Nuclear Symmetry Energy from Quantum Skyrmion Crystals

C. Adam
University of Santiago de Compostela and IGFAE

NuSym23, GSI, Darmstadt

September 20, 2023

Skyrme interactions (Skyrme functional)

$\neq$

Skyrme model

a nonlinear field theory supporting topological solitons (=Skyrmions)

Based on work in collaboration with
A. García Martín-Caro, M. Huidobro, R. Vázquez and A. Wereszczynski (AGMC, MH, RV, AW)

- *Dense matter equation of state and phase transitions from a generalized Skyrme model*, Phys. Rev. D105 (2022) 074019 [1]
- *Quantum skyrmion crystals and the symmetry energy of dense matter*, Phys. Rev. D106 (2022) 114031 [2]
- *Kaon condensation in skyrmion matter and compact stars*, Phys. Rev. D107 (2023) 074007 [3]
- Review: CA, AGMC, MH, AW, *Skyrme crystals, nuclear matter and compact stars*, Symmetry 15 (2023) 899 [4]

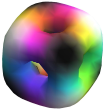
and partially on independent work by
D. Harland, P. Leask and M. Speight

- *Skyrme crystals with massive pions*, e-Print: 2305.14005 [HLS]

Outline

Skyrme model(s)

- Degrees of freedom of QCD:
  - high energy (fundamental): quarks and gluons
  - low energy (effective): hadrons
- $\Rightarrow$ low energy effective field theory (EFT) of hadrons (currently not derivable from QCD)
- Supported by large N_c: QCD = EFT of mesons
- One proposal: Skyrme model ↖
  - primary fields: *only* mesons ↑
  - baryons and nuclei realized as top. solitons called Skyrmions; topological charge = baryon number B (*integer* for finite E)



  - Numerical calc., e.g. $B = 4$
  - simplest case (two flavors): field space = chiral SU(2) matrix $U = \phi_0 \mathbf{1} + \mathrm{i}\vec{\phi} \cdot \vec{\tau}, \quad \phi_0^2 + \vec{\phi}^2 = 1$ (pions $\vec{\pi} = F_\pi \vec{\phi}$)

Skyrme model(s)

- Original Skyrme model (SkM), massive pions ($g^{\mu\nu} \to (+---)$):

$$\mathcal{L}_{024} = \mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_4 \quad , \quad \mathcal{L}_0 = -(c_0/2)\mathrm{Tr}(\mathbf{1} - U)$$

$$\mathcal{L}_2 = -c_2 g^{\mu\nu}\ \mathrm{Tr}\,(L_\mu L_\nu), \quad \mathcal{L}_4 = c_4\ g^{\mu\rho} g^{\nu\sigma} \mathrm{Tr}\,([L_\mu, L_\nu][L_\rho, L_\sigma])$$

$$L_\mu = U^\dagger \partial_\mu U \ \hookleftarrow \qquad \Lsh\ \mathcal{L}_{d>3}\ \ldots\ \text{necessary for Skyrmions}$$

$$c_0 = (F_\pi^2 m_\pi^2)/(4\hbar^3)\,,\ c_2 = F_\pi^2/(16\hbar)\,,\ c_4 = \hbar/(32 e_{\mathrm{Sk}}^2)\ \ldots\ \text{or free}$$

Skyrme model(s)

- Generalized Skyrme model (GSkM):

$$\mathcal{L}_{0246} = \mathcal{L}_{024} + \mathcal{L}_6 \,, \qquad \mathcal{L}_6 = -c_6 g^{\mu\nu} \mathcal{B}_\mu \mathcal{B}_\nu \,, \quad c_6 = g_\omega^2/(2m_\omega^2)$$

$$\mathcal{B}^\mu = \frac{1}{24\pi^2} \mathrm{Tr}\,(\epsilon^{\mu\nu\rho\sigma} L_\nu L_\rho L_\sigma) \,, \quad B = \int d^3x \mathcal{B}^0$$

- $\mathcal{L}_6$ ... ω-meson repulsion: indispensable at high density
  necessary for $M_{\max} \geq 2M_\odot$; for $c_s^2 \geq 1/3$
- Skyrmions: static config. $U(\vec{r})$ with fixed B (b.c.) $\Rightarrow$ minimize
  $E = -\int d^3r \mathcal{L}_{024(6)}$  ...starting point for nuclei/ons
- Relation to nuclei: Quantization ...... non-renormalizable!

$\Rightarrow$ Quantize finite # of DoF ... "Collective Coordinates (CC)"

... zero (spin, isospin) and nonzero (vibrational) modes

... nuclear spin, N, Z, nuclear spectra and resonances

Skyrme model(s)

- Conceptually attractive; naturally incorporates
  - chiral symmetry & breaking
  - baryon number conservation
  - Spin-statistics theorem
  - extended nucleons
  - $\Rightarrow$ no short distance sing. in NN interactions
- Weaknesses & Problems
  - Currently NOT quantitatively competitive
  - Main reason: Calculations hard (from the start)
  - Especially extended versions (more interactions; more mesons) – Exploration still in its infancy
  - BUT: significant recent progress
- Skyrme crystals (SkC) and nuclear matter (NM)
  - **Spoiler:** reasonable for $n_B > n_0$, still with problems for $n_B \leq n_0$
  - BUT: some very recent progress

Skyrme model(s)

Specific problems of original SkM

- Too large (classical) binding energies (B.E.)

⇒ generalized SkM: more mesons, more interactions

⇒ OR: vib. ZPE almost *exactly* cancel these B.E.

S.B. Gudnason, C. Halcrow, *Quantum binding energies in the Skyrme model*, arXiv:2307.09272

- Unrealistic shapes of Skyrmions

⇒ Inclusion of ρ mesons

C. Naya, P. Sutcliffe, *Skyrmions and clustering in light nuclei*, Phys. Rev. Lett. 121 (2018) 232002

- Only some (rotational) nuclear spectra explained

⇒ Inclusion of vibrational modes (*LEC-mf*)

C. Halcrow, C. King, N. Manton, *A dynamical α-cluster model of* $^{16}\mathrm{O}$, Phys. Rev. C95 (2017) 031303

- Insufficient description of NM by SkC for $n_B \leq n_0$

⇒ SkC with non-cubic unit cells [HLS]

⇒ Inclusion of ρ mesons unpublished

Skyrme crystals

Model nuclear matter (NM) with SkM?

- Skyrmions with $B >> 1$ (e.g., $B \sim 10^{58}$) . . . impossible
- **Simplifying assumption:** periodic lattice of unit cells $U(\vec{r} + \vec{X}) = U(\vec{r})$

$$\vec{X} \in \Lambda = \{n_1\vec{X}_1 + n_2\vec{X}_2 + n_3\vec{X}_3 : \vec{n} \in \mathbb{Z}^3\} , \quad \vec{X}_i \ldots \text{basis in } \mathbb{R}^3$$

$\Rightarrow$ minimize $E[U]$ in unit cell; $\exists$ solutions $\forall\ \vec{X}_i$

- **Simplifying assumption:** cubic unit cell $\vec{X}_i = \ell\hat{e}_i, \quad \ell = 2L$

  Until very recently *always* assumed; **not** true in general

  $\sim$ true for $m_\pi = 0,\ n_B \geq n_0$ or for $n_B >> n_0$
- Further (discrete) sym. $\Rightarrow$ FCC ($\rightarrow$ SC$_{\frac{1}{2}}$), $B_{\text{cell}} = 4$

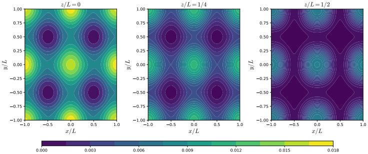
  or SC ($\rightarrow$ BCC$_{\frac{1}{2}}$), $B_{\text{cell}} = 8$

$\Rightarrow$ Skyrme crystal (SkC): necessary *technical* assumption. But: large B_{cell} (e.g. cubes of cubes $B_{\text{cell}} = n^3 B_{\text{min}}$); expensive

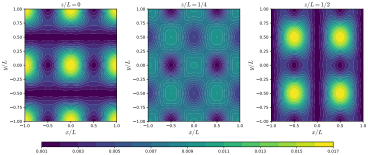
Skyrme crystals

Examples

- FCC $\to$ SC$_{\frac{1}{2}}$



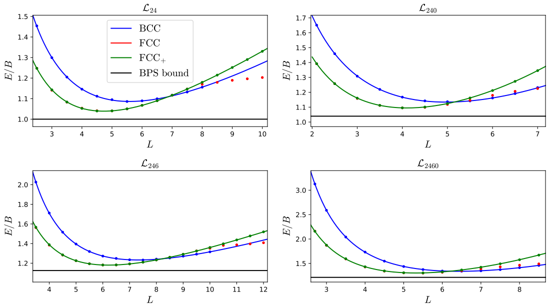
- SC $\to$ BCC$_{\frac{1}{2}}$



- Periodic $r_i \to r_i + 2L$; $\mathcal{E}$ and $\mathcal{B}^0$: accidental sym. $r_i \to r_i + L$
- Known for $\mathcal{L}_{(0)24}$, $\exists$ for $\mathcal{L}_{(0)246}$

Skyrme crystals

$(E/B)(L)$ curves, $n_B = (B_{\rm cell}/(8L^3))$



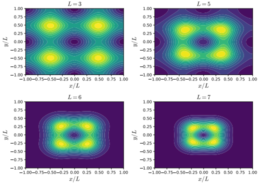
- min. at $L = L_0$, thermodyn. unstable for $L > L_0$ $\ldots$ $P < 0$
  $n_B(L_0) \equiv n_0$ $\sim$ symmetric NM
- $L \geq L_0$ might be relevant after corrections (SymEn, K-cond, phonon ZPE)
- But not FCC: too big curvature at L_0, too big *compression modulus* $\mathcal{K}$

Skyrme crystals

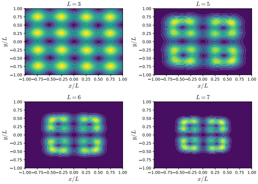
Possible improvements for $n_B \leq n_0$

- Unit cells with lesser symmetries and/or bigger B

Only cubic, $B_{\text{cell}} = 4$



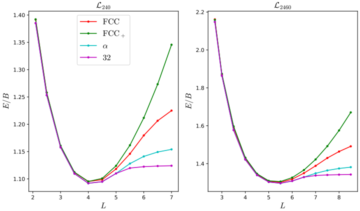
Only cubic, $B_{\text{cell}} = 32$



$\Rightarrow$ "nuclei" plus vacuum for $L >> L_0$

Skyrme crystals

$\Rightarrow$ Resulting $(E/B)(L)$ curves



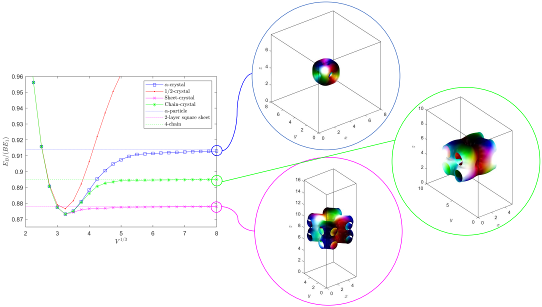
- *Much* lower energies for $L >> L_0$
- Somewhat lower $\mathcal{K}$ but insufficient decrease
- Like FCC for $L < L_0$

Skyrme crystals

Possible improvements for $n_B \leq n_0$

- Non-cubic unit cells,

  $B_{\rm cell} = 4$ [HLS]



- Again, lower B.E. and insufficient decrease of $\mathcal{K}$
- Nuclear pasta-like new phases: Chains (=spagetti), sheets (=lasagne)

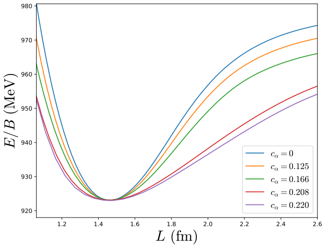
$\Rightarrow$ $B_{\rm cell} >> 4$ and further (quantum & Coulomb) corrections: very important, but expensive

Skyrme crystals

Possible improvements for $n_B \leq n_0$

- Inclusion of ρ meson $\mathcal{R}_\mu = i\vec{\rho}_\mu \cdot \vec{\tau},\ \mathcal{R}_{\mu\nu} = \partial_\mu \mathcal{R}_\nu - \partial_\nu \mathcal{R}_\mu$
  - Only simplest interaction $\mathcal{L}_I = \frac{\alpha}{2}\mathrm{Tr}\,\mathcal{R}_{\mu\nu}[L^\mu, L^\nu]$

    Cubic unit cell, $B_{\text{cell}} = 4,\ \mathcal{L}_{024}$ unpublished



  - Significant improvement from $\mathcal{K}/\mathcal{K}_{\text{ph}} \sim 4$ to $\mathcal{K}/\mathcal{K}_{\text{ph}} \sim 1.5$

Skyrme crystals

SkC: Indications for good behavior for large n_B

- Universal: different (improved) SkC converge to same FCC $E(L)$
- SkC EoS for $\mathcal{L}_{0246}$ coincides with Walecka model ... ω repulsion

  CA, M. Haberichter, AW, *Skyrme models and nuclear matter equation of state*, Phys. Rev. C 92 (2015), 055807
- Class. SkC EoS gives already good NS description (before full $\mathcal{L}_{0246}$ SkC)

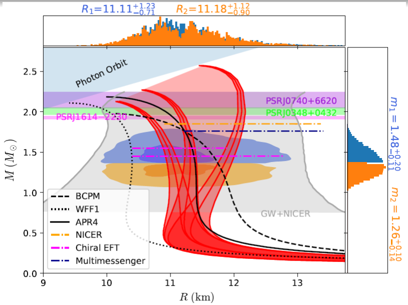
  CA, AGMC, MH, RV, AW, *A new consistent neutron star equation of state from a generalized Skyrme model*, Phys. Lett. B 811 (2020) 135928

  Idea: Hybrid EoS (NM = BCPM)

$$\begin{array}{ccccccccc} n_B: & > & n_2 & > & n_1 & > & n_0 & > & 0 \\ & \mathrm{EoS}_{\mathcal{L}_6} & & \mathrm{EoS}_{\mathcal{L}_{024}} & & \mathrm{EoS}_{\mathrm{NM}} & & \mathrm{EoS}_{\mathrm{NM}} & \end{array}$$

(Almost) independent of c_{0246} $\Rightarrow$

Skyrme crystals



Class. SkC . . . sym. NM; no E_{sym}; only EoS

Implicit assumptions for large n_B:

- Extended nucleons important
- Repulsive nuclear force most important
- No exotic (hyperon) contributions relevant in NS; can be included only for full SkC, not for EoS. Deconfinement not relevant for NS

Symmetry energy

Symmetry energy $\Leftrightarrow$ Isospin quantization

- $\mathcal{L}$ invar. under $U \to AUA^\dagger,\ A \in \mathrm{SU}(2)$ or $\vec{\phi} \to R(A)\vec{\phi},\ R_{ij}(A) = \frac{1}{2}\mathrm{Tr}(\tau_i A \tau_j A^\dagger)$
- $A \ldots$CC: $A \to A(t),\ U \to U_B(\vec{r})\ \Rightarrow\ \mathrm{L} \equiv \int d^3 r \mathcal{L} = \frac{1}{2}\omega_i \Lambda_{ij} \omega_j - M_B$

  $\omega_i = -i\mathrm{Tr}(\tau_i A^\dagger \dot{A})$ ... iso-ang. velocity

  Λ_{ij} ... iso-moment-of-inertia (IMoI) tensor; depends on $\mathcal{L}_{246}$

$\Rightarrow\ \mathrm{H}_{\mathrm{iso}} = \mathrm{H} - M_B = \frac{1}{2} K_i (\Lambda^{-1})_{ij} K_j,\quad K_i = \frac{\partial \mathrm{L}}{\partial \omega_i} = \Lambda_{ij}\omega_j$

  K_i ... body-fixed iso-angular momentum

  $I_i = -R_{ij}(A) K_j$ ... space-fixed iso-angular momentum

- Quantization: $K_i, I_i \to \hat{K}_i, \hat{I}_i$ ... $[\hat{K}_i, \hat{K}_j] = i\hbar\epsilon_{ijk}\hat{K}_k$ etc.
- $\mathrm{H}_{\mathrm{iso}} \to \hat{\mathrm{H}}_{\mathrm{iso}}$ acting on $|\Psi(q); i, i_3\rangle = \sum_{k_3=-i}^{i} c_{k_3}^{ii_3}(q)|i, i_3, k_3\rangle$ ... rigid rotor
- Not all $|i, i_3, k_3\rangle$ allowed ... FR constraints ... syms. of $U_B(\vec{r})$
- Now: $U_B(\vec{r}) \to U_{\mathrm{SkC}}(\vec{r})$ ... Skyrme crystal

Symmetry energy

- SkC: $V_{\rm cr} = N_{\rm cell} V_{\rm cell}$, $B_{\rm cr} = N_{\rm cell} B_{\rm cell}$, $N_{\rm cell} >> 1$
- Iso-moment of inertia: $(\Lambda_{\rm cr})_{ij} = N_{\rm cell}(\Lambda_{\rm cell})_{ij}$
- **Simplifying assumption:** $(\Lambda_{\rm cell})_{ij} = \Lambda_{\rm cell}\delta_{ij}$ ... true for many SkC

$\Rightarrow \hat{\rm H}_{\rm iso,cr} = \frac{\hat{\vec{K}}^2_{\rm cr}}{2N_{\rm cell}\Lambda_{\rm cell}} = \frac{\hat{\vec{I}}^2_{\rm cr}}{2N_{\rm cell}\Lambda_{\rm cell}}$, acts on $|\Psi\rangle_{\rm cr}$ ... not calculable

- **Assumptions:**
  - $|\Psi\rangle_{\rm cr} = \otimes_{N_{\rm cell}} |\psi\rangle_{\rm cell}$, and $|\Psi\rangle_{\rm cr}$ and $|\psi\rangle_{\rm cell}$ share same crystal syms.

  $\Rightarrow$ $|\psi\rangle_{\rm cell}$ calculable: $i_{\rm cell} = 0, \ldots, \frac{B_{\rm cell}}{2}$ and FR constraints
  - Mean-field approx. for isospin density $\hat{\mathcal{I}}^0_3 \to \langle \mathcal{I}^0_3 \rangle$

$$\langle \mathcal{I}^0_3 \rangle = \frac{\hbar\, i_3}{V_{\rm cr}} = \frac{\hbar \langle i_3 \rangle_{\rm cell}}{V_{\rm cell}}\ , \quad i_3 = \frac{Z-N}{2} = -\frac{1}{2}B_{\rm cr}(1-2\gamma_p) = -\frac{1}{2}B_{\rm cr}\delta$$

where $\gamma_p = (Z/(Z+N))$, $\delta = (N-Z)/(Z+N)$ and $Z + N = B_{\rm cr}$

- Isospin energy $E_{\rm iso,cr} = {}_{\rm cr}\langle \Psi | \hat{\rm H}_{\rm iso,cr} | \Psi \rangle_{\rm cr}$ minimal for $i = i_3$

Symmetry energy

- Isospin energy per baryon

$$E_{\mathrm{iso,cr}} = \frac{\hbar^2 i(i+1)}{2\Lambda_{\mathrm{cr}}} \sim \frac{\hbar^2 i_3^2}{2N_{\mathrm{cell}}\Lambda_{\mathrm{cell}}} = \frac{\hbar^2 B_{\mathrm{cr}} B_{\mathrm{cell}}}{8\Lambda_{\mathrm{cell}}}\delta^2$$

$$\Rightarrow \frac{E_{\mathrm{iso,cr}}}{B_{\mathrm{cr}}} = \frac{\hbar^2 B_{\mathrm{cell}}}{8\Lambda_{\mathrm{cell}}}\delta^2 = \frac{\hbar^2 V_{\mathrm{cell}} n_B}{8\Lambda_{\mathrm{cell}}}\delta^2$$

- Compare with B.E. per baryon of asymmetric NM:

$$\frac{E}{B}(n_B, \delta) = E_N(n_B) + S_N(n_B)\delta^2 + \mathcal{O}(\delta^3)$$

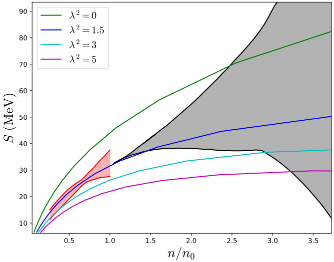
B.E. of symmetric NM ↲ ↳ Symmetry energy

- $S_N = \frac{\hbar^2 B_{\mathrm{cell}}}{8\Lambda_{\mathrm{cell}}}$ where Λ_{cell} implicitly depends on n_B.

Symmetry energy

Example [2]

- FCC SkC, parameters c_{024} from standard fit to p and Δ, $c_6 = \pi^4\lambda^2$ free, $n \equiv n_B$



- $\lambda^2 = 0 : \lim_{n_B\to\infty} S_N(n_B) = \infty$; $\lambda^2 > 0 : \lim_{n_B\to\infty} S_N(n_B) = S_\infty < \infty$
- qualitative good description of S_N for certain c_6, despite bad $\mathcal{K}$
- Why: IMoI $\Lambda = \int d^3r\,\tilde{\Lambda}$ is $\int$ of density ... insensitive to vacuum; to location of nucleons

Conclusions

- For a given SkC $\Rightarrow$ straight forward calculation of S_N [2]
- Works also beyond SkM (e.g. holographic QCD, Sakai-Sugimoto model): field theory, periodic NM solution $\Rightarrow$ isospin $\Rightarrow$ S_N

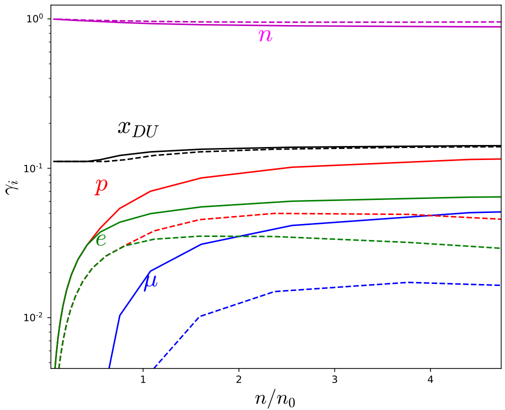
  L. Bartolini, S. Gudnason, *Symmetry energy in holographic QCD*, e-Print: 2209.14309
- Works also, e.g., for Kaon condensate [3]
- SkC as a model for NM? Current status
  - works reasonable for $n_B > n_0$ (universality).
  - Big problems for $n_B \leq n_0$.
  - Possible solutions:
    - Inclusion of further (vector) mesons
    - Non-cubic crystals
    - Bigger unit cells to detect non-crystalline solutions
- **Key question:** Skyrme matter = Skyrme crystal?
- Currently answer not known, difficult numerical problem $\Rightarrow$ future work.

Thank you

Backup

$$\lambda^2 = 1.5 : \; S_N(n_0) = S_0 = 31.9\,\text{MeV}, L = 46.4\,\text{MeV}, K_{\text{sym}} = -130\,\text{MeV}$$

Particle fractions
$\lambda^2 = 0$ ... continuous; $\lambda^2 = 1.5$ ... dashed



Symmetry energies for many parameters, many EoS