Gaussian Characterization of the Universal Window

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Outline

- Appearance of universal behavior
  $\rightarrow$ independence of the interaction details
  $\rightarrow$ equal long-range behavior but different short-range behavior
- Definition of the unitary window
  $\rightarrow$ Weakly bound systems
  $\rightarrow$ Correlation between bound and scattering states
- Dynamics governed by a few parameters (control parameters)
  $\rightarrow$ Continuous (or discrete) scale invariance

Interplay of two aspects

- Weakly bound systems are strongly correlated
- In the universal regime details of the interaction are not important
  $\rightarrow$ Effective interactions
  $\rightarrow$ Gaussian (or other) characterization
- Are correlated systems and universal properties compatible?
- Transition from universal to non-universal regime

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The unitary window

Low energy quantities

- We consider a short-range interaction: $V(r > r_0) \to 0$

  In this case low energy means $E = k^2\hbar^2/m < \hbar^2/mr_0^2$

- In this regime the s-wave phase shift is well described by the effective range expansion up to second order

  $k \cot \delta_0 = -1/a + r_e k^2/2 + \dots$

  with a the scattering length defined from the zero-energy Schrödinger equation, $H\phi_0 = 0$

  $$\phi_0(r \to \infty) \to u_0 = 1 - a/r$$

  and r_e the effective range

  $$r_e = \frac{2}{a^2}\int_0^\infty (\phi_0^2 - u_0^2) r^2 dr$$

The unitary window

The presence of a shallow bound (or virtual) state

A bound state correspond to the S-matrix pole $k \cot \delta_0 - ik = 0$
In general all terms in the expansion of $k \cot \delta_0$ are needed.
However, when a shallow state appears (fine tuning), we can use the expansion up to second order ($i\kappa = k$)

$$\kappa = 1/a + r_e \kappa^2/2 + \dots$$

which introduces a strict correlation between the low energy parameters. When $\kappa \to 0$, $a \to \infty$, and $r_e/a << 1$.

Defining the lengths $a_B = 1/\kappa$ and $r_B = a - a_B$, they are related up to second order by

$$r_e a = 2 r_B a_B$$

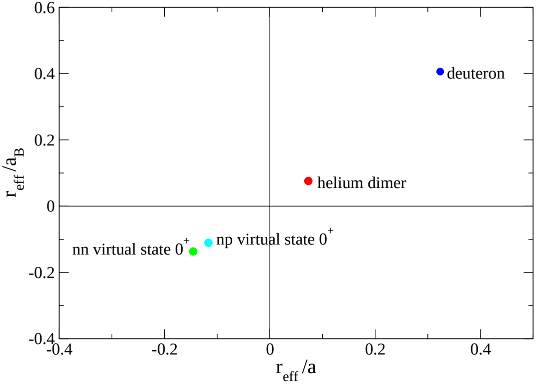
The unitary window

Protagonists of the story:

$a \rightarrow$ scattering length

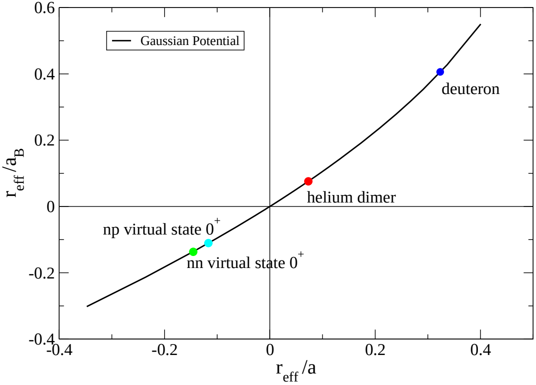
$r_{eff} \rightarrow$ effective range

$a_B \rightarrow E = \hbar^2/ma_B^2 \rightarrow$ energy length



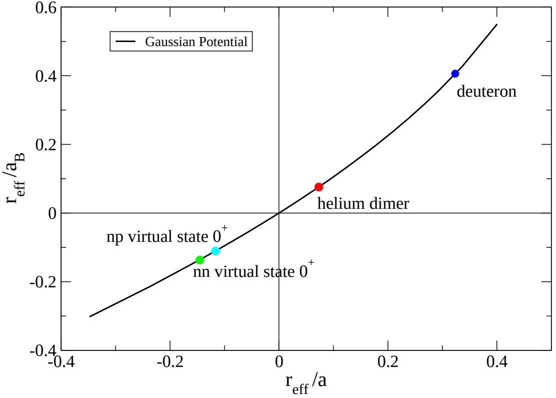
The Hero


G


Gaussian Potential
deuteron
helium dimer
np virtual state 0+
nn virtual state 0+
0.6
0.4
0.2
0
-0.2
-0.4
-0.4
-0.2
0
0.2
0.4
r_eff /a_B
r_eff /a

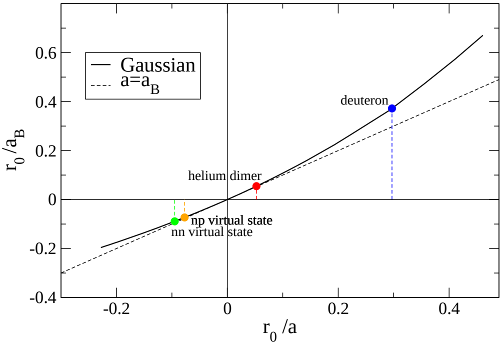
Universal behavior in few-body systems

- When a shallow state exists, the systems can be placed inside the universal window. The experimental points can be connected using a potential model (here a Gaussian potential).



Gaussian characterization of the universal window

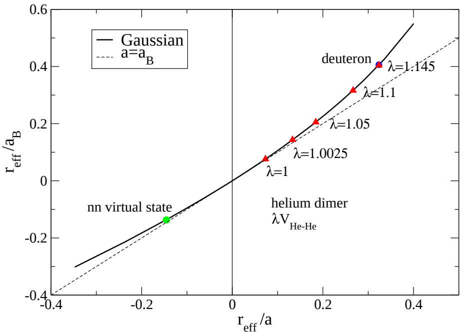
$V(1,2) = V_0 e^{-r^2/r_0^2}$



At $r_0/a = 0.2877$ a Gaussian potential describes the low-energy nuclear parameters a^1, r_e^1, a_B within 1% accuracy

Moving along the universal window

- By scaling the strength of the potential, systems can be (ideally) moved along the universal window
- The Gaussian charaterization can be used to continously connect systems inside the window



Moving along the universal window

- The scaled $\lambda V_{\mathrm{He-He}}$ potential can be followed by a gaussian with fixed range $V_0 e^{-(r/r_0)^2}$
- The running of the Gaussian strength is

$$V_0/(\hbar^2/mr_0^2) = C_0(1 + \alpha_1 \frac{r_0}{a} + \alpha_2 \left(\frac{r_0}{a}\right)^2 + \ldots)$$

- Two movements can be used to connect the systems:
  i) to change the strength V_0 maintaining r_0 fixed
  ii) to change the strength V_0 and range r_0 maintaining fixed r_{eff}/a
- Points having equal r_{eff}/a are related by a scale transformation:

$$\begin{cases} a \to \epsilon a \\ a_B \to \epsilon a_B \end{cases} \Rightarrow \begin{array}{l} r_B = a - a_B \to \epsilon r_B \\ r_{eff} = 2a_B r_B/a \to \epsilon r_{eff} \end{array}$$

The universal window

Observables

Observables are stricted correlated to the low-energy parameters (up to second order).
The mean square radius:

$$\langle r^2 \rangle = \frac{a^2}{8}\left[1 + \left(\frac{r_B}{a}\right)^2 + \cdots\right] = \frac{a_B^2}{8}\, e^{2r_B/a_B} = \frac{a_B^2}{8}\, f_{sc}$$

The asymptotic normalization constant

$$C_a^2 = \frac{2}{a_B}\frac{1}{1 - r_e/a_B} = \frac{2}{a_B}\, e^{2r_B/a_B} = \frac{2}{a_B}\, f_{sc}$$

the probability to be outside the interaction range

$$P_e = C_a^2 \int_{2r_B}^{\infty} e^{-2r/a_B} dr = e^{-2r_B/a_B} = \frac{1}{f_{sc}}$$

Universal behavior in few-body systems

Examples

- The helium dimer (as given by the LM2M2 potential):

  $a = 189.415$ a.u., $\rightarrow a_B = 182.221$ a.u.
  $r_e = 13.8447$ a.u., $\rightarrow r_B = 7.194$ a.u.
  $E_d = 1.303$ mk, $\rightarrow E(a, r_e) = 1.303$ mk

  $(r_e\, a)/(2 r_B a_B) = 1.0002$

- The deuteron:

  $a^1 = 5.419 \pm 0.007$ fm, $\rightarrow a_B^1 = 4.318$ fm
  $r_e^1 = 1.753 \pm 0.008$ fm, $\rightarrow r_B^1 = 1.101 \pm 0.007$ fm
  $E_d = 2.224575(9)$ MeV $\rightarrow E(a, r_e) = 2.223$ MeV
  $\langle r \rangle = 1.97535(85)$ fm $\rightarrow \langle r \rangle(a, r_e) = 1.970$ fm
  $C_a = 0.8781(44)$ fm$^{-1/2}$ $\rightarrow C_a(a, r_e) = 0.8782$ fm$^{-1/2}$

  $(r_e\, a)/(2 r_B a_B) = 0.9991$
  $(16/a_B^3)(\langle r^2 \rangle / C_a^2) = 1.005$

Other characterizations

The Gaussian potential is often used in the literature, however it is not the only one-parameter potential that can be used to characterize the universal window.

The Eckart potential

$$V_{Eckart}(r) = -2\frac{\hbar^2}{mr_0^2}\frac{W_0 e^{-r/r_0}}{(1+W_0 e^{-r/r_0})^2}$$

reproduces the effective range expansion up to second order exactly

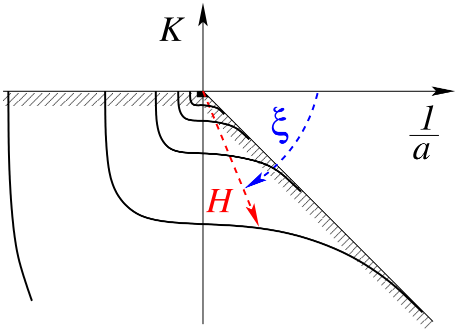
$$k\cot\delta_0 = -1/a + r_e k^2/2$$

with $a = 4W_0 r_0/(W_0 - 1)$ and $r_e = 4(a-2r_0)r_0/a$

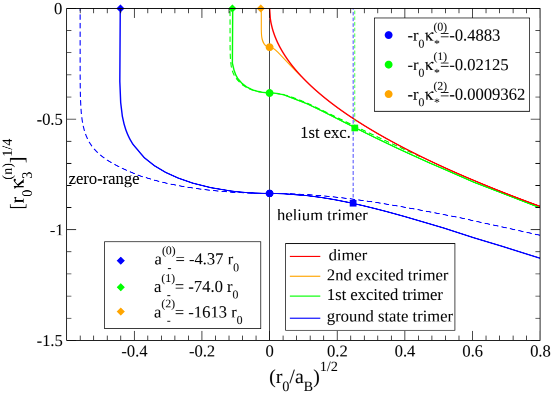
For $W_0 > 1$ there is a bound state with energy $E = \hbar^2/ma_B^2$ with $a_B = a - 2r_0$

The three-body system inside the universal window

- The energy spectrum of three equal bosons shows a discrete scale invariance (DSI)
- As the range of the interaction goes to zero, ($r_0 \to 0$) $\to$ Thomas collapse
- At the unitary limit $1/a \to 0$ $\to$ Efimov effect

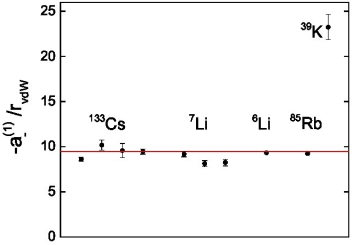


Gaussian characterization of the universal window for three bosons: $H = T + V_0 \sum_{i<j} e^{r_{ij}^2/r_0^2}$



A.K., M. Gattobigio, L. Girlanda and M. Viviani, Ann.Rev.Nucl.Part.Sci. 2021, 71:465

Footprint of universality



where the van der Waals length is $\ell_{vdW} = \frac{1}{2}(2mC_6/\hbar^2)^{1/4}$
For the Gaussian characterization this is encoded in the (almost) model independent relation $\kappa_*^{(0)} a_-^{(0)} = -4.37 r_0 \frac{0.4883}{r_0} = -2.14$

For van der Waals species $\kappa_*^{(0)} a_-^{(0)} \approx -2.2$

Moving along the universal window

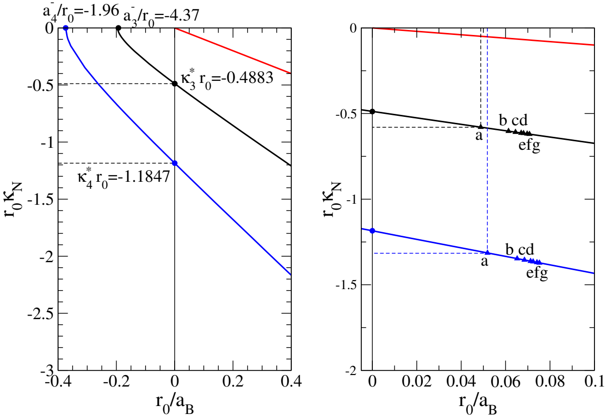
To study movements along the three-bosons universal window, we use the fact that different He-He interactions exist with different E_N values.

Potential	E_2(mK)	E_3(mK)	E_4(mK)	$r_0^{(3)}(a_0)$	$r_0^{(4)}(a_0)$
a: HFD-HE2	0.8301	117.2	535.6	11.146	11.840
b: LM2M2	1.3094	126.5	559.2	11.150	11.853
c: HFD-B3-FCH	1.4475	129.0	566.1	11.148	11.853
d: CCSAPT	1.5643	131.0	571.7	11.149	11.851
e: PCKLJS	1.6154	131.8	573.9	11.148	11.852
f: HFD-B	1.6921	133.1	577.3	11.149	11.854
g: SAPT96	1.7443	134.0	580.0	11.147	11.850

The Gaussian potential

$$V^{(N)} = V_0^{(N)} \sum_{i<j} e^{-(r_{ij}/r_0^{(N)})^2}$$

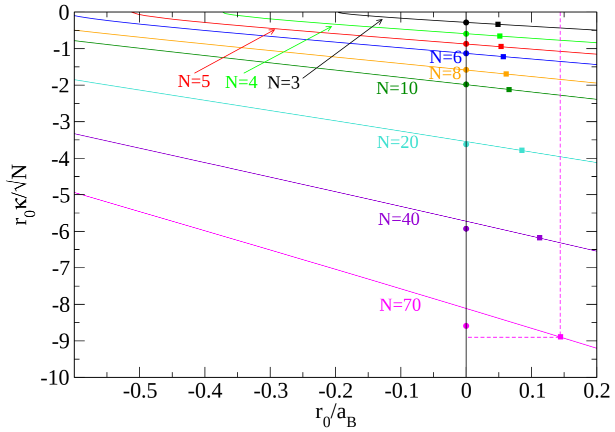
Moving along the universal window



$$E_N^* = (\hbar^2/m)(\kappa_N^*)^2 = [\hbar^2/m\ (r_0^{(N)})^2](\kappa_N^* r_0^{(N)})^2$$

$E_3^* = 83.05 \pm 0.05 mK$ and $E_4^* = 433.5 \pm 0.5 mK$

Appearing non universal beahvior



■ physical point (HFD-HE2)

• unitary point (HFD-HE2)

A.K. et al, PRA 102, 063320 (2020)

Some remarks

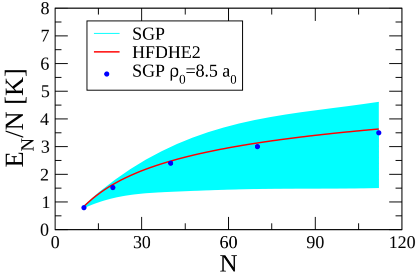
- The Gaussian characterization determines paths inside the universal window.
- When one scale emerges as a control parameter the system moves along the Gaussian path with fixed range
- In this case the Gaussian range depends on N
- This behavior is a manifestation of universal behavior
- It deteriorates as N increases
- In the following we consider a three-body force to incorporate the new scale

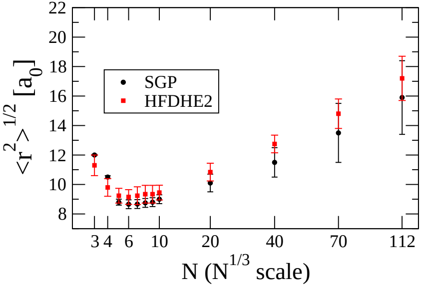
$$V = V_0 \sum_{i<j} e^{-r_{ij}^2/r_0^2} + W_0 \sum_{i<j<k} e^{-2\rho_{ijk}^2/\rho_0^2}$$

with $\rho_{ijk}^2 = (2/3)(r_{ij}^2 + r_{jk}^2 + r_{ki}^2)$.

- We study the effects of the three-body range ρ_0

The range of the three-body force





The two- plus three-body Gaussian interaction

	physical point		unitary point	
	SGP	HFD-HE2	SGP	HFD-HE2
$r_0[a_0]$	10.0485		10.0485	
V_0[K]	1.208018		1.150485	
$\rho_0[a_0]$	8.4853		8.4853	
W_0[K]	3.011702		3.014051	
E_4[K]	0.536	0.536	0.440	0.440
E_5[K]	1.251	1.266	1.076	1.076
E_6[K]	2.216	2.232	1.946	1.963
$E_{10}/10$[K]	0.792(2)	0.831(2)	0.714(2)	0.746(2)
$E_{20}/20$[K]	1.525(2)	1.627(2)	1.389(2)	1.491(2)
$E_{40}/40$[K]	2.374(2)	2.482(2)	2.170(2)	2.308(2)
$E_{70}/70$[K]	3.07(1)	3.14(1)	2.80(1)	2.92(1)
$E_{112}/112$[K]	3.58(2)	3.63(2)	3.30(2)	3.40(2)
$E_N/N(\infty)$[K]	7.2(3)	7.14(2)	6.8(3)	6.72(2)
HFD-B [K]		7.33(2)		6.73(2)

Gaussian characterization of the universal window for three 1/2 spin-isospin fermions

Two nucleon data

two nucleons	E_2 (MeV)	a^S (fm)	r_e^S (fm)
$np\ S=1$	2.2245	5.419	1.753
$np\ S=0$	0.0661	-23.740	2.77

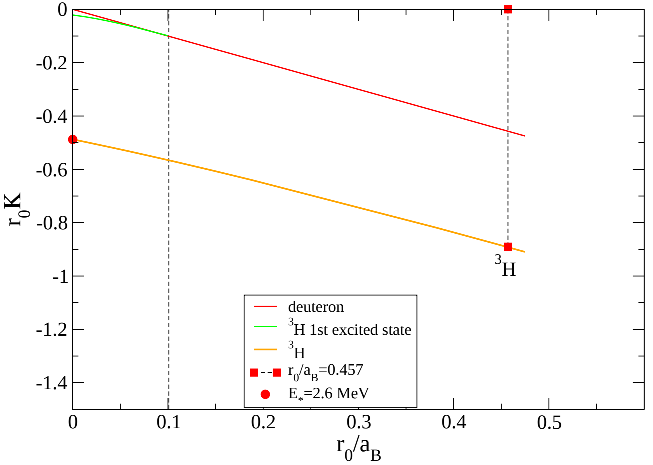
The Potential:

$$V(i,j) = V_0 e^{-r_{ij}^2/r_0^2}\mathcal{P}_{01} + V_1 e^{-r_{ij}^2/r_1^2}\mathcal{P}_{10}$$

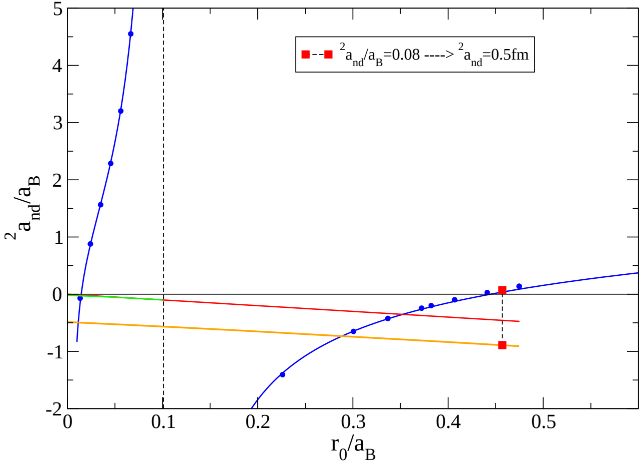
with $\mathcal{P}_{ST}$ the projector on the spin-isospin channels S, T

Walking along the nuclear cut:
Fixing $r_0 = r_1$, the strengths V_0 and V_1 are varied verifying $a^0/a^1 = -4.38$

Gaussian characterization of the universal window for three 1/2 spin-isospin fermions

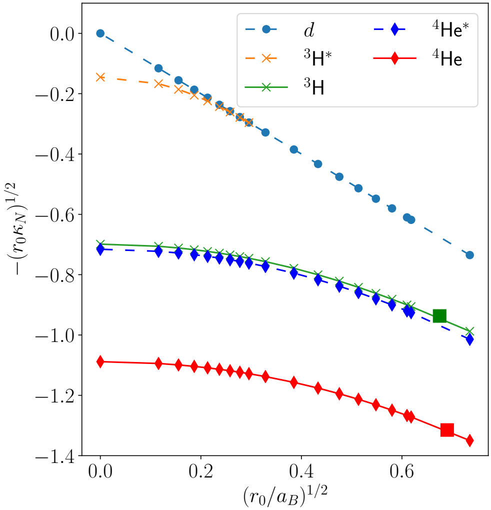


Correlations inside the universal window, the *nd* scattering length: ${}^2a_{nd} = 0.65 \pm 0.01$ fm



A. Deltuva, M. Gattobigio, A. K. and M. Viviani, PRC 102, 064001 (2020)

Gaussian characterization of the universal window for four 1/2 spin-isospin fermions



The ^{3}H and ^{4}He physical points

At the ^{3}H physical point $r_0/a_B = 0.457$ or $r_0 = 1.97$ fm
At the ^{4}He physical point $r_0/a_B = 0.483$ or $r_0 = 2.08$ fm
This implies that exist a potential

$$\sum_{i<j} \left(V_0 e^{-r_{ij}^2/r_0^2} \mathcal{P}_{01} + V_1 e^{-r_{ij}^2/r_0^2} \mathcal{P}_{10} \right)$$

that describes simultaneously ^{2}H and ^{3}H (for $r_0 = 1.97$ fm) and ^{2}H and ^{4}He (for $r_0 = 2.08$ fm).

At the unitary limit $E_*^3 = 2.5$ MeV and $E_*^4 = 13.5$ MeV

These potentials are a low-energy representation of the nuclear interaction.

The unitary limit with a realistic force

To see if the nuclear system moves along the Gaussian path we modify the tw-nucleon potential:

$$V(1,2) = \sum_{ST\nu} V_\nu^{ST}(r)\mathcal{O}_\nu^{ST}$$

with S, T the spin-isospin channels $01, 10, 00, 11$.

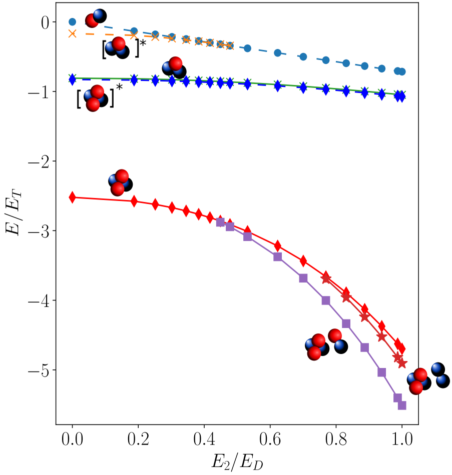
Taking the AV14 potential as example and multiplying

$$V^{01} \to 1.0633\,V^{01} \to {}^1a_{np} \to \infty$$

$$V^{10} \to 0.8\,V^{10} \to {}^3a_{np} \to \infty$$

with this calibration $E_3 \to 2.4\,\text{MeV}$

Gaussian characterization of the universal window for 1/2 spin-isospin fermions up to $A = 6$



Including the three-nucleon force

The potential is now

$$\sum_{i<j} V(i,j) + \sum_{i<j<k} W(i,j,k)$$

with

$$V(i,j) = \left(V_0 e^{-r_{ij}^2/r_0^2} \mathcal{P}_{01} + V_1 e^{-r_{ij}^2/r_0^2} \mathcal{P}_{10} \right)$$

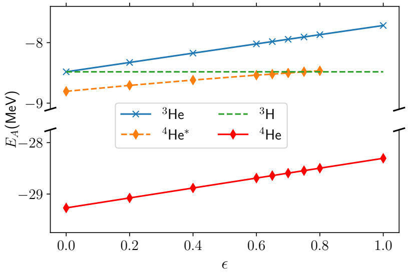
and

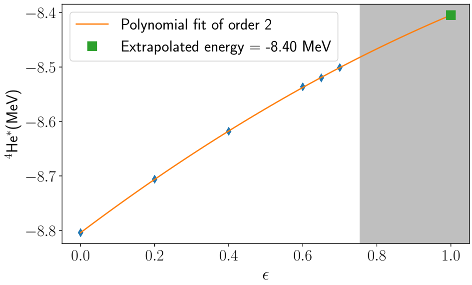
$$W(i,j) = W_0 e^{-r_{ijk}^2/\rho_0^2}$$

with $r_{ijk}^2 = r_{ij}^2 + r_{jk}^2 + r_{ki}^2$ and W_0 and ρ_0 fixed to describe simultaneously ^{3}H and ^{4}He.

The physical point turning on the Coulomb interaction

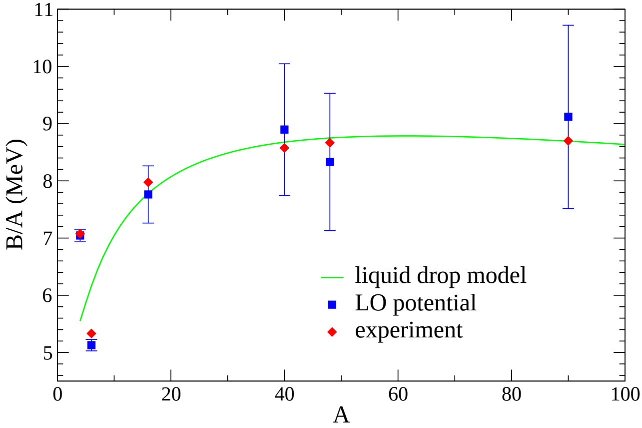
$$V_C(r) = \epsilon \frac{e^2}{r}$$





M. Gattobigio, A.K. and M. Viviani, PRC 100, 034004 (2019)

The nuclear chart with the two- plus three-body gaussian potential



R. Schiavilla et al., PRC 103, 054003 (2021)

The saturation point of nuclear matter

The two-body potential includes now the OPEP potential

$$\sum_{i<j}\left(V_0 e^{-r_{ij}^2/r_0^2}\mathcal{P}_{01} + V_1 e^{-r_{ij}^2/r_0^2}\mathcal{P}_{10} + V_\pi(r_{ij})\right) + \sum_{i<j<k} W(i,j,k)$$

where V_π is the regularized OPEP potential

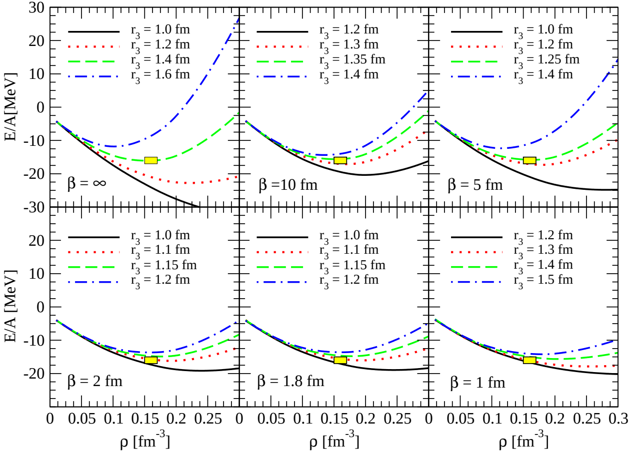
$$V_\pi(r) = \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left[\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 Y_\beta(r) + S_{12}T_\beta(r)\right]$$

with the central and tensor factors ($x = m_\pi r$)

$$Y_\beta(x) = \frac{g_A^2 m_\pi^3}{12\pi F_\pi^2}\frac{e^{-x}}{x}(1 - e^{-r^2/\beta^2})$$

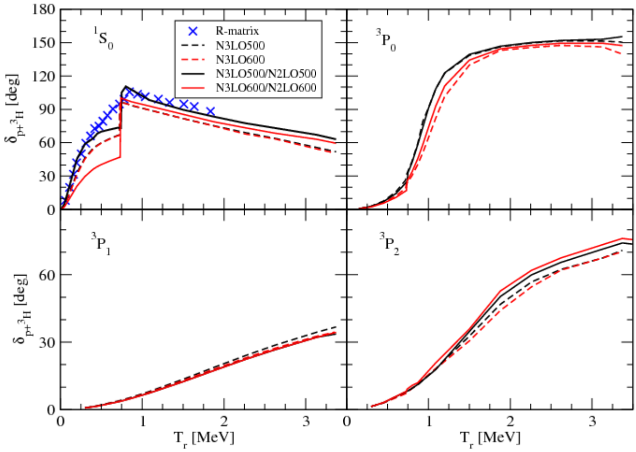
$$T_\beta(x) = \frac{g_A^2 m_\pi^3}{12\pi F_\pi^2}\frac{e^{-x}}{x}(1 + \frac{3}{x} + \frac{3}{x^2})(1 - e^{-r^2/\beta^2})^2 \quad .$$

The saturation point of nuclear matter



A. Kievsky et al., PRL 121, 072701 (2018)

The 0^+ resonance of ^{4}He



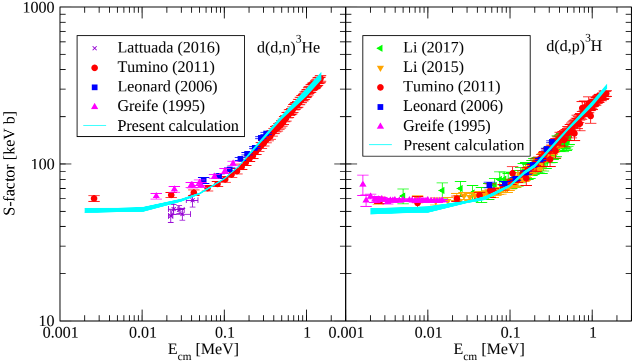
M. Viviani et al., PRC 102, 034007 (2020)

The 0^+ resonance of ^{4}He

Energy of the resonance and its width as extracted from the $p-{}^3$H phase-shifts using the HH method (M. Viviani et al, PRC 102, 034007 (2020)). The experimental values are extracted from the R-matrix analysis (D. R. Tilley, H. R. Weller, and G. M. Hale, Nucl. Phys. A541i, 1 (1992)).

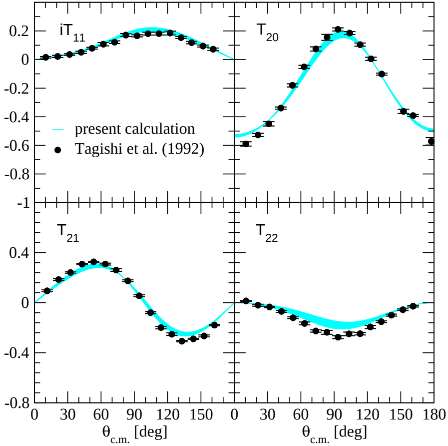
Interaction	E_R (MeV)	Γ (MeV)
N3LO500	0.126	0.556
N3LO600	0.134	0.588
N3LO500/N2LO500	0.118	0.484
N3LO600/N2LO600	0.130	0.989
N4LO450/N2LO450	0.126	0.400
N4LO500/N2LO500	0.118	0.490
N4LO550/N2LO550	0.130	0.740
Expt.	0.39	0.50

The $d(d,n)^3$He and $d(d,p)^3$H s-factor



M.Viviani, L.Girlanda, A.K., D.Logoteta, L.E.Marcucci, arXiv:nucl-th/2207.01433

The *dd* polarization observables



M.Viviani, L.Girlanda, A.K., D.Logoteta, L.E.Marcucci, arXiv:nucl-th/2207.01433

Conclusions

- Weakly bound systems can be located inside a window in which universal behavior emerges
- The universal behavior can be encoded in a one-parameter potential (as a Gaussian)
- Using this potential, trajectories (and correlations) can be studied along the universal window
- Non universal behavior has been identified as N increases
- A soft interaction has been used to reproduce the $N = 2, 3, 4$ energies and then used to predict binding energies as $N \to \infty$
- The Gaussian characterization can be used to analyse some aspects of EFTs

  • Its results are very similar to NLO pionless EFT though using a finite cutoff

  • When including the OPEP it looks very similar to chiral EFT at LO though a three-body force is included