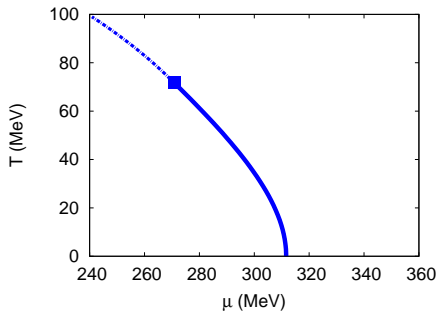


- ▶ One-dimensional solutions are the most favored!
- ▶ Extended “egg-carton” ansatz:

$$M(x, y) = \Delta \cos(Q_x x) \cos(Q_y y)$$

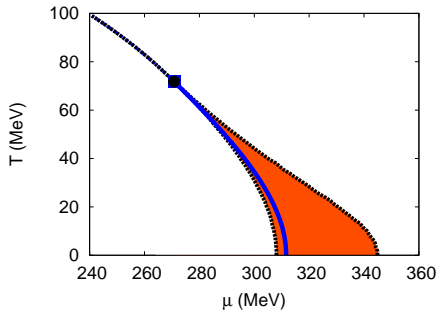


Phase diagram (chiral limit)



- ▶ Homogeneous only:
- ▶ **First order** phase transition
- ▶ ending at a **critical point**

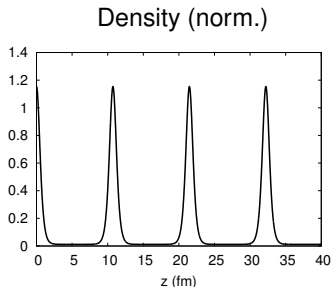
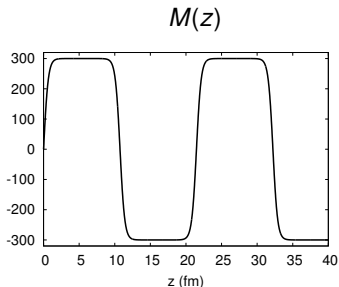
Phase diagram (chiral limit)



- ▶ Allow for 1D modulations like $M(z) = \Delta \sqrt{\nu} \operatorname{sn}(\Delta z|\nu)$
- ▶ **First order** transition line covered by **inhomogeneous phase**
- ▶ All phase transitions are **2nd order**
- ▶ **Critical point** \rightarrow **Lifshitz point**

Solitons: Mass and density profiles ($T = m = 0$)

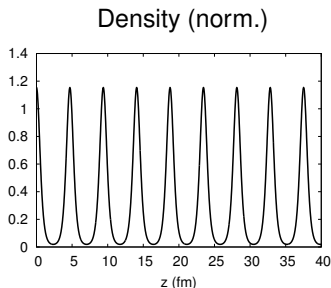
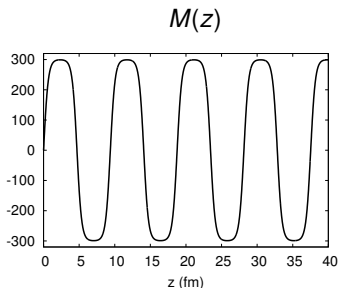
$$M(z) = \Delta\sqrt{\nu} \operatorname{sn}(\Delta z|\nu) = \begin{cases} \Delta \tanh(\Delta z) & \text{for } \nu \rightarrow 1 \\ \Delta\sqrt{\nu} \sin(\Delta z) & \text{for } \nu \rightarrow 0 \end{cases}$$



$$\mu = 307.5 \text{ MeV}$$

Solitons: Mass and density profiles ($T = m = 0$)

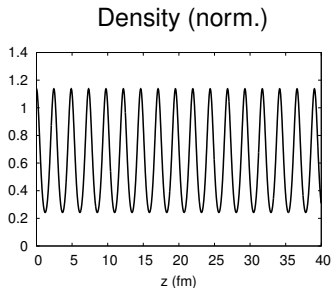
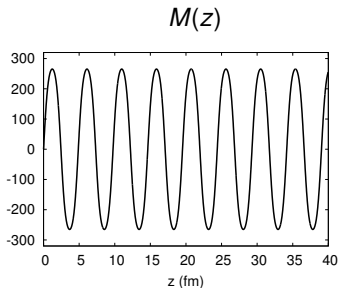
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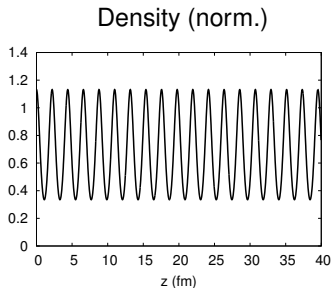
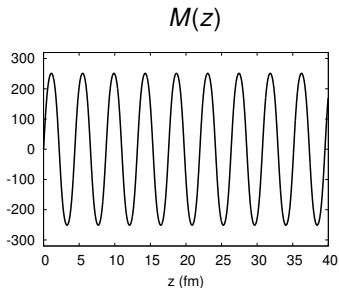
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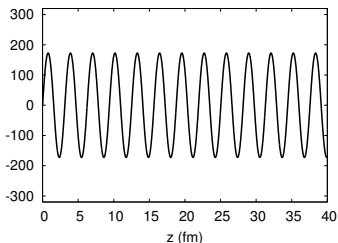


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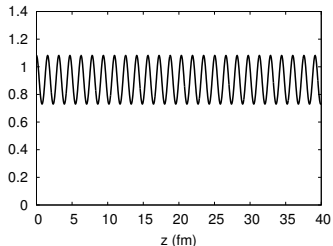
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$M(z)$



Density (norm.)

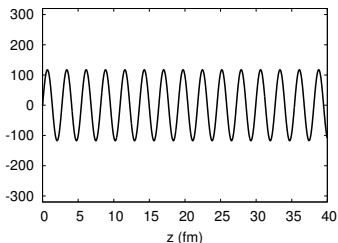


$$\mu = 320 \text{ MeV}$$

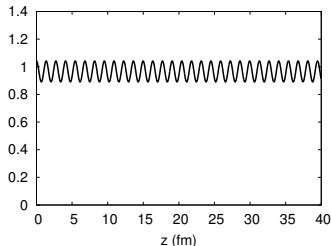
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$M(z)$



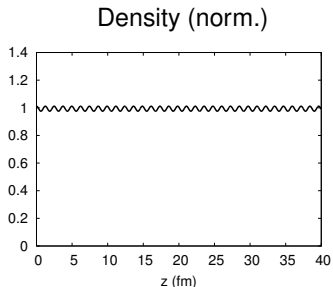
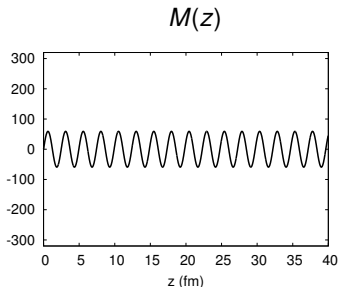
Density (norm.)



$$\mu = 330 \text{ MeV}$$

Solitons: Mass and density profiles ($T = m = 0$)

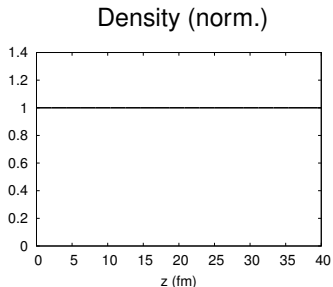
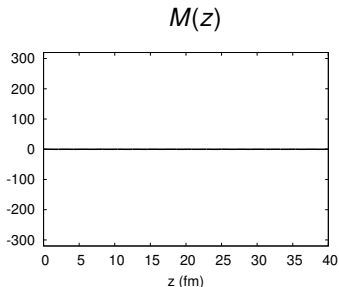
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$$\mu = 340 \text{ MeV}$$

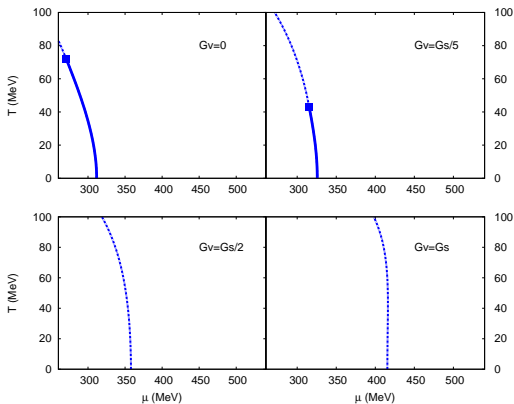
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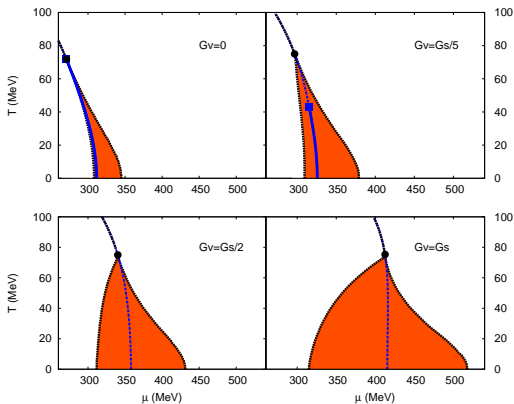
$$\mu = 345 \text{ MeV}$$

Vector interactions (Chiral limit)



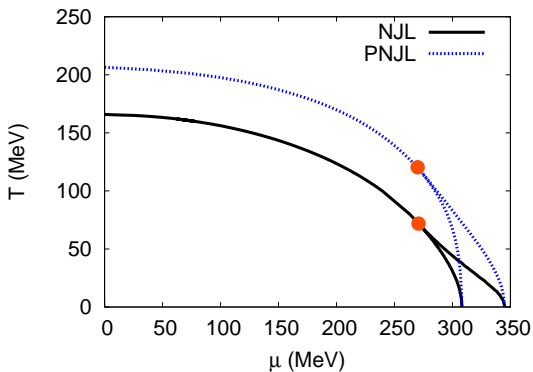
- ▶ Homogeneous:
- ▶ Shift towards higher μ
- ▶ Strong G_V -dependence of the critical point

Vector interactions (Chiral limit)



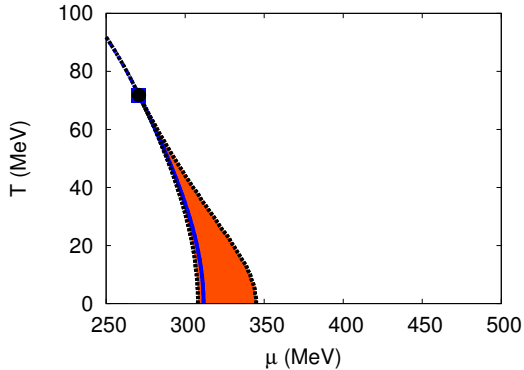
- ▶ Inhomogeneous:
- ▶ Stretch towards higher μ
- ▶ Lifshitz point at constant T
- ▶ Lifshitz and critical points split

PNJL (Chiral limit)

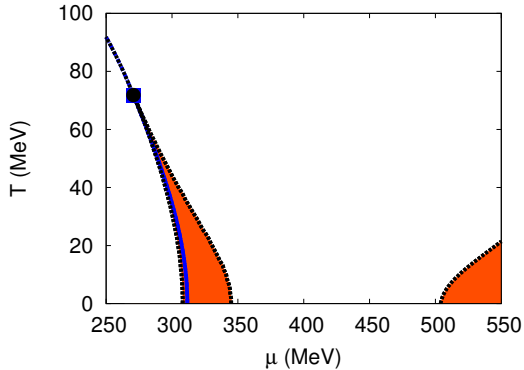


- ▶ Suppression of thermal effects
- ▶ Phase diagram stretched in T

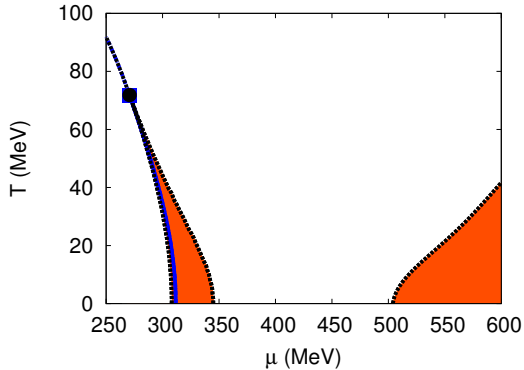
Islands and continents



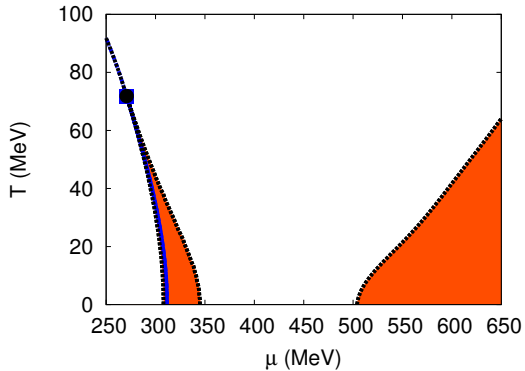
Islands and continents



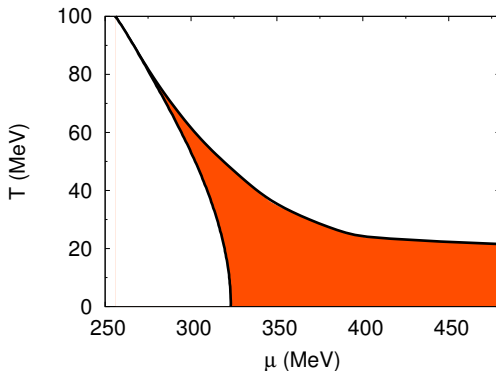
Islands and continents



Islands and continents



With a stronger coupling...



Regularization artifact ?



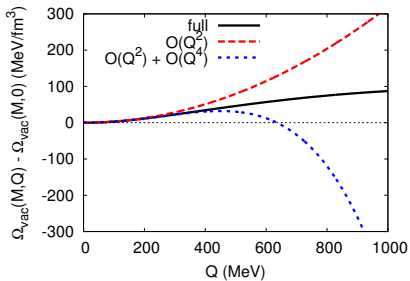
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- ▶ Present for both Pauli-Villars and proper time regularizations

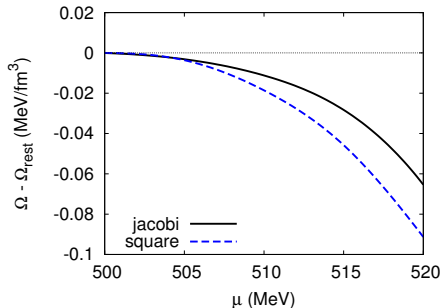
Regularization artifact ?

- ▶ Present for both Pauli-Villars and proper time regularizations
- ▶ Not present in Quark-Meson model !

- ▶ Difference NJL / QM ?
- ▶ Universal q^2 kinetic term
- ▶ Higher orders from NJL vacuum

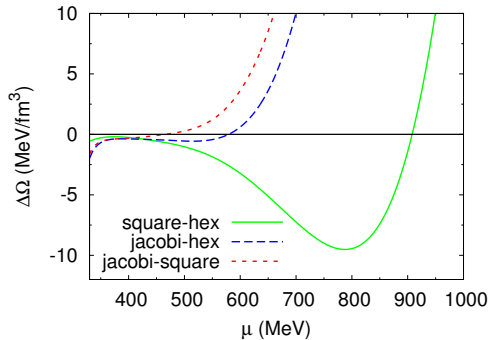


- ▶ Pushing the model to its validity limits..



- ▶ 2D modulations become favored over 1D ones at high μ .

2D crystals at high μ



- ▶ Inhomogeneous phase appears at intermediate densities and low temperatures

Conclusions

- ▶ Inhomogeneous phase appears at intermediate densities and low temperatures
- ▶ 2D modulations seem to be disfavored...

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(... except in the high μ region)

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- ▶ Model extensions enlarge the inhomogeneous window

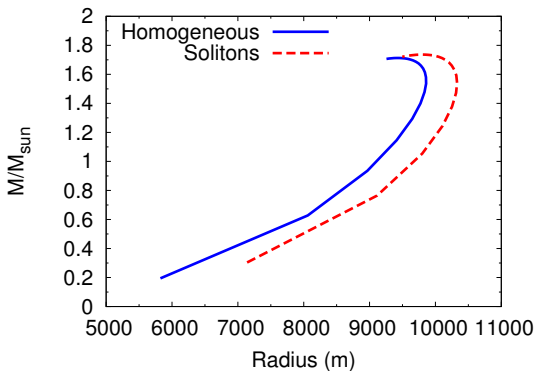
- ▶ Inhomogeneous phase appears at intermediate densities and low temperatures
- ▶ 2D modulations seem to be disfavored...
(... except in the high μ region)
(but it's kind of controversial)
- ▶ Model extensions enlarge the inhomogeneous window
- ▶ Many more things to try ..

more things



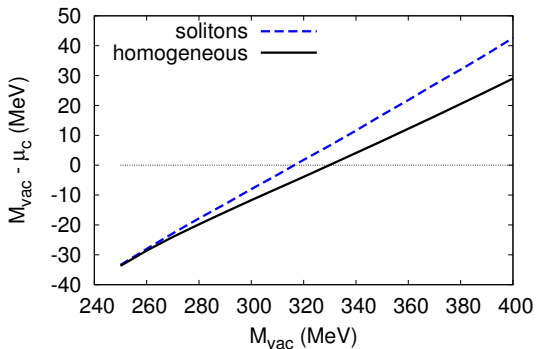
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Mass-radius plot (quark matter only!!)

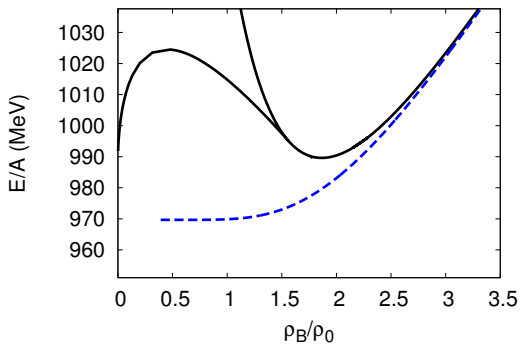


preliminary!

Coupling dependence - binding energy



preliminary!



preliminary!

- ▶ Restrict to lower-dimensional spatial modulations
- ▶ $[\mathcal{H}, P_{\perp}] = 0 \rightarrow$ Full spectrum from lower-dimensional eigenvalues λ
- ▶ Boost along the transverse directions:

$$(\lambda, \mathbf{0}) \rightarrow (\sqrt{p_{\perp}^2 + \lambda^2}, \mathbf{p}_{\perp})$$

$$\begin{aligned} \Omega(T, \mu; M(\vec{x})) &= -\frac{2TN_c}{V_{\parallel}} \sum_{\lambda} \int \frac{d\vec{p}_{\perp}}{(2\pi)^{d_{\perp}}} \ln \left(2 \cosh \left(\frac{\lambda \sqrt{1 + \vec{p}_{\perp}^2 / \lambda^2} - \mu}{2T} \right) \right) \\ &+ \frac{1}{V} \int_V \frac{|M(\vec{x}) - m|^2}{4G_s} + \text{const.} \end{aligned}$$

(D. Nickel, Phys.Rev.D80 074025, 2009 - arXiv:0906.5295)

One-dimensional modulations:

$$M(\vec{x}) \rightarrow M(z)$$

- ▶ Restrict to one-dimensional modulations

$$M(\vec{x}) \rightarrow M(z)$$

- ▶ The Hamiltonian becomes

$$\mathcal{H} \rightarrow \mathcal{H}_{1D} = \begin{pmatrix} H_{1D}(M(z)) & \\ & H_{1D}(M(z)^*) \end{pmatrix}$$

$$\mathcal{H}_{1D}(M(z)) = \begin{pmatrix} -i\partial_z & M(z) \\ M(z)^* & i\partial_z \end{pmatrix} \quad \text{Gross-Neveu Hamiltonian}$$



- ▶ One of the simplest interacting fermionic field theories
- ▶ Defined in 1+1 dimensions

$$\mathcal{L}_{GN} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + \frac{1}{2} g^2 (\bar{\psi} \psi)^2$$

- ▶ Hartree-Fock \rightarrow recover SUSY QM-like equation
- ▶ Real self-consistent solutions of the form

$$M(z) = \Delta \left(\nu \operatorname{sn}(b|\nu) \operatorname{sn}(\Delta z|\nu) \operatorname{sn}(\Delta z + b|\nu) + \frac{\operatorname{cn}(b|\nu) \operatorname{dn}(b|\nu)}{\operatorname{sn}(b|\nu)} \right)$$

- ▶ Eigenvalue spectrum well known

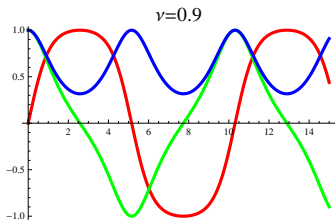
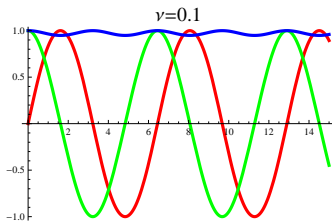
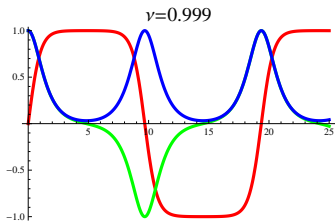
(M.Thies et al., Annals Phys. 314 (2004) 425-447, arXiv:hep-th/0402014)

Elliptic functions: $\text{sn}(z|\nu)$, $\text{cn}(z|\nu)$, $\text{dn}(z|\nu)$

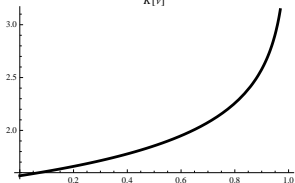
$\nu \in [0, 1]$



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Period $\propto K(\nu)$





$$\Omega_{MF}^{NJL}(T, \mu; \Delta, \nu, \delta) = -2N_c \int_0^\infty dE \tilde{\rho}(E; \nu, \Delta) \tilde{f}_{\text{bare}} \left(\sqrt{E^2 + \delta \Delta^2} \right) \\ + \frac{1}{4G_s P} \int_0^P dz |M(z) - m|^2 + C$$

$$\tilde{f}_{\text{bare}}(x) = \tilde{f}_{\text{vac}}(x) + \tilde{f}_{\text{medium}}(x)$$

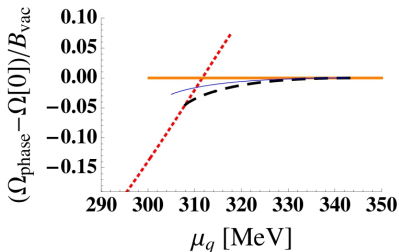
$$\tilde{f}_{\text{vac}}(x) = x$$

$$\tilde{f}_{\text{medium}}(x) = T \ln \left(1 + \exp \left(-\frac{x - \mu}{T} \right) \right) + T \ln \left(1 + \exp \left(-\frac{x + \mu}{T} \right) \right)$$

$$\tilde{f}_{UV}(x) \rightarrow \tilde{f}_{PV}(x) = \sum_{j=0}^3 c_j \sqrt{x^2 + j\Lambda^2} \quad (c_0 = 1, c_1 = -3, c_2 = 3, c_3 = -1)$$

Real VS complex modulations

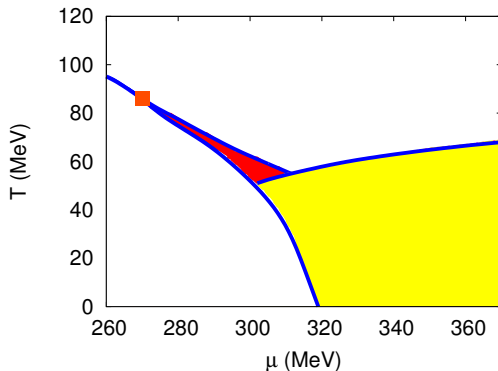
- ▶ Real modulations $\rightarrow P(x) = 0$
- ▶ Solitons: $M(z) \sim \Delta \sqrt{\nu} \operatorname{sn}(z|\nu)$
- ▶ Chiral density wave: $M(z) = \Delta e^{iqz}$
- ▶ Homogeneous broken: $M(z) = \Delta$



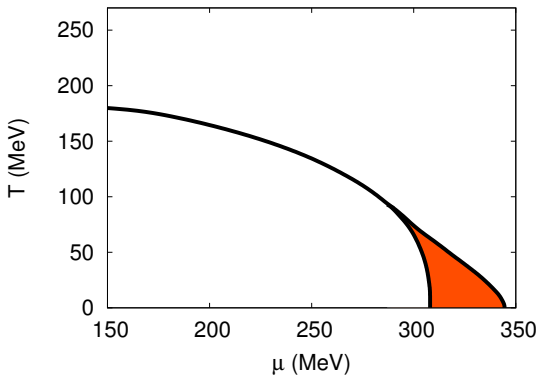
(D. Nickel, PRD 80)

- ▶ (Real) solitons are always favored over chiral density wave!

Comparison with (homogeneous) 2SC phase (with D. Nowakowski)

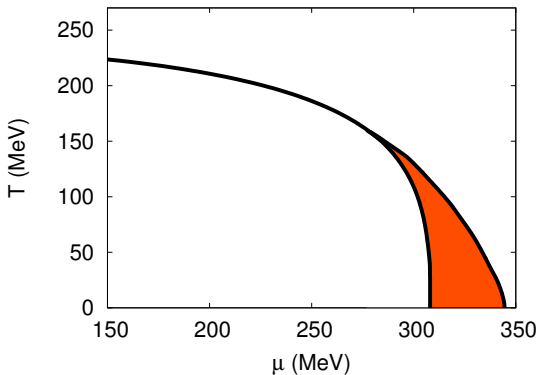


Preliminary!!

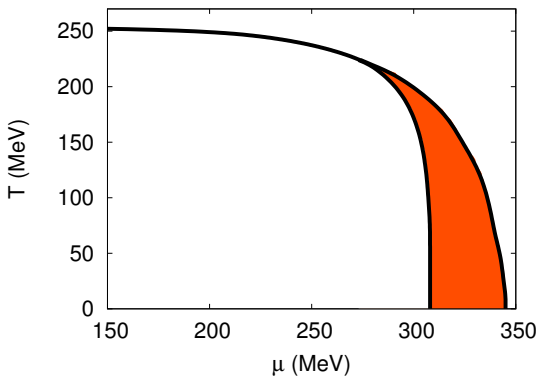


$$N_C = 3$$

PNJL - Large N_C results



$N_C = 10$



$N_C = 50$

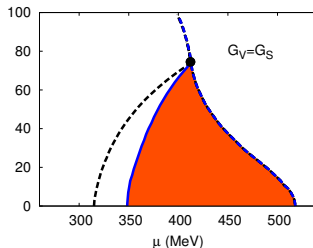
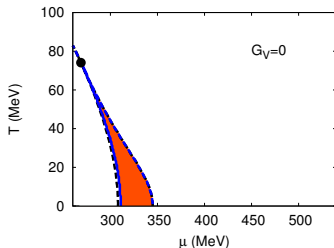


- ▶ Additional vector term: $\mathcal{L} = \mathcal{L}_{NJL} - G_V(\bar{\psi}\gamma^\mu\psi)^2$
- ▶ New mean field: $\bar{\psi}\gamma^\mu\psi \rightarrow \langle \bar{\psi}\gamma^\mu\psi \rangle \equiv n(\vec{x})\delta^{\mu 0}$ (density!)
- ▶ Introduce shifted chemical potential $\tilde{\mu}(\vec{x}) = \mu - 2G_V n(\vec{x})$
- ▶ Determine $\tilde{\mu}$ via $\frac{\delta\Omega}{\delta\tilde{\mu}} = 0$
- ▶ Sacrifice complete self-consistency: pick $\tilde{\mu} \equiv \langle \tilde{\mu} \rangle_z$ instead of $\tilde{\mu}(z)$
 - ▶ Most questionable in the inhomogeneous phase at low μ and T
 - ▶ More reliable close to the restored phase and the Lifshitz point

$$\Omega(T, \mu) \rightarrow \Omega(T, \tilde{\mu}) - \frac{(\mu - \tilde{\mu})^2}{4G_V}$$

Chiral density wave and vector interactions

- ▶ How good is our $\tilde{\mu}(z) \rightarrow \langle \tilde{\mu}(z) \rangle$ approximation ?
- ▶ Cross-check: **Chiral density wave** $\rightarrow M(z) = \Delta e^{iqz} \rightarrow n(z) = \text{const.}$



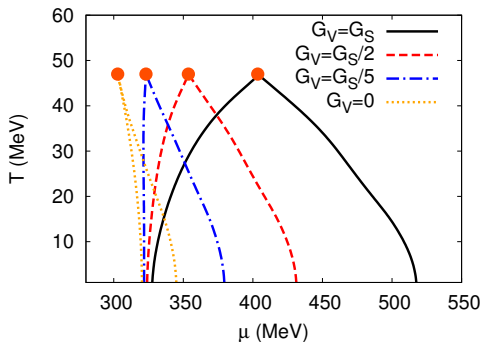
- ▶ Same qualitative behaviour as the solitonic solutions
- ▶ Lifshitz point at the same position
- ▶ Different (1st order) homogeneous \rightarrow inhomogeneous transition line

More phase diagrams: massive quarks

- ▶ Self-consistent solutions take the form

$$M(z) = \Delta \left(\sqrt{\nu} \operatorname{sn}(b|\nu) \operatorname{sn}(\Delta z|\nu) \operatorname{sn}(\Delta z + b|\nu) + \frac{\operatorname{cn}(b|\nu) \operatorname{dn}(b|\nu)}{\operatorname{sn}(b|\nu)} \right)$$

- ▶ Additional parameter: b
- ▶ Same qualitative features as $m = 0$
- ▶ Results for $m = 5$ MeV



- ▶ PNJL model:

$$\mathcal{L}_{PNJL} = \bar{\psi} (i\gamma^\mu D_\mu - \hat{m}) \psi + G_s \left((\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\tau^a\psi)^2 \right) - \mathcal{U}(L, \bar{L})$$

- ▶ Covariant derivative: $D_\mu = \partial_\mu + iA_0\delta_{\mu 0}$
- ▶ Polyakov loop: $L(\vec{x}) = \mathcal{P} \exp[i \int_0^{1/T} d\tau A_4(\tau, \vec{x})]$, $A_4(\tau, \vec{x}) = iA_0(t = -i\tau, \vec{x})$
- ▶ Expectation values: $\ell = \frac{1}{N_c} \langle \text{Tr}_c L \rangle$, $\bar{\ell} = \frac{1}{N_c} \langle \text{Tr}_c L^\dagger \rangle$
- ▶ **Assumption:** $\ell, \bar{\ell}$ space-time independent

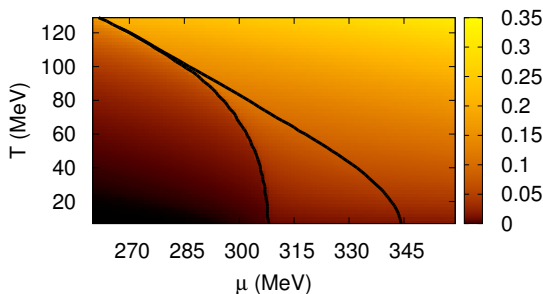
- ▶ Main effect:

$$N_c T \log \left(1 + e^{-\frac{E-\mu}{T}} \right) \rightarrow T \log \left(1 + e^{-3(E-\mu)/T} + 3\ell e^{-(E-\mu)/T} + 3\bar{\ell} e^{-2(E-\mu)/T} \right)$$

- ▶ Thermally excited quarks are suppressed at small $\ell, \bar{\ell}$

Polyakov loop expectation value

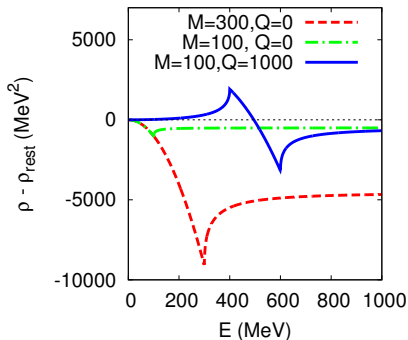
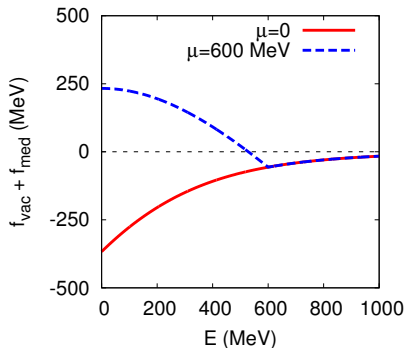
- ▶ How good is our approximation of constant l, \bar{l} ?



- ▶ Inhomogeneous regime: $l, \bar{l} \leq 0.2$
- ▶ Effects of neglecting spatial variations of l, \bar{l} presumably small

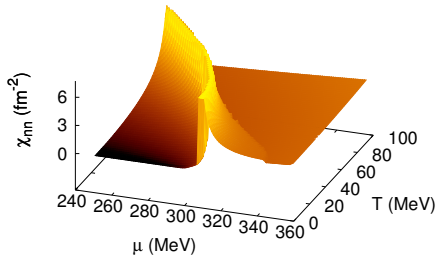
Continent - Origin? (Chiral density wave, $T = 0$)

$$\Omega - \Omega_{rest} = -2N_c \int_0^\infty dE [\rho(E; M, Q) - \rho_{rest}(E)] [f_{vac}(E) + f_{med}(E, \mu)] + \frac{M^2}{4G}$$

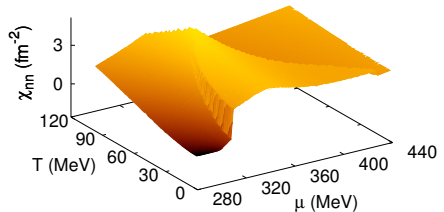


Quark number susceptibilities

$$\chi_{nn} = -\frac{\partial^2 \Omega}{\partial \mu^2} = \frac{\partial \bar{n}}{\partial \mu}$$



$G_V = 0$



$G_V = G_S/2$

Susceptibilities

