Generation of Transfer Maps

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Motivation

Goal: identification of magnetic field errors

- compare predicted to measured trajectories
- minimize discrepancy
  - vary multipole strengths
  - gradient-based optimization algorithms

Require

tracking differentiable w.r.t. multipole strengths

Hamiltonian Dynamics

Hamiltonian

$$H = c\sqrt{m^2c^2 + \left(\vec{p} - e\vec{A}(\vec{q})\right)^2} + e\phi(\vec{q}) \tag{1}$$

eqn. of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{2}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{3}$$

transfer map $\Leftrightarrow$ solution of eqn. of motion

Formal Solution of Eqn. of Motion

Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial p_i} \frac{\partial g}{\partial q_i} \tag{4}$$

express eqn. of motion via Poisson bracket

$$\vec{z} = [q_1, ..., q_n, p_1, ..., p_n]^T \tag{5}$$

$$\dot{\vec{z}} = -\{H, \vec{z}\} = -\underbrace{\{H, \cdot\}}_{\text{diff. operator}} \vec{z} \tag{6}$$

formal solution

$$\vec{z}(t) = e^{-t\{H,\cdot\}}\vec{z}(0) \tag{7}$$

Transfer Map

transfer map $\mathcal{M}$

$$\vec{z}(t) = \mathcal{M}\vec{z}(0) = e^{-t\{H,\cdot\}}\vec{z}(0) \tag{8}$$

Thin-lens Approximation

- factorize

$$e^{-t\{H,\cdot\}} \approx \prod_i e^{-tc_i\{H_{\text{drift}},\cdot\}} e^{-td_i\{H_{\text{kick}},\cdot\}}$$

- calculate each factor analytically

TPSA

- truncate to power N

$$e^{-t\{H,\cdot\}} \approx \sum_{k=0}^{N} \frac{1}{k!} \left(-t\{H,\cdot\}\right)^k$$

Dual Numbers

- extend real numbers $z_{\text{dual}} = a + \epsilon b$
  - $a, b \in \mathbb{R}$
  - special element $\epsilon \neq 0$: $\epsilon^2 = 0$
- all arithmetic operations defined
  - $(a + \epsilon b) + (c + \epsilon d) = (a + c) + \epsilon(b + d)$
  - $a + \epsilon b) * (c + \epsilon d) = (a + c) + \epsilon(ad + bc)$
  - etc...
- generalization to hyperdual numbers ϵ_1, ϵ_2, ϵ_3, ...

Dual Numbers: Application

let $f : \mathbb{R} \rightarrow \mathbb{R}$

- using dual numbers yields $f(a)$ and derivative $f'(a)$

$$f(a + \epsilon b) = f(a) + \epsilon f'(a)b \tag{9}$$

- hyperdual numbers: series expansion of f
- works for any computer program

differentiation defined

- $\partial \left[f(a) + \epsilon f'(a)\right] = f'(a)$
- implementation of differential operators $\Rightarrow$ Poisson bracket

Generation of Transfer Map

(hyper-) dual numbers

- calculation of

$$\mathcal{M} = e^{-t\{H(\vec{z}),\cdot\}}\vec{z} \approx \sum_{i=1}^{N}\left(\frac{-t\{H(\vec{z}),\cdot\}}{i!}\right)^{i}\vec{z} \tag{10}$$

  for some numerical $\vec{z}$

$\Rightarrow$ get $\mathcal{M}$ as a power series expanded around $\vec{z} = 0$

Outlook

- generate thick-lens based transfer map
  - differentiable w.r.t. to multipole components
  - enables application of gradient-based optimizers
  - successfully implemented
- alternative to thin-lens tracking
  - avoid choice of split scheme
  - potentially more computationally efficient
    - one map per element vs many drifts / kicks

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