Inhomogeneous phases

in low-dimensional strongly interacting matter

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1) Abelian Skyrme crystals in 1+1 d chirally symmetric models

2) Soliton crystal phases in Gross-Neveu models and
   conducting polymers

Hirschegg, August 30, 2012

1) Abelian Skyrme crystals in 1+1 d chirally symmetric models

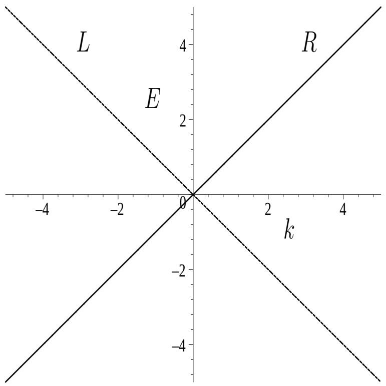
Massless Dirac fermions in 1+1 d

$$\mathcal{L}_0 = \bar{\psi} i \partial\!\!\!/ \psi \qquad (\partial\!\!\!/ = \gamma^\mu \partial_\mu)$$

$$\gamma_5 = \gamma^0 \gamma^1, \quad \{\gamma_5, \gamma^\mu\} = 0, \quad \gamma_5^2 = 1$$

"Chirality": Left/right movers

$$\psi_{R,L} = \frac{1 \pm \gamma_5}{2} \psi$$



Boost with rapidity ξ: "Helicity" 1/2

$$\psi_R \to e^{\xi/2} \psi_R, \quad \psi_L \to e^{-\xi/2} \psi_L$$

$$\mathcal{L}_0 = \bar{\psi}_R i \partial\!\!\!/ \psi_R + \bar{\psi}_L i \partial\!\!\!/ \psi_L$$

$U(1)_R \otimes U(1)_L$ chiral symmetry

$$\psi_R \to e^{i\beta} \psi_R, \quad \psi_L \to e^{i\alpha} \psi_L$$

Chirally invariant interactions?

i) Four-fermion interaction

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} \left[\left(\bar{\psi} \psi \right)^2 + \left(\bar{\psi} i \gamma_5 \psi \right)^2 \right]$$

$\cong$ Nambu–Jona-Lasinio model (1961) in 1+1 dimensions (NJL$_2$)

- 1+1 dimensions: $[\psi] = L^{-1/2}, \quad [g^2] = 1$ renormalizable
- U(N) flavor symmetry: $N \to \infty, \quad N g^2 = \text{const.}$
- asymptotic freedom, no confinement

ii) SU(N) gauge interactions

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - \frac{1}{2} \mathrm{Tr} F_{\mu\nu} F^{\mu\nu}$$

$$D_\mu = \partial_\mu + i g A_\mu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g \left[A_\mu, A_\nu\right]$$

't Hooft model (1974) $\cong$ QCD$_2$

- 1+1 dimensions: $[A_\mu] = 1, \quad [g] = M$ super-renormalizable
- SU(N) color: $N \to \infty, \quad N g^2 = \mathrm{const.}$
- confinement of quarks

Why large N limit? SSB of continuous symmetry viable in 1+1 dimensions.
Semiclassical methods become exact (here: relativistic Hartree-Fock).

NJL$_2$ model

Menu:

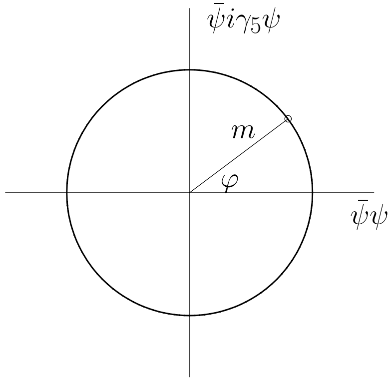
Vacuum $\rightarrow$ mesons $\rightarrow$ baryons $\rightarrow$ cold matter $\rightarrow$ hot dense matter

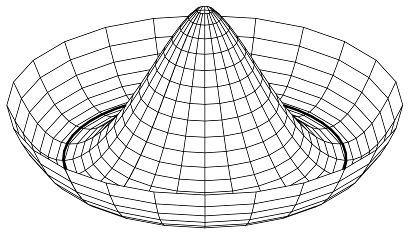
- Hartree-Fock vacuum

$$\left(-i\gamma_5\partial_x + \gamma^0 S + i\gamma^1 P\right)\psi_\alpha = \epsilon_\alpha \psi_\alpha$$

Two condensates

$$S = m\cos\varphi = -g^2\langle\bar{\psi}\psi\rangle$$
$$P = m\sin\varphi = -g^2\langle\bar{\psi}i\gamma_5\psi\rangle$$





- Mesons

RPA $\rightarrow$ scalar σ ($\mathcal{M}_\sigma = 2m$), pseudoscalar π ($\mathcal{M}_\pi = 0$) $\cong$ Goldstone boson
Unlike in 3+1 d, there are also massless “Goldstone baryons”

• Baryons and cold matter

Key: Global and local chiral rotations

Vacuum Dirac-Hartree-Fock equation

$$\left(-i\gamma_5\partial_x + \gamma^0 m e^{i\varphi\gamma_5}\right)\psi = \epsilon\psi$$

• Global chiral rotation

$$\psi = e^{-i\alpha\gamma_5}\tilde{\psi}$$

$$\left(-i\gamma_5\partial_x + \gamma^0 m e^{i(\varphi-2\alpha)\gamma_5}\right)\tilde{\psi} = \epsilon\tilde{\psi}$$

Chiral vacuum angle φ is irrelevant

• Local chiral rotation $\alpha \to \alpha(x)$

$$\left(-i\gamma_5\partial_x - \alpha'(x) + \gamma^0 m e^{i(\varphi-2\alpha(x))\gamma_5}\right)\tilde{\psi} = \epsilon\tilde{\psi}$$

Break translational invariance and generate unwanted vector potential

Interesting special case

$$\alpha = qx, \qquad \alpha' = q$$

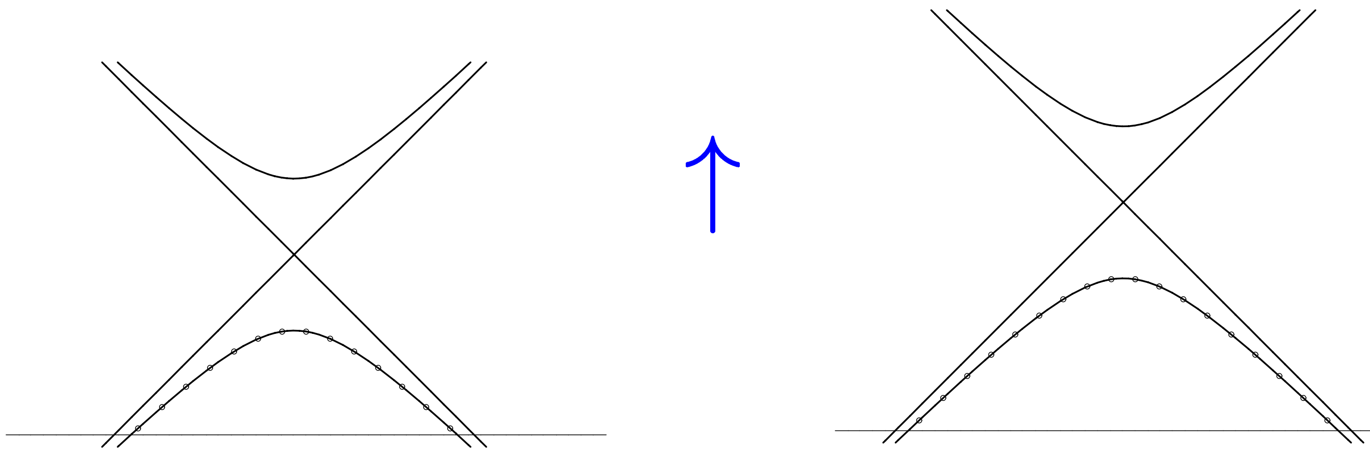
Condensate assumes helical structure — spectrum shifted rigidly

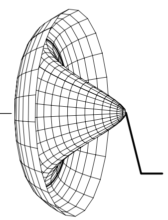
$$\left(-i\gamma_5\partial_x + \gamma^0 m e^{-2iqx\gamma_5}\right)\tilde{\psi} = (\epsilon + q)\tilde{\psi}$$

$$S(x) = m\cos 2qx, \qquad P(x) = -m\sin 2qx$$

New Hartree-Fock solution of the chiral Gross-Neveu model — physics?

Role of Dirac sea: Chiral anomaly





Evaluate fermion density and energy density, using cutoff $E > -\Lambda/2$

• Vacuum: Momentum cutoff $\pm\Lambda/2$

Fermion density

$$\frac{\rho_0}{N} = \int^{\Lambda} \frac{dk}{2\pi} = \frac{\Lambda}{2\pi}$$

Energy density

$$\frac{\mathcal{E}_0}{N} = -\int^{\Lambda} \frac{dk}{2\pi}\sqrt{k^2+m^2} = -\frac{\Lambda^2}{8\pi} - \frac{m^2}{4\pi} - \frac{m^2}{2\pi}\ln\frac{\Lambda}{m}$$

• Chirally twisted case: Momentum cutoff $\pm\Lambda'/2$ with $\Lambda' = \Lambda + 2q$

Fermion density

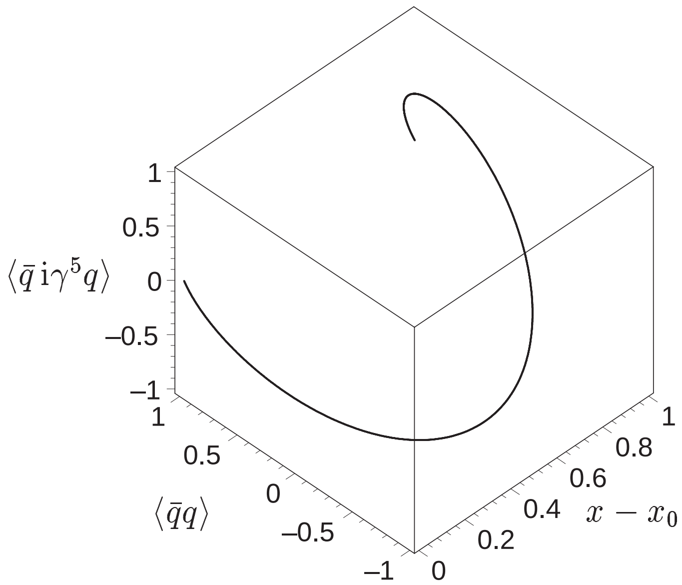
$$\frac{\rho}{N} = \int^{\Lambda'} \frac{dk}{2\pi} = \frac{\rho_0}{N} + \frac{q}{\pi}$$

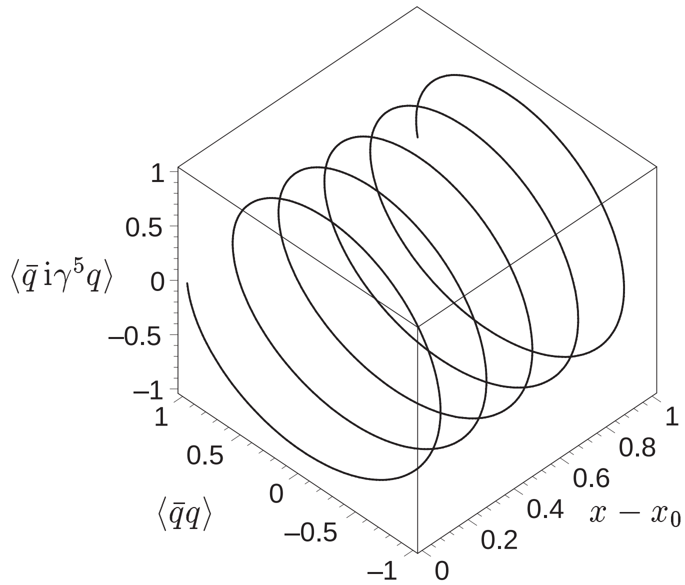
Energy density

$$\frac{\mathcal{E}}{N} = -\int^{\Lambda'} \frac{dk}{2\pi}\left(\sqrt{k^2+m^2} - q\right) = \frac{\mathcal{E}_0}{N} + \frac{q^2}{2\pi}$$

Mimics massless Fermi gas with $k_f = q$

Resulting Hartree-Fock solution at finite density: Crystal with helical order parameter — chiral spiral (Schön 2000)





Properties

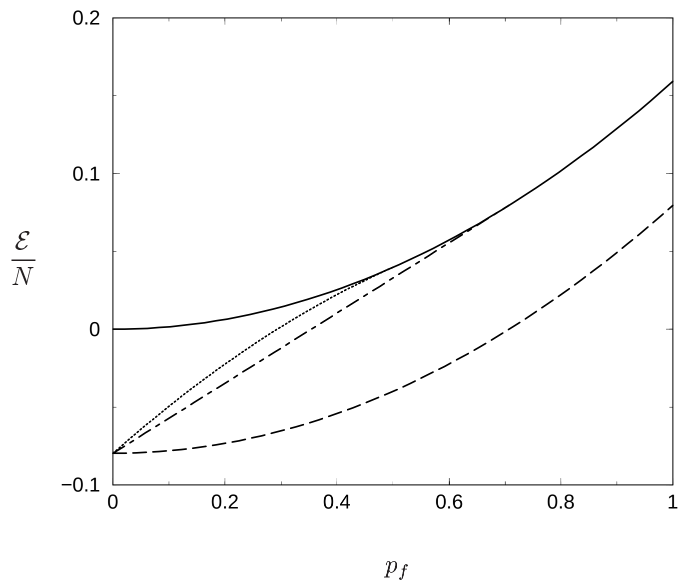
- Topological baryon number (Abelian Skyrmion and Skyrme crystal)
- Massless baryons ($L \to \infty$)
- Chiral spiral is true ground state of dense matter

$$S(x) = m \cos 2k_f x, \qquad P(x) = -m \sin 2k_f x$$

Compare energy density with homogeneous solutions

$$\frac{\rho}{N} = \frac{k_f}{\pi}$$

$$\frac{\mathcal{E}}{N} = -\frac{m^2}{4\pi} + \frac{k_f^2}{2\pi}$$



- Fermion density spatially constant — result of axial current conservation

$$0 = \partial_\mu \langle j_5^\mu \rangle = \partial_x \langle j_5^1 \rangle = \partial_x \langle \psi^\dagger \psi \rangle$$

- Translational and chiral symmetries broken, screw symmetry unbroken

$$P + k_f Q_5$$

- Mesons in dense matter $\leftrightarrow$ RPA on chiral spiral ground state:
  Only one gapless mode, pion-phonon hybrid (Riedl 2001)

• Hot dense matter

Phase diagram of the NJL_2 model?

Hartree-Fock at finite (T,μ): Start from finite temperature, $\mu = 0$ homogeneous Fermi gas

$$\left(-i\gamma_5\partial_x + \gamma^0 m(T)\right)\psi_\alpha = \epsilon_\alpha\psi_\alpha, \qquad m(T) = -Ng^2\sum_\alpha \bar{\psi}_\alpha\psi_\alpha\frac{1}{e^{\beta\epsilon_\alpha}+1}$$

Local chiral rotation

$$\psi(x) \to e^{i\mu x\gamma_5}\psi(x)$$

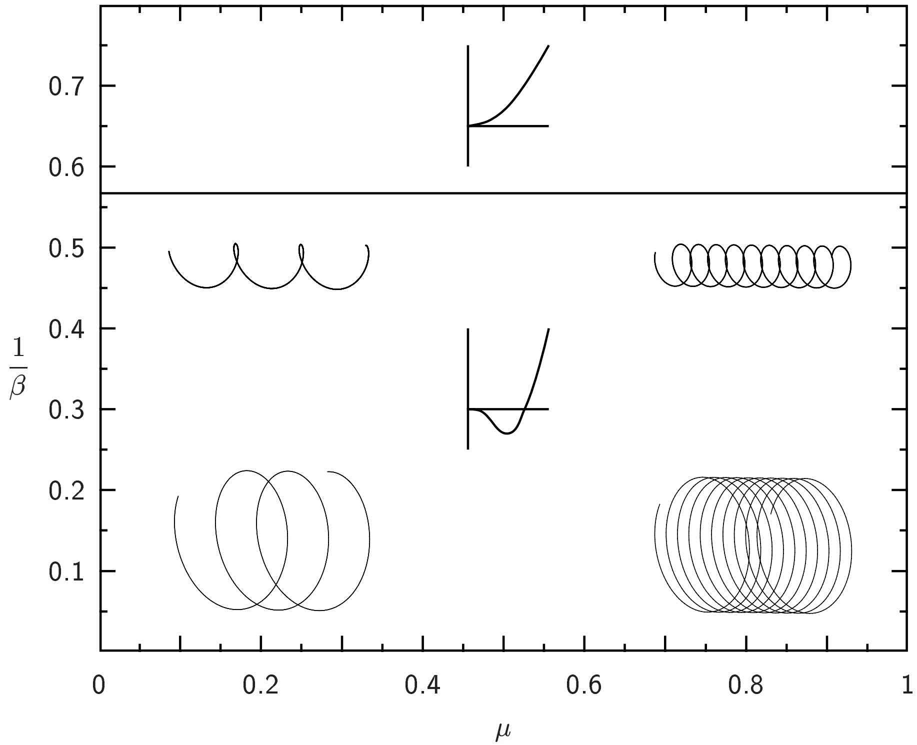
Chiral spiral condensate with radius $m(T)$ and spatial period π/μ. Shift in spectrum by $-\mu$ generates chemical potential
Radius depends only on T, pitch on μ

Renormalized grand canonical potential and fermion density

$$\begin{aligned} \psi(T,\mu)|_{\text{spir}} &= \psi(T,0) - N\frac{\mu^2}{2\pi} \\ \rho(T,\mu)|_{\text{spir}} &= N\frac{\mu}{\pi} \end{aligned}$$

Anomaly is UV effect $\to$ no T-dependence

Phase diagram of the NJL$_2$ model in the (μ, T) plane (Schön 2000)



’t Hooft model

Hartree-Fock approach — what is different?

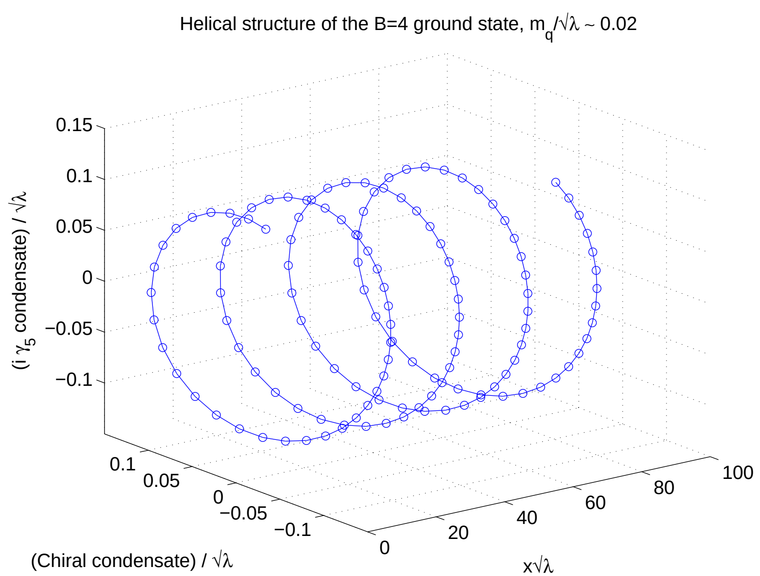
- Linear Coulomb potential, no transverse gluons
- Non-covariant momentum dependence of fermion self-energy (numerical)
- Chiral condensate not related to dynamical mass (Zhitnitskii)

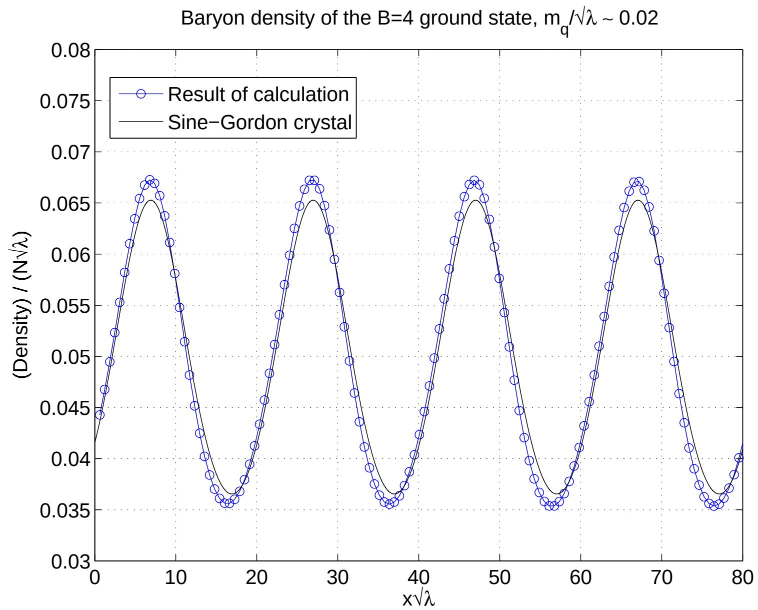
$$\langle \bar{q} q \rangle = -\frac{N}{\sqrt{12}} \left(\frac{N g^2}{2\pi} \right)^{1/2}$$

- Confinement in independent particle picture: divergent p-independent part of fermion self-energy
- Infinite tower of mesons ($q\bar{q}$ states)
- Goldstone boson, massless baryons and chiral spiral identical to NJL$_2$ model — interaction invariant under local chiral rotations. Universal behavior
- Analytic results confirmed by numerical Hartree-Fock studies on a lattice (Salcedo, Levit, Negele 1990, Bringoltz 2009)

• Massive model: LO chiral corrections → universal description in terms of sine-Gordon model (Salcedo et al. 1990, Schön 2000)

(Bringoltz 2009)





Finite temperature 't Hooft model

Large N limit: No temperature dependence to LO in $1/N$
Reason: Divergent quark self-energy (confinement)

No restoration of chiral symmetry — phase diagram in (μ, T) plane not universal

Upshot

• Universal behavior at $T = 0$ and near the chiral limit (sine-Gordon)

• Chiral symmetry governs mesons, baryons and dense matter in a radical way not seen in 3+1 d

• An Abelian version of Skyrme's picture is realized in the large N limit of a class of 1+1 dimensional models

2) Soliton crystal phases in Gross-Neveu models

and conducting polymers

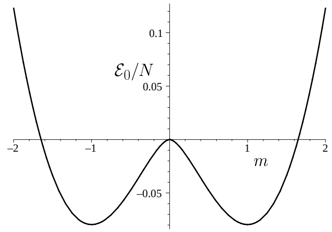
Lagrangian (Gross, Neveu 1974)

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} \left(\bar{\psi} \psi \right)^2$$

Similar to NJL_2 model, but only discrete Z_2 chiral symmetry

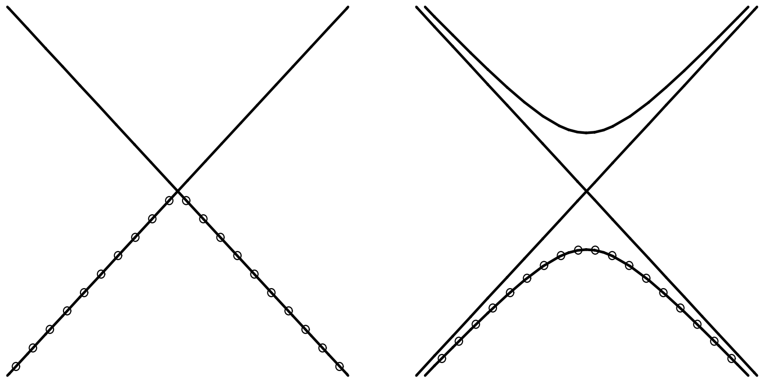
$$\psi \to \gamma_5 \psi, \qquad \bar{\psi}\psi \to -\bar{\psi}\psi$$

Vacuum: SSB, dynamical fermion mass $\quad S(x) = m$



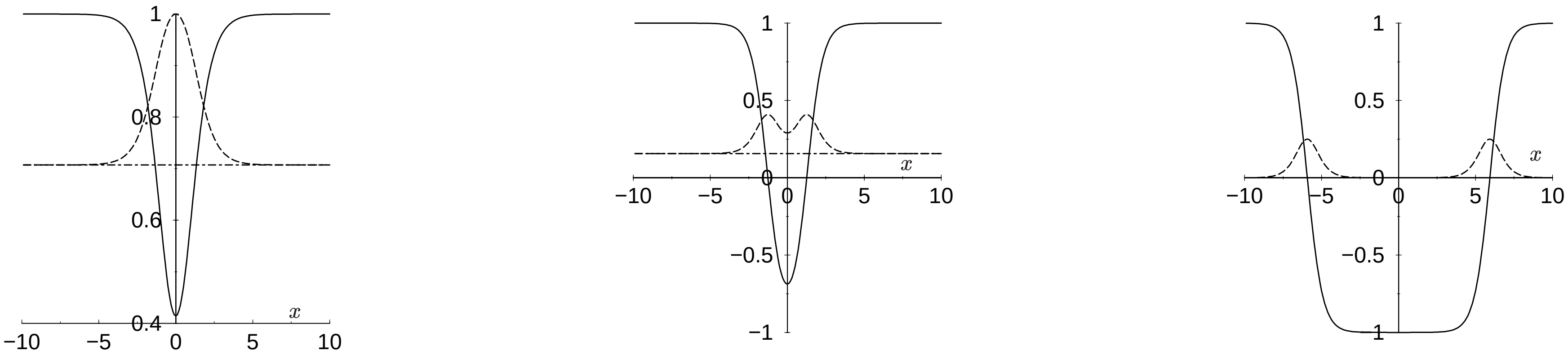
Relativistic version of Peierls instability (Peierls 1955)

$$\mathcal{E}_0(m) - \mathcal{E}_0(0) = -\frac{Nm^2}{4\pi}$$



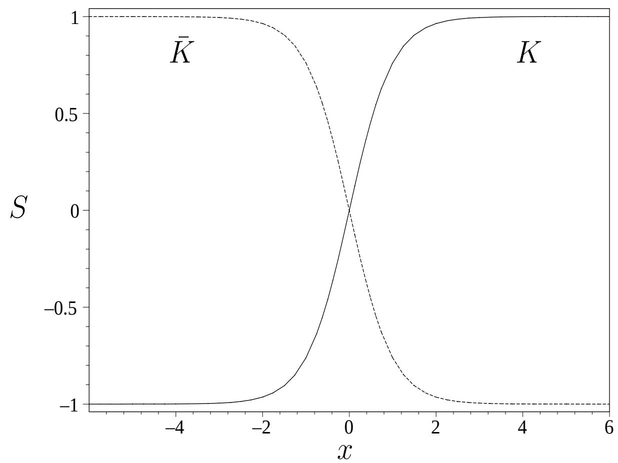
Baryons

- Kink-antikink (Dashen, Hasslacher, Neveu 1975)



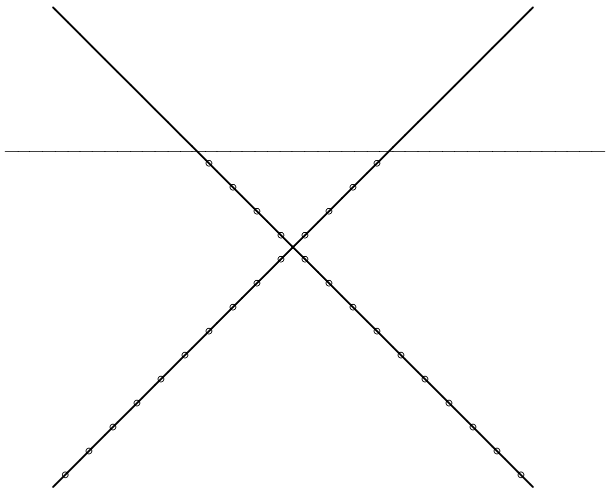
- Kink (Callan, Coleman, Gross, Zee) — reflection of Z_2 chiral symmetry

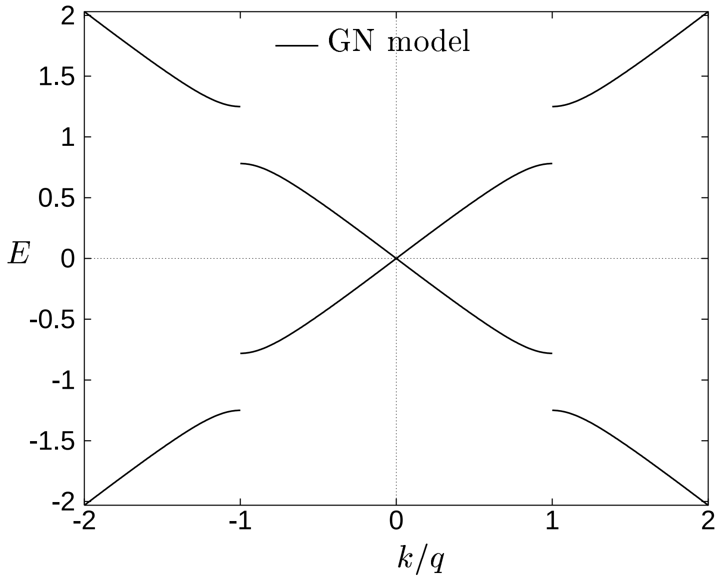
$$S(x) = \pm \tanh x$$



Soliton crystal

- Low density limit $\rightarrow$ array of isolated baryons
- Peierls instability



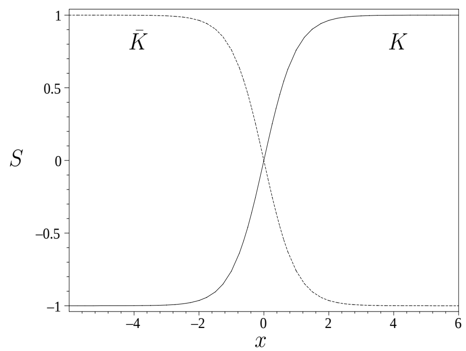


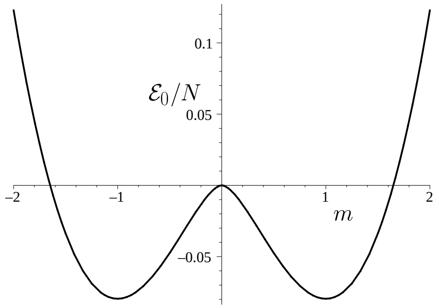
How to find self-consistent potential? How to solve Hartree-Fock equations?

$$\left(-i\gamma_5\partial_x + \gamma^0 S\right)\psi_\alpha = \epsilon_\alpha\psi_\alpha, \qquad S - m_0 = -Ng^2\sum_\alpha^{\text{occ}}\bar{\psi}_\alpha\psi_\alpha$$

CCGZ kink

$$S(x) = \pm \tanh x$$





• Distinguishing feature: reflectionless

Lattice of such kinks and antikinks

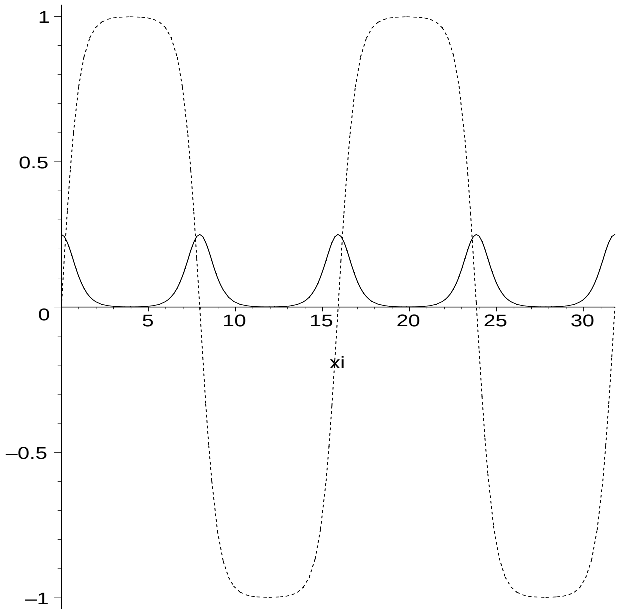
$$\sum_{n=-\infty}^{\infty} (-1)^n \tanh(x + an) = C \, \mathrm{sn}(Bx, \kappa)$$

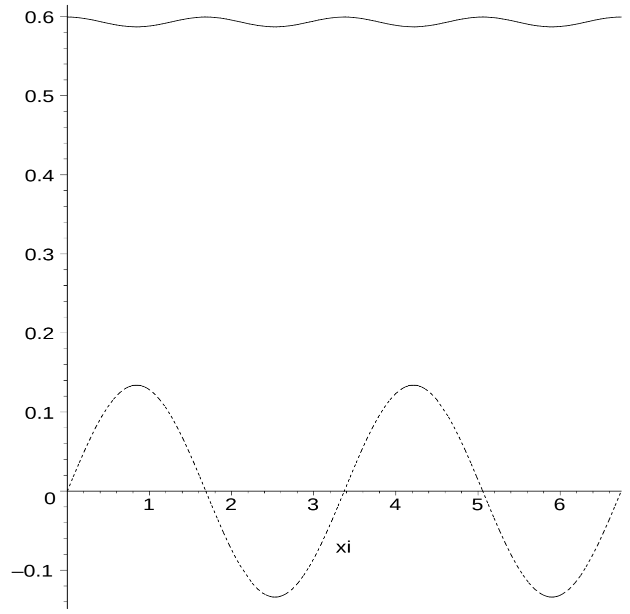
Dirac equation can be reduced to Lamé equation and solved exactly in terms of Jacobi elliptic functions (Whittaker, Watson)

• Distinguishing feature: finite band potential

Principal result: In the Gross-Neveu model the most general Dirac potential $S(x)$ leading to Lamé equation (3 parameter family) yields self-consistency for all T, μ, γ

Examples of shapes of $S(x)$ and fermion density ($T = 0$)

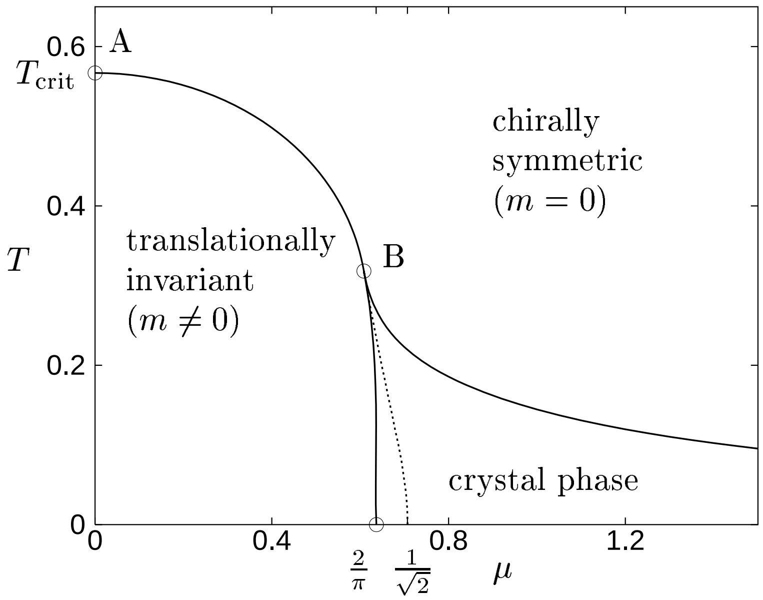




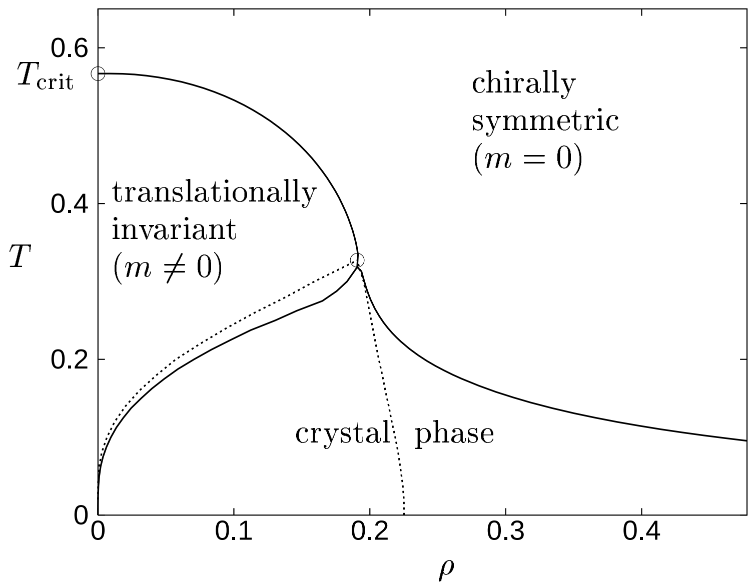
Phase diagram:

Expect 3 different phases: massless Fermi gas, massive Fermi gas, soliton crystal

Phase diagram of the Gross-Neveu model in the chiral limit
(Schnetz, Urlichs 2003/04)



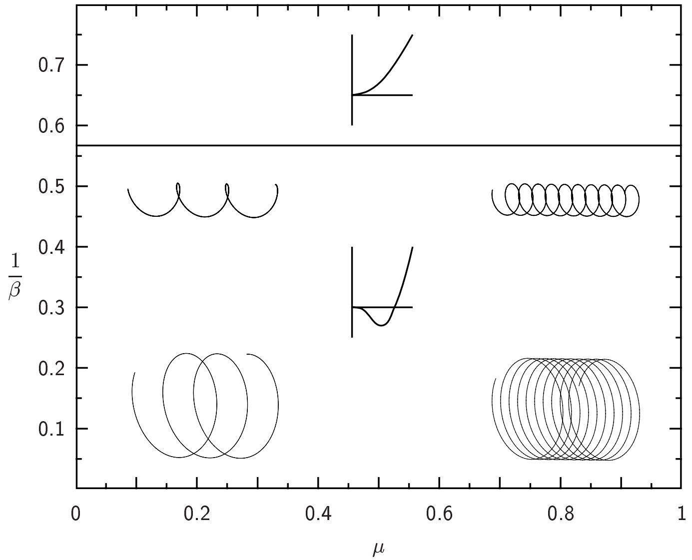
(T, μ)



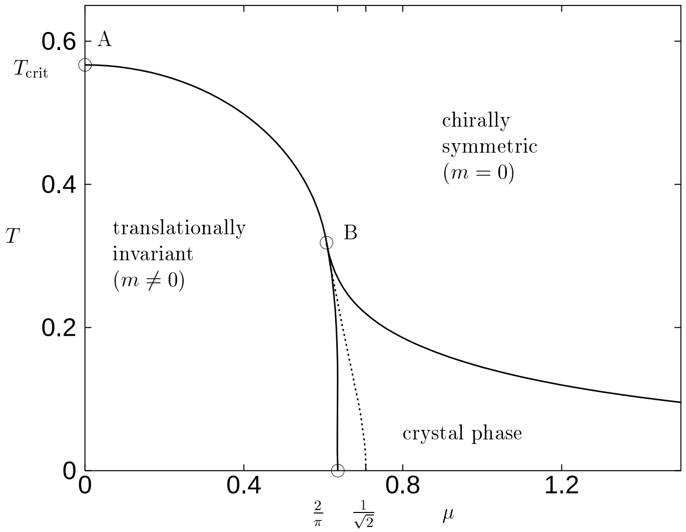
(T, ρ)

Comparison of NJL_2 and Gross-Neveu phase diagrams

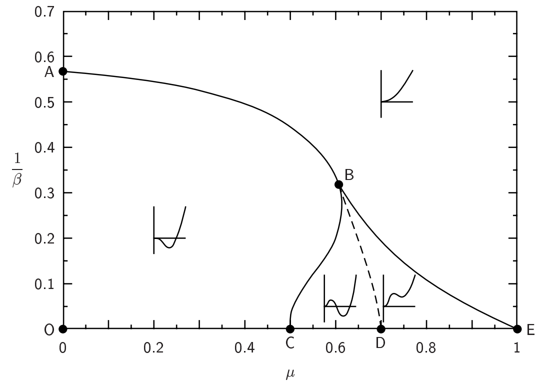
NJL_2



GN_2

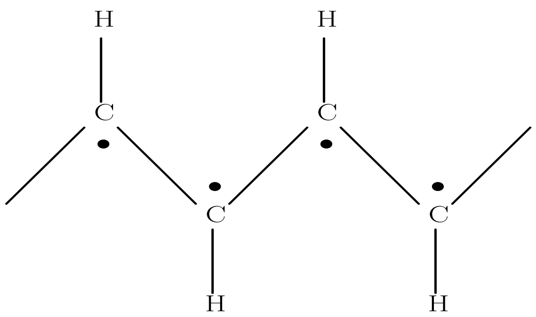


Homogeneous NJL_2/GN_2



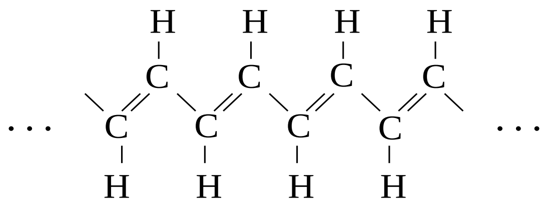
Relationship between GN_2 model and conducting polymers

$trans$-polyacetylene $(CH)_x$ — quasi one-dimensional system

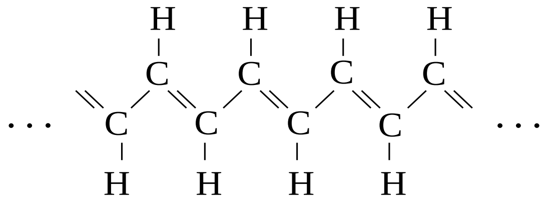


Unstable with respect to dimerization

a)



b)



Quantum mechanical model based on 1d lattice, electrons and phonons (Su, Schrieffer, Heeger 1979); Nobel prize in chemistry 2000 for Heeger, MacDiarmid, Shirakawa for conducting polymers

SSH Hamiltonian

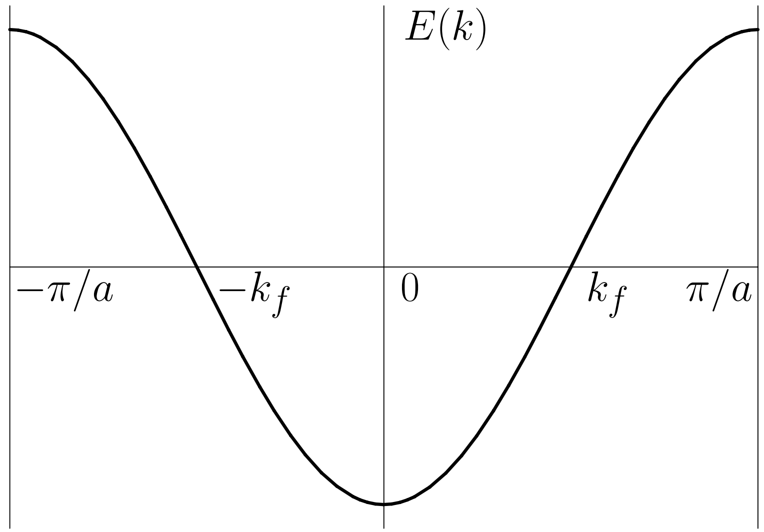
$$\begin{aligned} H &= \sum_n \left(\frac{p_n^2}{2M} + \frac{K}{2}(u_{n+1} - u_n)^2 \right) \\ &- \sum_{n,s} \left(t_0 - \alpha(u_{n+1} - u_n) \right) \left(c^\dagger_{n+1,s} c_{n,s} + c^\dagger_{n,s} c_{n+1,s} \right) \end{aligned}$$

- Hopping amplitude linearized in $(u_{n+1} - u_n)$
- Neglect kinetic energy of (CH) monomers (adiabatic approximation)
- Staggered displacement field $\phi_n = (-1)^n u_n$ is order parameter for dimerization (two degenerate ground states)

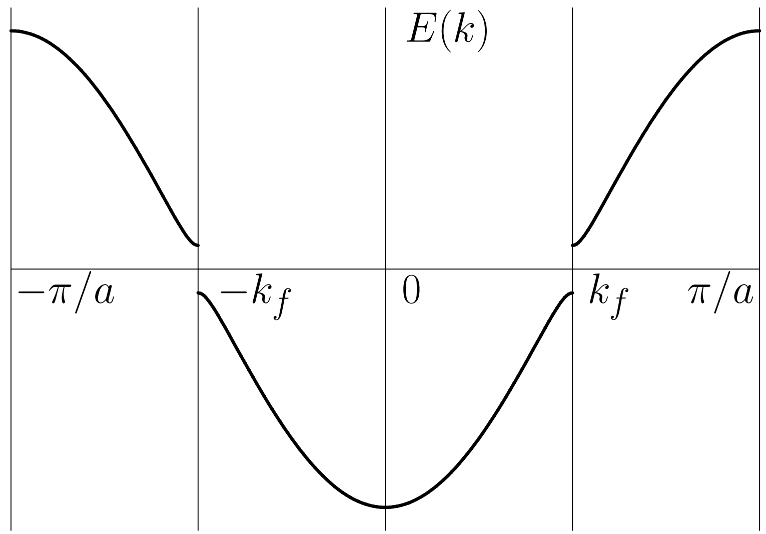
Half filled band: Gap at the Fermi surface (Peierls instability)

Electron spectra

symmetric phase



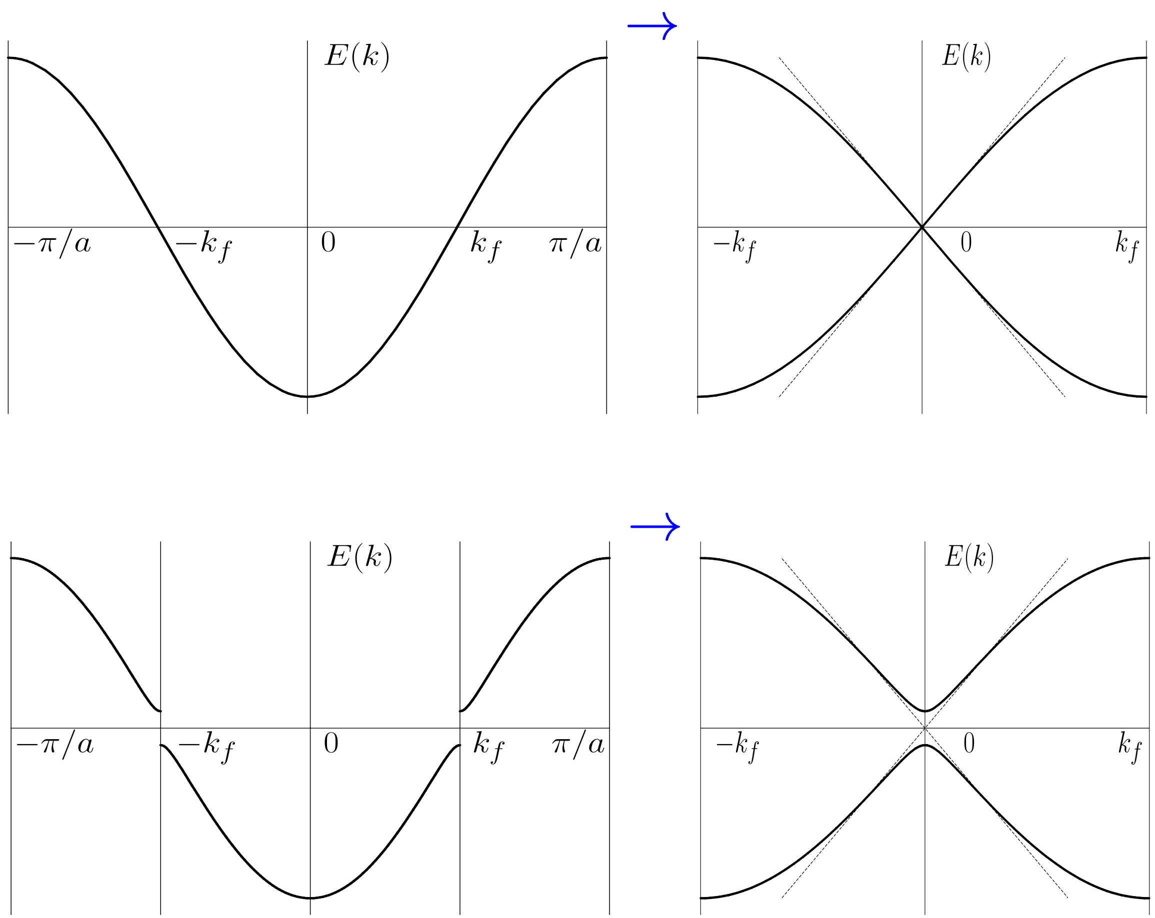
broken phase



Relation to relativistic field theory? Shift momenta by $\pm k_f = \pm \frac{\pi}{2a}$

symmetric → massless

broken → massive



Continuum theory (Brazovskii; Takayama, Lin-Liu, Maki)

$$H = \int dx \sum_s \Psi_s^\dagger(x) \left(-iv_f\sigma_3\partial_x + \sigma_1\Delta(x)\right) \Psi_s(x) + \frac{4K}{Mg^2}\int dx \Delta^2(x)$$

Spinor Ψ_s: Components ψ_1, ψ_2 correspond to fermions near $\pm k_f$

Minimize the g.s. energy

$$\begin{pmatrix} -iv_f\partial_x & \Delta(x) \\ \Delta(x) & iv_f\partial_x \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \epsilon \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Identical to Bogoliubov-de Gennes equations for 1d superconductors

— closely related to Dirac-Hartree-Fock equations for Gross-Neveu model

Dictionary (Bishop, Campbell)

$$\begin{aligned} v_f &\simeq c \\ \text{band width} &\simeq \text{cutoff} \\ \text{spin} &\simeq \text{flavor } (N=2) \\ \text{half} - \text{filled band} &\simeq \text{Dirac sea} \\ \text{dimerization} &\simeq \text{chiral symmetry breaking} \\ \text{doping} &\simeq \text{vary fermion number} \end{aligned}$$

Where is the 4-fermion interaction?

Two equivalent formulations of Gross-Neveu model

$$\begin{aligned} \mathcal{L} &= \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} \left(\bar{\psi} \psi \right)^2 \\ \frac{\partial \mathcal{L}}{\partial \bar{\psi}} &= 0 \quad \rightarrow \quad \left(i \partial\!\!\!/ + g^2 \bar{\psi} \psi \right) \psi = 0 \end{aligned}$$

Auxiliary scalar field (Hubbard-Stratonovitch transformation)

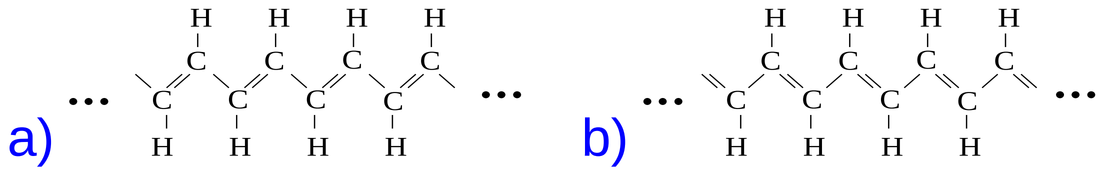
$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + g \sigma \bar{\psi} \psi - \frac{1}{2} \sigma^2$$

Euler-Lagrange equations

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \quad &\rightarrow \quad \left(i \partial\!\!\!/ + g \sigma \right) \psi = 0 \\ \frac{\partial \mathcal{L}}{\partial \sigma} = 0 \quad &\rightarrow \quad \sigma = g \bar{\psi} \psi \end{aligned}$$

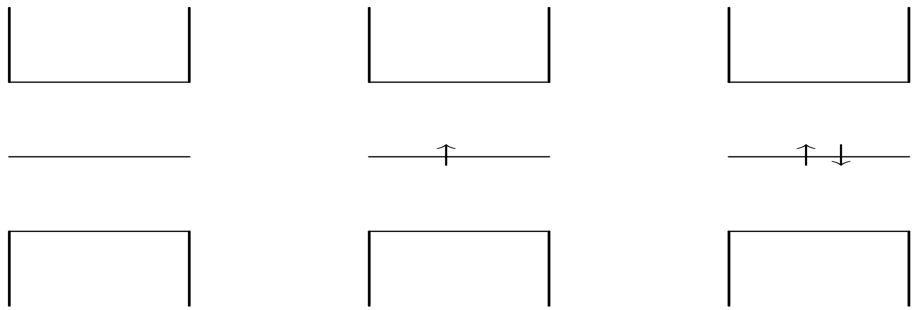
Phonon field corresponds to σ — equivalence only in the adiabatic limit

Role of DHN baryons in *trans*-polyacetylene?



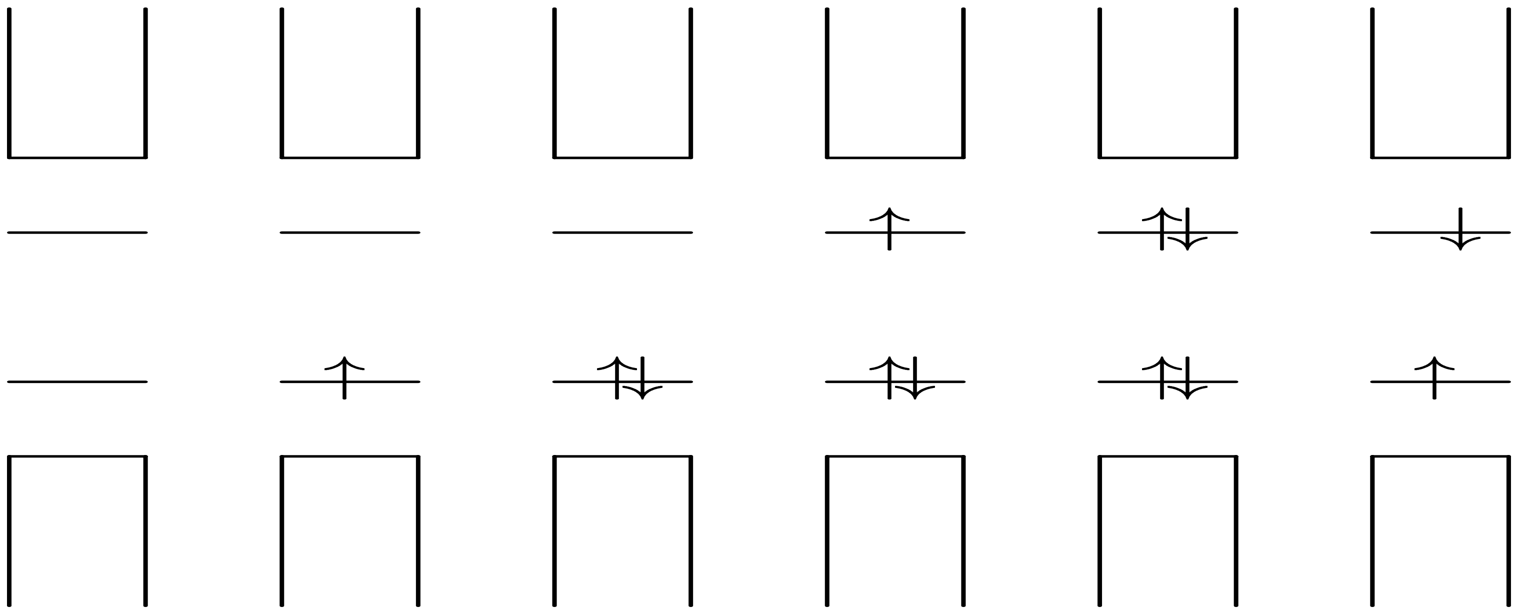
Kink or antikink $S(x) = \pm \mathtt{tanh}(x)$: Domain wall defect
— change from a) to b) over distance of 5-7 monomers ("Soliton")

Fractional fermion number of the kink ($N_f = n - N/2$) shows up through unnatural spin-charge assignments in polymers (Jackiw, Schrieffer)



(S, Q) = (0,1) $(\pm\frac{1}{2}$,0) (0,-1)

Kink-antikink baryon: “polaron” — natural spin-charge assignments (S, Q)



(0,2)* $(\pm\frac{1}{2}$,1) (0,0) $(\pm\frac{1}{2}, -1)$ (0,-2)* (0/1,0)†

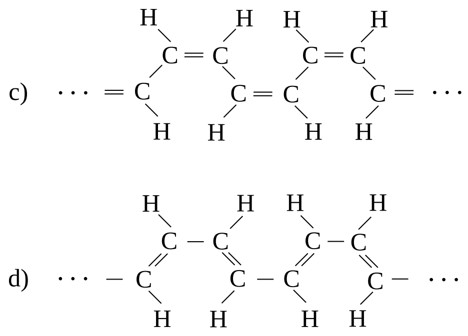
*) “Bipolarons” ($Q = \pm 2$) dissociate into kink-antikink pair in *trans*-PA (cf. limit $y \to 1$ of DHN baryon)

†) “Exciton” is excited state (charge or fermion number 0)

Polaron crystal Brazovskii (1980), Horovitz (1981) identical to baryon crystal in Gross-Neveu model

- Massive Gross-Neveu model?

cis-polyacetylene: Configuration c) has lower energy than d)



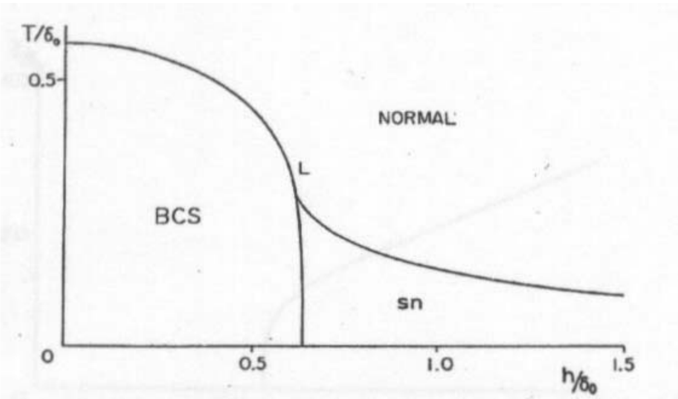
Theory by Brazovskii, Kirova (1981) equivalent to massive Gross-Neveu model

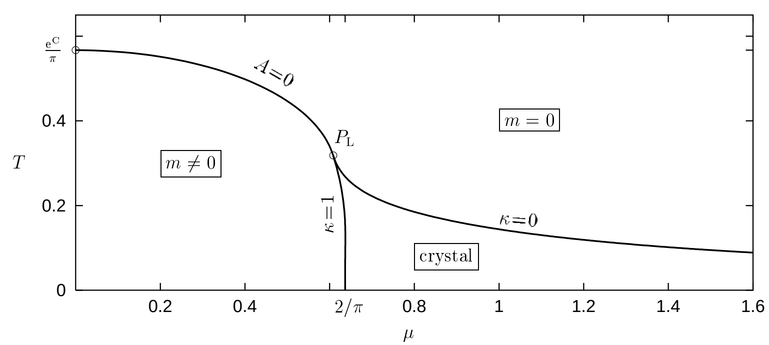
Polarons and bipolarons correspond to baryons

Parallel worlds

Mertsching and Fischbeck (1981) The incommensurate Peierls phase of the quasi-one dimensional Fröhlich model with a nearly half-filled band

Machida and Nakanishi (1984) Superconductivity under a ferromagnetic molecular field ($ErRh_4B_4$)

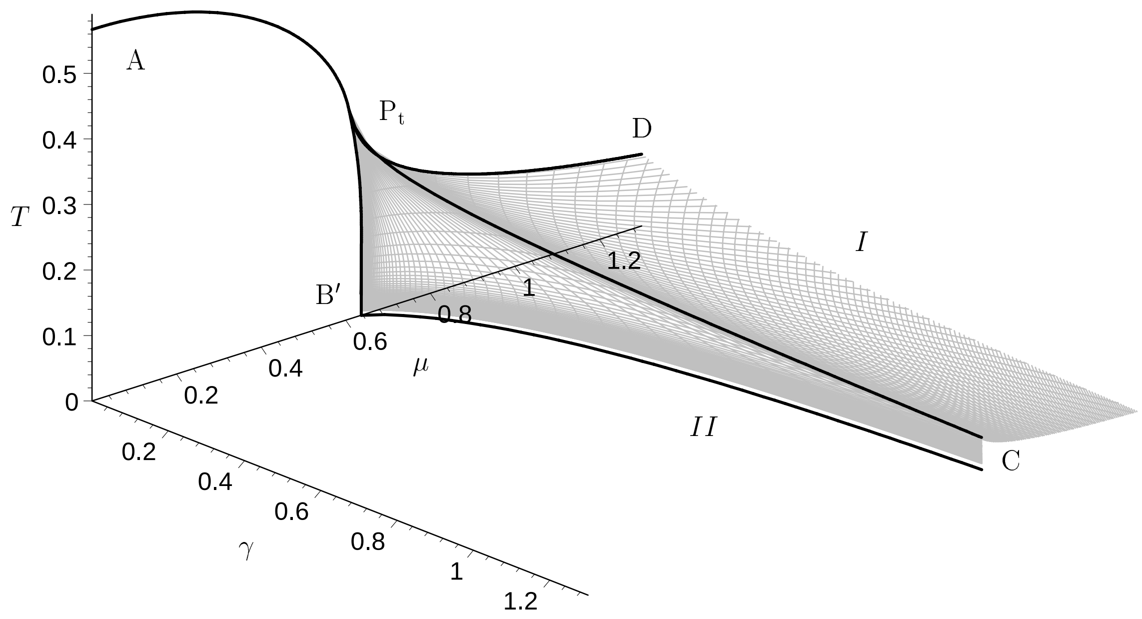




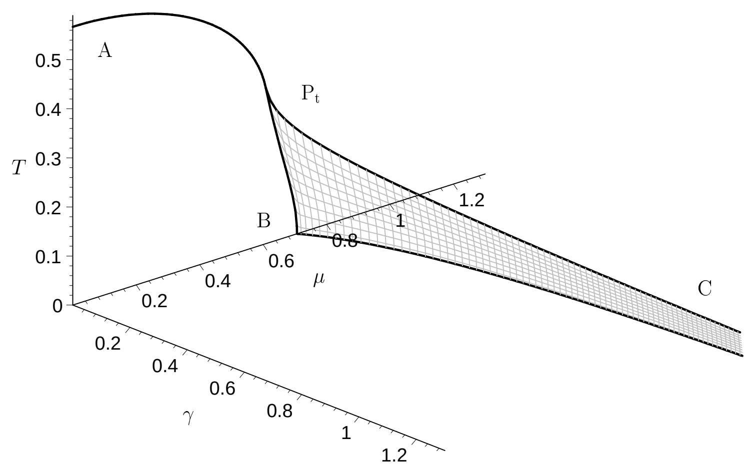
- Dictionary

Dirac HF equation — Bogoliubov-DeGennes equation
ultrarel. kinematics — lin. dispersion at Fermi surface
chiral condensate — Cooper pair condensate
chemical potential — magnetic field
baryon density — spin polarization

Revised phase diagram of the massive Gross-Neveu model (Schnetz, Urlichs 2006)

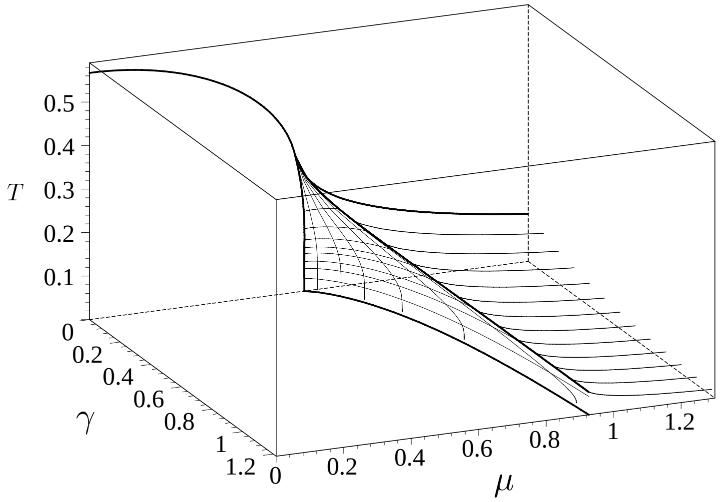


old

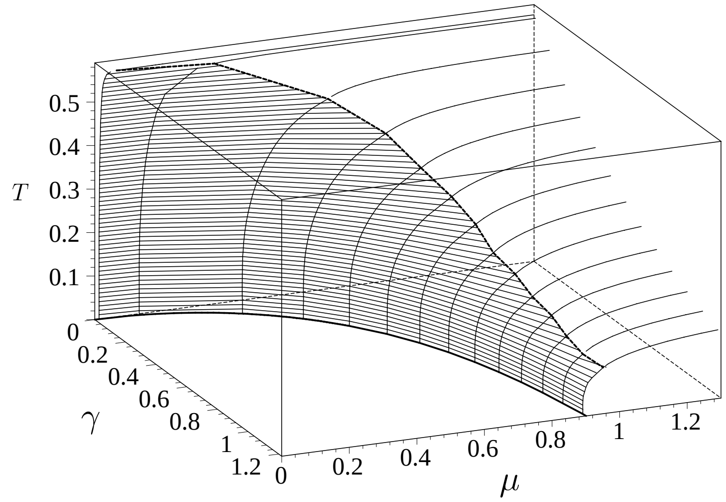


Phase diagram of the massive NJL_2 model (Boehmer, Fritsch, Kraus 2009)

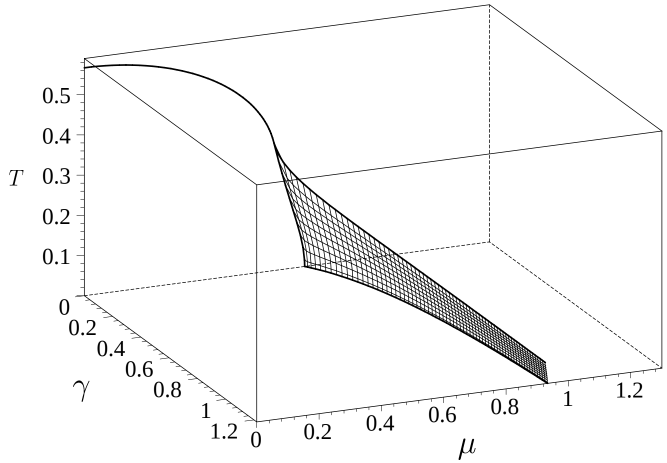
GN_2



NJL_2

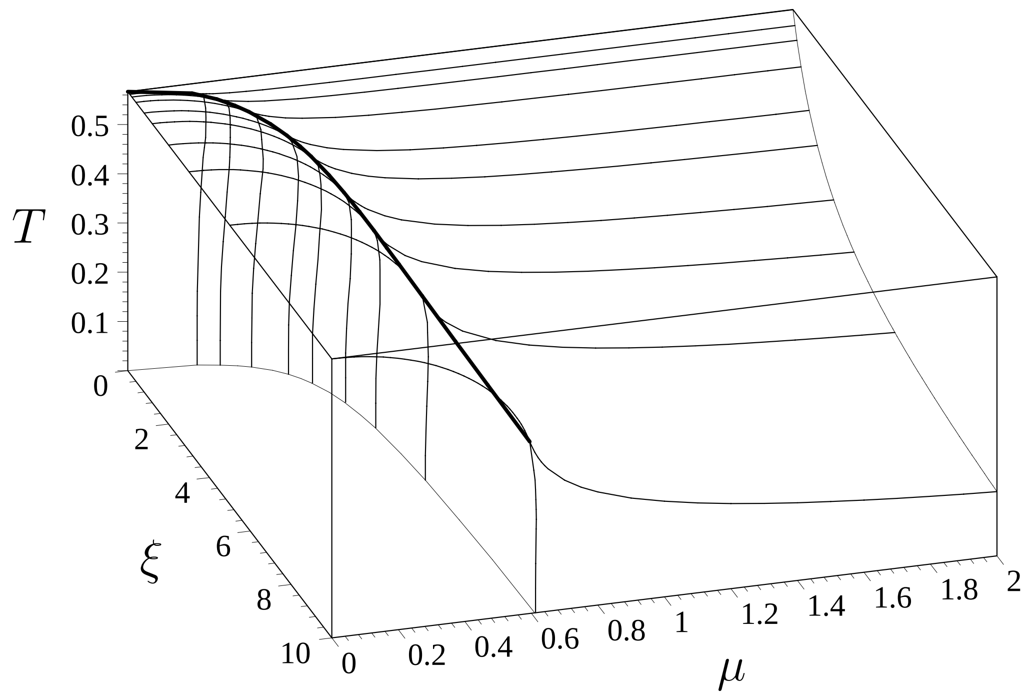


old



Interpolating between Gross-Neveu and NJL$_2$ models (Boehmer 2009)

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2}\left(\bar{\psi}\psi\right)^2 + \frac{G^2}{2}\left(\bar{\psi} i \gamma_5 \psi\right)^2$$



$$\xi = \frac{\pi}{N}\left(\frac{1}{G^2} - \frac{1}{g^2}\right)$$