The nuclear clustering phenomenon

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DE LA RECHERCHE À L'INDUSTRIE

EMMI Taskforce

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❶ Few words on nuclear structure theory

❷ Microscopic description of nuclear clustering

What is nuclear structure theory about ?

◉ How nucleons self-organize and become disorganized ?

Ground-state

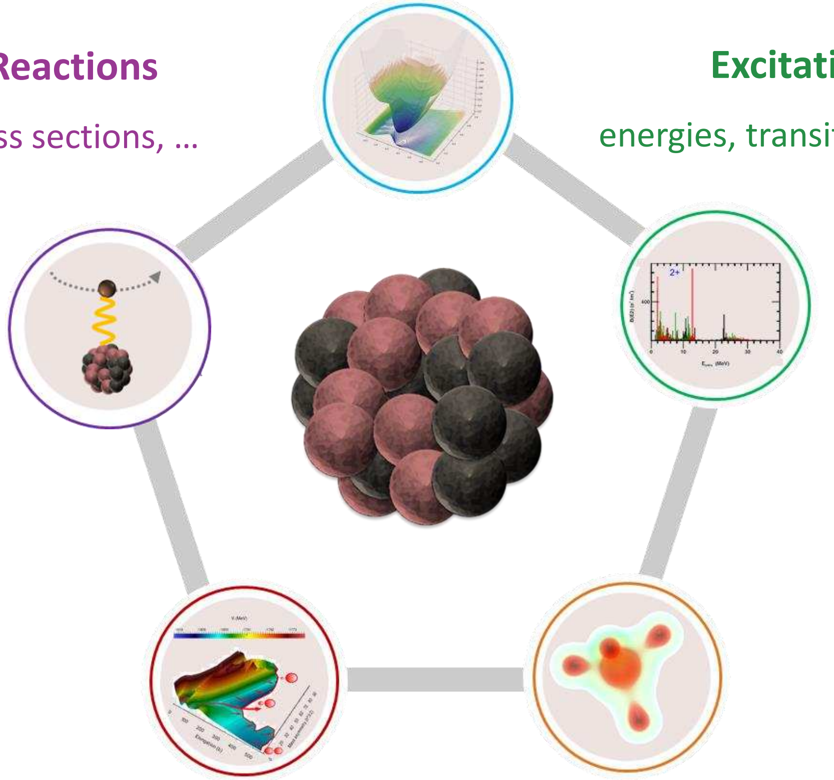
masses, radii, e.m. moments, …

Reactions

cross sections, …

Excitation spectra

energies, transition probabilities, …



Decay modes

lifetime, yields, …

Exotic structures

clusters, buble, halo, …

Nuclei are complex systems

◉ Many characteristic scales :

- --> p & n momenta ~ 100 MeV
- --> separation energies ~ 10 MeV
- --> vibration modes ~ 1MeV
- --> rotation modes ~ 0.01-5 MeV

◉ Strongly correlated:

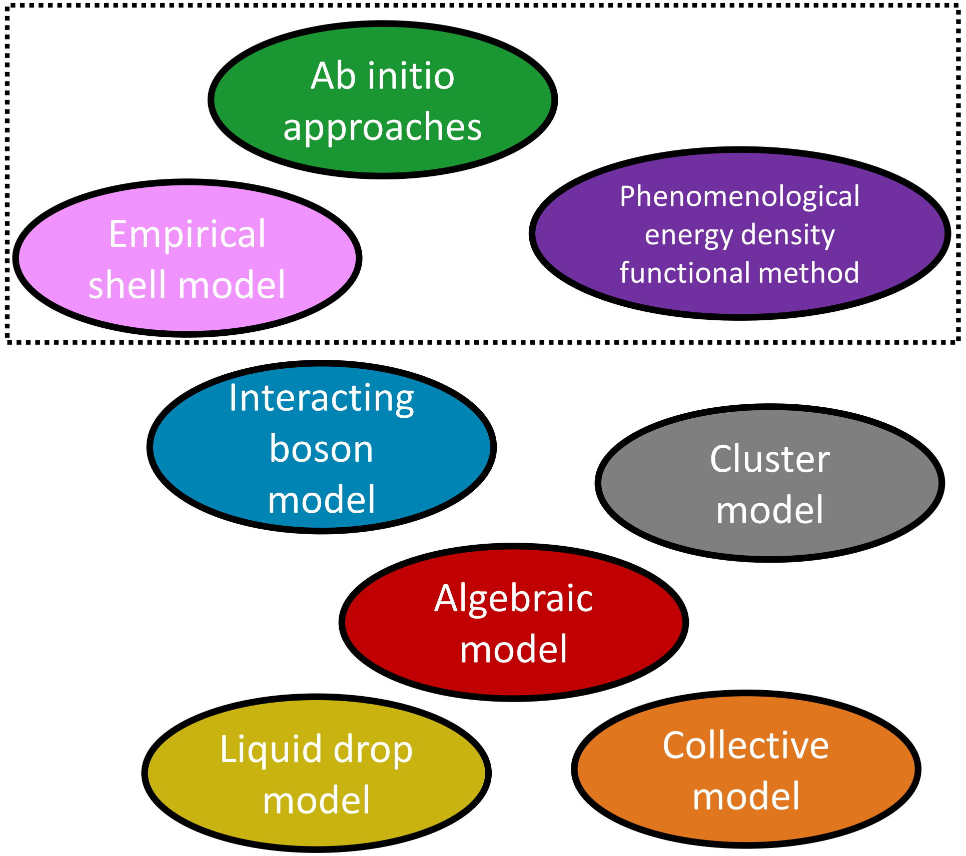
- --> angular correlations ⇒ deformation
- --> pairing correlations ⇒ superfluidity
- --> quartetting correlations ⇒ clustering

⇒ Rich diversity of nuclear phenomena, among which the clustering phenomenon

Best strategy ?

◉ What is the best strategy to achieve a **robust, predictive,** yet **computationally affordable** description of nuclear structure properties ?

⇢ Richness of nuclear phenomena propelled the formulation of a plethora of models, in general born from systematics of regularities in the behavior of nuclei



◉ Popular misconception : The more microscopic the more predictive

⇢ Confusion between the resolution of the language – **microscopic/coarse grain** – on the one hand and
the nature of the description – **effective theory/phenomenological model** – on the other hand

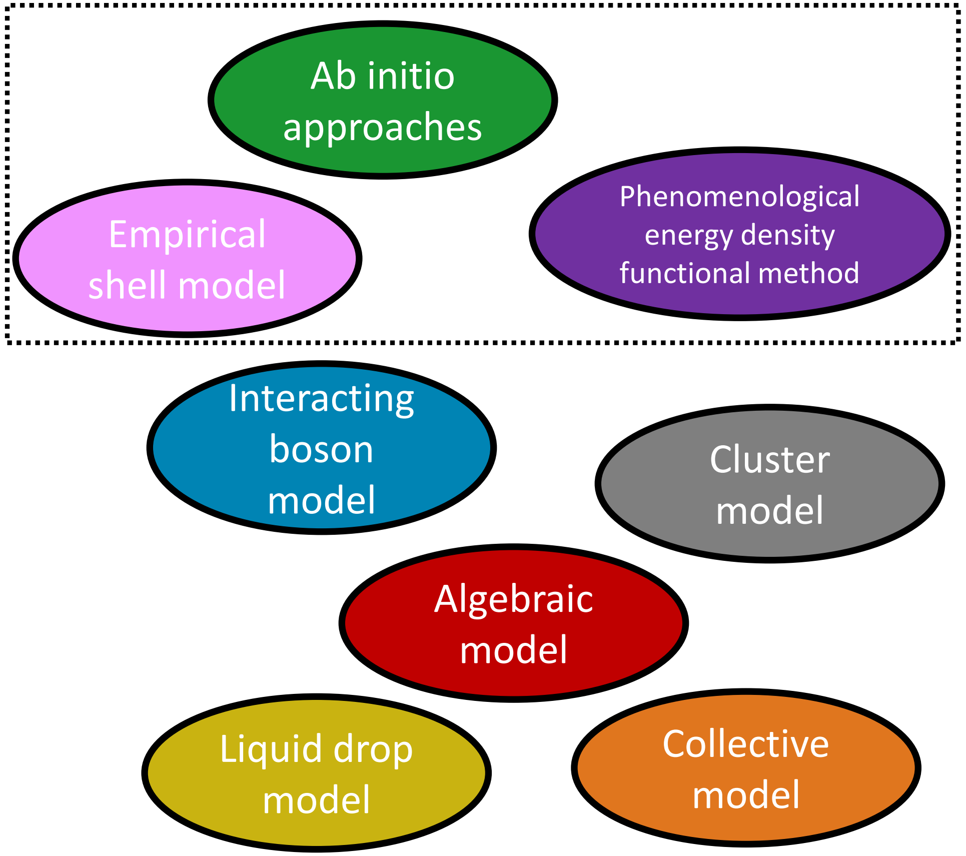
Precious empirical knowledge about various phenomena and associated relevant dofs

Best strategy ?

◉ What is the best strategy to achieve a **robust, predictive,** yet **computationally affordable** description of nuclear structure properties ?

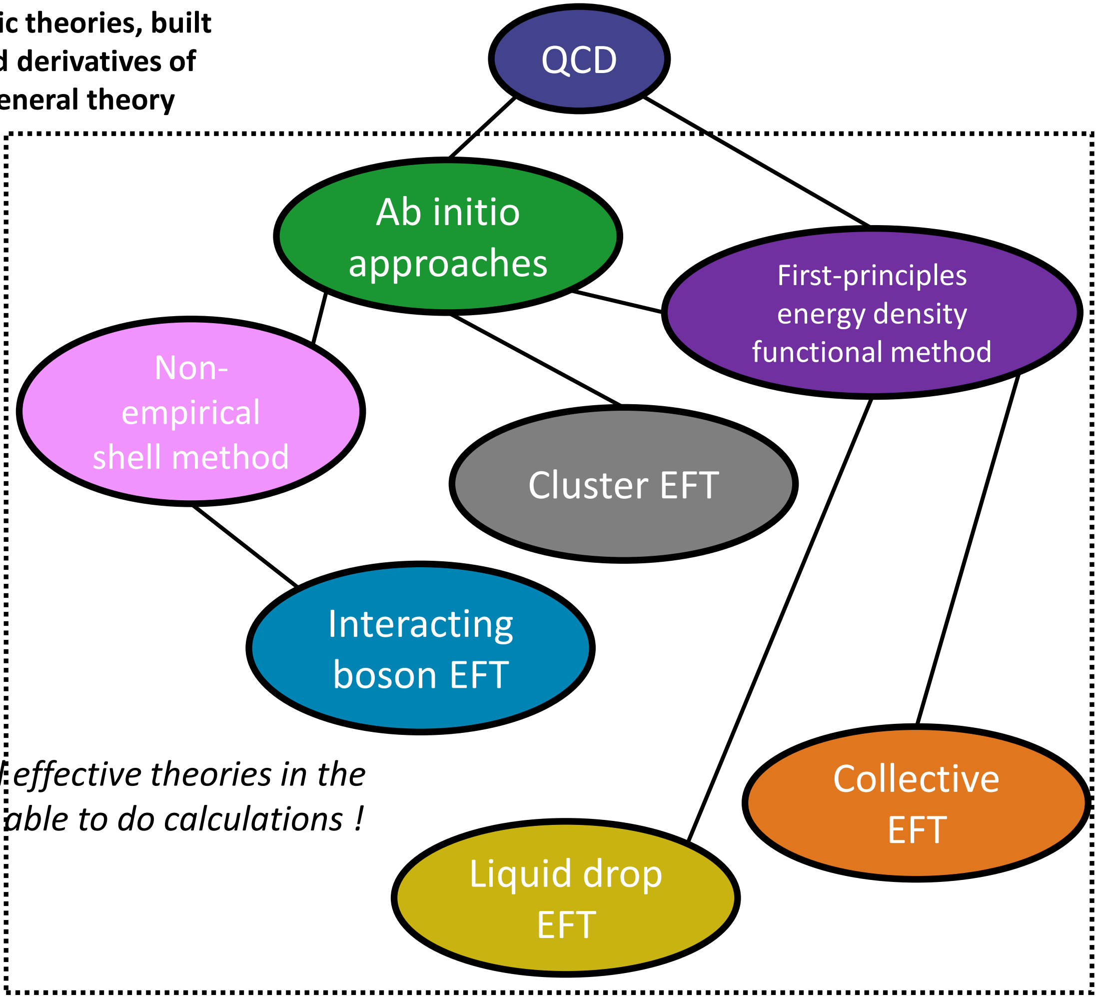
--> The more microscopic, the better ?

synthesizing



--> Web of interconnected E(F)Ts :
each blob provides a predictive description of specific features

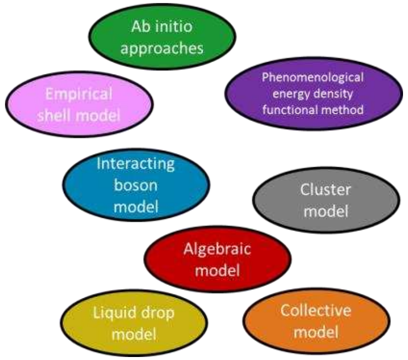
Go to more specific theories, built as derivatives (and derivatives of derivatives) of a general theory

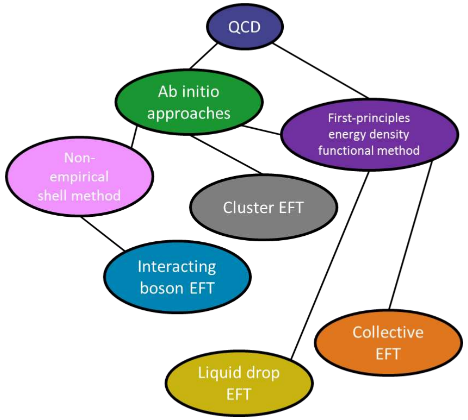


More powerful effective theories in the sense of being able to do calculations !

Best strategy ?

⦿ In what aspect(s) empirical models and EFTs differ ?





--> As we get more microscopic, things become harder to compute

--> Want the **simplest** framework that captures the essential physics
⇒ identification of relevant dofs + dynamics constrained by symmetry arguments

--> As we move up, it becomes harder to compute

--> Want the **simplest** framework that captures the essential physics
⇒ identification of relevant **scales** & dofs + dynamics constrained by symmetry arguments

--> At the same time, we don't want to give up anything , in the sense that even if we're giving up smthg at our LO description, we want to retain the ability to correct that LO description order by order in some expansions, so that it can be corrected to arbitrary precision

EFT (simplified) recipe

◉ To describe a physical system :

--> Determine the **relevant** dofs/scales *("everything should be made as simple as possible, but no simpler", Einstein)*

might be obvious or very tricky

--> Identify the symmetry pattern (global, gauged, accidental, spontaneously broken, anomalous, approximate, …) you want to be consistent with *(Totalitarian principle : "everything that is not forbidden is compulsory" Gell-Mann — Folk Theorem : "If one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with perturbative unitarity, analyticity, cluster decomposition, and the assumed symmetry properties" Weinberg)*

EFT can have more symmetries that the original theory

--> Specify what is (are) your expansion parameter(s) as well as your LO description

Ensures that only a finite number of terms contribute at any given order in an expansion $\frac{E}{\Lambda}$, and that we can decide upfront which terms to keep in the action based on the desired level of accuracy

--> Constrain LECs

--> Make predictions

In QFT language :

What fields ?

What interactions/dynamics ?

What power counting ?

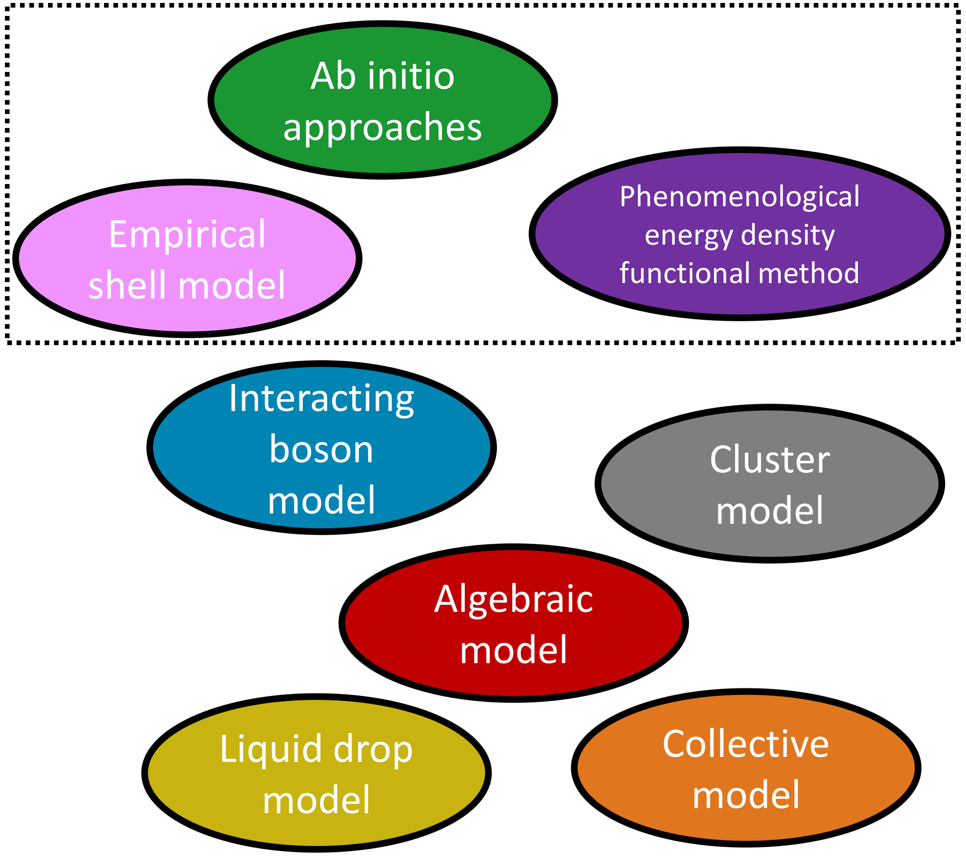
Matching

Best strategy ?

◉ What is the best strategy to achieve a **robust, predictive,** yet **computationally affordable** description of nuclear structure properties ?

--> The more microscopic, the better ?

synthesizing



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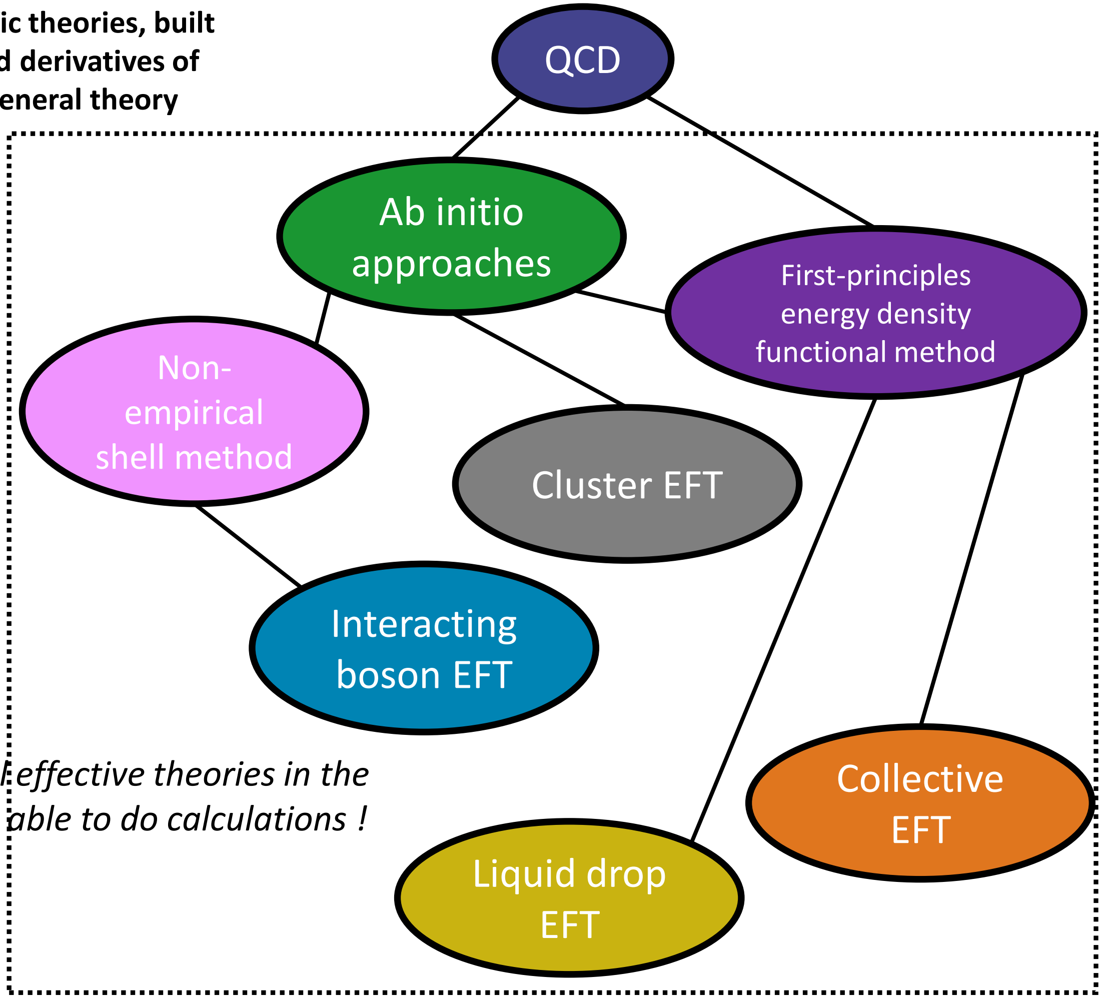

More powerful effective theories in the sense of being able to do calculations !

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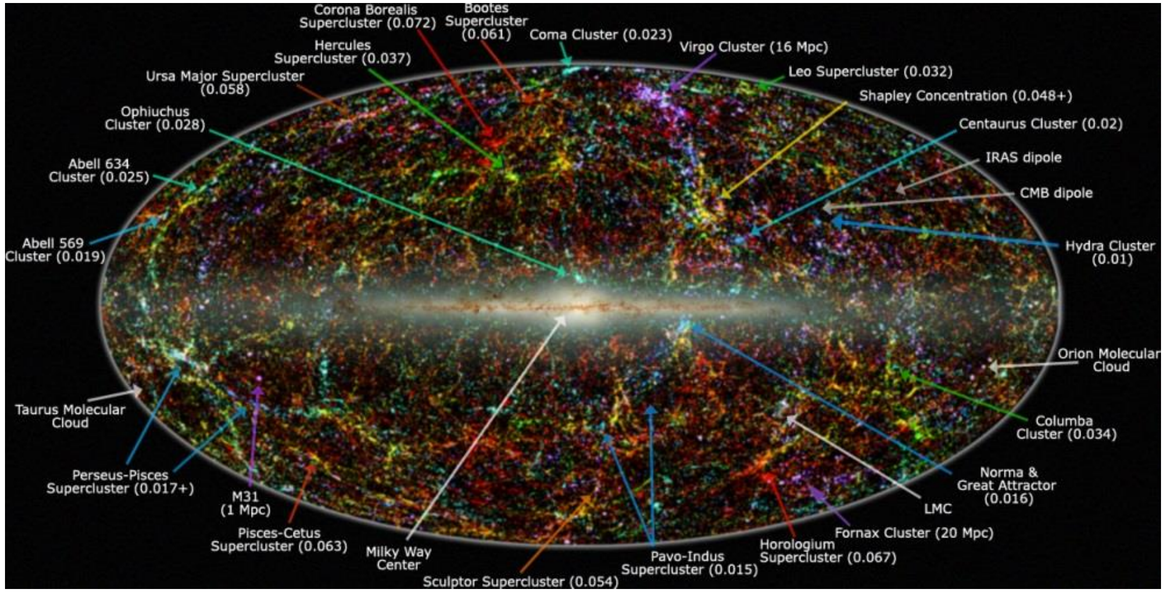
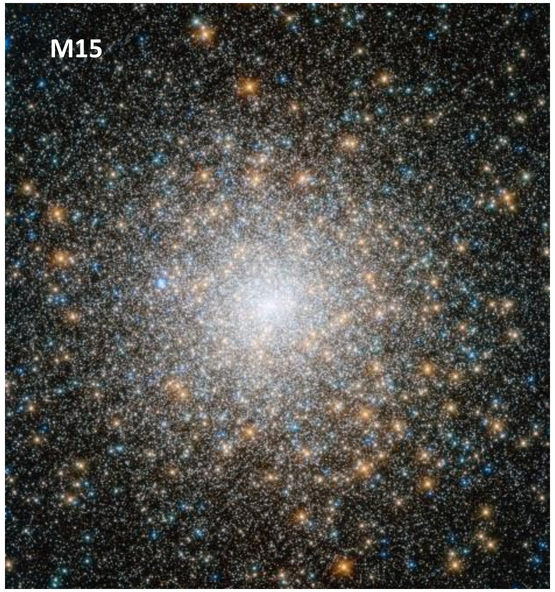
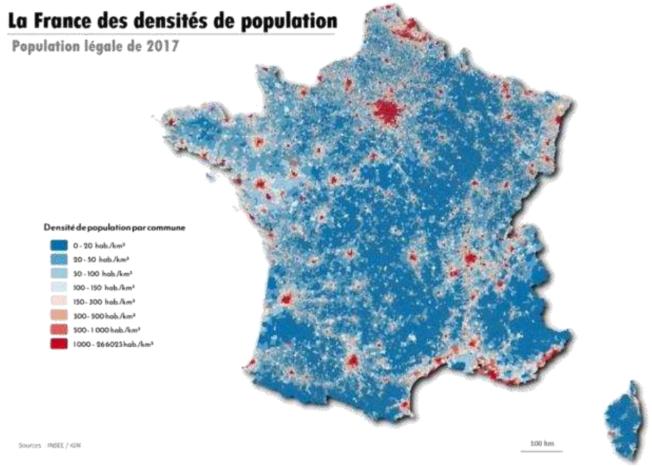
❷ Microscopic description of nuclear clustering

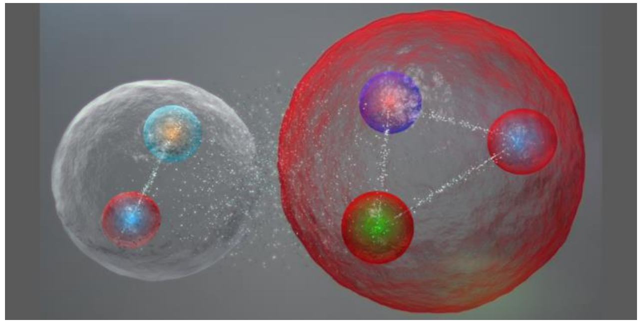
Clustering

- Clustering : an ubiquitous phenomenon



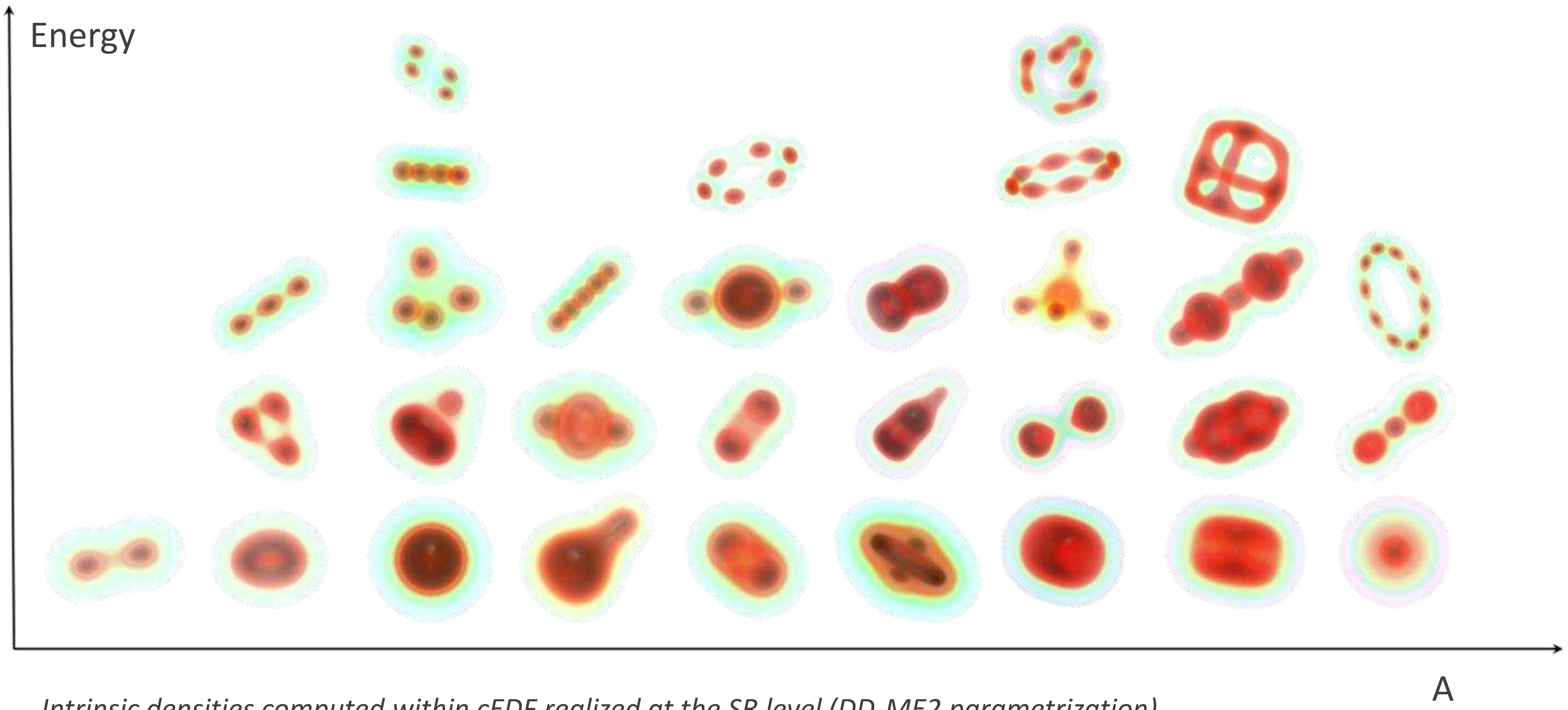






Nuclear clustering

- Nuclear clustering = nucleons clumping together into sub-groups within the nucleus



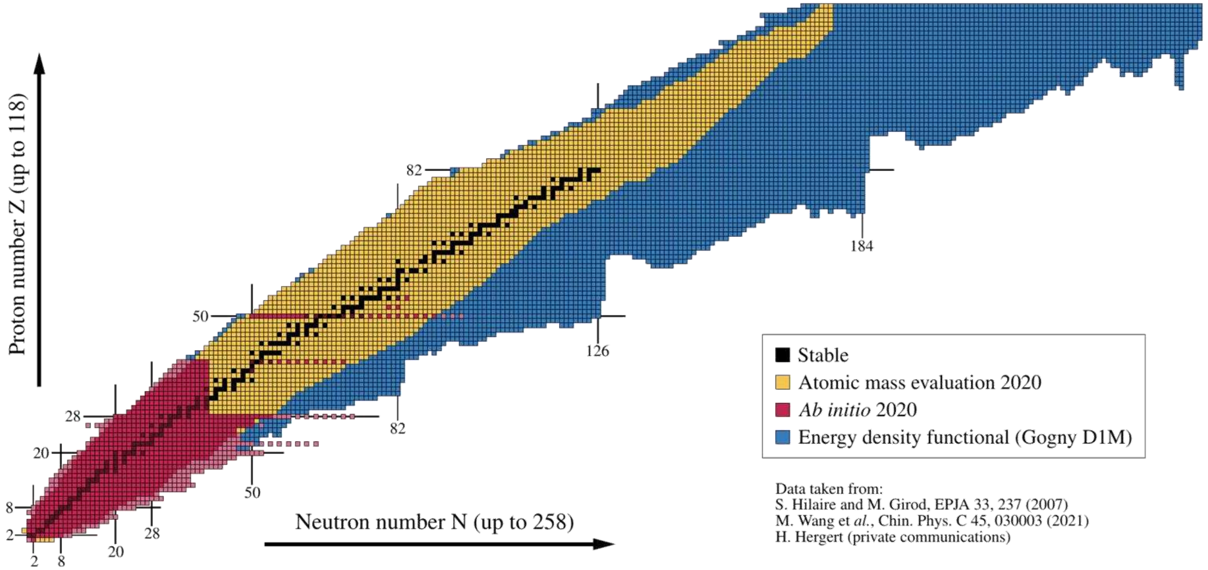
Intrinsic densities computed within cEDF realized at the SR level (DD-ME2 parametrization)

Microscopic treatment of nuclear systems

1) Nucleus: *A* interacting, structure-less nucleons
2) Structure & dynamic encoded in a Hamiltonien
3) Solve *A*-nucleon Schrödinger/Dirac equation to desired accuracy

$$H(\ ,\ , \ldots)|\Psi_{\mu,\sigma}\rangle = E_{\mu\tilde{\sigma}}\,|\Psi_{\mu,\sigma}\rangle$$

Strongly correlated wavefunction



Courtesy of B. Bally

Ab initio

- Chiral Hamiltonian rooted in QCD
- Bunch of many-body methods
  - --> *CI (full space diag.) : exponential scaling*
  - --> *Hybrids (valence space diag.) : mixed scaling*
  - --> *Expansion methods (partition, expand and truncate) : polynomial scaling*

EDF

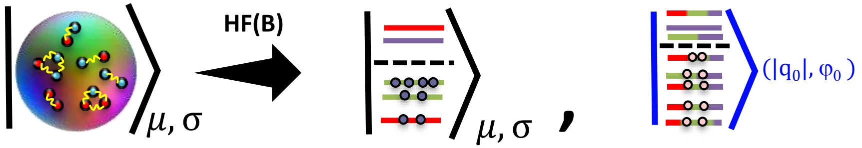
- Effective (empirical) pseudo-Hamiltonian

Original, matter free-space interactions	Effective in-medium interactions
	 *Phenomenological ansatz (Gogny, Skyrme, …)*
$\lvert\Psi_{\mu,\sigma}\rangle$	$\lvert\Theta_{\mu\sigma}\rangle$
Complicated wavefunction	Simplified wavefunction

- Various levels of realization
  - --> *Hartree-Fock-Bogoliubov (HFB)*
  - --> *Projected Generator Coordinate Method (PGCM)*
  - --> *Quasiparticule Random Phase Approximation (QRPA)*

◉ HFB treatment

--> *A*-nucleon problem ⟶ *A* 1-nucleon problems



Correlated A-nucleon wavefunction

Symmetry-conserving A independent nucleons wavefunction

Symmetry-breaking A independent nucleons wavefunction

--> SSB : Efficient way for capturing so-called static correlations

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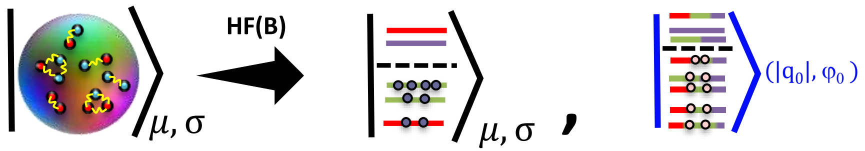

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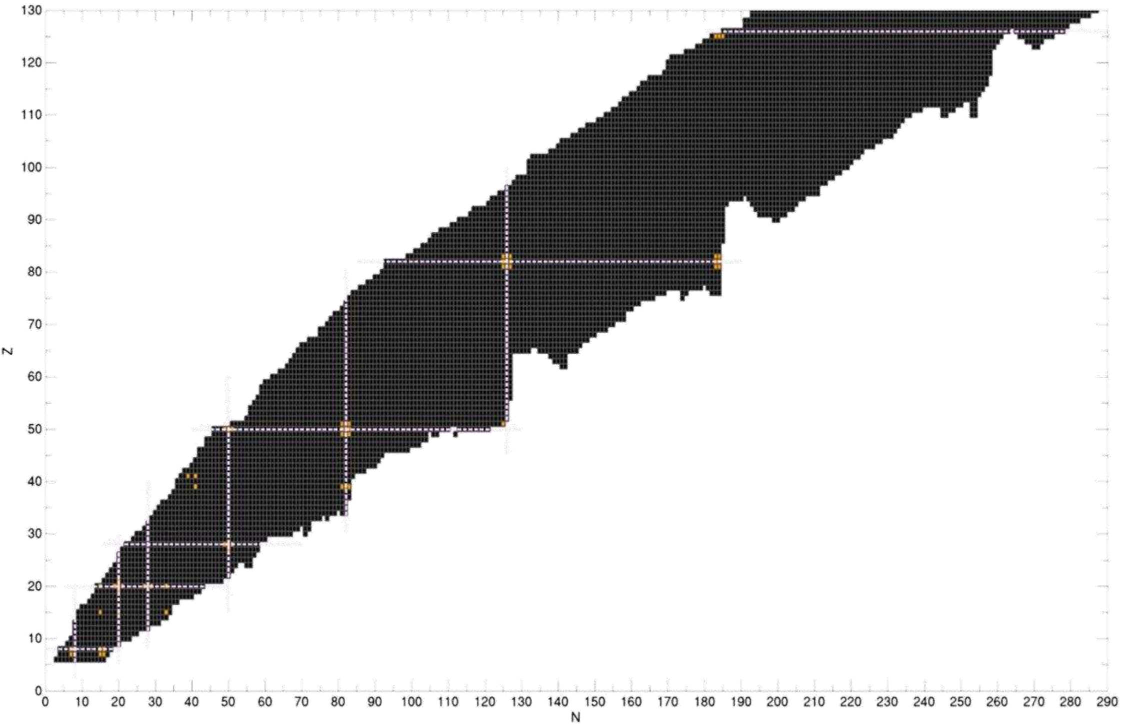
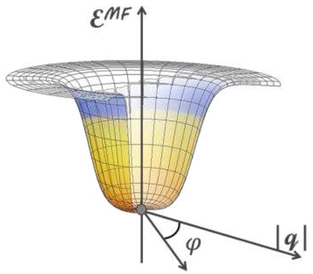
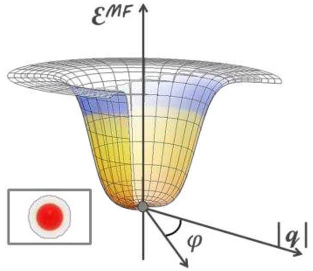


Symmetry-restriced HF : good description of GS of doubly closed-shell nuclei & neighbors (~30 nuclei)





The Energy Density Functional Method

◉ HFB treatment

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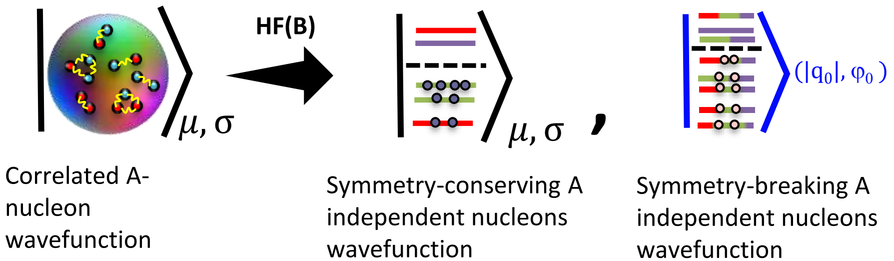
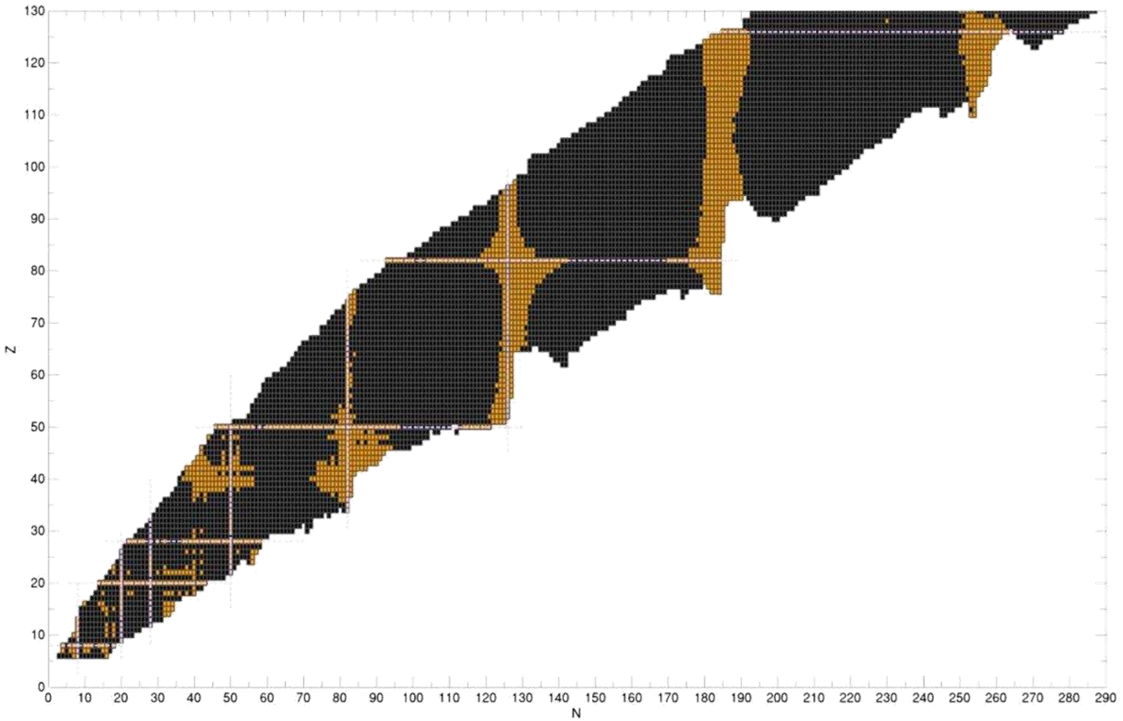
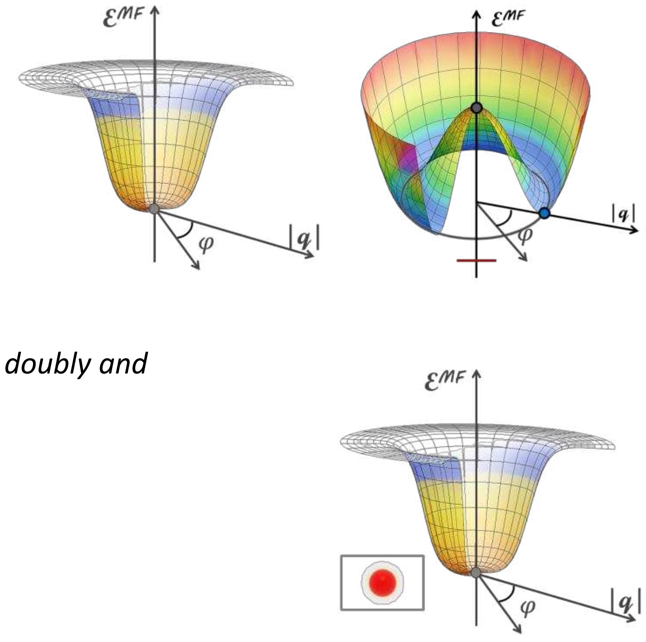


--→ SSB : Efficient way for capturing so-called static correlations



Spatial symmetry-restricted HFB: good description of GS of doubly and singly closed-shell nuclei & neighbors (~300 nuclei)



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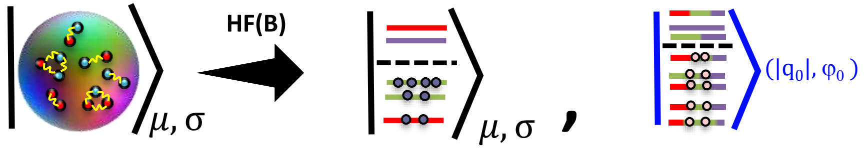

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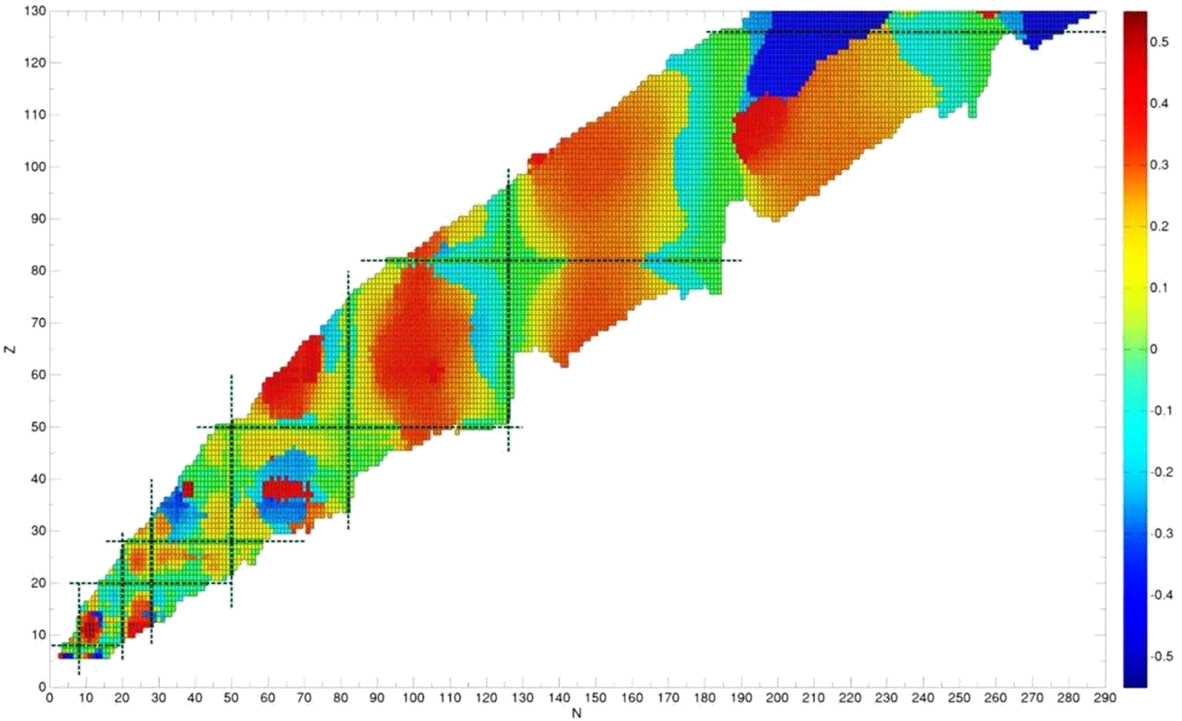
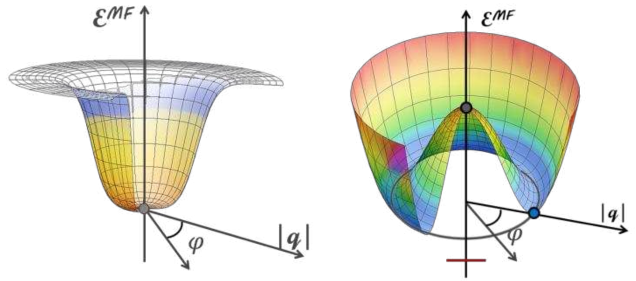
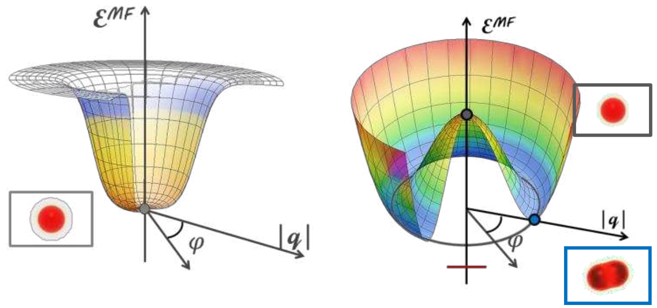




Symmetry-unrestricted HFB: good description of GS of all nuclei



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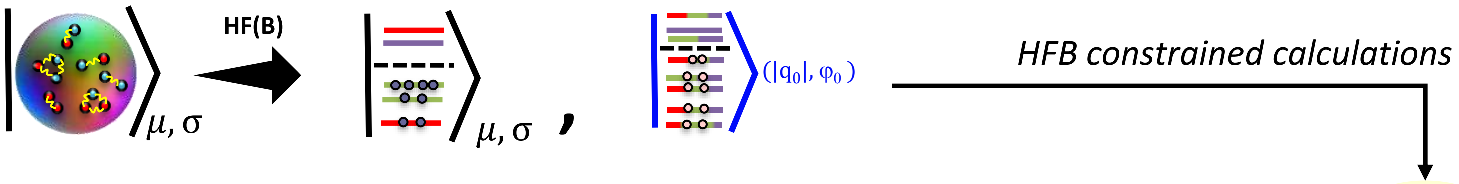
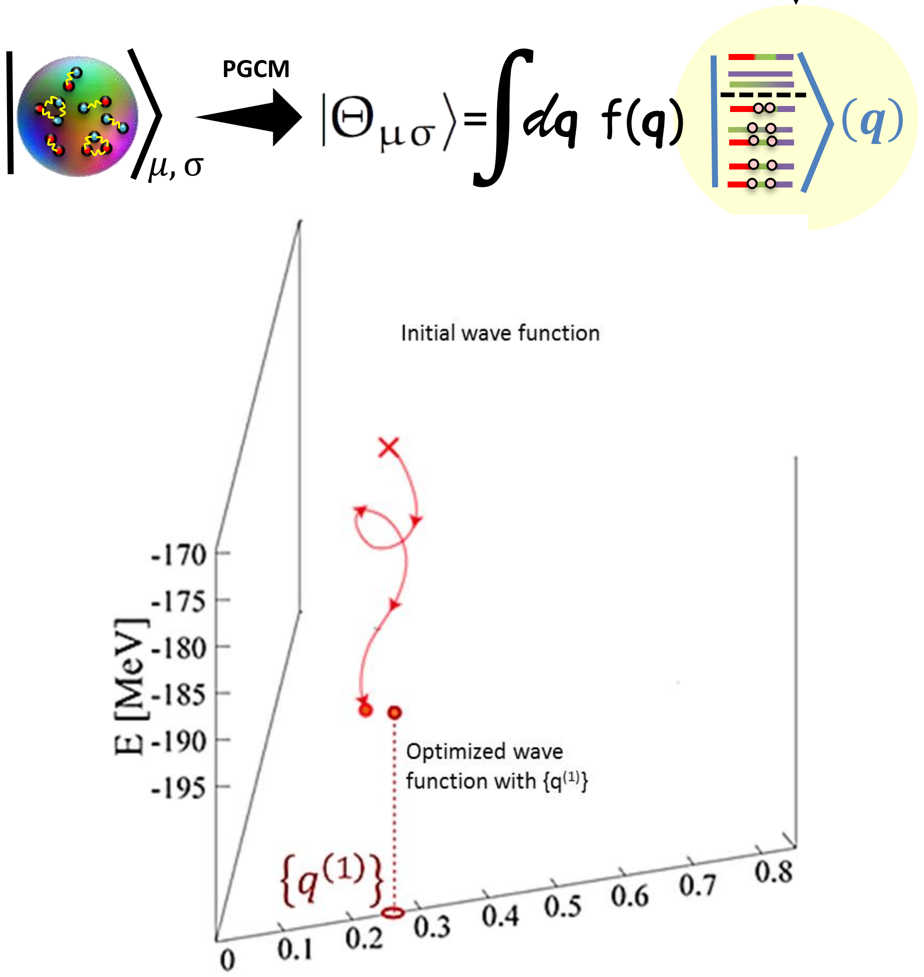


◉ Post-HFB treatment : PGCM

⇢ Symmetry-conserving (non orthogonal) mixture of symmetry-breaking HFB vacua



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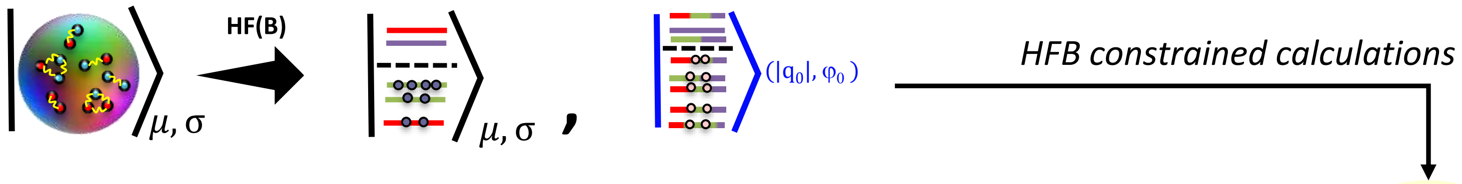

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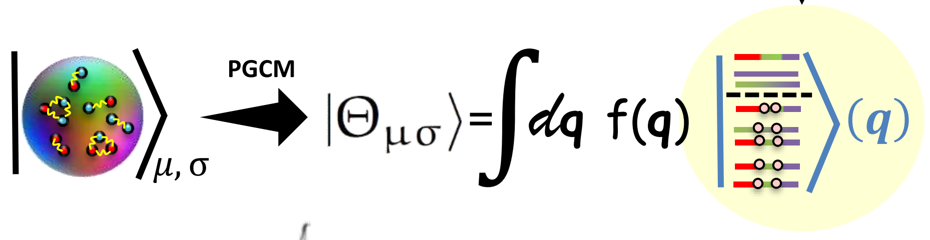

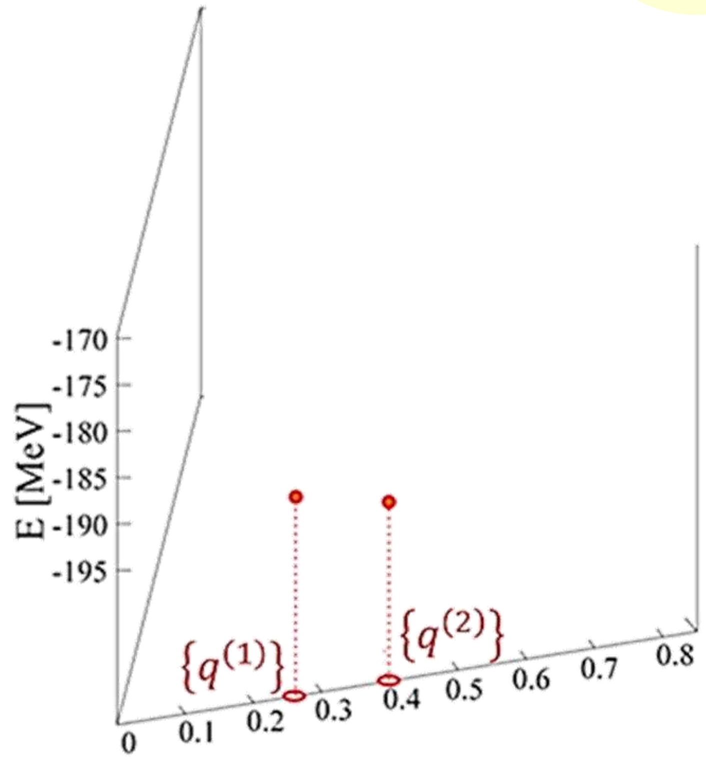

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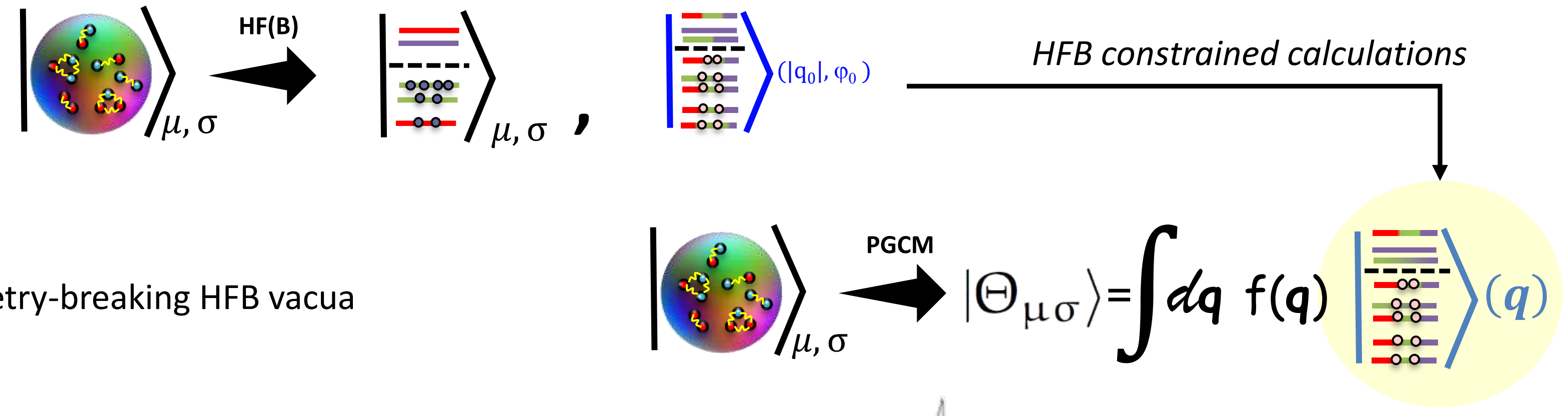

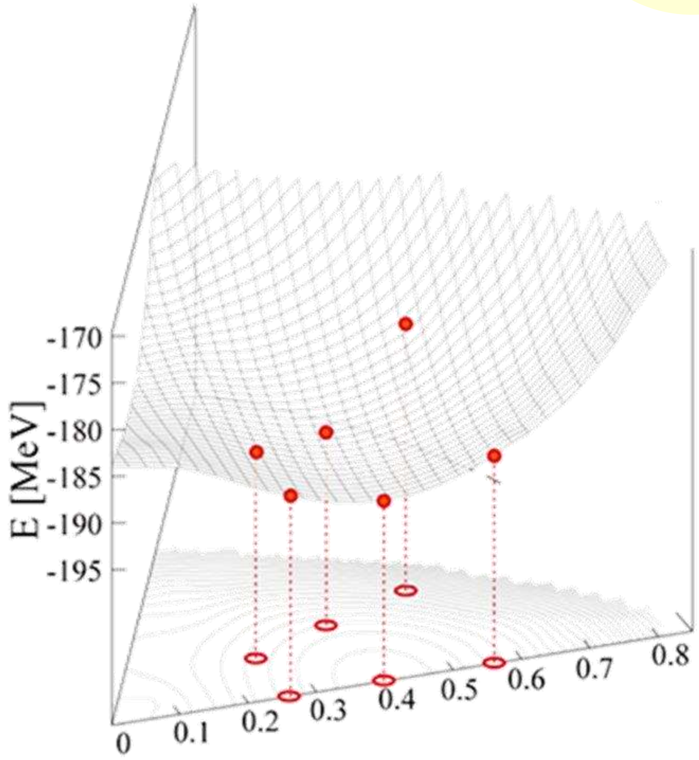

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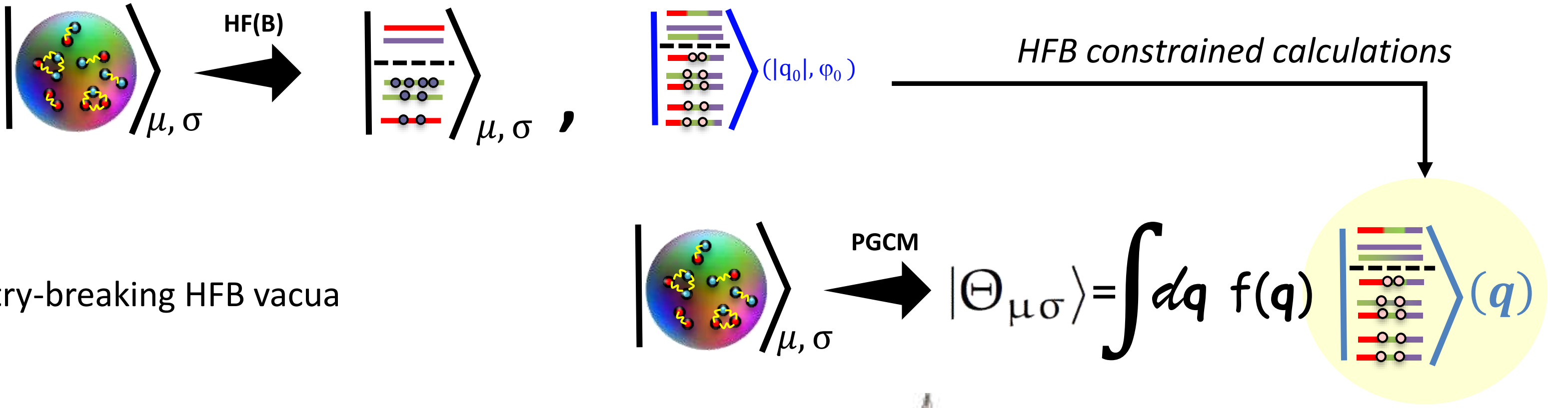

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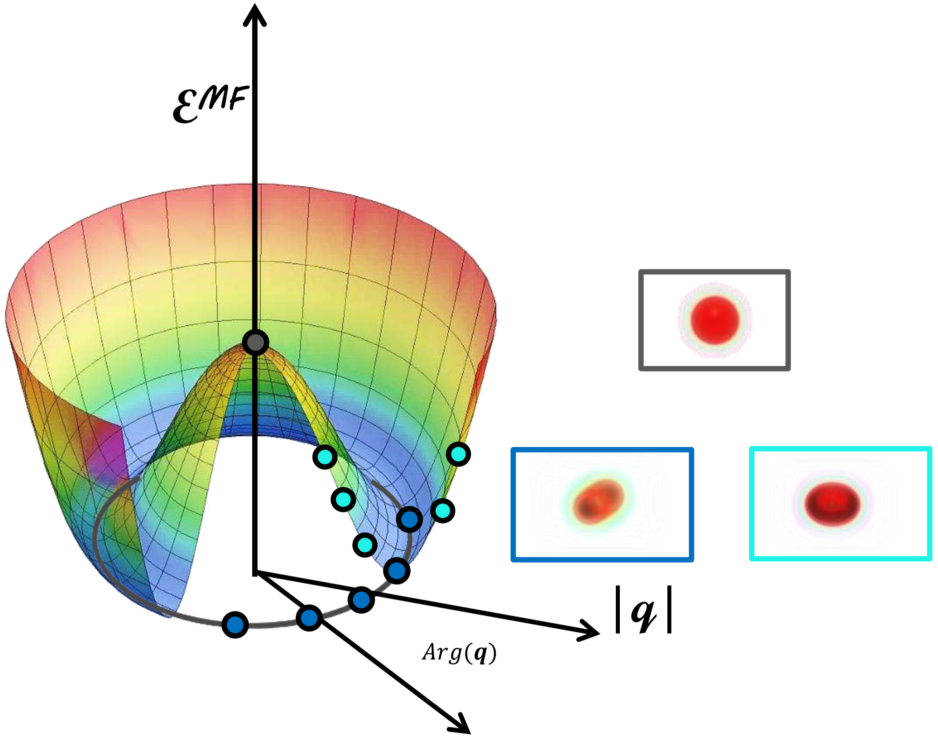

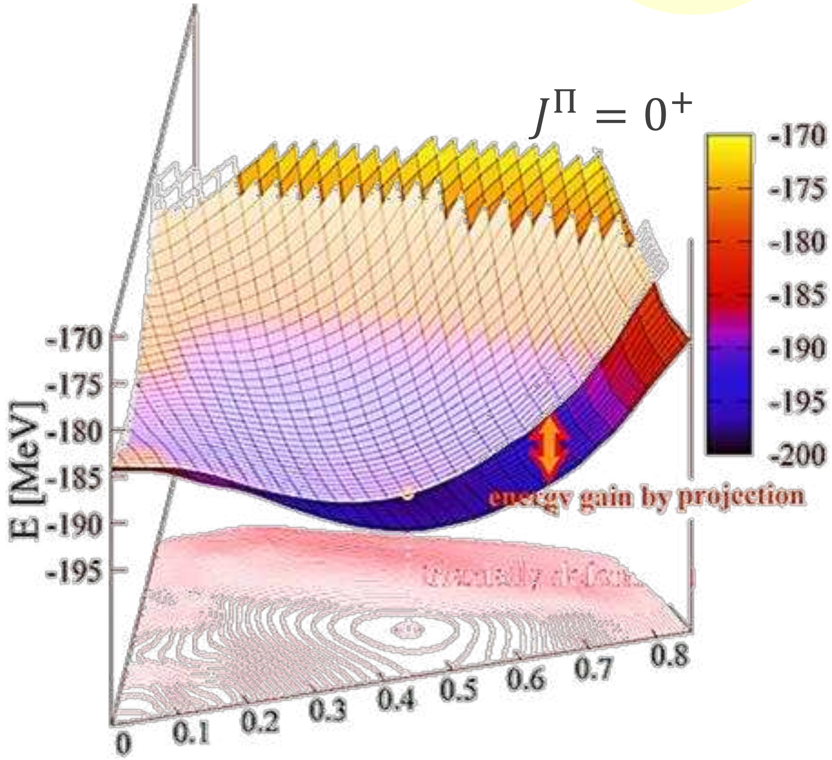

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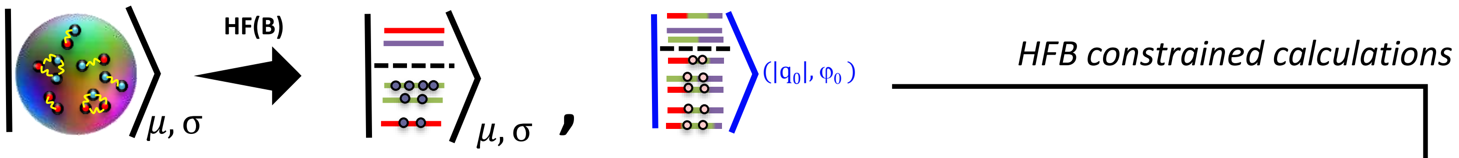

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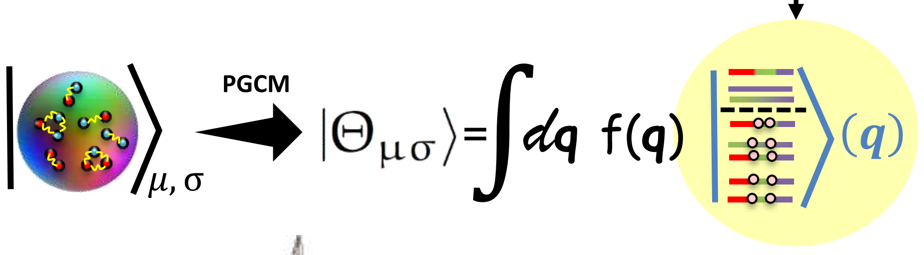

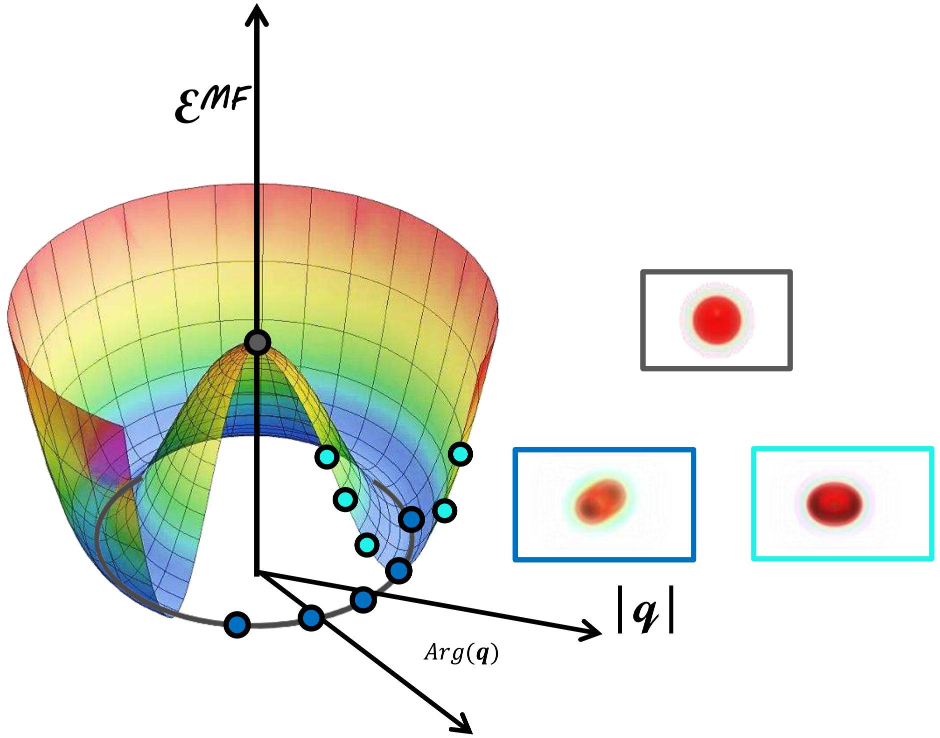

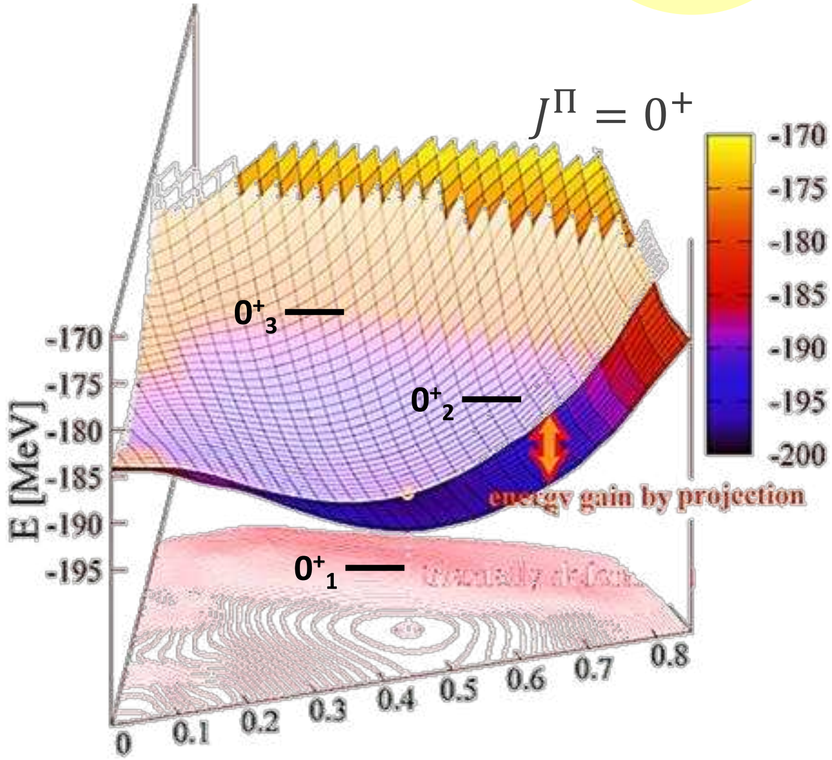

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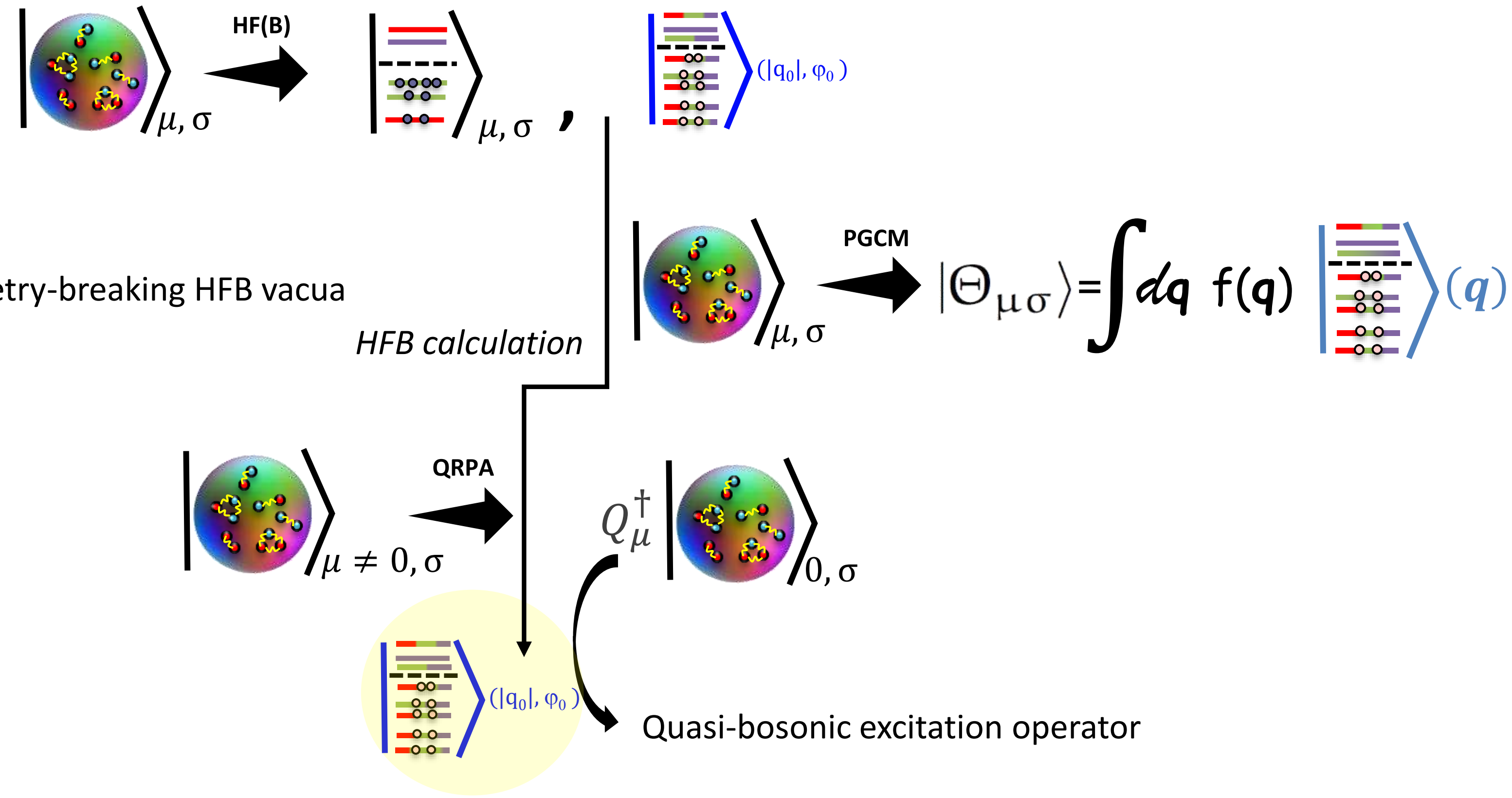
◉ Post-HFB : QRPA

--> Excitations = coherent mixture of 2-qp excitations

--> Harmonic limit of the GCM



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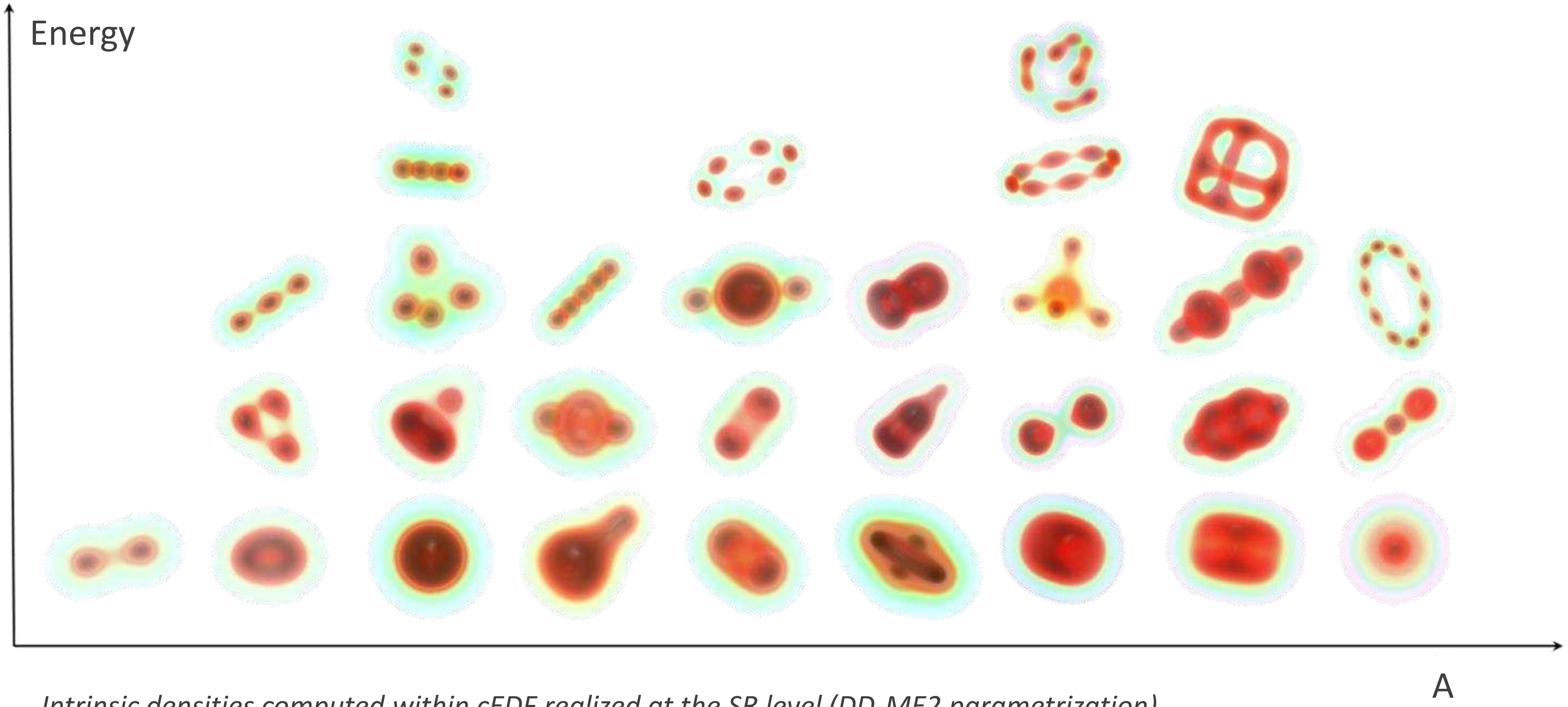
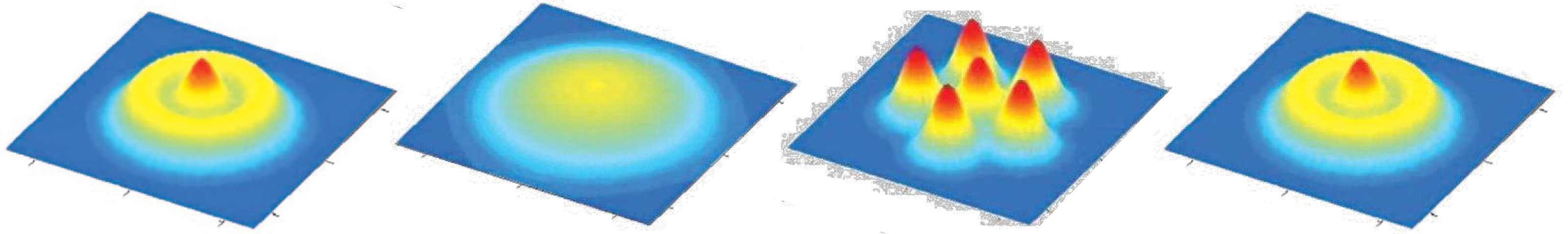
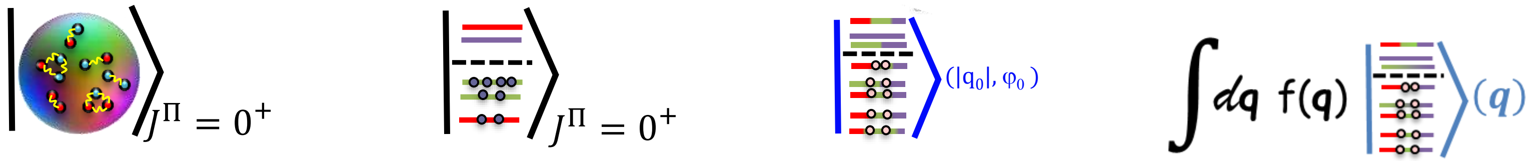


Intrinsic densities computed within cEDF realized at the SR level (DD-ME2 parametrization)

◉ Nuclear shapes : Take the case of a doubly open-shell system with strong angular correlation

Density profile



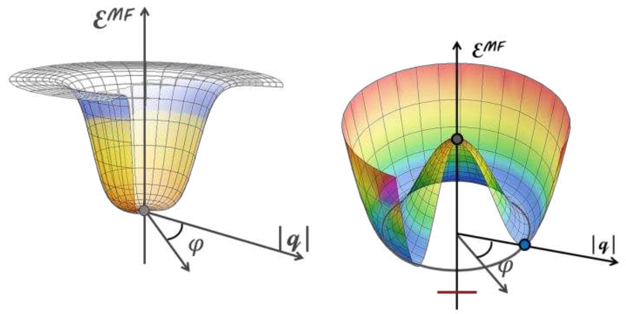


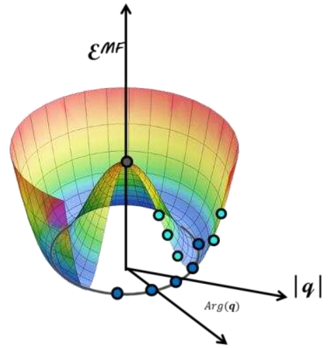
Exact WF

Approx : Symmetry-preserving HF WF

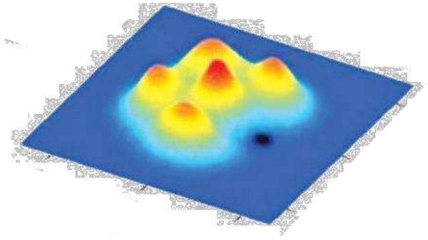
Approx : Symmetry-broken HFB WF

Approx : PGCM WF





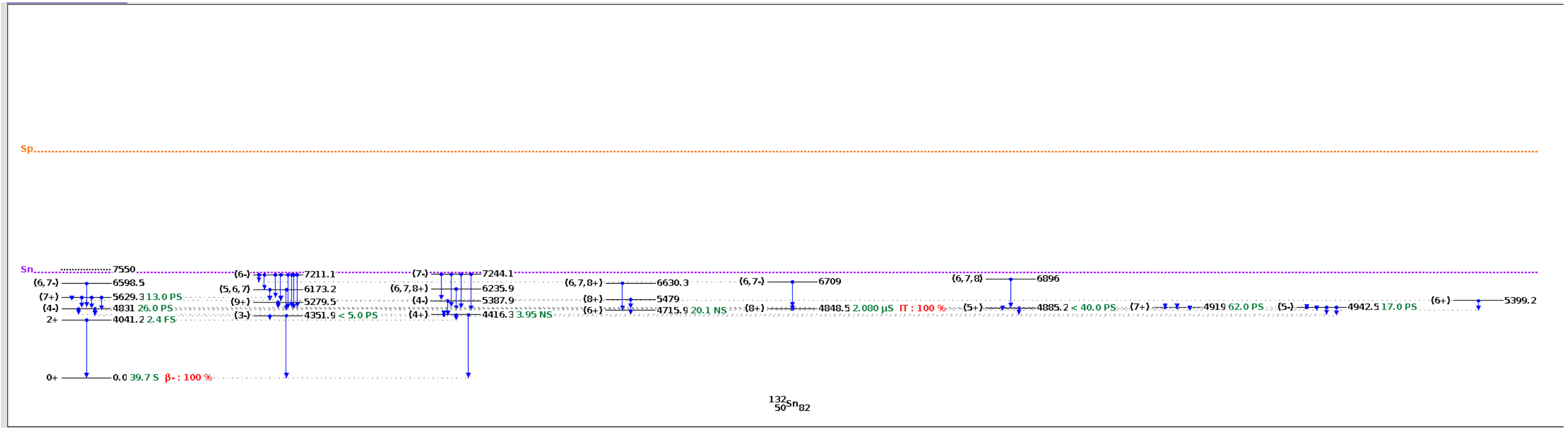
2-point correlation function

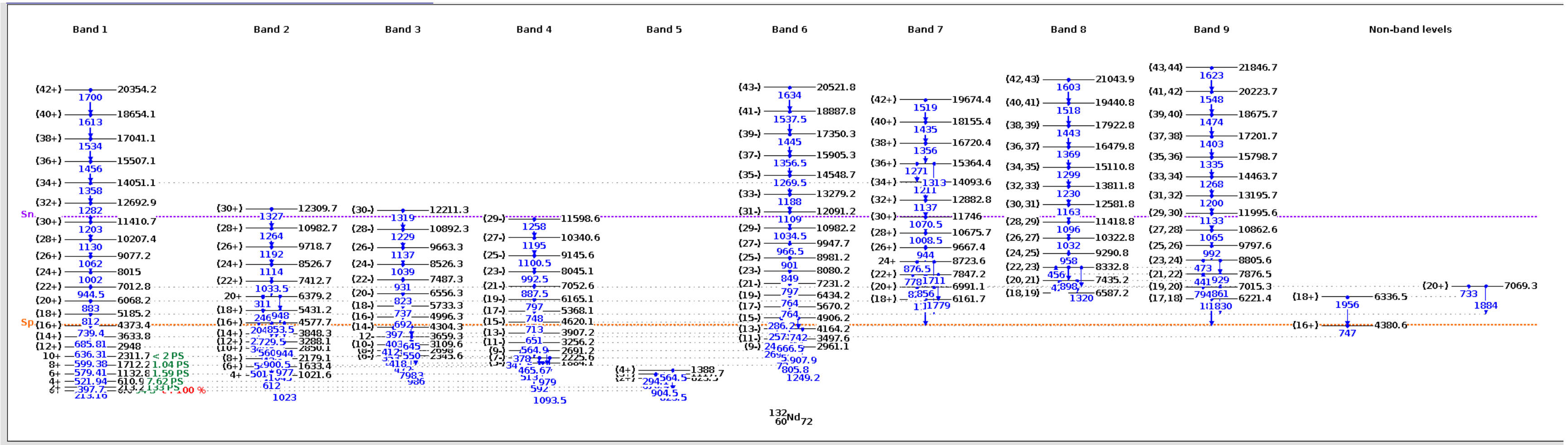


Spectroscopy

Yannouleas & Landman, 2017

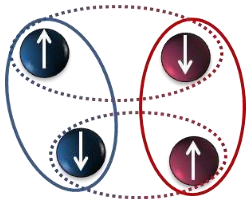
Nuclear clustering & PGCM





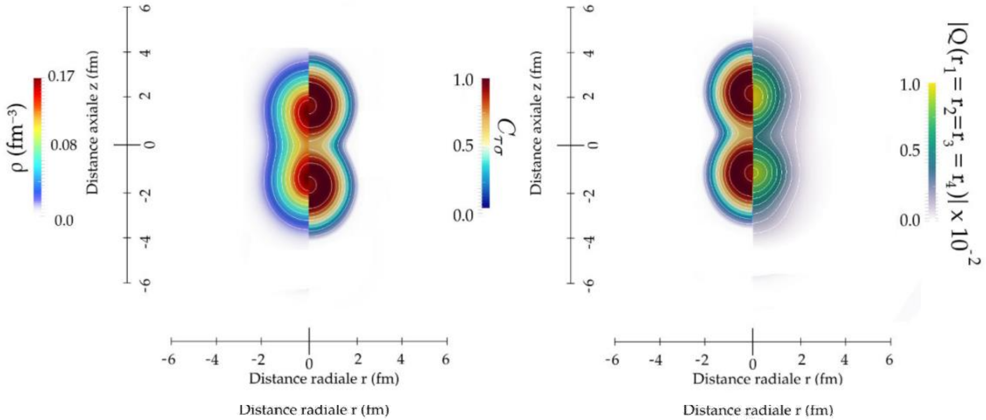
- How to account for correlations underpinning α-clustering ?

  i) Explicitly treat 4-nucleon correlations : RMF + QCM

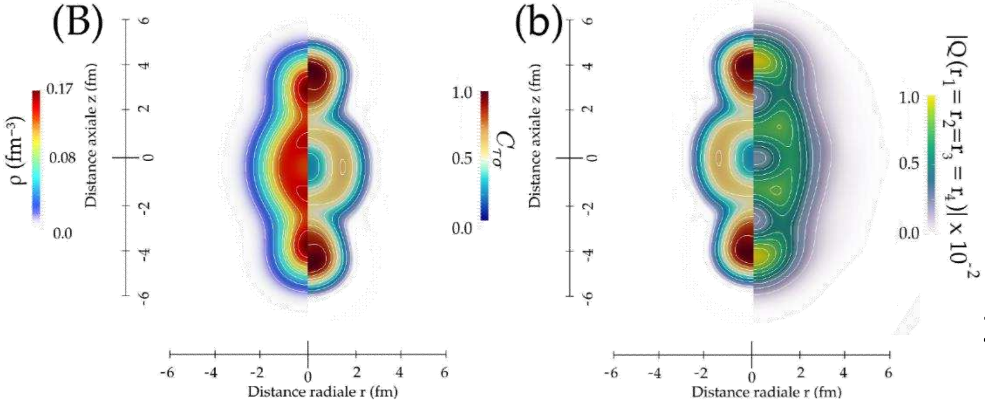


$$|\Psi\rangle = \left(\mathcal{Q}^\dagger\right)^{n_q}|0\rangle \qquad \mathcal{Q}^\dagger = 2\Gamma_1^\dagger\Gamma_{-1}^\dagger - \left(\Gamma_0^\dagger\right)^2 \qquad \Gamma_t^\dagger = \sum_k x_k P_{k,t}^\dagger$$

Lasseri, Ebran, Khan, Sandulescu

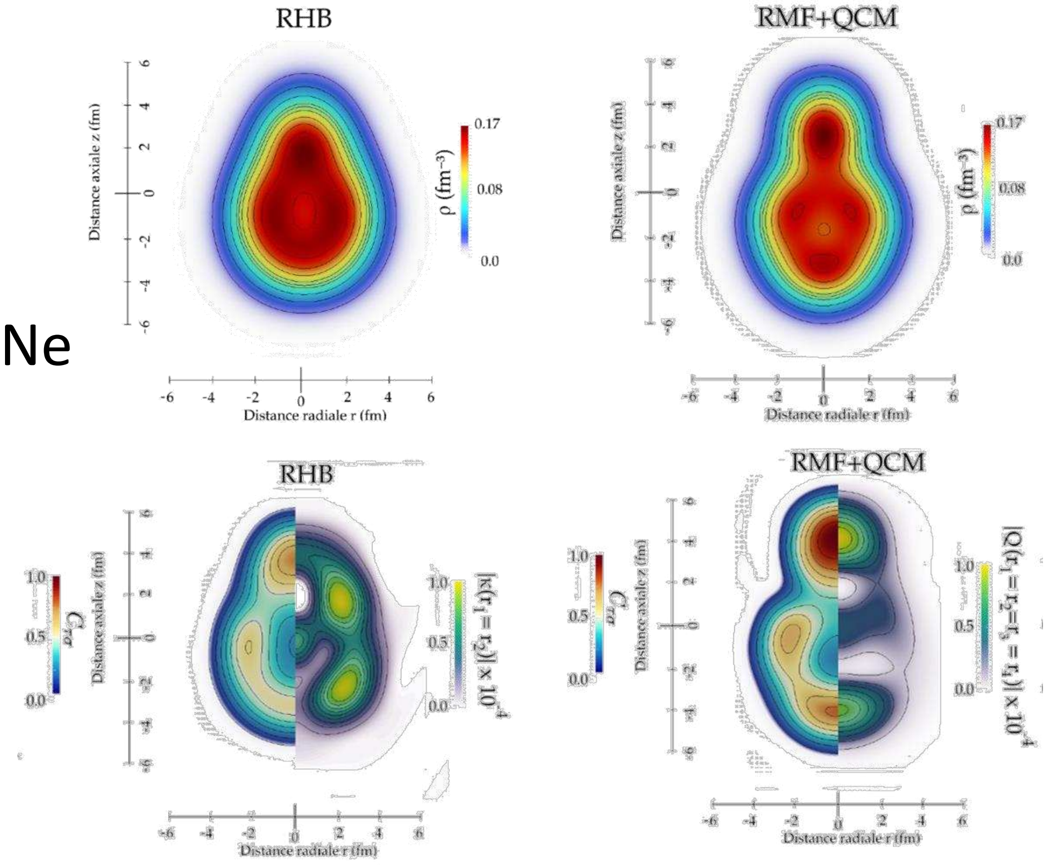


^{8}Be



^{24}Mg

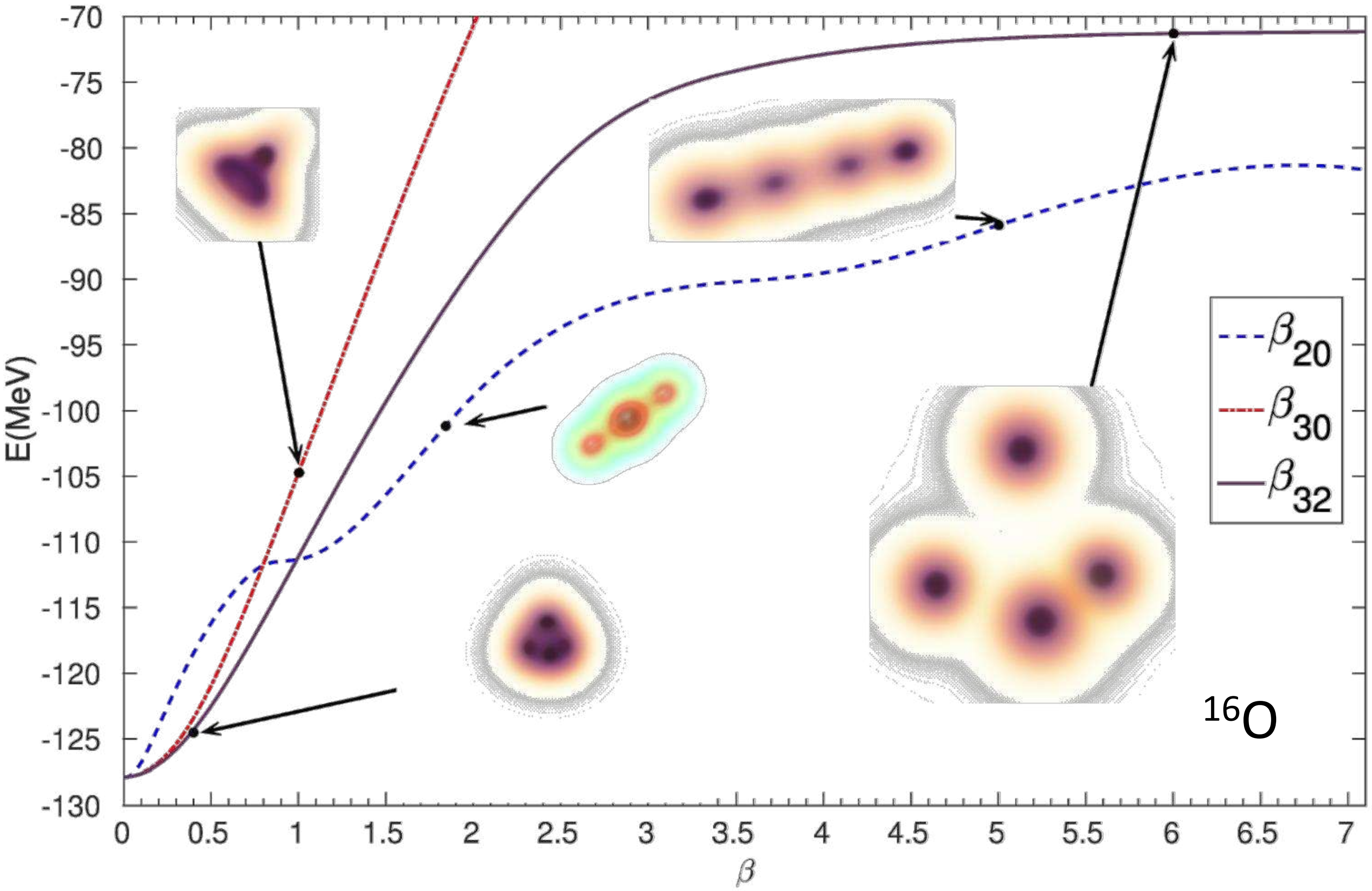
^{20}Ne



- How to account for correlations underpinning α-clustering ?

  i) Explicitly treat 4-nucleon correlations : RMF + QCM
  ii) Look for a collective field whose fluctuations cause nucleon to aggregate into α dofs

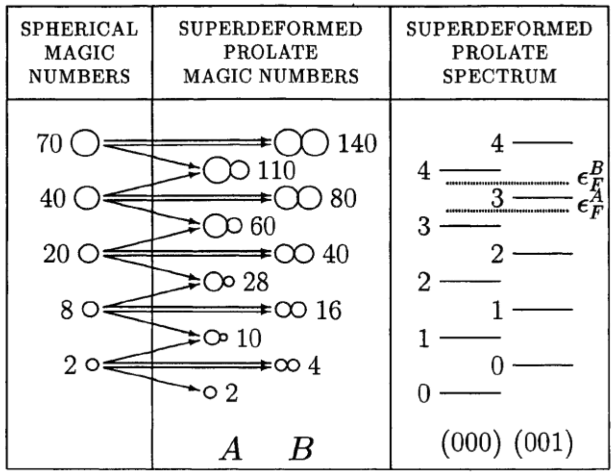
(Mott) transition from delocalized to totally localized nucleons takes the form of a transition from SO(3) (or continuous subgroup) to a discrete point-group

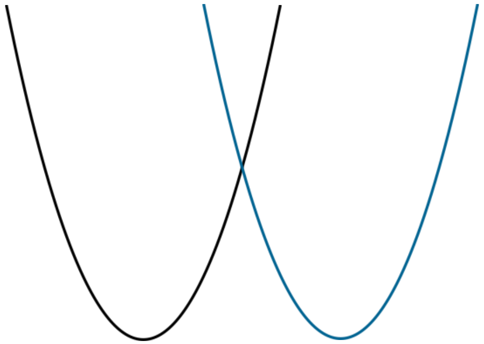


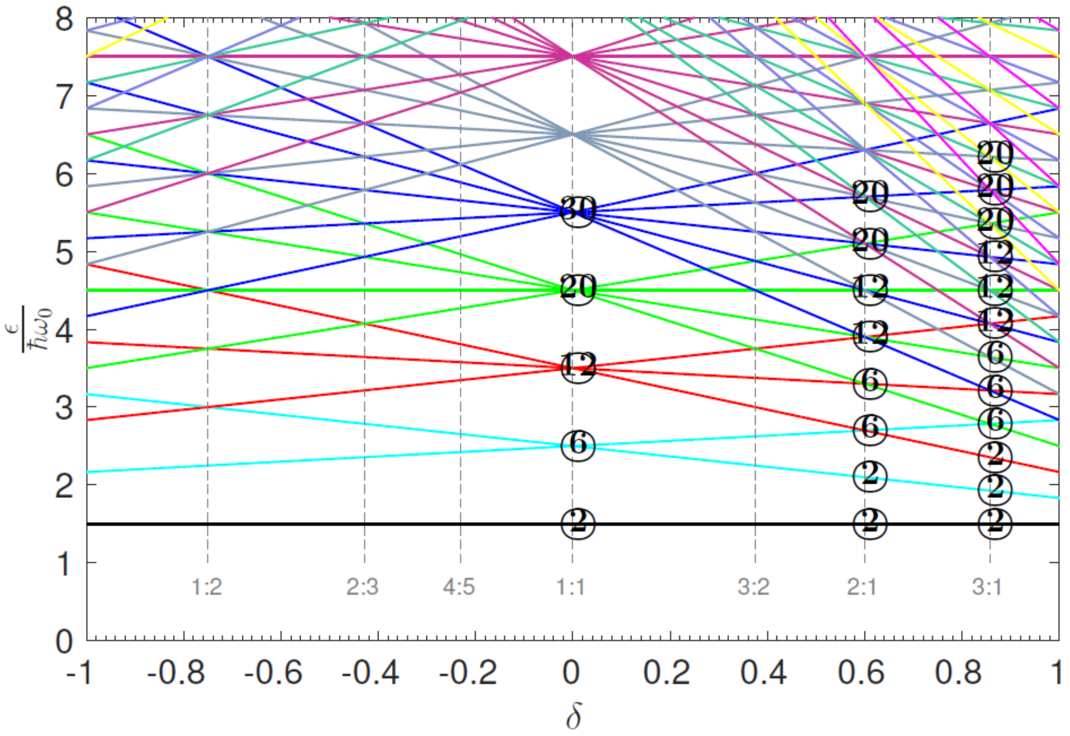
◉ Role of deformation

N-dimensional anisotropic HO with commensurate frequencies enjoys dynamical symmetries involving multiple independent copies of SU(N) irreps

Susceptibility of nucleons in deformed nuclei to arrange into multiple spherical fragments







Nazarewicz & *Dobaczewski*, PRL 1992

Deformation = necessary condition, but not a sufficient one

- Strength of correlations measured by dimensionless ratios

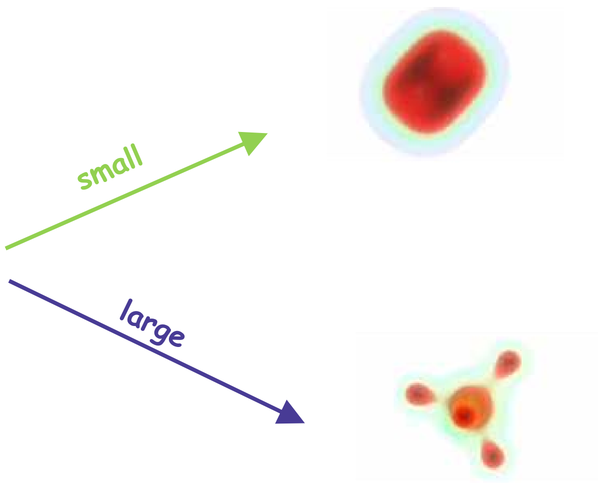
$$\sqrt{\Lambda} \equiv \sqrt{\frac{\langle V\rangle}{\langle T\rangle}} = \sqrt{2}\left(\frac{3}{4\pi}\right)^{\frac{1}{6}} (2MU)^{\frac{1}{4}} (An)^{-\frac{1}{6}} \sim \alpha_{loc}$$

Nucleon mass

Number of nucleons

Depth of the confining potential

Mean density



Ebran, Khan, Niksic & Vretenar Nature 2012
Ebran, Khan, Niksic & Vretenar PRC 2013

- Strength of correlations measured by dimensionless ratios

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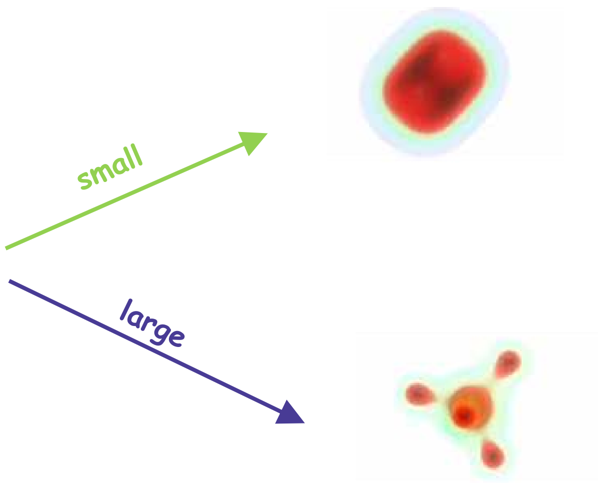
Nucleon mass (M)

Number of nucleons (A)

Depth of the confining potential (U)

Mean density (n)

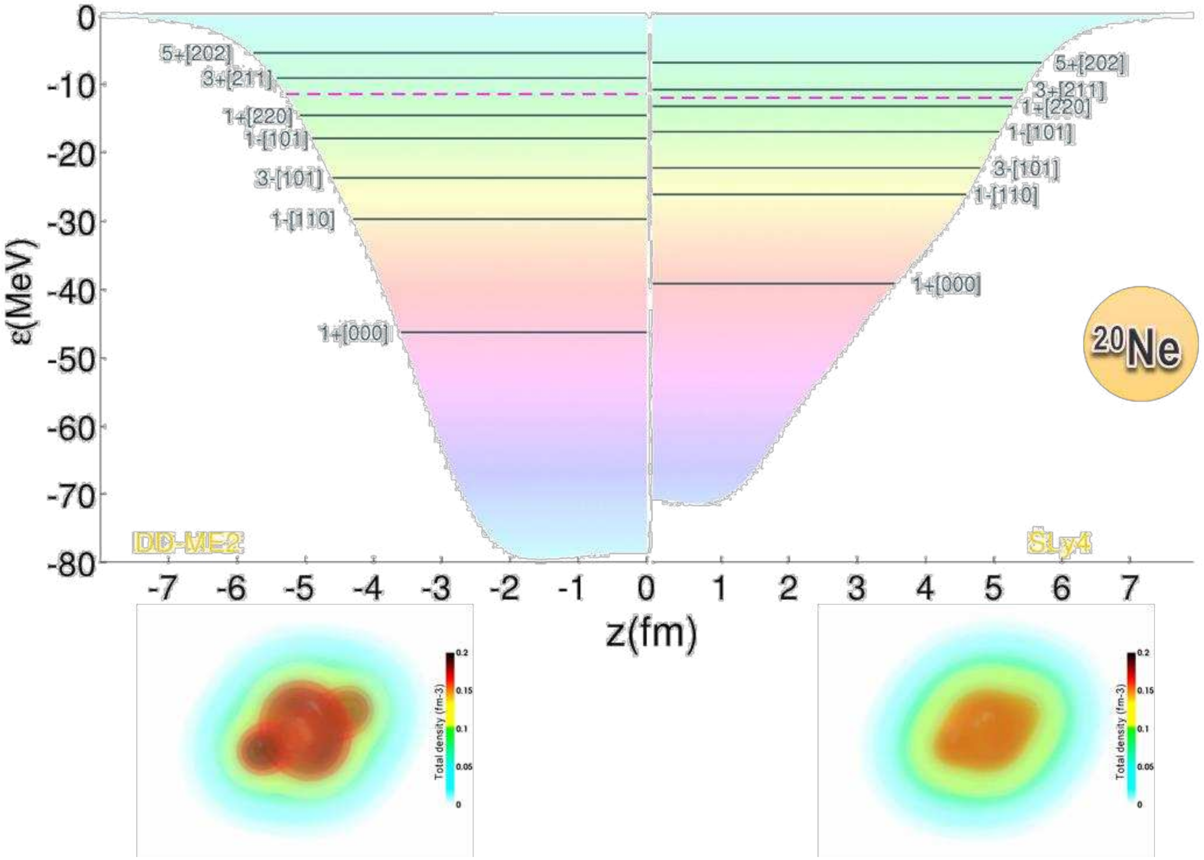


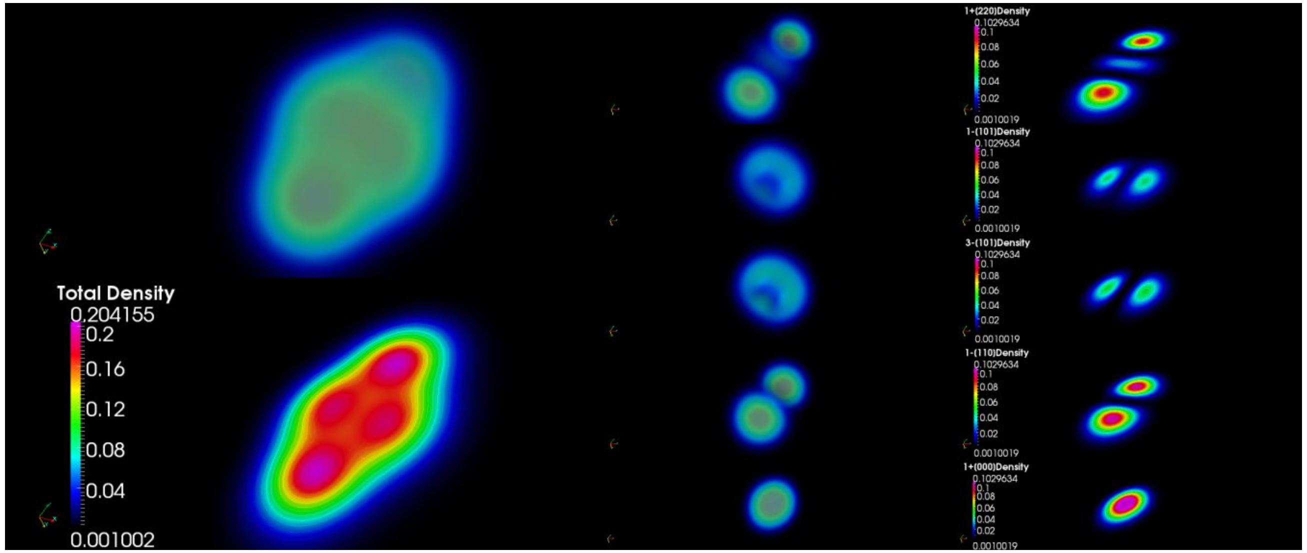
Clustering favored
- ⇢ For deep confining potential
- ⇢ For light nuclei
- ⇢ In regions at low-density

Ebran, Khan, Niksic & Vretenar Nature 2012
Ebran, Khan, Niksic & Vretenar PRC 2013

Effect of the depth of the confining potential

◉ Deeper potential yielding the same nuclear radii ⇒ more localized single-nucleon orbitals





◉ When Coulomb effects are not too important and owing to Kramers degeneracy, proton ↑, proton ↓, neutron ↑, neutron ↓ share the same spatial properties

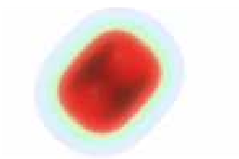
Strength of correlations

◉ Strength of correlations measured by dimensionless ratios

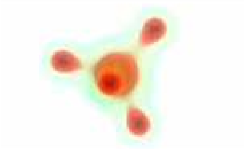
$$\sqrt{\Lambda} \equiv \sqrt{\frac{\langle V\rangle}{\langle T\rangle}} = \sqrt{2}\left(\frac{3}{4\pi}\right)^{\frac{1}{6}} (2MU)^{\frac{1}{4}} (An)^{-\frac{1}{6}} \sim \alpha_{loc}$$

Nucleon mass (M) · Number of nucleons (A) · Depth of the confining potential (U) · Mean density (n)

small →



large →



Clustering favored
⇢ For deep confining potential
⇢ For light nuclei
⇢ In regions at low-density

◉ Formation/dissolution of clusters : Mott parameter

$$\frac{R_X}{d^X_{Mott}} \sim 1 \Rightarrow n^X_{Mott} \sim \frac{\rho_{sat}}{A_X}$$

R_X: Size of the nucleus X

d^X_{Mott}: inter-nucleon average distance

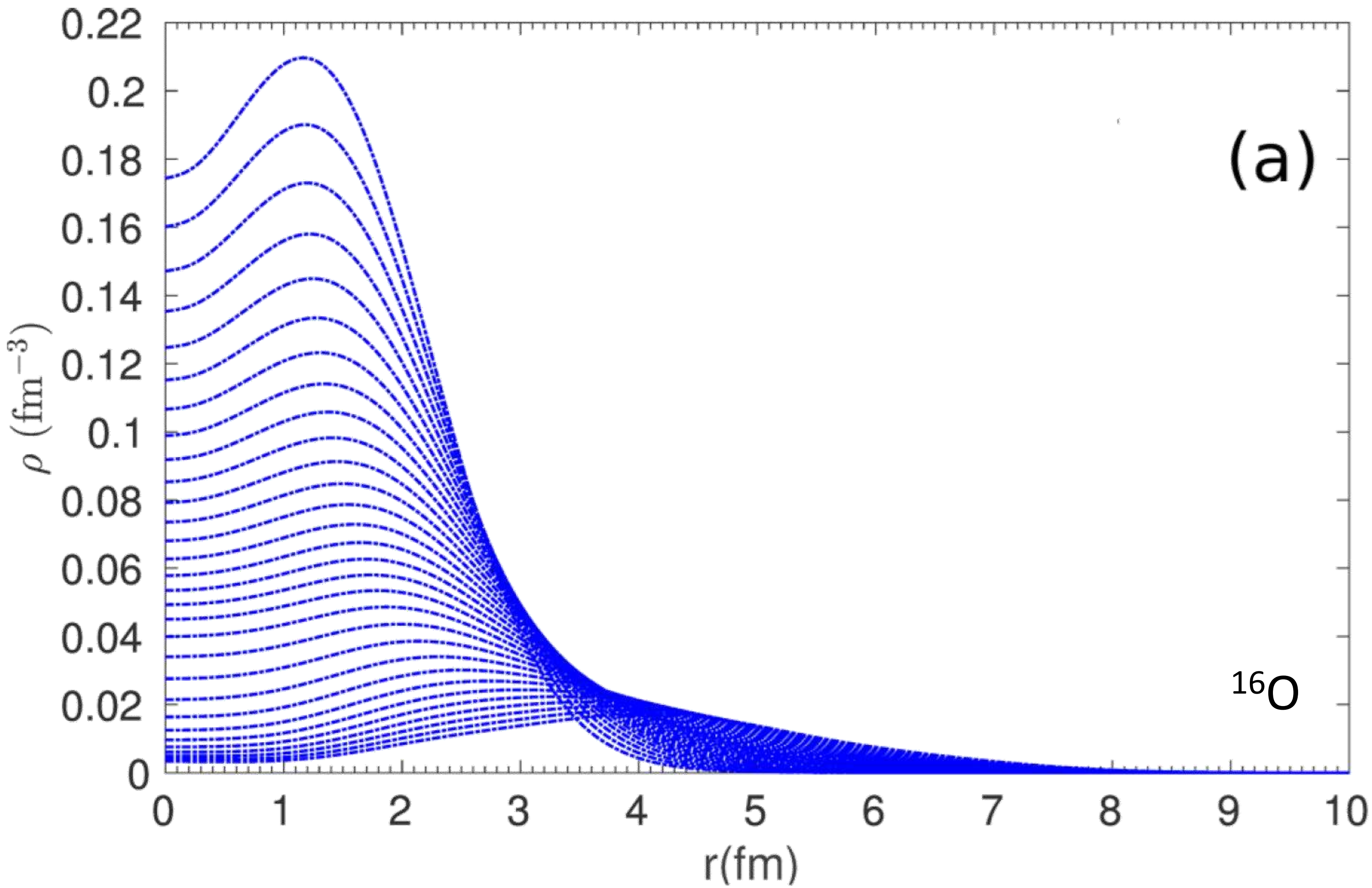
$n^{\alpha}_{Mott} \sim 0.25\rho_{sat}$ — Size of an α in free-space

$\sim \frac{\rho_{sat}}{3}$ — 0.9 size of an α in free-space

Ebran, Girod, Khan, Lasseri, Schuck, PRC 2020
Ebran, Khan, Niksic, Vretenar, PRC 2014

Effect of the density

- Isotropically inflate ^{16}O by constraining its r.m.s. radius while imposing a global quadrupole moment to be zero



Effect of the density

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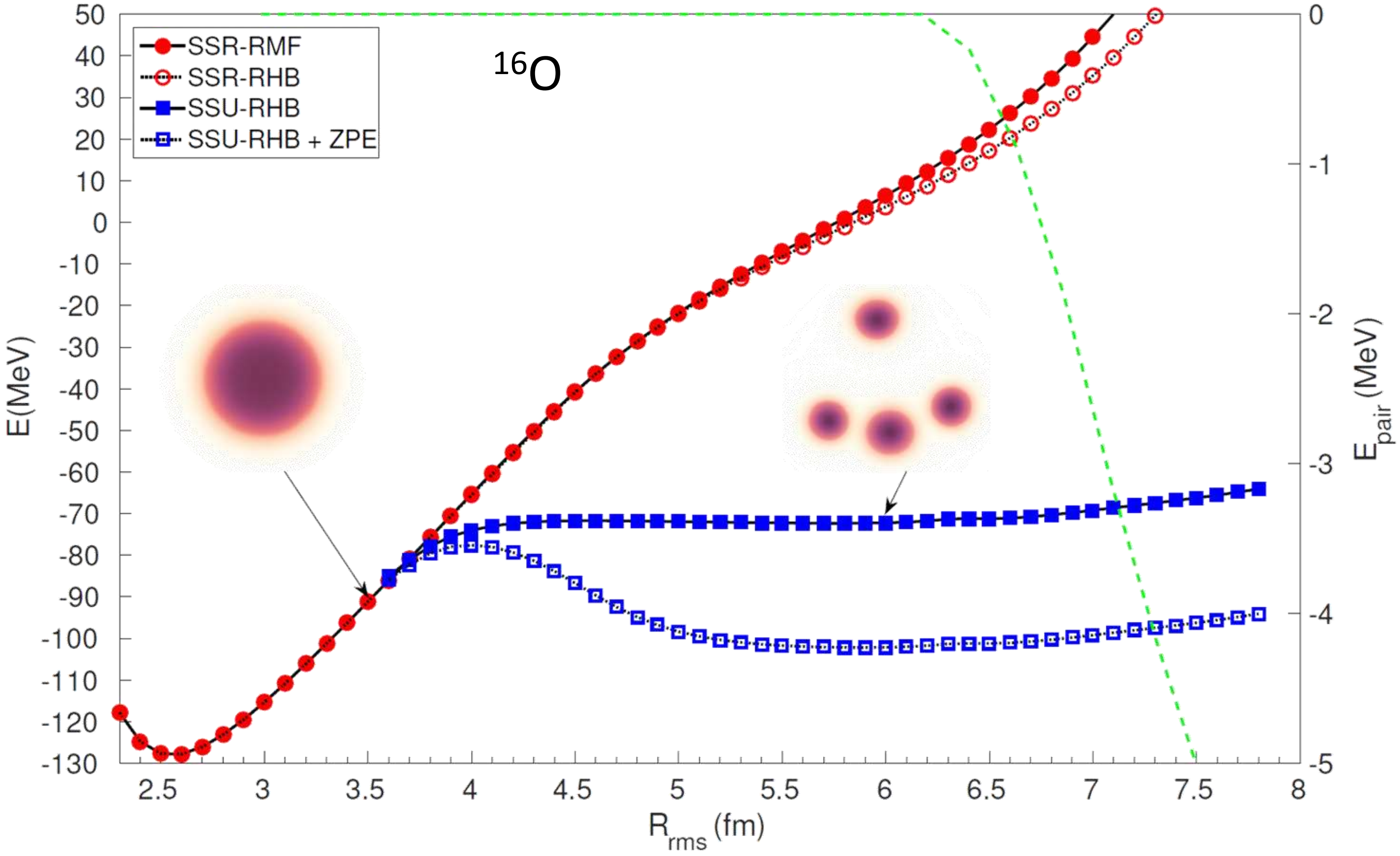

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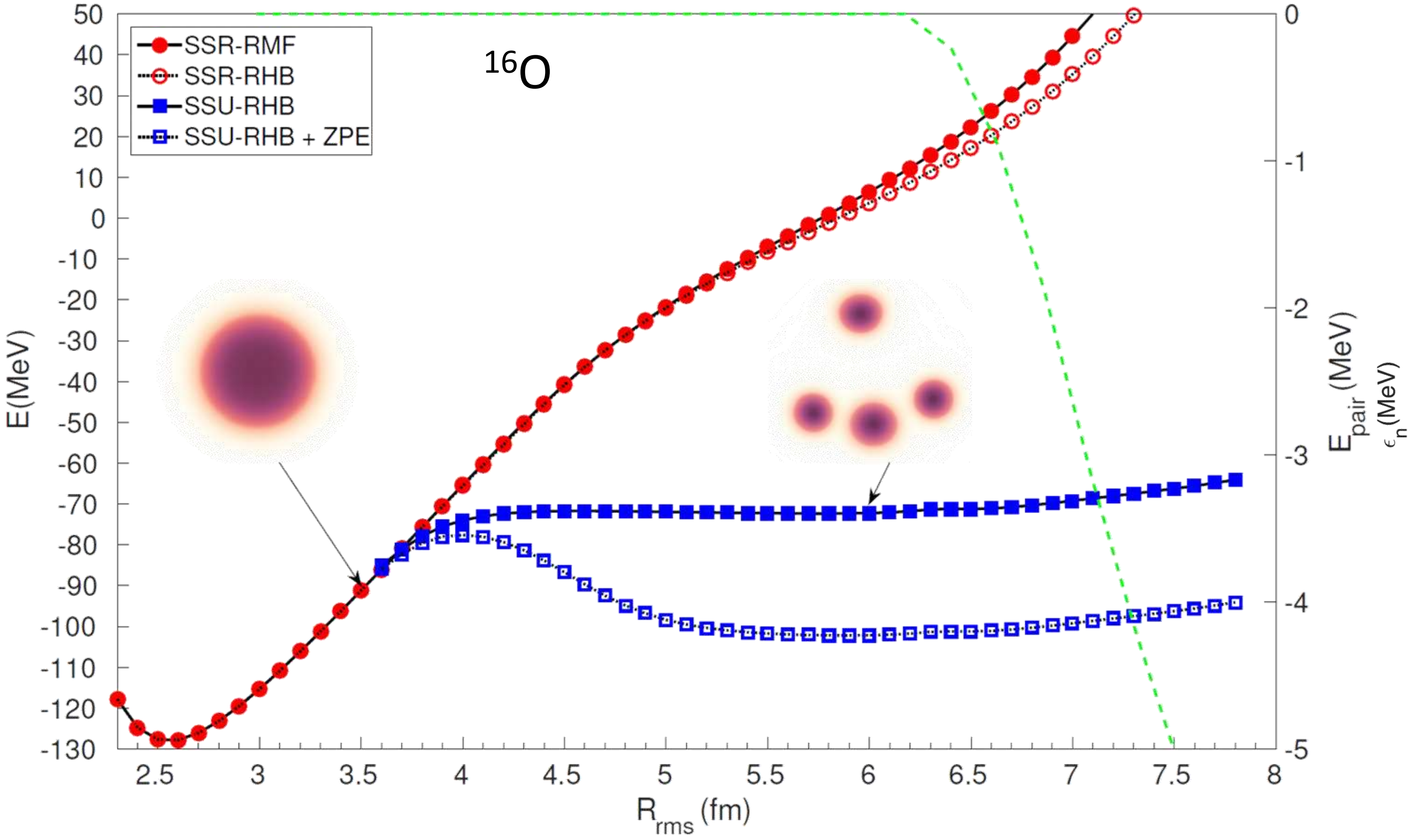

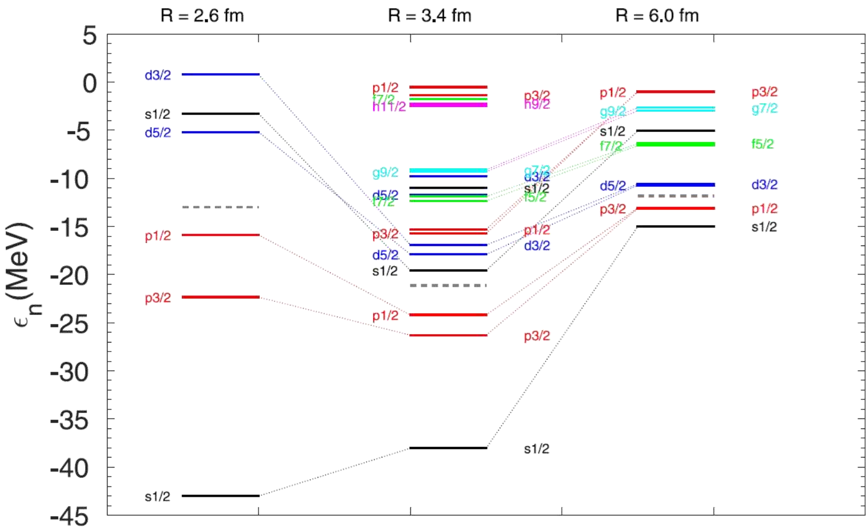

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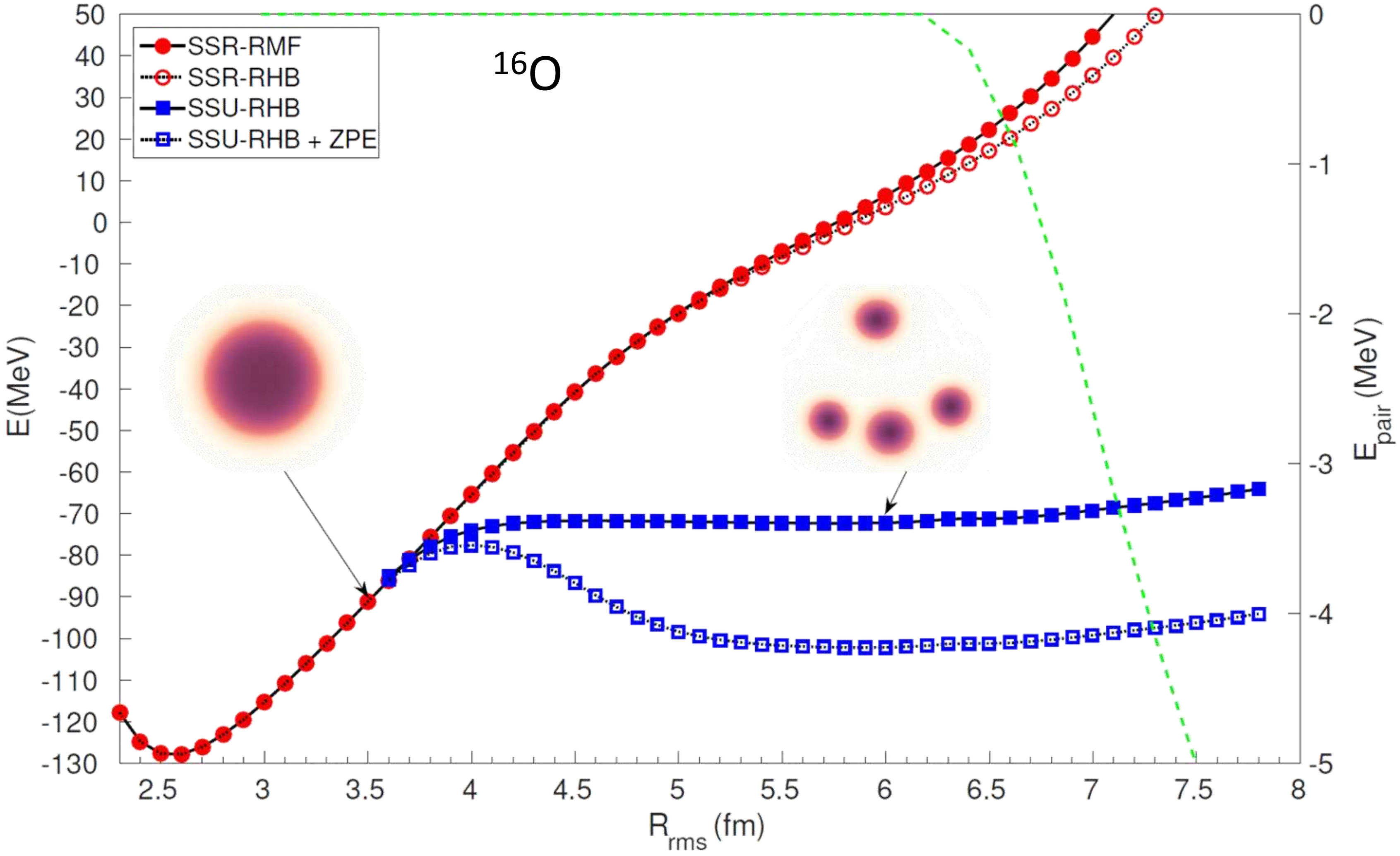
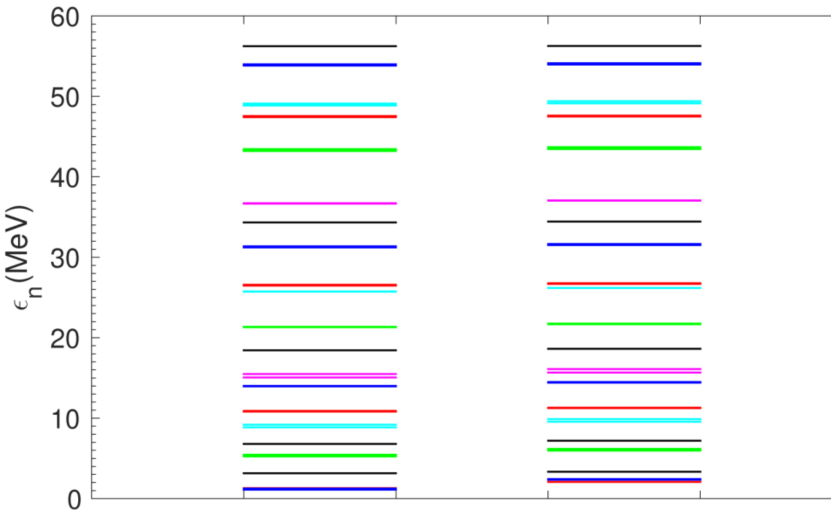
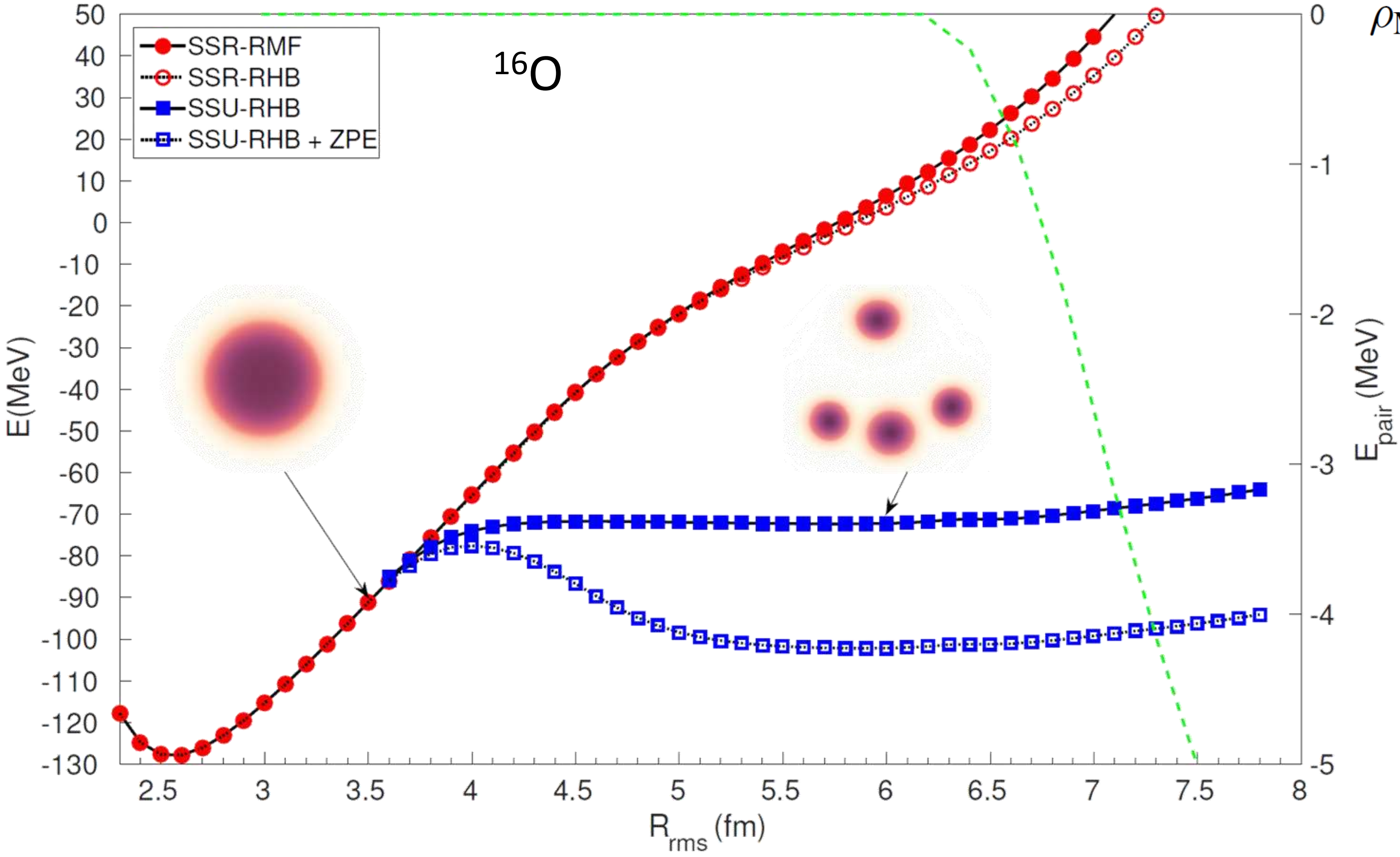


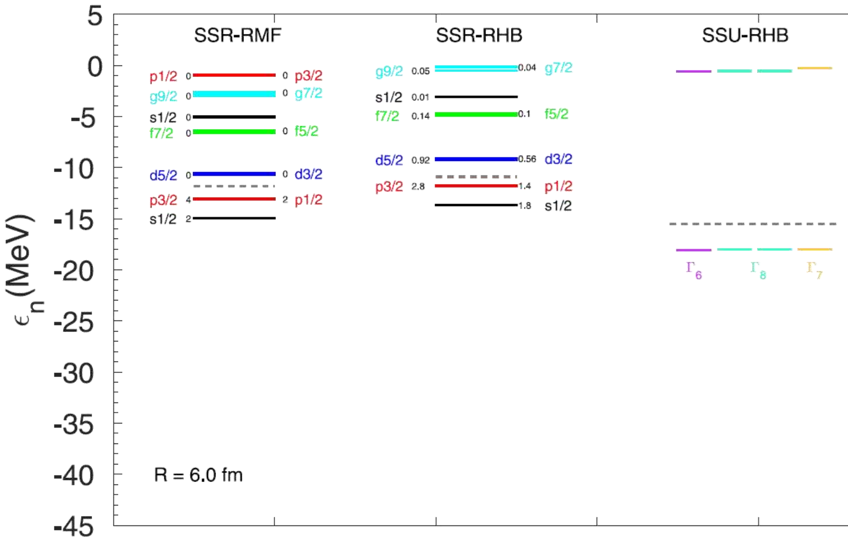


Effect of the density

◉ Isotropically inflate ^{16}O by constraining its r.m.s. radius while imposing a global quadrupole moment to be zero

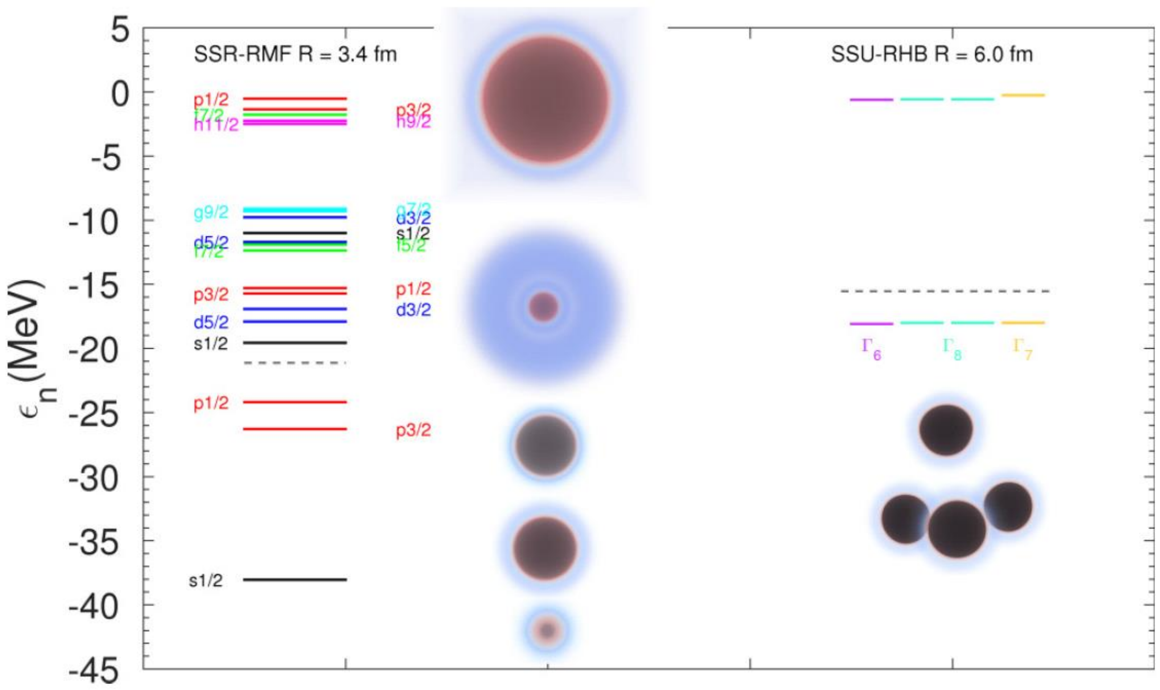


$$\rho_{\rm Mott}/\rho_0 = (R_{\rm eq}/R_c)^3 \approx \rho_0/3$$

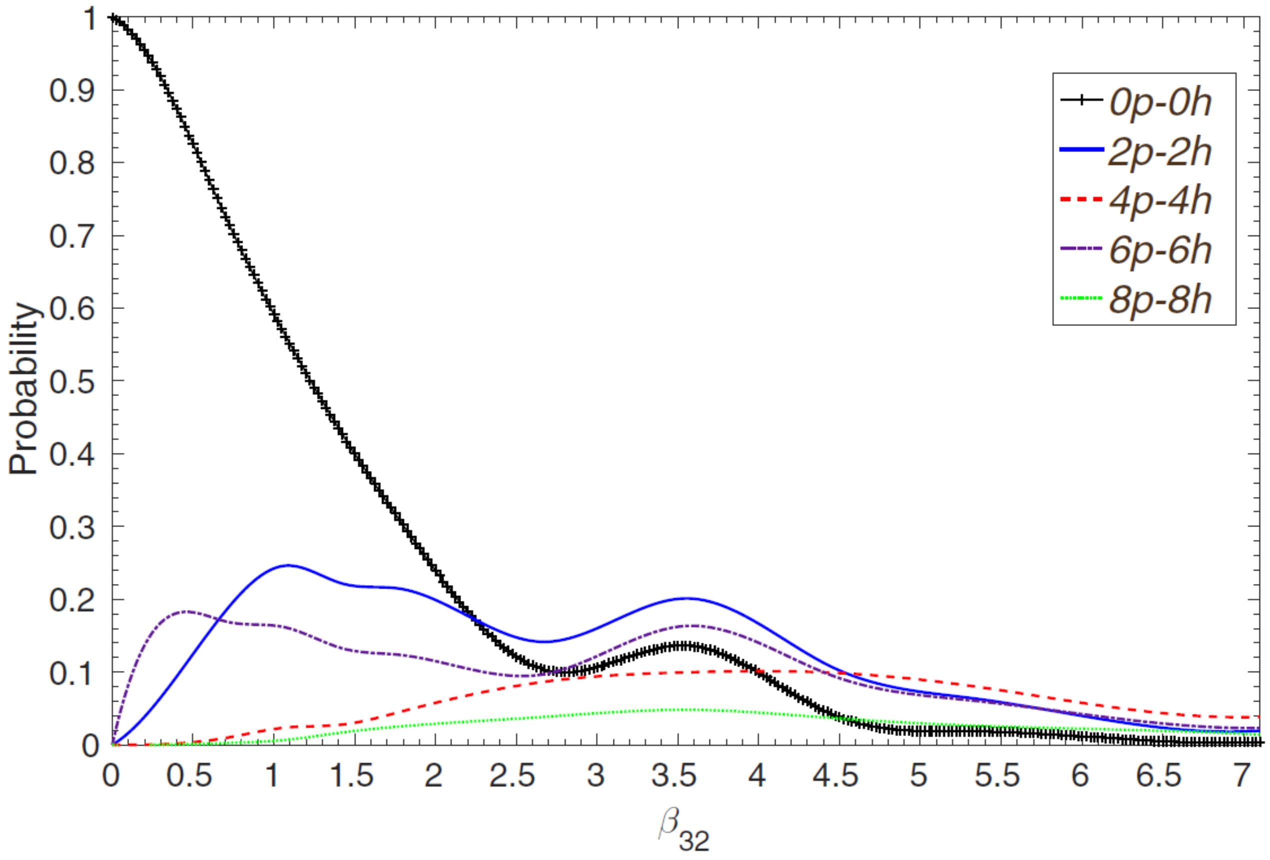


Effect of the density

- mp-mh content of a tetrahedrally-deformed Slater determinant



Symmetry breaking : O(3) → Td



- Borrowing the LCAO-MO language, on can think of the 16O thetrahedrally-deformed SD as a MO built from 4 1s α AOs

$$\psi_i = \sum_{j=1}^{4} f_j^i \phi_j$$

- Find the unknowns f in the Hückel approximation : $\mathcal{N}_{ij} = 0 \forall i,j$

$\epsilon \equiv \mathcal{H}_{ii}$; $-\mu \equiv \mathcal{H}_{ij}$ for adjacent i,j ; $\mathcal{H}_{ij} = 0$ otherwise

$$\begin{pmatrix} \epsilon & -\mu & -\mu & -\mu \\ -\mu & \epsilon & -\mu & -\mu \\ -\mu & -\mu & \epsilon & -\mu \\ -\mu & -\mu & -\mu & \epsilon \end{pmatrix} \begin{pmatrix} f_1^i \\ f_2^i \\ f_3^i \\ f_4^i \end{pmatrix} = E_i \begin{pmatrix} f_1^i \\ f_2^i \\ f_3^i \\ f_4^i \end{pmatrix}$$

$$\psi_1 = \frac{1}{2}(\phi_1 + \phi_2 + \phi_3 + \phi_4) \qquad E_1 = \epsilon - 3\mu$$

$$\psi_2 = \frac{1}{\sqrt{2}}(-\phi_1 + \phi_2) \qquad E_2 = \epsilon + \mu$$

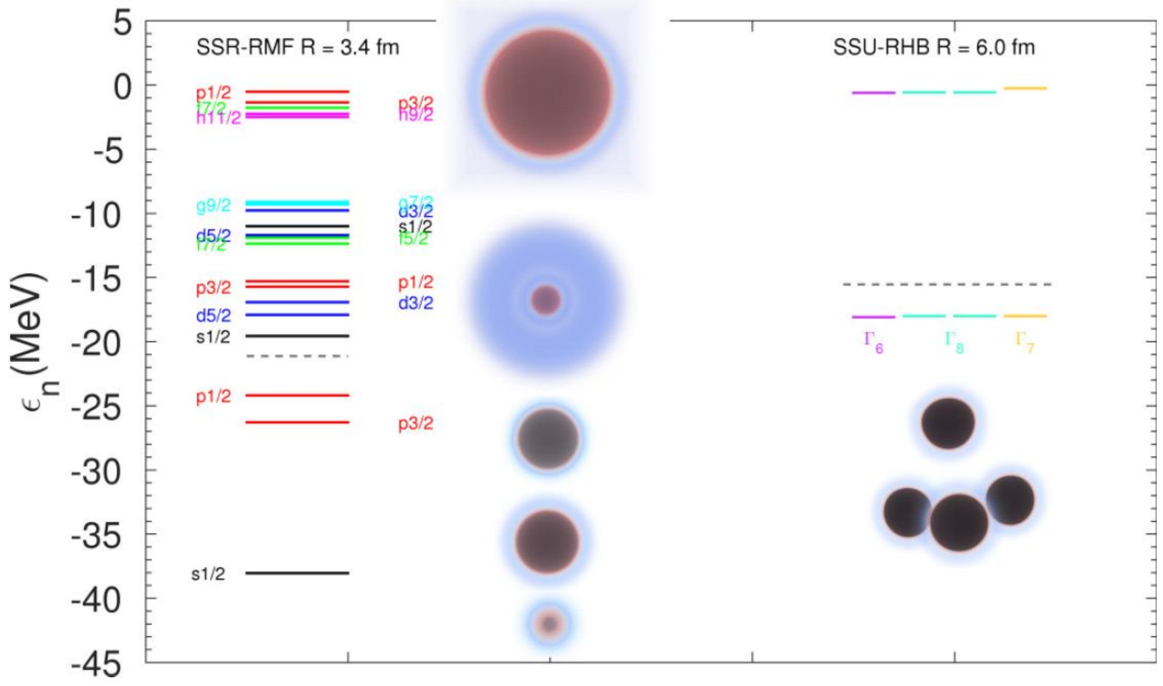
$$\psi_3 = \frac{1}{\sqrt{2}}(-\phi_1 + \phi_3) \qquad E_3 = E_2$$

$$\psi_4 = \frac{1}{\sqrt{2}}(-\phi_1 + \phi_4) \qquad E_4 = E_3 = E_2$$

$$\psi'_2 = \frac{1}{2}(\phi_1 - \phi_2 - \phi_3 + \phi_4),$$

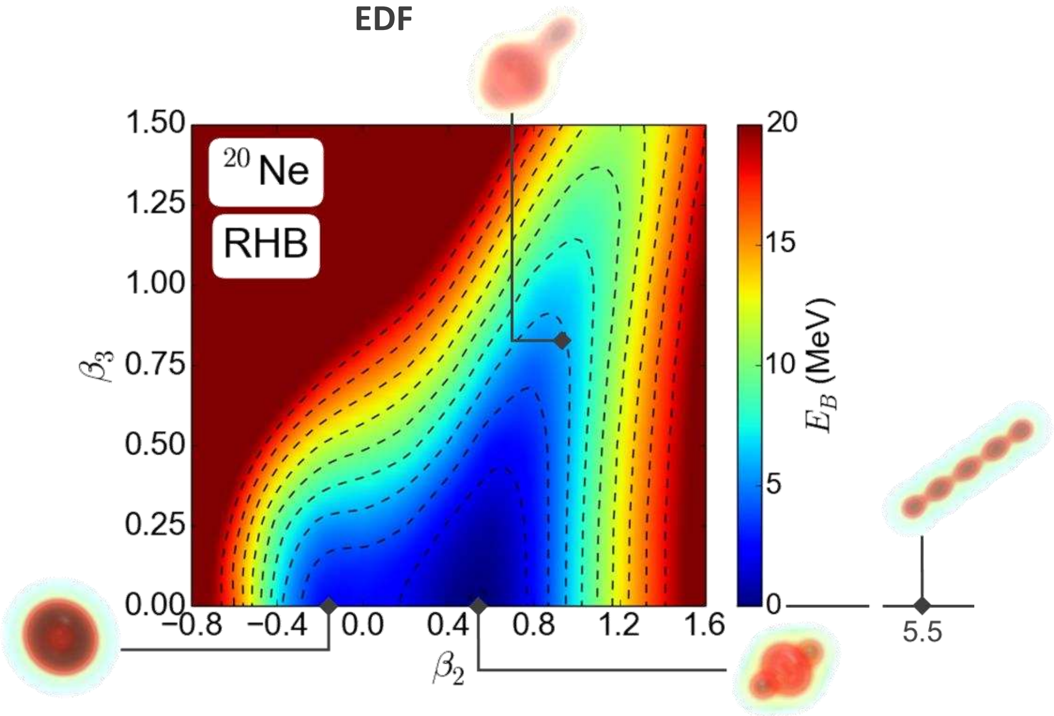
$$\psi'_3 = \frac{1}{2}(\phi_1 + \phi_2 - \phi_3 - \phi_4),$$

$$\psi'_4 = \frac{1}{2}(-\phi_1 + \phi_2 - \phi_3 + \phi_4).$$



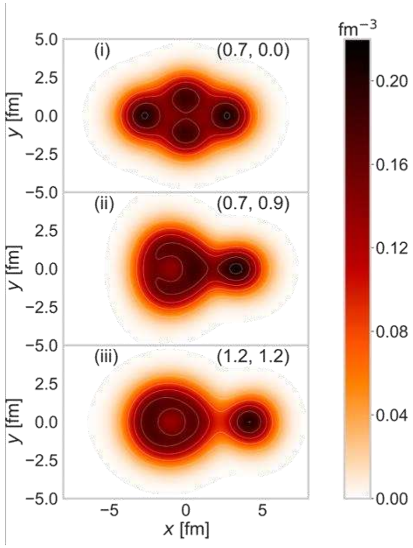
◉ Correlated GS

EDF

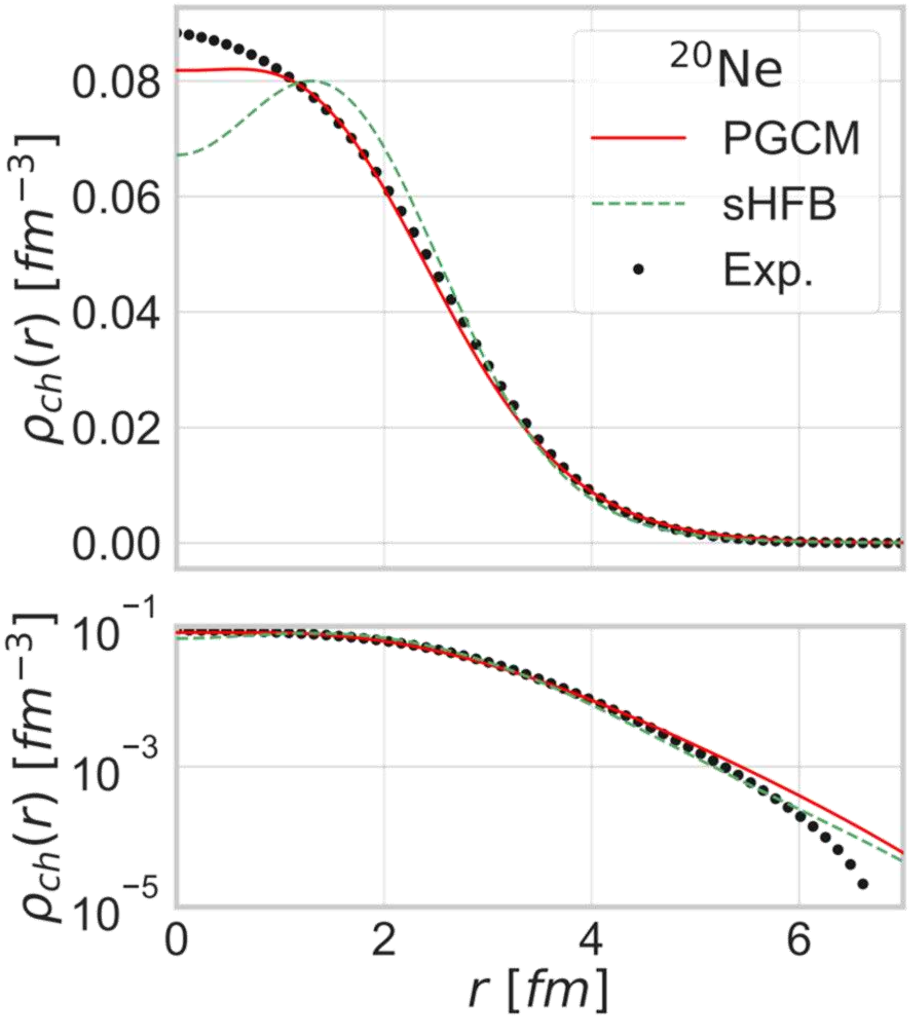


Marevic, Ebran, Khan, Niksic, Vretenar, PRC 97 (2018)

Ab initio



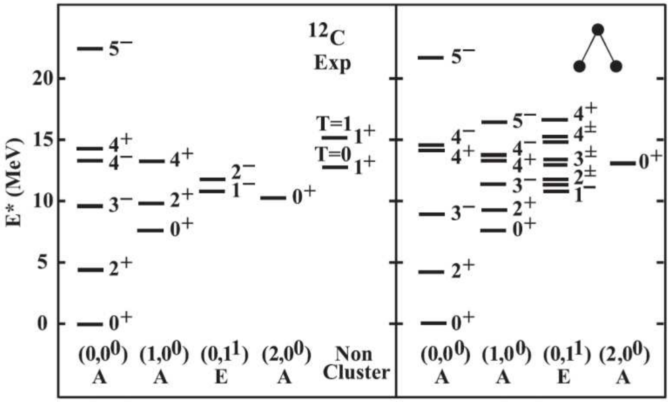
Frosini, Duguet, Ebran, Somà

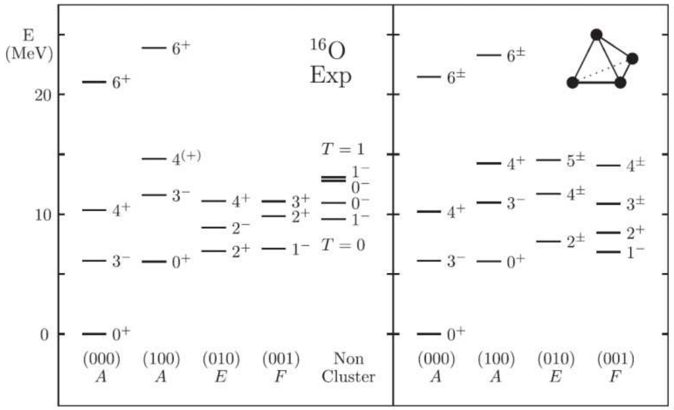


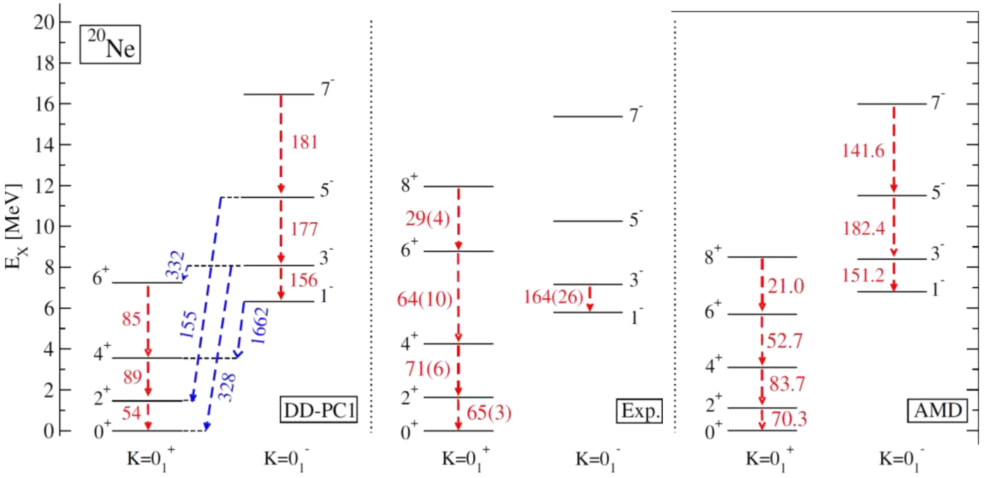
Nuclear clustering & PGCM

◉ Spectroscopy

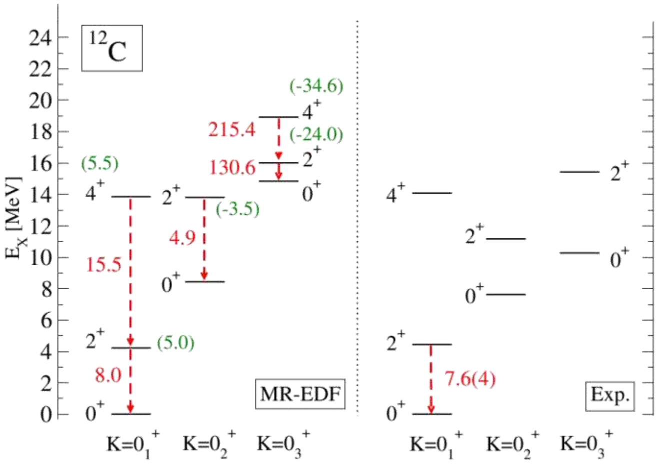
Bijker (2016)

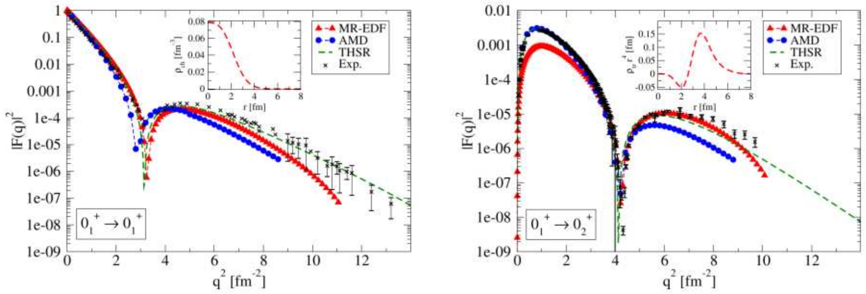




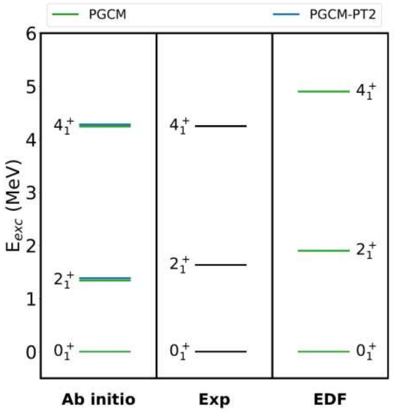


Marevic, Ebran, Khan, Niksic, Vretenar, PRC 2018



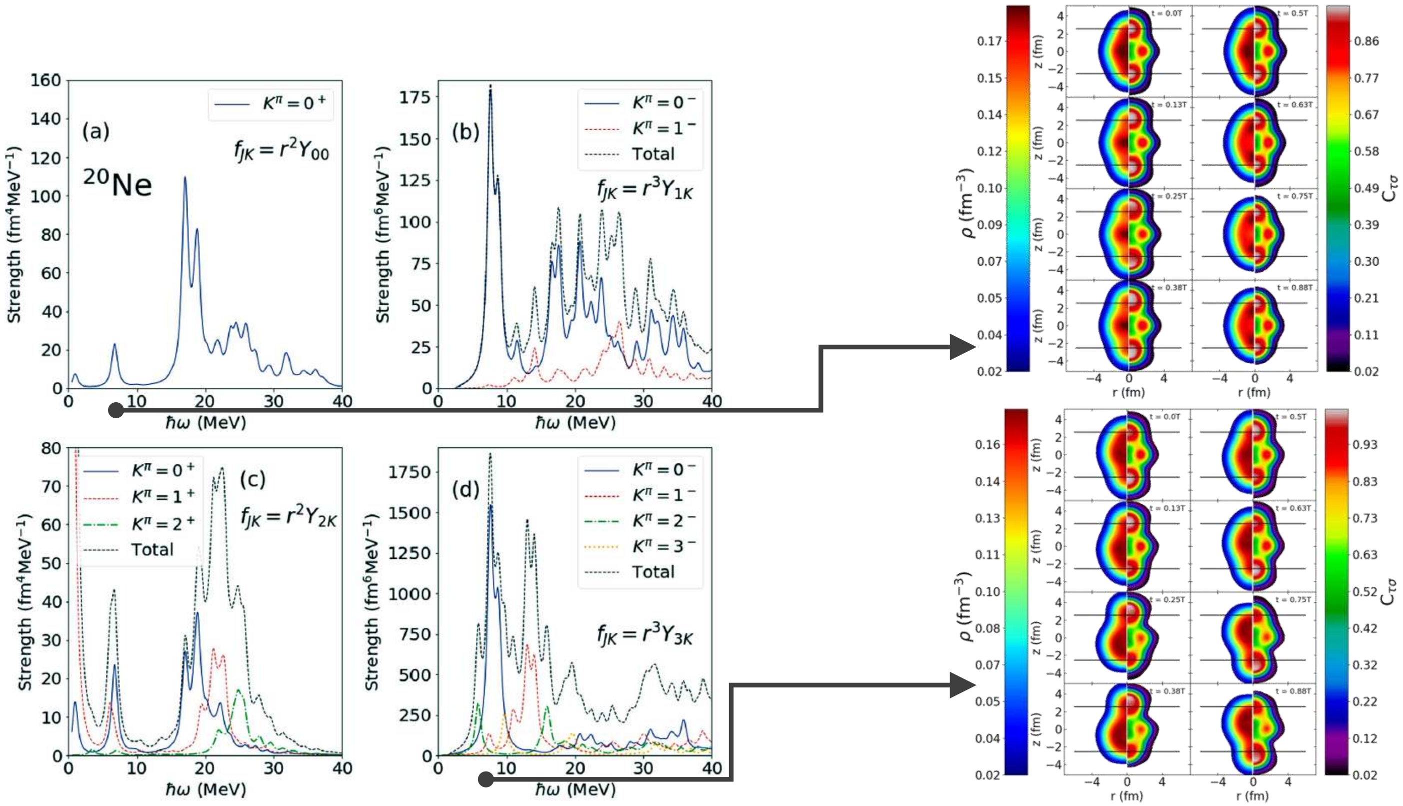


Marević, Ebran, Khan, Nikšić, and Vretenar PRC 2019



Frosini, Duguet, Ebran, Somà, EPJA 2022

◉ Cluster vibration

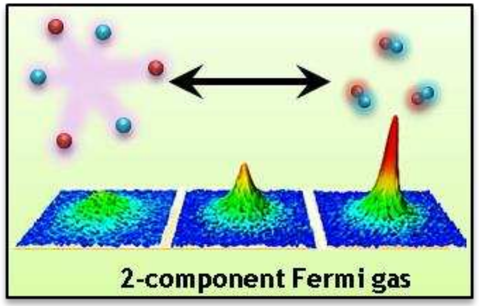


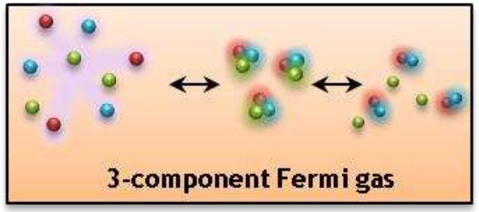
Mercier, Bjelčić, Nikšić, Ebran, Khan,Vretenar PRC 2021
Mercier, Ebran, Khan PRC 2022

Thank you for your attention

N-component Fermi systems

◉ BCS/BEC crossover + phases stabilized by internal dofs





◉ How does this translate in nuclei = 4-component Fermi systems ?

proton
neutron

- Schematic Hamiltonian : $H = H_0 + \mathcal{V}_{\text{res}}$

$$H_0 = \int d^3r \sum_\alpha \varepsilon_\alpha \psi^\dagger_\alpha(\boldsymbol{r})\psi_\alpha(\boldsymbol{r})$$

$$\mathcal{V}_{\text{res}} \sim V_{\text{pair}} = -\int d^3r \left[g^{\text{T}=1} \sum_{\nu=\pm1,0} P^\dagger_\nu(\boldsymbol{r})P_\nu(\boldsymbol{r}) + g^{\text{T}=0} \sum_{\mu=\pm1,0} Q^\dagger_\mu(\boldsymbol{r})Q_\mu(\boldsymbol{r}) \right]$$

Correlated pair operators

$$P^\dagger_\nu(\boldsymbol{r}) \equiv \sqrt{\frac{1}{2}} \sum_l \sqrt{2l+1} \left\{ \psi^\dagger_{l,s=\frac{1}{2},t=\frac{1}{2}}(\boldsymbol{r}) \psi^\dagger_{l,s=\frac{1}{2},t=\frac{1}{2}}(\boldsymbol{r}) \right\}^{(L=0,S=0,T=1)}_{M_L=0,M_S=0,M_T=\nu}$$

$$Q^\dagger_\mu(\boldsymbol{r}) \equiv \sqrt{\frac{1}{2}} \sum_l \sqrt{2l+1} \left\{ \psi^\dagger_{l,\frac{1}{2},\frac{1}{2}}(\boldsymbol{r}) \psi^\dagger_{l,\frac{1}{2},\frac{1}{2}}(\boldsymbol{r}) \right\}^{(L=0,S=1,T=0)}_{M_L=0,M_S=\mu,M_T=0}$$

- One-to-one correspondence with a system of spin-3/2 fermions with the Hamiltonian

$$H = \int d^3r \left\{ \sum_\alpha \varepsilon_\alpha \varphi^\dagger_\alpha(\boldsymbol{r})\varphi_\alpha(\boldsymbol{r}) - g_0 S^\dagger_{0,0}(\boldsymbol{r}) S_{0,0}(\boldsymbol{r}) - \sum_{m=\pm2,\pm1,0} g_{2,m} D^\dagger_{2,m}(\boldsymbol{r}) D_{2,m}(\boldsymbol{r}) \right\}$$

Singlet (S=0) pairing operator $\quad S^\dagger_{0,0} = \sum_{\alpha\beta} \langle \tfrac{3}{2}\tfrac{3}{2};00|\tfrac{3}{2}\tfrac{3}{2}\alpha\beta\rangle \varphi^\dagger_\alpha \varphi^\dagger_\beta$

Quintet (S=2) pairing operator $\quad D^\dagger_{2,m} = \sum_{\alpha\beta} \langle \tfrac{3}{2}\tfrac{3}{2};2m|\tfrac{3}{2}\tfrac{3}{2}\alpha\beta\rangle \varphi^\dagger_\alpha \varphi^\dagger_\beta$

with $S^\dagger_{0,0} = P^\dagger_0,\ D^\dagger_{2,0} = Q^\dagger_0,\ D^\dagger_{2,\pm1} = P^\dagger_{\pm1}$ and $D^\dagger_{2,\pm2} = Q^\dagger_{\pm1}$

Group theory considerations

proton
neutron

- Sp(4) ~ SO(5) symmetry without fine tuning the coupling constants

- Generators of $\mathfrak{so}(5)$

$$\Gamma^{ab} \equiv -\frac{i}{2}\left[\Gamma^a, \Gamma^b\right] \quad (1 \leq a, b \leq 5)$$

$$\Gamma^1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad \Gamma^{2,3,4} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

- Bilinears of fermions can be classified according to their behavior under SO(5)

Particle-hole channel

$$\begin{aligned} n(\boldsymbol{r}) &= \sum_\alpha \varphi^\dagger_\alpha(\boldsymbol{r})\varphi_\alpha(\boldsymbol{r}), \\ n_a(\boldsymbol{r}) &= \frac{1}{2}\sum_{\alpha\beta} \varphi^\dagger_\alpha(\boldsymbol{r})\Gamma^a_{\alpha\beta}\varphi_\beta(\boldsymbol{r}), \\ L_{ab}(\boldsymbol{r}) &= -\frac{1}{2}\sum_{\alpha\beta} \varphi^\dagger_\alpha(\boldsymbol{r})\Gamma^{ab}_{\alpha\beta}\varphi_\beta(\boldsymbol{r}). \end{aligned}$$

Particle-particle channel

$$\begin{aligned} \eta^\dagger(\boldsymbol{r}) &= \frac{1}{2}\sum_{\alpha\beta} \varphi^\dagger_\alpha(\boldsymbol{r})C_{\alpha\beta}\varphi^\dagger_\beta(\boldsymbol{r}), \\ \xi^\dagger_a(\boldsymbol{r}) &= -\frac{i}{2}\sum_{\alpha\beta} \varphi^\dagger_\alpha(\boldsymbol{r})\left(\Gamma^a C\right)_{\alpha\beta}\varphi^\dagger_\beta(\boldsymbol{r}), \end{aligned}$$

$$C = \Gamma^1\Gamma^3$$

$$S^\dagger_{0,0} = -\frac{\eta^\dagger}{\sqrt{2}} \quad D^\dagger_{2,0} = -i\frac{\xi^\dagger_4}{\sqrt{2}}, \quad D^\dagger_{2,\pm1} = -\frac{\xi^\dagger_3 \mp i\xi^\dagger_2}{\sqrt{2}}, \quad D^\dagger_{2,\pm2} = \frac{\mp\xi^\dagger_1 + i\xi^\dagger_5}{\sqrt{2}}$$

C. Wu PRL 2005

Group theory considerations

proton
neutron

$$H = \int d^3r \left\{ \sum_\alpha \varepsilon_\alpha \varphi^\dagger_\alpha(\boldsymbol{r})\varphi_\alpha(\boldsymbol{r}) - g_0 S^\dagger_{0,0}(\boldsymbol{r}) S_{0,0}(\boldsymbol{r}) - \sum_{m=\pm2,\pm1,0} g_{2,m} D^\dagger_{2,m}(\boldsymbol{r}) D_{2,m}(\boldsymbol{r}) \right\}$$

- If $g_0 = g_2 \equiv g$, singlet and quintet pairing states are degenerate and can be recast into a sextet pairing state $\Rightarrow$ SU(4) symmetry

- 2 different superfluid orders : i) Sp(4)-singlet BCS pairing phase : $\eta^\dagger(\boldsymbol{r})$

  ii) SU(4) molecular superfluid phase formed from bound states of 4 fermions: $A^\dagger(\boldsymbol{r}) \equiv \varphi^\dagger_{\frac{3}{2}}(\boldsymbol{r})\varphi^\dagger_{\frac{1}{2}}(\boldsymbol{r})\varphi^\dagger_{-\frac{1}{2}}(\boldsymbol{r})\varphi^\dagger_{-\frac{3}{2}}(\boldsymbol{r})$

- Competition manifested by a $\mathbb{Z}_2$ discrete symmetry (coset between the center of SU(4) and the center of Sp(4)) $\mathcal{U}_n = e^{in_4\pi}$

$$\begin{aligned} \eta^\dagger &\mapsto \mathcal{U}_n \eta^\dagger \mathcal{U}_n^{-1} = -\eta^\dagger, \\ A^\dagger &\mapsto \mathcal{U}_n A^\dagger \mathcal{U}_n^{-1} = A^\dagger. \end{aligned}$$

$\mathbb{Z}_2$ needs to be spontaneously broken to stabilize the BCS quasi-long range order.

$\mathbb{Z}_2$ remaining unbroken $\Rightarrow$ strong quantum fluctuations in the spin channel suppressing Cooper pairing (2 fermions can't form a $\mathbb{Z}_2$ singlet) $\Rightarrow$ leading superfluid instability = quartetting