Many-body perturbation theory and beyond with nuclear contact interactions

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Hirschegg 2023 - Effective fields theories for nuclei and nuclear matter

- Introduction: Many-fermion systems with large scattering lengths a
- Resummation of in-medium ladder diagrams to all orders
- Nonperturbative construction of complex single-particle potential
- Results for $U(p, k_f) + i\,W(p, k_f)$ at order a^n and in strong coupling limit
- Resummation of fermionic ladder diagrams in two spatial dimensions
- Third-order ring and ladder diagrams with $O(p^2)$-contact interaction

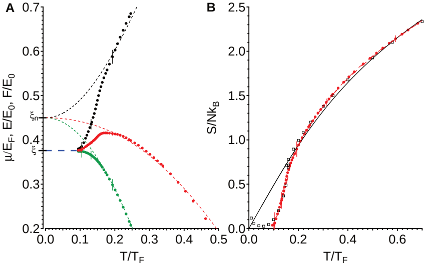
Publications: N. Kaiser, NPA 860 ('11) 41; EPJA 49 ('13) 140; EPJA 53 ('17) 140

Dilute many-fermion systems with large scattering lengths

- Low-density nuclear or neutron star matter, $a_{nn} \simeq 19\,\mathrm{fm}$
- Ultracold atom gases: interactions tunable via Feshbach resonances
- Of particular interest: "unitary limit", $a \to \infty$, boundstate at zero energy

$$\textit{scale invariance}: \qquad \bar{E}(k_f) = \frac{3k_f^2}{10M}\,\xi\,, \qquad \xi = \textit{Bertsch parameter}$$

- Calculation of ξ an intrinsically non-perturbative problem
- Quantum Monte-Carlo simulations [PRA 84, 061602 ('11)]: $\xi = 0.372 \pm 0.005$
- Experimental determination [MIT-Harvard, M. Zwierlein et al.]: $\xi = 0.37 \pm 0.01$



fermionic ^{6}Li atoms at Feshbach resonance

normal Bertsch parameter $\xi_n \simeq 0.45$ extrapolate over pairing transition at $T_c \simeq T_F/6$

- Effective field theory: low-density expansion for fermionic systems with short-range interactions [H.-W. Hammer, R. Furnstahl, NPA 678, 277 ('00)]

$$\bar{E}(k_f) = \frac{k_f^2}{2M}\left\{\frac{3}{5} + (g-1)\left[-\frac{2}{3\pi}ak_f + \frac{4}{35\pi^2}(11-2\ln 2)(ak_f)^2 -(0.0756+0.0574(g-3))(ak_f)^3\right] + \frac{a^2 r_s k_f^3}{10\pi} + (g+1)\frac{a_p k_f^3}{5\pi} +(g-1)(g-2)\frac{16}{27\pi^3}(4\pi-3\sqrt{3})(ak_f)^4 \ln|ak_f| + \dots\right\}$$

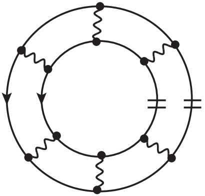
- a = s-wave scattering length ($a > 0$ attraction), density $\rho = g\, k_f^3/6\pi^2$
- Non-analytical $(ak_f)^4 \ln|ak_f|$ term from three-particle scattering
- Resummation of particle-particle ladders in form of geometrical series gives value $\xi_n^{(pp)} \simeq 0.237$ [T. Schäfer et al., NPA 762, 82 ('05)]
- Hole-hole ladder series starts at a^3, unitary limit $a \to \infty$ does not exist
- Beyond that: large classes of diagrams with mixed pp- and hh-ladders
- First mixed ladder at order a^4: J. Steele's numerical result [nucl-th/0010066] has to be corrected by a factor 2 (confirmed by H.-W. Hammer)
- Here: Evaluate and resum all ladder diagrams $\sim a^n$

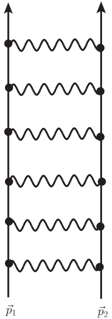
Preparation: in-medium propagator

- Key to solution: different organization of many-body calculation
- Keep propagating particles and holes together, measure difference to propagation in vacuum by a “medium-insertion” (on-shell, $|\vec{p}\,| < k_f$)

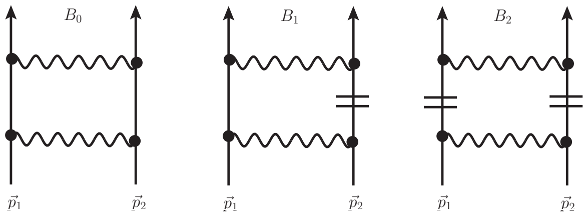
$$
\begin{aligned}
G(p_0, \vec{p}\,) &= i\left(\frac{\theta(|\vec{p}\,| - k_f)}{p_0 - \vec{p}\,^2/2M + i\epsilon} + \frac{\theta(k_f - |\vec{p}\,|)}{p_0 - \vec{p}\,^2/2M - i\epsilon} \right) \\
&= \frac{i}{p_0 - \vec{p}\,^2/2M + i\epsilon} - 2\pi\,\delta(p_0 - \vec{p}\,^2/2M)\,\theta(k_f - |\vec{p}\,|)
\end{aligned}
$$

- Consider a closed ladder diagram
- Minimal pair of medium-insertions on adjacent positions of double-ring
- After opening: planar ladder diagram with kinetic energy denominators only
- Balances M factors from interact. $\frac{4\pi a}{M}$
- Further medium-ins. on internal lines
- Multiloops = power of in-medium loop





In-medium loop generated by contact interaction



- External momenta $|\vec{p}_{1,2}| < k_f$, introduce $\vec{P} = \frac{1}{2}(\vec{p}_1 + \vec{p}_2)$ and $\vec{q} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$
- Contributions from zero, one, and two medium-insertions: $B_0 + B_1 + B_2$

$$B_0 = 4\pi a \int \frac{d^3l}{(2\pi)^3} \frac{1}{\vec{l}^{\,2} - \vec{q}^{\,2} - i\epsilon} = \frac{2a}{\pi} \int_0^\infty dl \left(1 + \frac{\vec{q}^{\,2}}{l^2 - \vec{q}^{\,2} - i\epsilon}\right) = 0 + i\,a|\vec{q}\,|$$

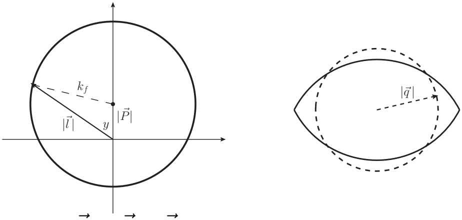
- Rescatterings in vacuum: unitarized scattering length approximation

$$f = a\left\{1 + ia|\vec{q}\,| + (ia|\vec{q}\,|)^2 + \dots\right\} = \frac{1}{a^{-1} - i|\vec{q}\,|} = \frac{1}{|\vec{q}\,|(\cot\delta_0 - i)}$$

- Leads to (well-known) relation for s-wave phase shift: $\tan\delta_0 = a|\vec{q}\,|$

- B_1 from diagrams with one medium-insertion (two equal contributions)

$$B_1 = -4\pi a \int \frac{d^3l}{(2\pi)^3} \frac{1}{\vec{l}^{\,2} - \vec{q}^{\,2} - i\epsilon} \Big\{ \theta(k_f - |\vec{P} - \vec{l}\,|) + \theta(k_f - |\vec{P} + \vec{l}\,|) \Big\}$$



- Shift loop momentum $\vec{l} \to \vec{l} \pm \vec{P}$, Fermi spheres centered at the origin

$$\mathrm{Re}\, B_1 = -\frac{a k_f}{\pi} R(s,\kappa)$$

$$R(s,\kappa) = 2 + \frac{1}{2s}[1-(s+\kappa)^2] \ln\frac{1+s+\kappa}{|1-s-\kappa|} + \frac{1}{2s}[1-(s-\kappa)^2] \ln\frac{1+s-\kappa}{1-s+\kappa}$$

- where $s = |\vec{p}_1 + \vec{p}_2|/2k_f$ and $\kappa = |\vec{p}_1 - \vec{p}_2|/2k_f$, constraint $s^2 + \kappa^2 < 1$
- Equal to sum of pp- and hh-bubble $R(s,\kappa) = F_{pp}(s,\kappa) + F_{pp}(-s,\kappa)$

In-medium loop generated by contact interaction

- B_2 purely imaginary contribution, total imaginary part of in-medium loop

$$\mathrm{Im}(B_0+B_1+B_2)=4\pi^2 a\int\frac{d^3l}{(2\pi)^3}\delta(\vec{l}^{\,2}-\vec{q}^{\,2})\Big\{\Big[1-\theta(k_f-|\vec{P}-\vec{l}\,|)\Big]\times\Big[1-\theta(k_f-|\vec{P}+\vec{l}\,|)\Big]+\theta(k_f-|\vec{P}-\vec{l}\,|)\,\theta(k_f-|\vec{P}+\vec{l}\,|)\Big\}$$

- First $(1-\theta)(1-\theta)$ term: phase space Pauli-blocked, energy conservation

$$\mathrm{Im}(B_0+B_1+B_2)=\frac{B_2}{2i}=a k_f\, I(s,\kappa)$$

$$I(s,\kappa)=\begin{cases}\kappa & , \quad 0<\kappa<1-s\\ (1-s^2-\kappa^2)/2s & , \quad 1-s<\kappa<\sqrt{1-s^2}\end{cases}$$

- Complex-valued in-medium loop, if B_2 taken out: Im-part changes sign

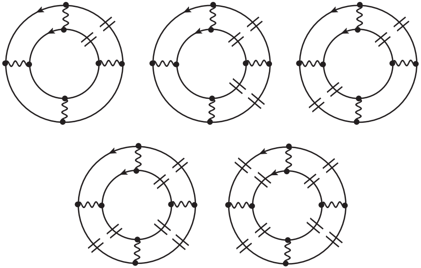
$$B_0+B_1+B_2=-\frac{a k_f}{\pi}\Big\{R(s,\kappa)-i\pi\, I(s,\kappa)\Big\}$$

$$B_0+B_1=-\frac{a k_f}{\pi}\Big\{R(s,\kappa)+i\pi\, I(s,\kappa)\Big\}$$

- Crucial property to derive correct expression for energy per particle $\bar{E}(k_f)$

Energy per particle at order a^n and resummation to all orders

- Construct energy density at order a^n: close open lines and integrate over $|\vec{p}_{1,2}| < k_f$, but complex quantity $(R - i\pi I)^{n-1}$ cannot be correct integrand
- Diagrams with repeated double medium-insert. have symmetry factors, not respected by binomial expansion $(R - i\pi I)^{n-1} = [(R + i\pi I) + (-2i\pi I)]^{n-1}$



- Example: $(R+i\pi I)^3 + (R+i\pi I)^2(-2\pi i I)(1+1/2) + (R+i\pi I)(-2\pi i I)^2 + (-2\pi i I)^3/4$ $= R(R^2 - \pi^2 I^2)$ is real-valued in the end
- This crucial amendment leads to summation formula

$$\sum_{j=0}^{n-1}(R+i\pi I)^{n-1-j}(-2i\pi I)^j \binom{n-1}{j}\frac{1}{j+1} = \frac{1}{2i\pi I\, n}\Big\{(R+i\pi I)^n - (R-i\pi I)^n\Big\}$$

Energy per particle at order a^n and resummation to all orders

- Coefficient $\binom{n-1}{j}/(j+1) = \binom{n}{j+1}/n$ number of possibilities to attach $j+1$ double medium-insert. on ring with n segments, divided by n rotations
- Now sum up series $\sum_{n=1}^{\infty}[-ak_f(R \pm i\pi I)/\pi]^n/n$ via a (complex) logarithm
- Complete Hartree term by Fock-exchange term $(1-1/g=1/2)$

$$\bar{E}(k_f) = -\frac{24k_f^2}{\pi M}\int_0^1 ds\, s^2 \int_0^{\sqrt{1-s^2}} d\kappa\,\kappa\, \arctan\frac{ak_f I(s,\kappa)}{1+\pi^{-1}ak_f R(s,\kappa)}$$

- Usual branch of arctangent function: $\arctan(-x) = -\arctan x$, in order to respect weak coupling limit $a \to \pm 0$
- When denominator passes through zero: discontinuity by amount $-\pi$
- $R(s,\kappa)$ unbounded: radius of convergence of power series in ak_f is zero
- Double-integral representation obtained with help of master formula

$$\int\limits_{|\vec{p}_{1,2}|<k_f} \frac{d^3p_1\, d^3p_2}{(2\pi)^6} F(s,\kappa) = \frac{2k_f^6}{\pi^4}\int_0^1 ds\, s^2 \int_0^{\sqrt{1-s^2}} d\kappa\,\kappa\, I(s,\kappa) F(s,\kappa)$$

$$s = |\vec{p}_1 + \vec{p}_2|/2k_f\,, \qquad \kappa = |\vec{p}_1 - \vec{p}_2|/2k_f$$

Expansion in powers of ak_f and unitary limit

- Check against low-density expansion (terms from pp and hh-ladders)

$$\bar{E}(k_f) = \frac{k_f^2}{2M}\Big\{-\frac{2}{3\pi}ak_f + \frac{4}{35\pi^2}(11-2\ln 2)a^2k_f^2 \\ -0.0755733\, a^3k_f^3 + 0.0524813\, a^4k_f^4 + \dots\Big\}$$

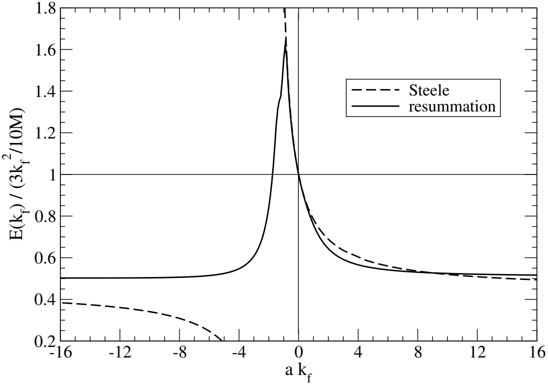
- At a^3: partition $R^2 - \pi^2 I^2/3$ versus twofold pp and twofold hh scattering, $0.0861836 - 0.0106103 = 0.0640627 + 0.0115106$
- At a^4: here $R^3 - R\pi^2 I^2$ vs. triple pp, triple hh, and mixed pphh scattering, $0.0671902 - 0.0147089 = 0.0383116 - 0.0006851 + 6 \cdot 0.0024758$
- For resummed ladder series the limit $a \to \infty$ is straightforward

$$\xi_n = 1 - \frac{80}{\pi}\int_0^1 ds\, s^2 \int_0^{\sqrt{1-s^2}} d\kappa\, \kappa \arctan\frac{\pi I(s,\kappa)}{R(s,\kappa)} = 0.5067$$

- More than twice as large as result from resummation of pp ladders only $\xi_n^{(pp)} = 1 - 80\int_0^1 ds\, s^2 ⨍_0^{\sqrt{1-s^2}} d\kappa\, \kappa\, I(s,\kappa) F_{pp}^{-1}(s,\kappa) \simeq 0.237$
- Smooth extrapolations of unitary Fermi gas over pairing transition at T_c give for "normal" Bertsch parameter $\xi_n \simeq 0.45$

Non-perturbative resummation

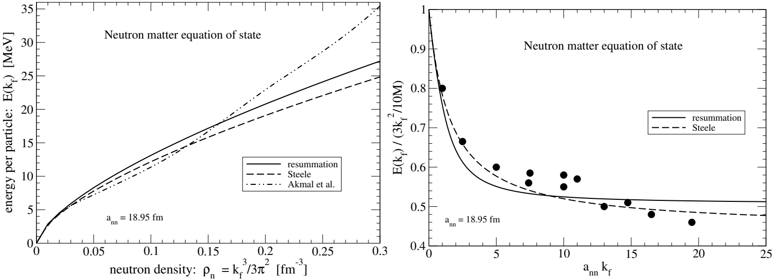
- $\bar{E}(k_f)$ from resummed <u>ladder</u> diagrams plus kinetic energy $3k_f^2/10M$



- Sharp peak with maximum value of 1.62 at $ak_f \simeq -0.9$
- Outside $|ak_f| > 6$ curve is almost flat ($\rightarrow$ unitary limit reached)
- Repulsive scattering length ($a < 0$) gives attraction for $ak_f < -1.8$
- Steele's suggested geometric series: $\bar{E}(k_f)^{(\mathrm{St})} = -ak_f^3[3M(\pi + 2ak_f)]^{-1}$
- Negative compressibility $K = 9dP/d\rho = 3k_f^2/M + k_f^2\bar{E}''(k_f) + 4k_f\bar{E}'(k_f)$ in region $-\pi/2 < ak_f < -4\pi/9$ of coupling parameter

Application to neutron matter equation of state

- Very large nn-scattering length $a_{nn} = (19.0 \pm 0.4)$ fm
- Low density neutron matter supposed to be close to a unitary Fermi gas
- Apply result $\bar{E}(k_f)$ for resummation of ladder diagrams to neutron matter



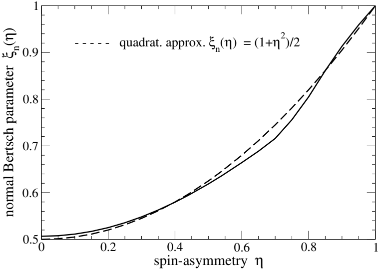
- Fair agreement with realistic n-matter calculations up to quite high ρ_n
- Quantum Monte Carlo calc. of neutron matter [Armani, Gezerlis, Pederiva,...]: at low densities ρ_n kinetic energy gets reduced to about 1/2
- Extension to P-waves and effective range by J.M. Alarcon and J.A. Oller, several long papers, arXiv:2106.02652, arXiv:2107.0805, arXiv:2212.05092.

Spin-polarized Fermi gas or neutron matter

- Generalize resummation to spin-asymmetric situation $k_{\uparrow,\downarrow} = k_f(1 \pm \eta)^{1/3}$
- Four channels $(\uparrow\uparrow, \downarrow\downarrow, \uparrow\downarrow, \downarrow\uparrow)$, interaction only in spin-singlet state $S = 0$
- Work with diagonalized 4×4 matrices to perform resummation

$$\begin{aligned}\bar{E}(k_\uparrow, k_\downarrow) &= \frac{3}{M(k_\uparrow^3 + k_\downarrow^3)} \bigg\{ \frac{k_\uparrow^5 + k_\downarrow^5}{10} - \frac{16}{\pi} \int_0^{(k_\uparrow + k_\downarrow)/2} dP\, P^2 \int_{q_{\min}}^{q_{\max}} dq\, q \\ &\qquad \arctan \frac{a\, \Phi(P, q, k_\uparrow, k_\downarrow)}{1 + \frac{a}{2\pi}\Big[k_\uparrow R(P/k_\uparrow, q/k_\uparrow) + k_\downarrow R(P/k_\downarrow, q/k_\downarrow)\Big]} \bigg\}\end{aligned}$$

- $\Phi(P, q, k_\uparrow, k_\downarrow)$ generalization of imaginary part (three different branches)



- $a \to \infty$: Bertsch parameter $\xi_n(\eta)$
- spin asymmetry energy: $\bar{E}(k_\uparrow, k_\downarrow) = \bar{E}(k_f) + \eta^2 S(k_f) + \dots$
- $\bar{E}(k_f)^{(\infty)} \simeq \frac{3k_f^2}{10M} \cdot \frac{1}{2}$ (reduction)

  $S(k_f)^{(\infty)} \simeq \frac{k_f^2}{6M} \cdot \frac{4}{3}$ (enhancement)

- Polarized neutron matter $\simeq$ free Fermi gas, EFT calc. [T. Krüger et al. ('15)]

Nonperturbative construction of single-particle potential

- Single-particle potential reveals more details of many-body dynamics
- First functional derivative: add a "test-particle" with $\vec{p}$ to filled Fermi sea

$$\theta(k_f - |\vec{p}_j|) \;\to\; N_\eta(\vec{p}_j, \vec{p}\,) = \theta(k_f - |\vec{p}_j|) + 4\pi^3 \eta\, \delta^3(\vec{p}_j - \vec{p}\,)$$

- To linear order in η energy density of system changed as:

$$\frac{k_f^3}{3\pi^2} \bar{E}(k_f) \;\to\; \frac{k_f^3}{3\pi^2} \bar{E}(k_f) + \eta \Big[U(p, k_f) + i\, W(p, k_f) \Big]$$

- Modification of in-medium loop: real part

$$\mathrm{Re}\, B_1 = -4\pi a \int \frac{d^3 l}{(2\pi)^3}\, \frac{1}{\vec{l}^{\,2} - \vec{q}^{\,2}} \Big\{ N_\eta(\vec{P} + \vec{l}, \vec{p}\,) + N_\eta(\vec{P} - \vec{l}, \vec{p}\,) \Big\}\,.$$

- Twice the same energy denominator, average over the directions of $\vec{p}$

$$\mathrm{Re}\, \bar{B}_1 = -\frac{a k_f}{\pi} \Big\{ R(s,\kappa) + \tilde{\eta}\, \widehat{R}(s,\kappa,x) \Big\}\,, \qquad \tilde{\eta} = \eta\, \pi^2 / k_f^3$$

$$\widehat{R}(s,\kappa,x) = \frac{1}{sx} \ln \frac{|(s+x)^2 - \kappa^2|}{|(s-x)^2 - \kappa^2|}\,, \qquad x = p/k_f$$

- Imaginary part of in-medium loop

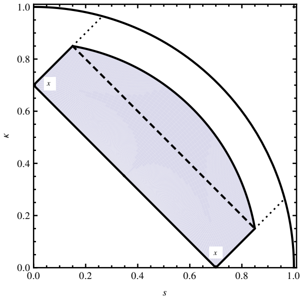
$$\mathrm{Im}(B_0 + B_1 + B_2) = 4\pi^2 a \int \frac{d^3 l}{(2\pi)^3}\, \delta(\vec{l}^{\,2} - \vec{q}^{\,2}) \Big\{ N_\eta(\vec{P} + \vec{l}, \vec{p}\,)\, N_\eta(\vec{P} - \vec{l}, \vec{p}\,)$$
$$+ \Big[1 - N_\eta(\vec{P} + \vec{l}, \vec{p}\,)\Big] \Big[1 - N_\eta(\vec{P} - \vec{l}, \vec{p}\,)\Big] \Big\}$$

Nonperturbative construction of single-particle potential

- Averaging over directions of $\vec{p}$: $\mathrm{Im}(B_0+\bar{B}_1+\bar{B}_2) = ak_f\{I(s,\kappa)+\tilde{\eta}\,\widehat{I}(s,\kappa,x)\}$

$$I(s,\kappa) = \frac{1}{2s}\min\Big(2s\kappa, |s^2+\kappa^2-1|\Big)\,, \qquad s,\kappa \textit{ unconstrained}$$

$$\widehat{I}(s,\kappa,x) = \frac{1}{sx}\,\theta(s+\kappa-x)\,\theta(x-|s-\kappa|)\,\mathrm{sign}(1+x^2-2(s^2+\kappa^2))$$



- Diagram with two medium insertions: $\bar{B}_2 = 2iak_f\{I_*(s,\kappa)+\tilde{\eta}\,\widehat{I}_*(s,\kappa,x)\}$

$$I_*(s,\kappa) = I(s,\kappa)\,\theta(1-s^2-\kappa^2)\,, \qquad \textit{restriction to } s^2+\kappa^2<1$$

$$\widehat{I}_*(s,\kappa,x) = \frac{1}{sx}\,\theta(s+\kappa-x)\,\theta(x-|s-\kappa|)\,\theta(1+x^2-2(s^2+\kappa^2))$$

Nonperturbative construction of single-particle potential

- Interaction density of resummed ladder diagrams

$$V(\tilde{\eta}) = \frac{4\pi a}{M\bar{B}_2} \ln \frac{1 - B_0 - \bar{B}_1 - \bar{B}_2}{1 - B_0 - \bar{B}_1}$$

- Two contributions to complex single-particle potential $U(p, k_f) + i\, W(p, k_f)$
  i) $\tilde{\eta}$ from "last two" momentum space integrations $\to$ weight fct. $\widehat{I}_*(s, \kappa, x)$
  ii) $\tilde{\eta}$ from expansion of $V(\tilde{\eta})$ to linear order $\to$ weighting function $I_*(s, \kappa)$
- Analyze $\widehat{I}_*\, V(0) + I_*\, V'(0)$ for $x < 1$ and $x > 1$ in domains of $s\kappa$ plane: obtain real and imaginary potential as well as continuation across $p = k_f$
- Single-particle potential <u>inside</u> Fermi sphere $p < k_f$ (for a hole-state)

$$U(p, k_f) = \frac{8ak_f^3}{M} \int_0^1 ds\, s^2 \int_0^{\sqrt{1-s^2}} d\kappa\, \kappa \left\{ \frac{ak_f \widehat{R}(s, \kappa, x) I(s, \kappa) - \widehat{I}_*(s, \kappa, x)[\pi + ak_f R(s, \kappa)]}{[\pi + ak_f R(s, \kappa)]^2 + [ak_f \pi\, I(s, \kappa)]^2} - \frac{1}{ak_f} \widehat{R}(s, \kappa, x)\, \delta\left(\frac{\pi}{ak_f} + R(s, \kappa) \right) \right\}$$

- Origin of δ-function term is very subtle: it arises from differentiating the discontinuity (by $-\pi$) of the arctangent-function at infinity

- δ-term is crucial for (numerical) validity of Hugenholtz-Van-Hove theorem

$$U(k_f,k_f)=\bar{E}(k_f)+\frac{k_f}{3}\frac{\partial\bar{E}(k_f)}{\partial k_f}$$

- Imaginary part of single-particle potential: $W(k_f,k_f)=0$

$$W(p,k_f)=\frac{8\pi a^2k_f^4}{M}\int_0^1 ds\,s^2\int_0^{\sqrt{1-s^2}} d\kappa\,\kappa\,\frac{[\widehat{I}_*(s,\kappa,x)-\widehat{I}(s,\kappa,x)]\,I(s,\kappa)}{[\pi+ak_fR(s,\kappa)]^2+[ak_f\pi\,I(s,\kappa)]^2}>0$$

- Single-particle potential <u>outside</u> Fermi sphere $p>k_f$ (for a particle-state)

$$U(p,k_f)=\frac{8ak_f^3}{M}\int_0^{(x+1)/2} ds\,s^2\int_0^{(x+1)/2} d\kappa\,\kappa\left\{\frac{ak_f\widehat{R}(s,\kappa,x)I_*(s,\kappa)-\widehat{I}_*(s,\kappa,x)[\pi+ak_fR(s,\kappa)]}{[\pi+ak_fR(s,\kappa)]^2+[ak_f\pi\,I(s,\kappa)]^2}\right.$$
$$\left.-\frac{\theta(1-s^2-\kappa^2)}{ak_f}\widehat{R}(s,\kappa,x)\,\delta\left(\frac{\pi}{ak_f}+R(s,\kappa)\right)\right\},$$

$$W(p,k_f)=-\frac{8\pi a^2k_f^4}{M}\int_0^{(x+1)/2} ds\,s^2\int_0^{(x+1)/2} d\kappa\,\kappa\,\frac{\widehat{I}_*(s,\kappa,x)I(s,\kappa)\,\theta(s^2+\kappa^2-1)}{[\pi+ak_fR(s,\kappa)]^2+[ak_f\pi\,I(s,\kappa)]^2}<0$$

- Continuation of potentials across $p=k_f$ is continuous, but <u>not smooth</u>

Perturbative expansion of single-particle potential

- Expansion of single-particle potential in powers of $-ak_f$

$$U(p,k_f)+i\,W(p,k_f)=\frac{k_f^2}{2M}\sum_{n=1}^{\infty}(-ak_f)^n\Big[\Phi_n(x)+i\,\Omega_n(x)\Big]$$

- δ-function term drops out in this expansion, a truly nonperturbative term
- First order Hartree-Fock mean-field result: $\Phi_1(x)=4/3\pi,\;\;\Omega_1(x)=0$
- Second order in scattering length a: V.M. Galitskii's classical work (1958)

$$\Phi_2(x)=\frac{4}{15\pi^2}\left\{11-2x^4\ln\frac{1-x^2}{x^2}+\frac{10}{x}(1-x^2)\ln\frac{1+x}{1-x}-\frac{2}{x}(2-x^2)^{5/2}\ln\frac{1+x\sqrt{2-x^2}}{1-x^2}\right\},$$

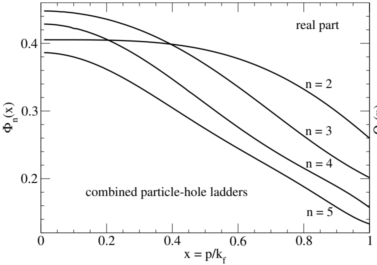
$$\Phi_2(x)=\frac{4}{15\pi^2}\left\{11-2x^4\ln\frac{x^2-1}{x^2}+\frac{10}{x}(1-x^2)\ln\frac{x+1}{x-1}-\frac{2}{x}\Big[\theta(\sqrt{2}-x)\,(2-x^2)^{5/2}\right.$$
$$\left.\cdot\ln\frac{1+x\sqrt{2-x^2}}{x^2-1}+\theta(x-\sqrt{2})\,(x^2-2)^{5/2}\arcsin\frac{1}{x^2-1}\Big]\right\},\qquad x>1\,,$$

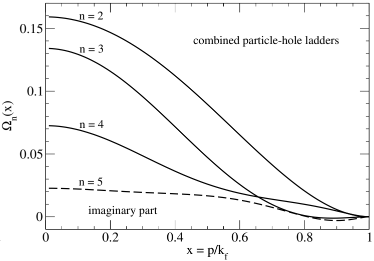
$$\Omega_2(x)=\frac{\theta(1-x)}{2\pi}\,(1-x^2)^2+\frac{2\,\theta(x-1)}{15\pi x}\left\{7-5x^2-2(2-x^2)^{5/2}\,\theta(\sqrt{2}-x)\right\}$$

- Perfect numerical reproduction of these exact analytical results
- Crucial check of construction of $U(p,k_f)+i\,W(p,k_f)$ with a complex in-medium loop that includes corrections linear in η from a "test-particle"

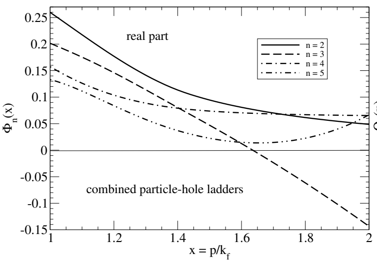
Perturbative results for single-particle potential

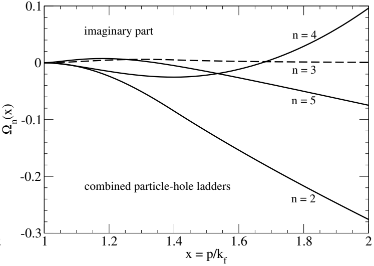
- Expansion up to fifth order in the scattering length *a*:
- Single-particle potential inside the Fermi sphere $p < k_f$





- Single-particle potential outside the Fermi sphere $p > k_f$

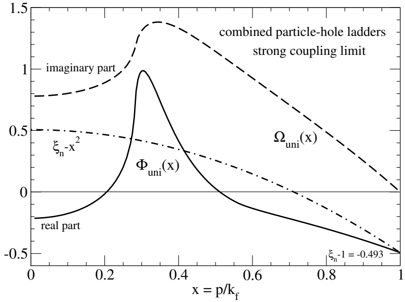




- Strong coupling limit $a \to \infty$ can be performed straightforwardly

$$U(p,k_f)^{(\infty)} + i\,W(p,k_f)^{(\infty)} = \frac{k_f^2}{2M}\Big[\Phi_{\rm uni}(x) + i\,\Omega_{\rm uni}(x)\Big]$$

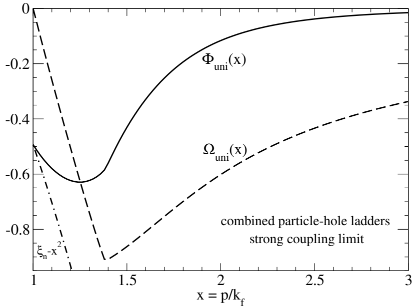
- $\Phi_{\rm uni}(0) = -0.214\,, \quad \Phi_{\rm uni}(1) = \xi_n - 1 = -0.493$ (Hugenholtz-Van-Howe)
- $\Phi_{\rm uni}(1) = -0.031 - 0.462$ comes almost entirely from δ-function term



- Strong p-dependence of $\Phi_{\rm uni}(x)$: instability against (topological) phase transition to state with separation in momentum space (Sarma phase)
- Single-particle states above Fermi energy $\xi_n k_f^2/2M$ are not occupied $\to$ empty shell in Fermi sphere, density gets reduced to $0.95\,k_f^3/3\pi^2$
- Bubble formation in n-matter: π^0-condensation [Pankratov,...], large a_{nn}

Results in strong coupling limit $a \to \infty$

- Continuation of $U(p,k_f)^{(\infty)} + i\,W(p,k_f)^{(\infty)}$ outside the Fermi surface



- Particle excitations with high p weakly attracted by fermionic medium
- Imaginary potential $W(p,k_f)$ vanishes linearly at Fermi surface $p = k_f$

$$\Omega'(1) = -\frac{8\pi s_1^2(1-s_1^2)}{2+\pi(ak_f)^{-1}s_1^2}\,, \qquad R(s_1,\sqrt{1-s_1^2}) = -\frac{\pi}{ak_f}\,, \quad \begin{cases} ak_f > 0 \\ ak_f < -\pi/2 \end{cases}$$

- Numerator in representation of $W(p,k_f)$ gives quadratic behavior $(1-x)^2$
- Denominator introduces singular region into double-integral $\to 1-x$
- Nonperturb. counterexample to “Luttinger theorem”: $W(p,k_f) \sim \pm(k_f-p)^2$

Resummation of ladder diagrams in two dimensions

- In 2 dimensions: density $\rho_2 = k_f^2/2\pi$, free Fermi gas $\bar{E}(k_f)^{(0)} = k_f^2/4M$
- Contact coupling $2\pi\alpha/M$, first-order contribution $\bar{E}(k_f)^{(1)} = -\alpha\, k_f^2/4M$
- Vacuum loop in two dimensions:

$$B_0' = 2\pi\alpha \int \frac{d^2l}{(2\pi)^2} \frac{1}{\vec{l}^{\,2} - \vec{q}^{\,2} - i\epsilon} = \alpha \left(\frac{i\pi}{2} - \ln \frac{|\vec{q}\,|}{\Lambda} \right)$$

- Bound-state pole in resummed vacuum scattering amplitude $\alpha/(1 - B_0')$ determines cutoff $\Lambda = q_b\, e^{1/\alpha}$, two-body binding energy $E_b = -q_b^2/M < 0$
- Medium corrections from Pauli-blocking:

$$\mathrm{Re}\, B_1' = \alpha \left\{ \ln\kappa - H(s,\kappa) \right\}, \qquad \mathrm{Im}(B_0' + B_1' + B_2') = \frac{B_2'}{2i} = \alpha\, J(s,\kappa)$$

- Two-dimensional in-medium loop functions:

$$H(s,\kappa) = 2\,\theta(1-s-\kappa) \ln \frac{\sqrt{1-(s+\kappa)^2} + \sqrt{1-(s-\kappa)^2}}{2\sqrt{\kappa}} + \theta(s+\kappa-1)\ln s$$

$$J(s,\kappa) = \frac{\pi}{2}\,\theta(1-s-\kappa) + \theta(s+\kappa-1) \arcsin \frac{1-s^2-\kappa^2}{2s\kappa}, \qquad s^2+\kappa^2 < 1$$

- $J(s,\kappa)$ is weighting function for integrals over two Fermi discs $|\vec{p}_{1,2}| < k_f$

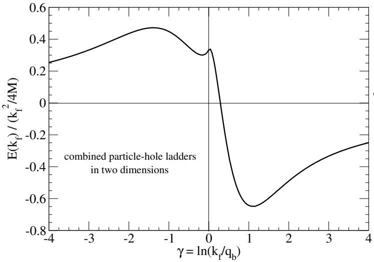
Resummation of ladder diagrams in two dimensions

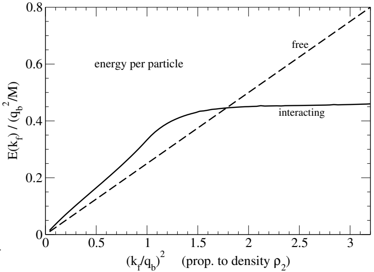
- Resummed energy per particle for a two-dimensional Fermi gas:

$$\bar{E}(k_f) = -\frac{8k_f^2}{\pi M}\int_0^1 ds\, s \int_0^{\sqrt{1-s^2}} d\kappa\, \kappa \arctan \frac{J(s,\kappa)}{H(s,\kappa) + \ln(k_f/q_b)}$$

- Parameter α has dropped out in above ratio: $\alpha^{-1} + \ln(k_f/\Lambda) = \ln(k_f/q_b)$
- Binding momentum q_b remains as a single scale in the equation of state

$$\bar{E}(k_f) = \frac{k_f^2}{4M}\left\{ -\gamma^{-1} + \left(\frac{3}{4} - \ln 2\right)\gamma^{-2} - 0.16079\,\gamma^{-3} + \ldots \right\}, \qquad \gamma = \ln\frac{k_f}{q_b}$$





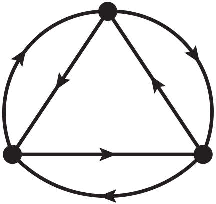
- Weak repulsion at low densities ↔ sizeable attraction at high densities
- S. Beane et al., Toward precision Fermi liquid theory in flatland, arXiv:2212.05177

3rd order ph-ring diagrams with contact-interactions

- Known from low-density expansion: 3rd order ph-contribution from contact-interaction proportional to S-wave scattering length a

$$\bar{E}(k_f)^{3ph} = (g-1)(3-g)\,\frac{a^3 k_f^5}{\pi^4 M}\cdot 2.7950523$$

g is spin-degeneracy factor, density $\rho = g k_f^3/6\pi^2$, $a > 0$ attraction

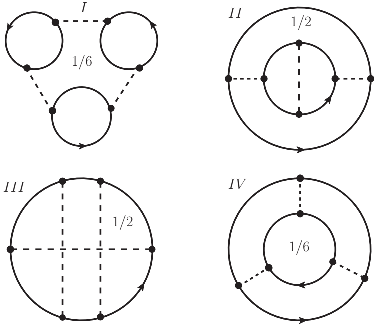


- Nuclear Fermi gas with two different scattering lengths: a_s and a_t

$$\bar{E}(k_f)^{3ph} = 1.0481446\,\frac{(a_s + a_t)k_f^5}{\pi^4 M}\left(5{a_s}^2 + 5{a_t}^2 - 14 a_s a_t\right)$$

- Extend this result to general $\mathcal{O}(p^2)$ NN-contact interaction (9 parameters) in NLO chiral NN-potential named: C_S, C_T and $C_1, \ldots, C_7$

- Direct and exchange-type 3-ring diagrams: $I+II+III+IV = (dir-exc)^3/6$



- Antisymmetrized Galilei-invariant contact-interaction (a la Skyrme)

$$V_{\rm Sk} - P_\sigma P_\tau V_{\rm Sk}\Big|_{\vec q_{\rm out}\to -\vec q_{\rm out}} = (1-P_\sigma P_\tau)\Big\{t_0(1+x_0P_\sigma) + \frac{t_1}{2}(1+x_1P_\sigma)(\vec q_{\rm out}^{\,2}+\vec q_{\rm in}^{\,2})\Big\}$$
$$+(1+P_\sigma P_\tau)\, t_2(1+x_2P_\sigma)\,\vec q_{\rm out}\cdot\vec q_{\rm in} + (1+P_\tau)\, iW_0(\vec\sigma_1+\vec\sigma_2)\cdot(\vec q_{\rm out}\times\vec q_{\rm in})$$

Spin and isospin exchange operators: $P_\sigma=(1+\vec\sigma_1\cdot\vec\sigma_2)/2$, $P_\tau=(1+\vec\tau_1\cdot\vec\tau_2)/2$, $\vec q_{\rm in}=(\vec p_1-\vec p_2)/2$, $\vec q_{\rm out}=(\vec p_1\,'-\vec p_2\,')/2$ momentum differences in initial/final state, completed to general $\mathcal{O}(p^2)$ contact-interaction by adding 2 tensor terms

3rd order ph ring-diagrams from contact interactions

- Triple spin- and isospin-traces:

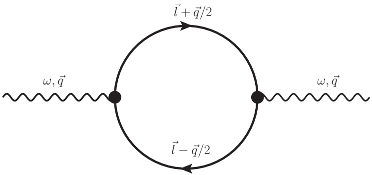
$$\frac{1}{64}\mathrm{tr}_1\mathrm{tr}_2\mathrm{tr}_3\Big\{(A+B\vec\sigma_1\cdot\vec\sigma_2+C\vec\tau_1\cdot\vec\tau_2+D\vec\sigma_1\cdot\vec\sigma_2\vec\tau_1\cdot\vec\tau_2)(A'+B'\vec\sigma_2\cdot\vec\sigma_3+C'\vec\tau_2\cdot\vec\tau_3+D'\vec\sigma_2\cdot\vec\sigma_3\vec\tau_2\cdot\vec\tau_3)$$
$$\times(A''+B''\vec\sigma_3\cdot\vec\sigma_1+C''\vec\tau_3\cdot\vec\tau_1+D''\vec\sigma_3\cdot\vec\sigma_1\vec\tau_3\cdot\vec\tau_1)\Big\} = AA'A''+3BB'B''+3CC'C''+9DD'D''$$

- Resulting interaction product, exploiting permutational symmetry (123)

$$
\begin{aligned}
&12t_0^3(1-6x_0^2)+9t_0^2t_1(1-2x_0^2-4x_0x_1)(\vec l_{12}^{\,2}+\vec q^{\,2})+9t_0^2t_2\,[5+4x_2+2x_0^2(1+2x_2)]\,(\vec l_{12}^{\,2}-\vec q^{\,2})\\
&+\frac{9}{4}t_0t_1^2(1-4x_0x_1-2x_1^2)(\vec l_{12}^{\,2}\vec l_{13}^{\,2}+2\vec l_{12}^{\,2}\vec q^{\,2}+\vec q^{\,4})+\frac{9}{2}t_0t_1t_2\,[5+4x_2+2x_0x_1(1+2x_2)]\\
&\times(\vec l_{12}^{\,2}\vec l_{13}^{\,2}-\vec q^{\,4})+\frac{9}{4}t_0t_2^2(5+8x_2+2x_2^2)(\vec l_{12}^{\,2}\vec l_{13}^{\,2}-2\vec l_{12}^{\,2}\vec q^{\,2}+\vec q^{\,4})\\
&+\frac{9}{16}t_1^2t_2\,[5+2x_1^2+4x_2(1+x_1^2)]\,(\vec l_{12}^{\,2}\vec l_{13}^{\,2}\vec l_{23}^{\,2}+\vec l_{12}^{\,2}\vec l_{13}^{\,2}\vec q^{\,2}-\vec l_{12}^{\,2}\vec q^{\,4}-\vec q^{\,6})\\
&+\frac{9}{16}t_1t_2^2(5+8x_2+2x_2^2)(\vec l_{12}^{\,2}\vec l_{13}^{\,2}\vec l_{23}^{\,2}-\vec l_{12}^{\,2}\vec l_{13}^{\,2}\vec q^{\,2}-\vec l_{12}^{\,2}\vec q^{\,4}+\vec q^{\,6})\\
&+\frac{3}{16}t_1^3(1-6x_1^2)(\vec l_{12}^{\,2}\vec l_{13}^{\,2}\vec l_{23}^{\,2}+3\vec l_{12}^{\,2}\vec l_{13}^{\,2}\vec q^{\,2}+3\vec l_{12}^{\,2}\vec q^{\,4}+\vec q^{\,6})\\
&+\frac{t_2^3}{16}(35+84x_2+78x_2^2+28x_2^3)(\vec l_{12}^{\,2}\vec l_{13}^{\,2}\vec l_{23}^{\,2}-3\vec l_{12}^{\,2}\vec l_{13}^{\,2}\vec q^{\,2}+3\vec l_{12}^{\,2}\vec q^{\,4}-\vec q^{\,6})\\
&+9W_0^2(\vec l_{12}\times\vec q\,)\cdot(\vec l_{13}\times\vec q\,)\{4t_0(1+x_0)+t_1(1+x_1)(\vec l_{23}^{\,2}+\vec q^{\,2})+5t_2(1+x_2)(\vec l_{23}^{\,2}-\vec q^{\,2})\}
\end{aligned}
$$

$\vec l_{ij}=\vec l_i-\vec l_j$ difference of loop-momenta, $\vec q$ flows through polarization-bubbles

Euclidean polarization functions



- Finite-temperature formalism in limit $T \to 0$ gives the representation:

$$\Pi(\omega,\vec{q}\,) = \int \frac{d^3l}{(2\pi)^3}\,\frac{1}{i\omega + \vec{l}\cdot\vec{q}/M}\Big\{\theta(k_f - |\vec{l}-\vec{q}/2|) - \theta(k_f - |\vec{l}+\vec{q}/2|)\Big\}$$

Fermionic Matsubara-sum yields Fermi-distributions $\to$ step-functions

- Euclidean polarization function: $\Pi[1](\omega,\vec{q}\,) = Mk_f\, Q_0(s,\kappa)/(4\pi^2 s)$, setting $|\vec{q}\,| = 2sk_f$, $\omega = 2s\kappa k_f^2/M$, agrees with alternative derivations

$$Q_0(s,\kappa) = s - s\kappa \arctan\frac{1+s}{\kappa} - s\kappa\arctan\frac{1-s}{\kappa} + \frac{1}{4}(1-s^2+\kappa^2)\ln\frac{(1+s)^2+\kappa^2}{(1-s)^2+\kappa^2}$$

- For contact-interaction: 3-loops factorize into "cube" of 1-loop

$$\Pi[\vec{l}\,] = -\frac{Mk_f^2}{4\pi^2 s}\, i\kappa\, Q_0(s,\kappa)\,\hat{q}\,, \quad \Pi[l_i l_j] = \frac{Mk_f^3}{4\pi^2 s}\left\{\frac{\delta_{ij}}{3}\,Q_1(s,\kappa) + \left(\hat{q}_i\hat{q}_j - \frac{\delta_{ij}}{3}\right)Q_2(s,\kappa)\right\}$$

- Translate products of scalar-products into cubic expressions in $Q_j(s,\kappa)$
- Resulting 3-ring energy per particle for isospin-symmetric nuclear matter

$$\bar{E}(k_f)^{3\mathrm{ph}} = \frac{M^2 k_f^5}{32\pi^7}\Big\{ t_0^3(1-6x_0^2)\mathcal{N}_1 + k_f^2\, t_0^2 t_1(1-2x_0^2-4x_0x_1)\mathcal{N}_2$$
$$+k_f^2\, t_0^2 t_2\,[5+4x_2+2x_0^2(1+2x_2)]\mathcal{N}_3 + k_f^4\, t_0 t_1^2(4x_0x_1+2x_1^2-1)\mathcal{N}_4$$
$$+k_f^4\, t_0 t_1 t_2\Big[\frac{5}{2}+x_0x_1(1+2x_2)+2x_2\Big]\mathcal{N}_5 + k_f^4\, t_0 t_2^2\Big[\frac{5}{2}+4x_2+x_2^2\Big]\mathcal{N}_6$$
$$+k_f^6\, t_1^2 t_2\Big[\frac{5}{2}+x_1^2+2x_2(1+x_1^2)\Big]\mathcal{N}_7 + k_f^6\, t_1 t_2^2\Big[\frac{5}{2}+4x_2+x_2^2\Big]\mathcal{N}_8$$
$$+k_f^6\, t_1^3(1-6x_1^2)\mathcal{N}_9 + k_f^6\, t_2^3\Big[\frac{5}{4}+3x_2+\frac{39}{14}x_2^2+x_2^3\Big]\mathcal{N}_{10}$$
$$+k_f^4\, W_0^2\Big[t_0(1+x_0)\mathcal{N}_{11} + k_f^2\, t_1(1+x_1)\mathcal{N}_{12} + k_f^2\, t_2(1+x_2)\mathcal{N}_{13}\Big]\Big\}$$

- For neutron matter isospin-factors change, no $\mathcal{N}_{11,12}$ terms ($^3S_1$-state)

$$\bar{E}_n(k_n)^{3\mathrm{ph}} = \frac{M^2 k_n^5}{96\pi^7}\Big\{ t_0^3(x_0-1)^3\mathcal{N}_1 + k_n^2\, t_0^2 t_1(x_0-1)^2(x_1-1)\mathcal{N}_2$$
$$+k_n^2\, t_0^2 t_2(x_0-1)^2(x_2+1)3\mathcal{N}_3 + k_n^4\, t_0 t_1^2(1-x_0)(x_1-1)^2\mathcal{N}_4$$
$$+k_n^4\, t_0 t_1 t_2(x_0-1)(x_1-1)(x_2+1)\frac{3\mathcal{N}_5}{2} + k_n^4\, t_0 t_2^2(1-x_0)(1+x_2)^2\frac{3\mathcal{N}_6}{2}$$
$$+k_n^6\, t_1^2 t_2(1-x_1)^2(1+x_2)\frac{3\mathcal{N}_7}{2} + k_n^6\, t_1 t_2^2(1-x_1)(1+x_2)^2\frac{3\mathcal{N}_8}{2}$$
$$+k_n^6\, t_1^3(x_1-1)^3\mathcal{N}_9 + k_n^6\, t_2^3(1+x_2)^3\frac{45\mathcal{N}_{10}}{28} + k_n^6\, W_0^2 t_2(1+x_2)\frac{8\mathcal{N}_{13}}{5}\Big\}$$

- For fourth loop-integral $\int d\omega \int d^3q/(2\pi)^4$ introduce polar coordinates in $s\kappa$-plane: $s = r\cos\varphi$, $\kappa = r\sin\varphi$

$$\mathcal{N}_1 = 12\int_0^\infty dr \int_0^{\pi/2} d\varphi\, r\,[Q_0(s,\kappa)]^3 = 4.1925784$$

- Dimensional regularization of $\mathcal{N}_j$: subtract power divergences $\int_0^{r_{\max}} dr\, r^{2n}$

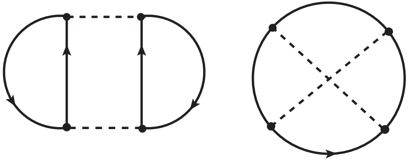
$$\mathcal{N}_2 = \int_0^\infty dr \int_0^{\pi/2} d\varphi \Big\{ 18r\,Q_0^2\,[Q_1 + (2s^2+\kappa^2)Q_0] - \frac{16}{3}\cos^3\varphi\,(2+\cos 2\varphi)\Big\} = -0.4633512$$

$$\mathcal{N}_3 = \int_0^\infty dr \int_0^{\pi/2} d\varphi \Big\{ 18r\,Q_0^2\,[Q_1 + (\kappa^2 - 2s^2)Q_0] + \frac{16}{3}\cos^3\varphi\,\cos 2\varphi\Big\} = -2.259163$$

- Expand integrands up to r^{-4} and include $a_2/r_{\max} + a_4/3r_{\max}^3$ for outside region $r > r_{\max}$, <u>accurate</u> and well converged results for $20 < r_{\max} < 40$
- Method verified by rederiving analytical results for 2nd order contribut.
- Remaining four-loop coefficients in dimensional regularization:

$$\mathcal{N}_4 = 2.902123\,,\quad \mathcal{N}_5 = 2.12658\,,\quad \mathcal{N}_6 = 0.438970\,,\quad \mathcal{N}_7 = 0.48756\,,$$
$$\mathcal{N}_8 = -0.27614\,,\quad \mathcal{N}_9 = -1.01924\,,\quad \mathcal{N}_{10} = 0.315484\,,$$
$$\mathcal{N}_{11} = -2.244200\,,\quad \mathcal{N}_{12} = -2.30577\,,\quad \mathcal{N}_{13} = 2.53887$$

Recovering second-order contributions



- 2nd order particle-particle ladder diagram = 2-ring particle-hole diagram, factor 1/2 not to double-count direct + exchange term via $(dir - exc)^2$

$$\bar{E}(k_f)^{\text{2nd}} = \frac{3Mk_f^4}{32\pi^5}\Big\{t_0^2(1+x_0^2)\mathcal{Z}_1 + k_f^2\, t_0 t_1(1+x_0x_1)\mathcal{Z}_2 + k_f^4\, t_1^2(1+x_1^2)\mathcal{Z}_3 + k_f^4\, t_2^2(5+8x_2+5x_2^2)\mathcal{Z}_4 + k_f^4\, W_0^2\,\mathcal{Z}_5\Big\}$$

- Three-loop coefficients $\mathcal{Z}_j$ in dimensional regularization:

$$\mathcal{Z}_1 = -8\int_0^\infty dr\int_0^{\pi/2} d\varphi\Big\{3rs\,Q_0^2 - \frac{4}{3}\cos^3\varphi\Big\} = 3.451697 = \frac{4\pi}{35}(11-2\ln 2)\,,$$

$$\mathcal{Z}_2 = -24\int_0^\infty dr\int_0^{\pi/2} d\varphi\Big\{rs\,Q_0[Q_1+(2s^2+\kappa^2)Q_0]\Big\}_{\text{reg}} = 3.99902 = \frac{8\pi}{945}(167-24\ln 2)\,,$$

$$\mathcal{Z}_3 = 1.37573 = \frac{\pi}{10395}(4943-564\ln 2)\,,\qquad \mathcal{Z}_4 = 0.0931718 = \frac{\pi}{31185}(1033-156\ln 2)\,,$$

$$\mathcal{Z}_5 = 128\int_0^\infty dr\int_0^{\pi/2} d\varphi\Big\{rs^3\,Q_0(Q_2-Q_1)\Big\}_{\text{reg}} = 2.70935 = \frac{16\pi}{10395}(631-102\ln 2)$$

Tensorial contact-terms

- Contact-interaction of $\mathcal{O}(p^2)$ gets completed by adding two tensor terms

$$V_{\rm ten} - P_\sigma P_\tau V_{\rm ten}\Big|_{\vec q_{\rm out}\to -\vec q_{\rm out}} = (1-P_\tau)\, t_4 \Big\{\vec\sigma_1\cdot\vec q_{\rm out}\,\vec\sigma_2\cdot\vec q_{\rm out} + \vec\sigma_1\cdot\vec q_{\rm in}\,\vec\sigma_2\cdot\vec q_{\rm in} - \frac{1}{3}\vec\sigma_1\cdot\vec\sigma_2(\vec q_{\rm out}^{\,2}+\vec q_{\rm in}^{\,2})\Big\}$$
$$+(1+P_\tau)\, t_5 \Big\{\vec\sigma_1\cdot\vec q_{\rm out}\,\vec\sigma_2\cdot\vec q_{\rm in} + \vec\sigma_1\cdot\vec q_{\rm in}\,\vec\sigma_2\cdot\vec q_{\rm out} - \frac{2}{3}\vec\sigma_1\cdot\vec\sigma_2\,\vec q_{\rm out}\cdot\vec q_{\rm in}\Big\}$$

- Tensor contributions to 3-ring energy per particle (t_4 absent in n-matter)

$$\begin{aligned}\bar E(k_f)^{\rm 3ph} &= \frac{M^2 k_f^9}{32\pi^7}\Big\{k_f^2\, W_0^2\,[t_4\,\mathcal{N}_{14} + t_5\,\mathcal{N}_{15}]\\ &+t_4^2\,[t_0(x_0-2)\mathcal{N}_{16} + k_f^2\, t_1(x_1-2)\mathcal{N}_{17} + k_f^2\, t_2(x_2+2)\mathcal{N}_{18}]\\ &+t_4 t_5\,[t_0 x_0\,\mathcal{N}_{19} + k_f^2\, t_1 x_1\,\mathcal{N}_{20} + k_f^2\, t_2 x_2\,\mathcal{N}_{21}]\\ &+t_5^2\,[t_0(3x_0-2)\mathcal{N}_{22} + k_f^2\, t_1(3x_1-2)\mathcal{N}_{23} + k_f^2\, t_2(3x_2+2)\mathcal{N}_{24}]\\ &+k_f^2\,[t_4^3\,\mathcal{N}_{25} + t_4^2 t_5\,\mathcal{N}_{26} + t_4 t_5^2\,\mathcal{N}_{27} + t_5^3\,\mathcal{N}_{28}]\Big\}\,,\\ \bar E_n(k_n)^{\rm 3ph} &= \frac{M^2 k_n^9}{24\pi^7}\Big\{k_n^2\,\frac{2t_5}{5}\,[W_0^2\,\mathcal{N}_{15} + t_5^2\,\mathcal{N}_{28}]\\ &+t_5^2\,[t_0(x_0-1)\mathcal{N}_{22} + k_n^2\, t_1(x_1-1)\mathcal{N}_{23} + k_n^2\, t_2(x_2+1)\mathcal{N}_{24}]\Big\}\end{aligned}$$

- Spin-traces $\to$ triple scalar-products $\to$ cubic expressions in $Q_j(s,\kappa)$
- Accurate four-loop coefficients calculated in dimensional regularization:
  $\mathcal{N}_{14} = 0.8722$, $\mathcal{N}_{15} = -5.0175$, $\mathcal{N}_{16} = -2.9160$, $\mathcal{N}_{17} = -2.7834$, $\mathcal{N}_{18} = 0.42202$,
  $\mathcal{N}_{19} = 10.154$, $\mathcal{N}_{20} = 8.0564$, $\mathcal{N}_{21} = -0.7457$, $\mathcal{N}_{22} = -1.0762, \ldots, \mathcal{N}_{28} = 5.4015$

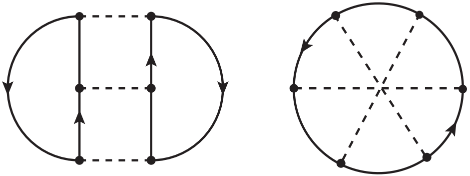
- Known from low-density expansion: contributions up to 3rd order ladder diagrams from contact-interaction prop. to S-wave scattering length a

$$\bar{E}(k_f)^{\mathrm{lad}} = (g-1)\frac{k_f^2}{M}\left\{ -\frac{ak_f}{3\pi} + \frac{2}{35}(11 - 2\ln 2)\left(\frac{ak_f}{\pi}\right)^2 - 1.1716223\left(\frac{ak_f}{\pi}\right)^3 \right\}$$

- Extend this result to general $\mathcal{O}(p^2)$ contact-interaction (7+2 parameters)

$$V_{\mathrm{cont}} = t_0(1 + x_0 P_\sigma) + \frac{t_1}{2}(1 + x_1 P_\sigma)(\vec{q}_{\mathrm{out}}^{\,2} + \vec{q}_{\mathrm{in}}^{\,2}) + t_2(1 + x_2 P_\sigma)\,\vec{q}_{\mathrm{out}}\cdot\vec{q}_{\mathrm{in}}$$
$$+ iW_0(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot(\vec{q}_{\mathrm{out}}\times\vec{q}_{\mathrm{in}}) + V_{\mathrm{ten}}(t_4, t_5)$$

- Third-order Hartree (direct) and Fock (exchange) ladder diagrams

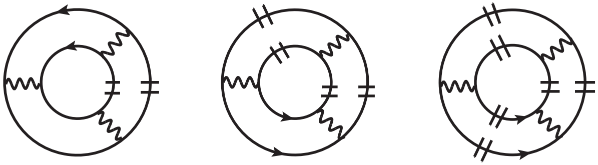


- Twice-iterated interaction in medium integrated over two Fermi spheres

- P-wave interactions introduce factor $l_i l_j$ in loop integrals, decompose this tensor into a transversal and a longitudinal part

$$\begin{aligned}
\text{real part:} \quad & \frac{Mk_f^3}{12\pi^2}\Big\{R_\perp(s,\kappa)\big[\delta_{ij}-\hat{P}_i\hat{P}_j\big]+R_\|(s,\kappa)\hat{P}_i\hat{P}_j\Big\},\\
\text{imaginary part:} \quad & \frac{Mk_f^3}{12\pi^2}\Big\{I_\perp(s,\kappa)\big[\delta_{ij}-\hat{P}_i\hat{P}_j\big]+I_\|(s,\kappa)\hat{P}_i\hat{P}_j\Big\},\\
\text{relations:} \quad & 2R_\perp+R_\| = 4+3\kappa^2 R\,, \quad 2I_\perp+I_\| = 3\kappa^2 I
\end{aligned}$$

- Mixing terms of S- and P-wave interactions vanish in a medium with one single Fermi momentum k_f, i.e. without isospin- or spin-asymmetries, weight functions $\theta(k_f-|\vec{P}\pm\vec{l}|)$ in loop integral are even under $\vec{l}\to-\vec{l}$
- Proper real-valued integrand for energy density $\rho\bar{E}$ at third order



$$(R-i\pi I)^2+(R-i\pi I)(2i\pi I)+\frac{1}{3}(2i\pi I)^2 = R^2-\frac{\pi^2}{3}I^2$$

Evaluation of 3rd order ladder diagrams: S-wave interactions

- Results in pure neutron matter (only ${}^1S_0, P_\sigma \to -1$), density $\rho_n = k_n^3/3\pi^2$

$$\bar{E}_n(k_n)^{\rm 3lad} = \frac{M^2 k_n^5}{64\pi^6}\Big\{t_0^3(1-x_0)^3 B_1 + k_n^2\, t_0^2 t_1(1-x_0)^2(1-x_1) B_2 + k_n^4\, t_0 t_1^2(1-x_0)(1-x_1)^2 B_3 + k_n^6\, t_1^3(1-x_1)^3 B_4\Big\}$$

- Four-loop coefficients with high numerical accuracy, $t_0(x_0-1) = 4\pi a/M$

$$B_1 = 8\int_0^1 ds\, s^2 \int_0^{\sqrt{1-s^2}} d\kappa\, \kappa\, I(s,\kappa)\big[3R^2 - \pi^2 I^2\big] = 1.1716223\,,$$

$$B_2 = 8\int_0^1 ds\, s^2 \int_0^{\sqrt{1-s^2}} d\kappa\, \kappa\, I\big[3\kappa^2(3R^2-\pi^2 I^2)+8R\big] = 1.9893144\,,$$

$$B_3 = 1.360736\,, \qquad B_4 = 0.3344923$$

- Result in isospin-symmetric nuclear matter, density $\rho = 2k_f^3/3\pi^2$

$$\bar{E}(k_f)^{\rm 3lad} = \frac{3M^2 k_f^5}{64\pi^6}\Big\{t_0^3(1+3x_0^2) B_1 + k_f^2\, t_0^2 t_1(1+x_0^2+2x_0x_1) B_2 + k_f^4\, t_0 t_1^2(1+2x_0x_1+x_1^2) B_3 + k_f^6\, t_1^3(1+3x_1^2) B_4\Big\}$$

- Sum contributions from isotriplet ${}^1S_0$- and isosinglet ${}^3S_1$-states

$$\big[3t_0^3(1-x_0)^3 + 3t_0^3(1+x_0)^3\big]/2 = 3t_0^3(1+3x_0^2)\,,$$

$$\big[3t_0^2 t_1(1-x_0)^2(1-x_1) + 3t_0^2 t_1(1+x_0)^2(1+x_1)\big]/2 = 3t_0^2 t_1(1+x_0^2+2x_0x_1)$$

- Results for neutron matter (only $^3P_{0,1,2}$) and symmetric nuclear matter

$$\bar{E}_n(k_n)^{\rm 3lad} = \frac{M^2 k_n^{11}}{64\pi^6}\Big\{t_2^3(1+x_2)^3 B_5 + t_2(1+x_2)W_0^2 B_6 + W_0^3 B_7\Big\},$$

$$\bar{E}(k_f)^{\rm 3lad} = \frac{M^2 k_f^{11}}{64\pi^6}\Big\{\frac{t_2^3}{3}(5+12x_2+15x_2^2+4x_2^3)B_5 + \frac{3t_2}{2}(1+x_2)W_0^2 B_6 + \frac{3}{2}W_0^3 B_7\Big\}$$

- Spin-traces: Hartree $\mathrm{tr}_{\sigma_1}\mathrm{tr}_{\sigma_2}$, Fock $\mathrm{tr}_{\sigma_1=\sigma_2}$ taking care of ordering
- t_2^3-term: $9(1+x_2)^3 \to \big[3(1-x_2)^3+27(1+x_2)^3\big]/2 = 3(5+12x_2+15x_2^2+4x_2^3)$
- Isotriplet ^3P_J-interactions: 3(Hart+Fock)=4Hart+2Fock, thus: Hart=Fock
- Pertinent four-loop coeffcients computed from double-integrals

$$B_5 = \frac{8}{9}\int_0^1 ds\, s^2 \int_0^{\sqrt{1-s^2}} d\kappa\, \kappa \Big[2I_\perp(3R_\perp^2 - \pi^2 I_\perp^2) + I_\parallel(3R_\parallel^2 - \pi^2 I_\parallel^2)\Big] = 0.06699116\,,$$

$$B_6 = \frac{128}{9}\int_0^1 ds\, s^2 \int_0^{\sqrt{1-s^2}} d\kappa\, \kappa \Big\{I_\perp\Big[3R_\perp^2 + 2R_\perp R_\parallel + R_\parallel^2 - \frac{\pi^2}{3}(3I_\perp^2 + 2I_\perp I_\parallel + I_\parallel^2)\Big]$$
$$+ I_\parallel\Big[R_\perp(R_\perp + 2R_\parallel) - \frac{\pi^2}{3} I_\perp(I_\perp + 2I_\parallel)\Big]\Big\} = 1.327456\,,$$

$$B_7 = \frac{128}{9}\int_0^1 ds\, s^2 \int_0^{\sqrt{1-s^2}} d\kappa\, \kappa \Big\{2I_\perp\Big(R_\perp R_\parallel - \frac{\pi^2}{3} I_\perp I_\parallel\Big) + I_\parallel\Big(R_\perp^2 - \frac{\pi^2}{3} I_\perp^2\Big)\Big\} = 0.4527642$$

Tensor contact-interactions up to third order

- Tensor contributions in neutron matter (only ^3P_J) and symmetric N-matter

$$\bar{E}_n(k_n)^{\rm 3lad} = \frac{M^2 k_n^{11}}{64\pi^6}\Big\{t_5 W_0^2 B_8 + t_5^2 t_2(1+x_2) B_9 + t_5^2 W_0 B_{10} + t_5^3 B_{11}\Big\}\,,$$

$$\bar{E}(k_f)^{\rm 3lad} = \frac{M^2 k_f^{11}}{64\pi^6}\Big\{\frac{3}{2} t_5 W_0^2 B_8 + \frac{3}{2} t_5^2 t_2(1+x_2) B_9 + \frac{3}{2} t_5^2 W_0 B_{10} + \frac{3}{2} t_5^3 B_{11} + k_f^{-2}\, t_4^2 t_0(1+x_0) B_{12} + t_4^2 t_1(1+x_1) B_{13}\Big\}$$

- t_5-term acts in ^3P_J-states, $t_5 t_2^2$ and $t_5 t_2 W_0$ interferences give spin-trace $=0$
  $B_8 = -2.243263\,,\quad B_9 = 2.042028\,,\quad B_{10} = 6.66812\,,\quad B_{11} = -1.655323$
- Tensorial t_4-term responsible for $^3S_1\,^3D_1$-mixing: $P_\sigma \to 1$, param. $t_{0,1}(1+x_{0,1})$
- Spin-traces give: Hart = Fock, relevant in nuclear matter: 4 Hart - 2 Fock
- Pertinent four-loop coefficients: $B_{12} = 2.421103\,,\quad B_{13} = 1.559127$

Summary: Semi-analytical many-body calculations

- Resummation of in-medium ladder diagrams $\sim a$ to all orders: arctan
- Construction of complex single-particle potential $U(p,k_f) + i\,W(p,k_f)$
- Third-order particle-hole ring diagrams from $\mathcal{O}(p^2)$ contact-interaction
- Third-order ladder diagrams from general $\mathcal{O}(p^2)$ NN-contact interaction
- Four-loop coefficients computed accurately from dim-reg. double-integral