# $\eta$, $\eta'$ mesons and the Witten-Veneziano formula from lattice QCD

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# Introduction

Quarks cannot be observed directly but are bound in hadrons (at low energies):

- The lightest hadrons $\pi^{\pm}$, $\pi^0$, $K^{\pm}$, $K^0$, $\bar{K}^0$, $\eta$ ("octet mesons") have masses from 135 MeV to 548 MeV.
- In addition there is a "flavor-singlet", the $\eta'$.
- For exact flavor symmetry ($m_u = m_d = m_s$) all 9 mesons should have the same mass.

However: $M_{\eta'} \approx 958\,\mathrm{MeV} \gg M_{octet}$

Solution to this puzzle:

- Large mass of the $\eta'$ is caused by the QCD vacuum structure and the $U(1)_A$ anomaly.
  *Weinberg (1975), Belavin et al. (1975), t'Hooft (1976), Witten (1979), Veneziano (1979)*
- The $U(1)$ axial current is anomalously broken, i.e. even for $m_q = 0$:
  *Adler (1969), Jackiw and Bell (1969)*
  $$\partial_\mu A^0_\mu = \frac{N_f g^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} \neq 0$$
- Instantons with non-trivial topology provide non-perturbative explanation.
  *Belavin et al. (1975) , t'Hooft (1976)*
- The flavor-singlet $\eta'$ is **not** a Goldstone boson (not even in the chiral limit).

**Is it possible to reproduce $M_\eta$, $M'_\eta$ from first principles?**

# Mixing

For exact $SU(3)$ flavor symmetry one expects

- Flavor octet state $|\eta_8\rangle = \frac{1}{\sqrt{6}}(|\bar{u}u\rangle + |\bar{d}d\rangle - 2|\bar{s}s\rangle)$ (Pseudo-Goldstone boson)
- Flavor singlet state $|\eta_0\rangle = \frac{1}{\sqrt{3}}(|\bar{u}u\rangle + |\bar{d}d\rangle + |\bar{s}s\rangle)$ (related to $U(1)_A$ anomaly)

However, $SU(3)$ flavor symmetry is broken by large $m_s \gg m_u \approx m_d \equiv m_l$:

- Physical $\eta$, $\eta'$ states are not flavor eigenstates but **mixtures**, e.g.

$$\begin{pmatrix} |\eta\rangle \\ |\eta'\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi_l & -\sin\phi_s \\ \sin\phi_l & \cos\phi_s \end{pmatrix} \begin{pmatrix} |\eta_l\rangle \\ |\eta_s\rangle \end{pmatrix}$$

  in the **quark flavor basis** $|\eta_l\rangle = \frac{1}{\sqrt{2}}(|\bar{u}u\rangle + |\bar{d}d\rangle)$, $|\eta_s\rangle = |\bar{s}s\rangle$.

- In nature further mixing possible, e.g. with $\pi^0$ ($m_u \neq m_d$), $\eta_c$ (including $c$ quark)
- How did nature arrange the mixing pattern?

**Can we determine the mixing parameters?**

# Witten-Veneziano formula

The Witten-Veneziano formula

$$\chi_\infty = \frac{f_0^2}{4N_f}(M_{\eta'}^2 + M_\eta^2 - 2M_K^2)$$

establishes a relation between:

- Topological susceptibility $\chi_\infty$ in pure Yang-Mills gauge theory
- Meson masses $M_K$, $M_\eta$, $N_{\eta'}$
- Singlet decay constant $f_0$ (a parameter of the $\eta$,$\eta'$–mixing)

**Remark:** From a modern perspective the formula is LO $\chi$PT for a combined power counting scheme:

$$\mathcal{O}(p^2) \sim \mathcal{O}(m_q) \sim \mathcal{O}(1/N_c)$$

All these quantities can be computed from first principles in lattice QCD (LQCD), i.e. using:

- Dedicated simulations in quenched LQCD for $\chi_\infty$ (for $N_c = 3$)
- Ensembles generated with $N_f = 2+1+1$ dynamical quark flavors.

**Can we test the Witten-Veneziano formula on the lattice?**

# Outline

1. A few details on the lattice setup and analysis
2. Physical results for $M_{\eta,\eta'}$ and mixing parameters
   *JHEP 11 (2012) 048, PRL 111 (2013) 18, 181602, PRD 97 (2018) 5, 054508*
3. The Witten-Veneziano formula from LQCD
   *JHEP 09 (2015) 020, PoS CD2018 (2019) 077*
4. Newer / ongoing work directly at physical quark mass.
   *PRD 99 (2019) 3, 034511*

**Contributions by many collaborators over the years:**

| | |
|---|---|
| Krzysztof Cichy | Petros Dimopoulos |
| Elena Garcia-Ramos | Christopher Helmes |
| Karl Jansen | Christian Jost |
| Bastian Knippschild | Bartosz Kostrzewa |
| Liuming Liu | Chris Michael |
| Marcus Petschlies | Siebren Reker |
| Carsten Urbach | Urs Wenger |
| Markus Werner | Falk Zimmermann |

European Twisted Mass Collaboration

# Lattice setup

To obtain physical results from LQCD we have to:

- **Control discretization effects:**
  - Simulate at different (small) values of $a$
  - Perform continuum extrapolation
  - Three lattice spacings $a_A = 0.086\,\mathrm{fm}$, $a_B = 0.078\,\mathrm{fm}$ and $a_D = 0.061\,\mathrm{fm}$
- **Correct for unphysical quark masses:**
  - Simulate at several pion masses
  - Perform chiral extrapolation
  - Pion masses from $\sim 230\,\mathrm{MeV}$ to $\sim 500\,\mathrm{MeV}$
  - However, bare $m_s$, $m_c$ usually **fixed for each** $a$
  - Few dedicated ensembles with different $m_s$
- **Control finite size effects:**
  - Simulate different physical volumes

Use 17 gauge ensembles with $N_f = 2+1+1$ dynamical flavors of Wilson twisted-mass quarks provided by ETMC

*JHEP 0108:058 (2001), Nucl. Phys. Proc. Suppl.1258 (2004), JHEP 1006:111 (2010)*

# $\eta, \eta'$ on the lattice

Extract masses and amplitudes from suitable meson two-point correlation functions:

$$\mathcal{C}_{ij}(t) \sim \sum_{\vec{x}} \langle 0 | O_i(x) O_j(0) | 0 \rangle$$

- Interpolating operators $O_{i,j}$ need to couple to the desired states
- For $\eta$, $\eta'$ use local pseudoscalar operators (**quark flavor basis**):

$$\eta_l = \frac{1}{\sqrt{2}}(\bar{u} i\gamma_5 u + \bar{d} i\gamma_5 d), \quad \eta_s = \bar{s} i\gamma_5 s, \quad \eta_c = \bar{c} i\gamma_5 c$$

- Considering $i = j$:

$$\mathcal{C}(t) = \sum_n \frac{|\langle 0 | O_i | n \rangle|^2}{2M_n} \exp(-M_n t) \overset{t \gg 0}{\to} \frac{|\langle 0 | O_i | \eta \rangle|^2}{2M_\eta} \exp(-M_\eta t)$$

→ Ground state mass $M_\eta$ can be extracted from $aM(t) = \log \frac{\mathcal{C}(t)}{\mathcal{C}(t+1)}$

→ Decay constants / mixing parameters from physical amplitudes $A_i^n = \langle 0 | O_i | n \rangle$.

→ Higher states: diagonalize $\mathcal{C}_{ij}$ (GEVP) + correlated fits to principal correlators

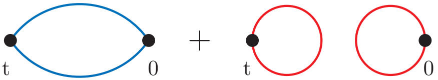
# Quark disconnected diagrams

- Consider $O_i = O_j = \eta_l$:

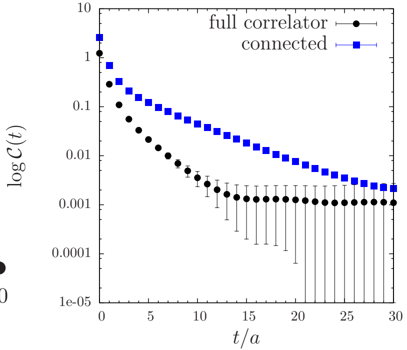
$$\mathcal{C}_{ll}(t) \sim \sum_{\vec{x}} \langle 0|\, \bar{u}(x) i\gamma_5 u(x) \bar{u}(0) i\gamma_5 u(0)\, |0\rangle$$

$$\sim \mathrm{tr}\left[D_{0t}^{-1}\gamma_5 D_{t0}^{-1}\gamma_5\right] + \mathrm{tr}\left[D_{tt}^{-1}\gamma_5\right]\mathrm{tr}\left[D_{00}^{-1}\gamma_5\right]$$

- Quark connected and disconnected pieces:




- Lattice Dirac operator $D_{xy}$ is a very large $(3\cdot 4\cdot L^3\cdot T)\times(3\cdot 4\cdot L^3\cdot T)$ – matrix




**Full** and connected-only correlators;
$M_\pi = 270\,\mathrm{MeV}$, $a = 0.078\,\mathrm{fm}$

- Disconnected diagrams need **all-to-all** propagator $D_{xx}^{-1} \Rightarrow$ prohibitively expensive
- Use stochastic method + one-end trick instead (still not cheap; required $\gtrsim 10^8$ core hours)

$\rightarrow$ **Severe signal-to-noise problem; signal typically lost at $t \gtrsim 1\,\mathrm{fm}$ ...**

# How to tackle the signal-to-noise problem?

Assumption:

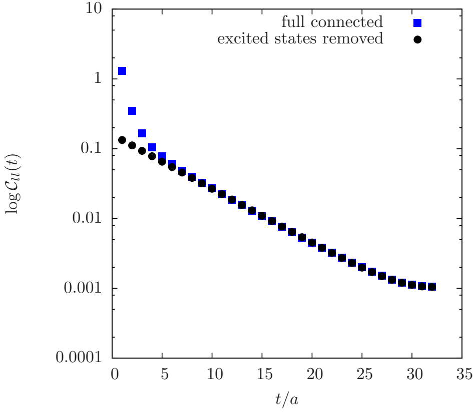
**Disconnected diagrams couple only to $\eta$, $\eta'$:**

- Ignore charm quark. (contributions are negligibly small)
- No signal-to-noise problem in quark-connected contribution
- Replace connected contributions by respective ground state contributions

  *PRD 64 (2001), 114509,* *EPJ C58 (2008), 261-269*

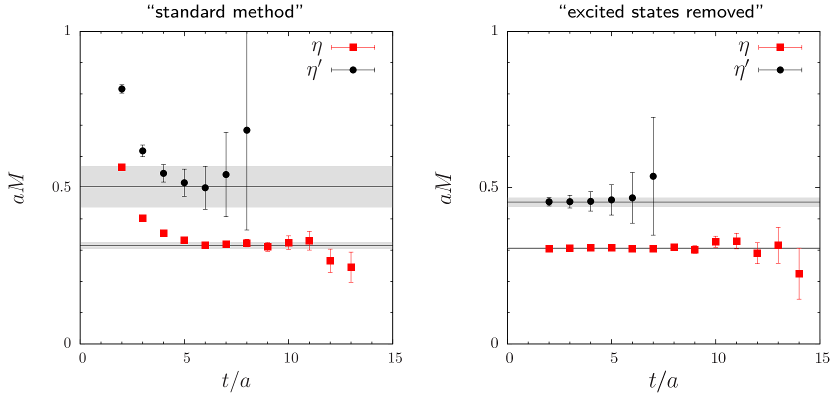
  *PRL 111 (2013) 18, 181602*

**If this assumption is correct we should see a plateau at very small values of $t/a$ ...**




Connected contribution with and w/o excited states
$M_\pi = 270\,\mathrm{MeV}$, $a = 0.78\,\mathrm{fm}$

# Removal of excited state




- **... We observe plateaux in both states starting at** $t/a = 2$
- $M_\eta$ agrees very well with previous result
- Significant improvement in the statistical error for $M_{\eta'}$
- Requires to check validity of assumption from Monte-Carlo data

# Topological finite volume effect (I)

In **finite volume** and for **fixed top. charge** $Q_t$ one finds

$$\langle \omega(x)\omega(0)\rangle_{Q_t=\text{fixed}} \to \frac{1}{V}\left(\chi_t - \frac{Q_t^2}{V} + \frac{c_4}{2V\chi_t}\right) + \dots,$$

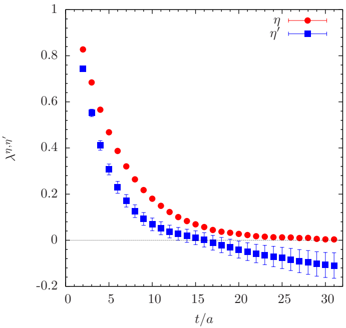
for correlators of winding number densities $\omega(x)$ at large $|x|$.

*S. Aoki et al., Phys.Rev. D76, 054508 (2007)*

$\Rightarrow$ Expect constant offset in $\eta'$ ($\eta$) correlator **at large** $t$:

$$< \lambda^{\eta'}(t) >_{Q_t=\text{fixed}} \to \sim \frac{a^5}{T}\left(\chi_t - \frac{Q_t^2}{V} + \frac{c_4}{2V\chi_t}\right).$$

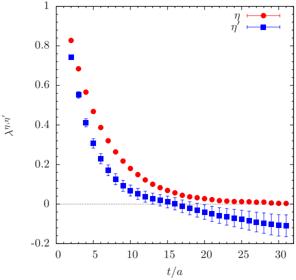
*G. S. Bali et al., Phys.Rev. D91 (2015) 1, 014503*



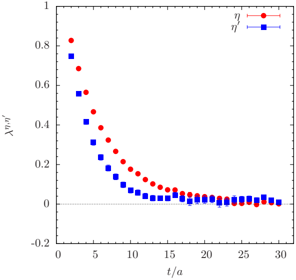

Principal correlators
$M_\pi = 375\,\text{MeV}$, $M_\pi \cdot L = 3.8$, $L = 2\,\text{fm}$

- Always present for finite volume + finite statistics.
- **Often masked by statistical point errors!**
- Noise in $\eta'$-signal largely due to fluctuation + autocorrelation of this constant.

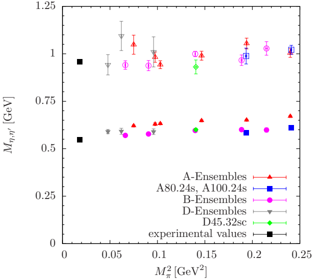
# Topological finite volume effect (II)




principal correlators
$M_\pi = 375\,\mathrm{MeV}$, $L = 2\,\mathrm{fm}$




Same, but from time-derivative
$M_\pi = 375\,\mathrm{MeV}$, $L = 2\,\mathrm{fm}$




Lattice results for $M_\eta$, $M_{\eta'}$

**Simple but efficient way to correct for this effect:**

- Remove constant using discrete derivative ("time-shifted") correlator:

$$\mathcal{C}(t) \to \tilde{\mathcal{C}}(t) = \mathcal{C}(t) - \mathcal{C}(t + \Delta t)$$

- Resulting data are much less correlated.
- Remaining analysis (GEVP, CCF extrapolation) can be carried out in standard way...

# Physical results for $M_\eta$, $M_{\eta'}$

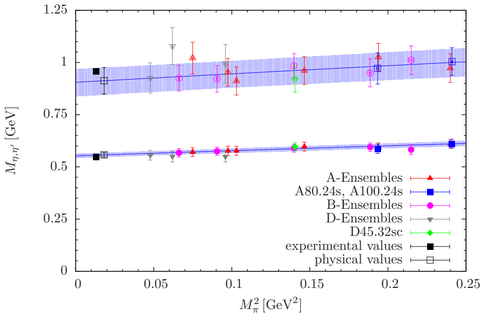
We use a LO $\chi$PT fit ansatz ($P = \eta, \eta'$)

$$(r_0 M_P)^2 = (r_0 \mathring{M}_P)^2 + \sum_{i=\pi,K} L_i (r_0 M_i)^2 + L_\beta \left(\frac{a}{r_0}\right)^2 .$$

**Final results:**

$$M_\eta = 557(11)_{\mathrm{stat}}(03)_{\chi PT}\,\mathrm{MeV}$$
$$M'_\eta = 911(64)_{\mathrm{stat}}(03)_{\chi PT}\,\mathrm{MeV}$$

PRD 97 (2018) 5, 054508




Data corrected for physical $m_s$, continuum limit.
Point errors highly correlated.

- **Results in good agreement with experiment.**
- Compatible with results from older analysis, but better control of systematics for $\eta'$. PRL 111 (2013) 18, 181602
- Chiral and continuum behavior mild.
- Scale setting using Sommer parameter $r_0 = 0.474(14)\,\mathrm{fm}$. Nucl.Phys. B887 (2014) 19-68

# Mixing

Decay constants $f_{\mathrm{P}}^{i}$ are defined from axial-vector matrix elements (amplitudes)

$$\langle 0| A_{\mu}^{i} |\mathrm{P}(p)\rangle = i f_{\mathrm{P}}^{i} p_{\mu}\,, \qquad \mathrm{P} = \eta, \eta'\,,$$

On the lattice: **quark flavor basis** (i=l,s) is a more "natural" choice

$$A_{\mu}^{l} = \frac{1}{\sqrt{2}}(\bar{u}\gamma_{\mu}\gamma_5 u + \bar{d}\gamma_{\mu}\gamma_5 d)\,, \qquad A_{\mu}^{s} = \bar{s}\gamma_{\mu}\gamma_5 s\,.$$

$\eta$ and $\eta'$ are not flavor eigenstates; most general parametrization:

$$\begin{pmatrix} f_{\eta}^{l} & f_{\eta}^{s} \\ f_{\eta'}^{l} & f_{\eta'}^{s} \end{pmatrix} = \begin{pmatrix} f_l \cos\phi_l & -f_s \sin\phi_s \\ f_l \sin\phi_l & f_s \cos\phi_s \end{pmatrix}$$

From $\chi$PT one expects $|\phi_l - \phi_s|$ to be small, i.e. $\frac{|\phi_l - \phi_s|}{|\phi_l + \phi_s|} \ll 1$

- Small difference in one basis does **NOT** imply small difference in another basis!
- Expect that only one angle $\phi \approx \phi_l \approx \phi_s$ is required: $\tan^2(\phi) = -\frac{f_l^{\eta'} f_s^{\eta}}{f_l^{\eta} f_s^{\eta'}}$.

# Mixing

However, we find the axial vector too noisy to determine $\phi/\phi_{l,s}$ and $f_{l,s}$ directly.

Consider pseudoscalar matrix elements

$$h^i_{\mathrm{P}} = 2m_i < 0|P^i|\mathrm{P} > , \quad \mathrm{P} = \eta, \eta' ,$$

which can be related to axial vector ones via the **anomaly equation** using $\chi$PT:

$$\begin{pmatrix} h^l_\eta & h^s_\eta \\ h^l_{\eta'} & h^s_{\eta'} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \mathrm{diag}\left( f_l M_\pi^2 , f_s \left( 2M_K^2 - M_\pi^2 \right) \right) .$$

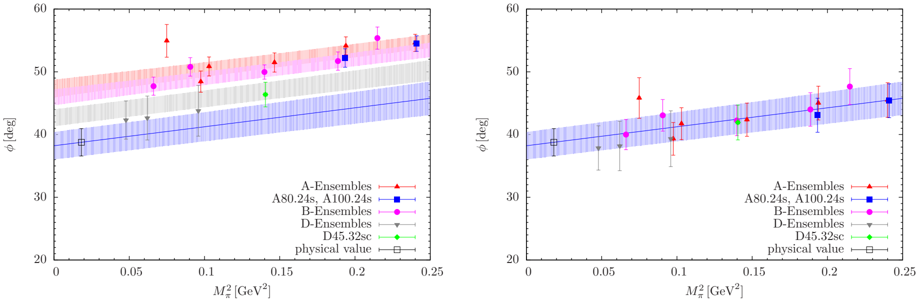
*Th. Feldmann et al., PRD 58 (1998), 114006*

*Th. Feldmann et al., Phys.Lett. B449 (1999) 339-346*

- This expression holds to LO $\chi$PT.
- Mixing angle(s) do not depend on renormalization.
- Can check whether $|\phi_l - \phi_s|$ is small!

$\rightarrow$ **Residual $\chi$PT-dependence compared to axial-vector approach**

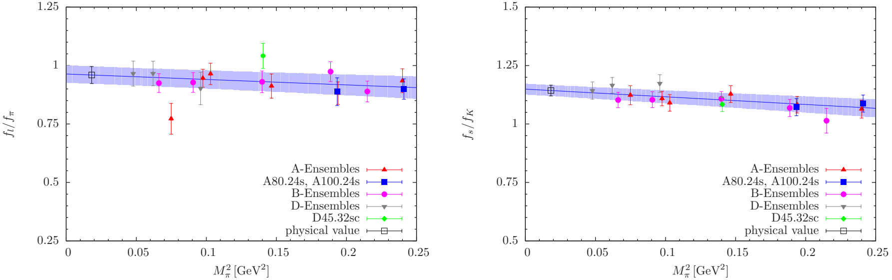
# Physical result for $\phi$




**Final result:** $\phi_{\text{phys}} = 38.8^\circ(2.2)_{\text{stat}}(2.4)_{\chi PT}$ *PRD 97 (2018) 5, 054508*

- Good agreement with phenomenology: $\phi^{\text{pheno}} = 39.3^\circ(1.0)$. *Th. Feldmann, Int.J.Mod.Phys. A15 (2000) 159-207*
- **Significant effect from continuum extrapolation.**
- Data compatible with requirement $\phi_{\text{SU}(3)_F} \approx 54.7^\circ$.
- $|\phi_l - \phi_s| = 2.8(1.1)_{\text{stat}}(2.6)^\circ_{\text{sys}}$ confirms smallness of NLO (OZI) corrections.

# Results for $f_l$, $f_s$: Chiral and continuum extrapolation of $f_l/f_\pi$ and $f_s/f_K$




- $f_l$, $f_s$ rather difficult to fit individually.
- Ratios $f_l/f_\pi$ and $f_s/f_K$ cancel most of the quark mass, lattice spacing and volume-dependence.

**Final results:**

$$(f_l/f_\pi)_{\rm phys} = 0.960(37)_{\rm stat}(46)_{\chi PT} \rightarrow f_{l,\rm phys} = 0.125(5)_{\rm stat}(6)_{\chi PT}\ \mathrm{GeV}$$
$$(f_s/f_{\rm K})_{\rm phys} = 1.143(23)_{\rm stat}(04)_{\chi PT} \rightarrow f_{s,\rm phys} = 0.178(4)_{\rm stat}(1)_{\chi PT}\ \mathrm{GeV}$$

Averages from phenomenology: $f_l/f_\pi = 1.07(2)$ and $f_s/f_{\rm K} = 1.12(6)$ *Th. Feldmann, Int.J.Mod.Phys. A15 (2000) 159-207*

# The Witten-Veneziano formula

The Witten-Veneziano formula

$$\chi_\infty^{\mathrm{YM}} = \frac{f_0^2}{4N_f}(M_{\eta'}^2 + M_\eta^2 - 2M_K^2)$$

*Nucl. Phys. B 156 (1979) 269*
*Nucl. Phys. B 159 (1979) 213*

connects:

- Topological susceptibility $\chi_\infty^{\mathrm{YM}}$ in pure Yang-Mills gauge theory,
- Singlet decay constant $f_0$,
- Meson masses $M_K$, $M_\eta$, $N_{\eta'}$.

To test it from first principles, calculate all quantities (for $N_c = 3$) using:

- Results from our dynamical $N_f = 2+1+1$ tmLQCD simulations (as shown before),
- Dedicated simulations in the quenched setup for $\chi_\infty^{\mathrm{YM}}$.

*JHEP 1509 (2015) 020*

# Quenched computation of $\chi_\infty$

Compute $\chi_\infty$ from (stochastically estimated) density chains

$$\chi_t = m_1 \cdot \ldots m_5 \cdot a^{16} \sum_{x_1,\ldots,x_4} \langle P_{31}(x_1) S_{12}(x_2) S_{23}(x_3) \times P_{54}(x_4) S_{45}(0) \rangle_c$$

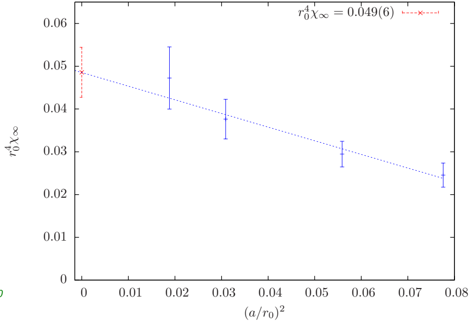
JHEP 0903 (2009) 013

where $S_{ij}$, $P_{ij}$ denote scalar and pseudoscalar densities, respectively.

$\rightarrow$ **Theoretical sound definition**; only multiplicative renormalization

- Use Wilson twisted mass valence quarks
  $\Rightarrow$ Automatic $\mathcal{O}(a)$–improvement
- Box length fixed to 2.8 fm
- Four values of $a$: $0.07\,\mathrm{fm}$ to $a = 0.14\,\mathrm{fm}$
- Linear scaling in $a^2$ as expected
- Continuum limit: $r_0^4 \chi_\infty^{\mathrm{YM}} = 0.049(6)_{\mathrm{stat+sys}}$

JHEP 1509 (2015) 020



# Computation of $f_0$

- $f_0$ is defined in **octet-singlet basis**:

$$A_\mu^0 = \frac{1}{\sqrt{6}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s)\,,$$
$$A_\mu^8 = \frac{1}{\sqrt{3}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d - 2\bar{s}\gamma_\mu\gamma_5 s)\,.$$

- $f_0$ can be related by LO continuum $\chi$PT to $f_l$, $f_s$ and $f_\pi$, $f_K$ e.g.

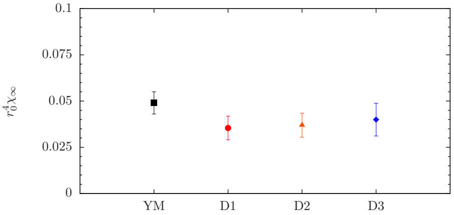
$$f_0^2 = -7/6 f_\pi^2 + 2/3 f_K^2 + 3/2 {f_l}^2\,, \qquad \textbf{(D1)}$$
$$f_0^2 = +1/3 f_\pi^2 - 4/3 f_K^2 + {f_l}^2 + {f_s}^2\,, \qquad \textbf{(D2)}$$
$$f_0^2 = +10/3 f_\pi^2 - 16/3 f_K^2 + 3{f_s}^2\,. \qquad \textbf{(D3)}$$

- **Not unambiguous**; they have different systematics:

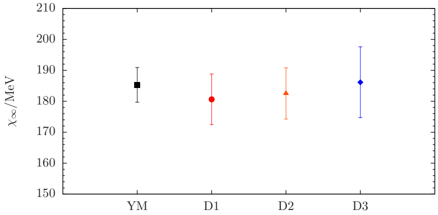
$$f_{0,\mathrm{phys}} = 0.141(06)_{\mathrm{stat}}\ \mathrm{GeV} \qquad \textbf{(D1)}$$
$$f_{0,\mathrm{phys}} = 0.144(07)_{\mathrm{stat}}\ \mathrm{GeV} \qquad \textbf{(D2)}$$
$$f_{0,\mathrm{phys}} = 0.149(13)_{\mathrm{stat}}\ \mathrm{GeV} \qquad \textbf{(D3)}$$

# Results

Putting everything together:




Results in units of $r_0$.




Results in physical units.

- Weighted average for dynamical simulations: $r_0^4\chi_\infty^{\mathrm{dyn}} = 0.037(7)_{\mathrm{stat+sys}}$.
- Comparison in physical units problematic because in general $r_0^{dyn} \neq r_0^{\mathrm{YM}}$.
- $r_0^{\mathrm{YM}} = 0.5\,\mathrm{fm}$ and $r_0^{\mathrm{dyn}} = 0.474(14)\,\mathrm{fm}$ yields good agreement: *Nucl.Phys. B887 (2014) 19-68*

$$\chi_\infty^{\mathrm{YM}} = (185.3(5.6)_{\mathrm{stat+sys}}\,\mathrm{MeV})^4 \quad \text{vs.} \quad \chi_\infty^{\mathrm{dyn}} = (182.6(8.3)_{\mathrm{stat+sys}}\,\mathrm{MeV})^4$$

*PoS CD2018 (2019) 077*

# Towards the physical point ($N_f = 2$)

Most important issue for simulating $\eta$, $\eta'$ at the physical point:

**How do the errors scale?**

- Results shown so far use ensembles with $M_\pi \gtrsim 230\,\mathrm{MeV}$ for $N_f = 2+1+1$.
- In the last years $N_f = 2+1+1$ simulations by ETMC became available at physical $M_\pi$.

Isospin symmetric $N_f = 2$ theory conceptually and technically more simple:

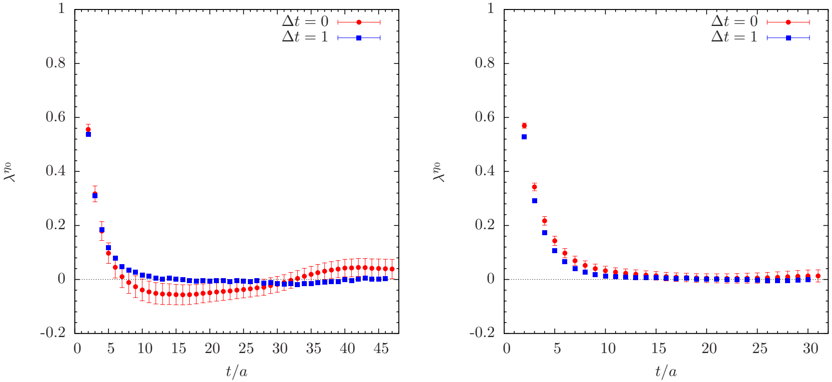
- Only three (degenerate) pions and one singlet field $\eta_0$ related to the anomaly; no mixing.
- No GEVP required for computing the mass, can analyze ground state directly.
- Provides testbed for simulating $\eta$, $\eta'$ at physical $M_\pi$ (error scaling, topological FV effect...)

| ensemble | $T/a$ | $L/a$ | $L/\mathrm{fm}$ | $M_\pi\,[\mathrm{MeV}]$ | $M_\pi \cdot L$ | $N_\mathrm{conf}$ |
|---|---|---|---|---|---|---|
| **cA2.09.48** | 96 | 48 | 4.5 | **132** | 3.0 | 615 |
| cA2.30.48 | 96 | 48 | 4.5 | 240 | 5.4 | 352 |
| cA2.30.24 | 48 | 24 | 2.2 | 245 | 2.8 | 352 |
| cA2.60.32 | 64 | 32 | 3.0 | 337 | 5.0 | 337 |
| cA2.60.24 | 48 | 24 | 2.2 | 340 | 3.8 | 424 |

- Ensembles with $N_f = 2$ Wilson twisted mass + clover fermions

*ETMC, Lat15 (2015)*
*ETMC, Phys.Rev. D95 (2017) no.9, 094515*

# How much noise?

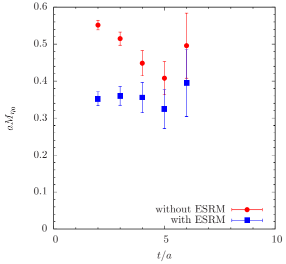



cA2.09.48, $M_\pi = 135\,\mathrm{MeV}$, $a = 0.093\,\mathrm{fm}$ cA2.60.32, $M_\pi = 340\,\mathrm{MeV}$, $a = 0.093\,\mathrm{fm}$

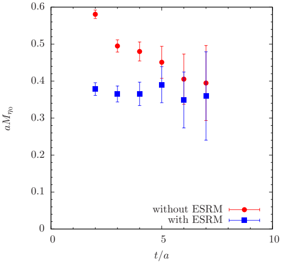
- Massive increase in noise towards physical point in flavor-singlet correlator.
- Correlation and point errors drastically reduced in derivative correlator.
- Resulting shift due to topological FV effect not too severe.

$\rightarrow$ **Reasonable signal quality; analysis is possible.**

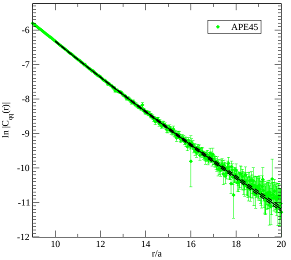
# Masses




cA2.09.48, $M_\pi = 135\,\mathrm{MeV}$, $a = 0.093\,\mathrm{fm}$




cA2.30.48, $M_\pi = 240\,\mathrm{MeV}$, $a = 0.093\,\mathrm{fm}$




Fit to gluonic correlator

cA2.09.48, $M_\pi = 135\,\mathrm{MeV}$, $a = 0.093\,\mathrm{fm}$

- Excite state removal in connected piece again crucial to improve result at small $M_\pi$
- Can also determine $M_{\eta_0}$ from fit to (smeared) gluonic correlation function

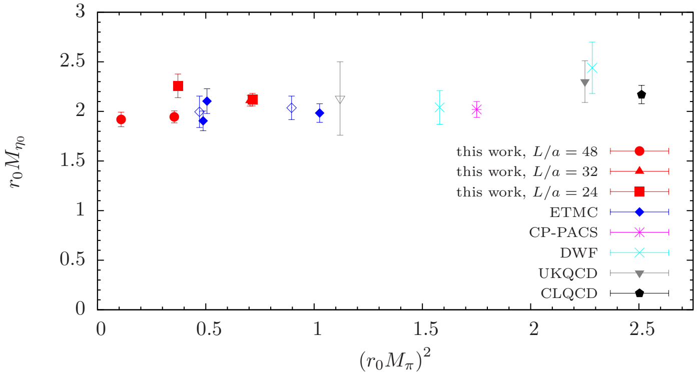
$$C_{qq}(r) \sim \sqrt{\frac{M_{\eta_0}}{r^{3/2}}}\, e^{-M_{\eta_0} r} \left(1 + \mathcal{O}\left(\frac{1}{M_{\eta_0} r}\right)\right) \quad \text{at large separations } r\,.$$

- Fermionic and gluonic results at physical point in excellent agreement:

$$M_{\eta_0}^{\mathrm{ferm}} = 772(18)_{\mathrm{stat}}\,\mathrm{GeV} \quad \text{vs.} \quad M_{\eta_0}^{\mathrm{glue}} = 781(21)_{\mathrm{stat}}\,\mathrm{GeV}$$

*PRD 99 (2019) 3, 034511*

# Results




- Data in agreement with existing lattice results.
- Chiral behavior very flat; similar to $N_f = 2+1+1$.
- Errors are small due to removal of excited states; $\eta_0$ being ground state

$\rightarrow$ **It is possible to simulate $\eta$,$\eta'$ directly at physical quark mass for $N_f = 2+1+1$.**

**Work in progress...**

# Summary and outlook

**First lattice study of $\eta$, $\eta'$ with $N_f = 2+1+1$ dynamical quark flavors and controlled systematics:**

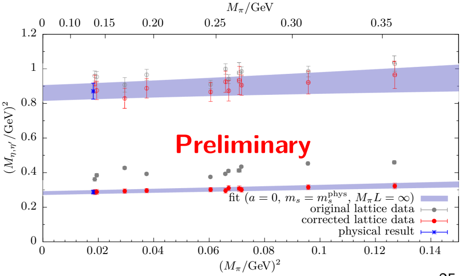
- Physical extrapolations for all observables.
- Several improvements in the analysis since 2013 lead to reduced systematic and statistical errors.
- Large mass of $\eta'$ reproduced from first principles; $M_{\eta,\eta'}$ in agreement with experiment.

**First systematic lattice study of $\eta$,$\eta'$–mixing:**

- Full chiral and continuum extrapolations for $\phi$, $f_l$ and $f_s$.
- Results in good agreement with phenomenology, with competitive errors.
- **Confirmed the validity of the Veneziano-Witten formula directly using lattice data.**

**Future plans:**

- Use "new" ETMC $N_f = 2+1+1$ ensembles (four lattice spacings, three boxes at $m_q^{\text{phys}}$)
- Study properties of $\eta$, $\eta'$ directly at physical quark mass.
- Removing the need for chiral extrapolation.
- Production almost complete, analysis in progress …



# Backup slides

# Ensembles ($N_f = 2+1+1$)

| ensemble | $\beta$ | T/a | L/a | $a\mu_l$ | $a\mu_\sigma$ | $a\mu_\delta$ | $N_{\rm conf}$ | $\Delta N$ | $N_S$ |
|---|---|---|---|---|---|---|---|---|---|
| A30.32 | 1.90 | 64 | 32 | 0.0030 | 0.150 | 0.190 | 1363 | 4 | 24 |
| A40.32 | 1.90 | 64 | 32 | 0.0040 | 0.150 | 0.190 | 863 | 4 | 24 |
| A40.24 | 1.90 | 48 | 24 | 0.0040 | 0.150 | 0.190 | 1877 | 4 | 32 |
| A60.24 | 1.90 | 48 | 24 | 0.0060 | 0.150 | 0.190 | 1248 | 4 | 128 |
| A80.24 | 1.90 | 48 | 24 | 0.0080 | 0.150 | 0.190 | 2449 | 2 | 32 |
| A100.24 | 1.90 | 48 | 24 | 0.0100 | 0.150 | 0.190 | 2514 | 2 | 32 |
| A80.24s | 1.90 | 48 | 24 | 0.0080 | 0.150 | 0.197 | 2489 | 2 | 32 |
| A100.24s | 1.90 | 48 | 24 | 0.0100 | 0.150 | 0.197 | 2312 | 2 | 32 |
| B25.32 | 1.95 | 64 | 32 | 0.0025 | 0.135 | 0.170 | 1467 | 4 | 24 |
| B35.32 | 1.95 | 64 | 32 | 0.0035 | 0.135 | 0.170 | 1251 | 4 | 24 |
| B55.32 | 1.95 | 64 | 32 | 0.0055 | 0.135 | 0.170 | 4996 | 4 | 48 |
| B75.32 | 1.95 | 64 | 32 | 0.0075 | 0.135 | 0.170 | 922 | 8 | 24 |
| B85.24 | 1.95 | 48 | 24 | 0.0085 | 0.135 | 0.170 | 573 | 10 | 32 |
| D15.48 | 2.10 | 96 | 48 | 0.0015 | 0.120 | 0.1385 | 1034 | 2 | 24 |
| D20.48 | 2.10 | 96 | 48 | 0.0020 | 0.120 | 0.1385 | 429 | 4 | 24 |
| D30.48 | 2.10 | 96 | 48 | 0.0030 | 0.120 | 0.1385 | 458 | 8 | 24 |
| D45.32sc | 2.10 | 64 | 32 | 0.0045 | 0.0937 | 0.1077 | 1074 | 4 | 48 |

# Technical aside: $\eta,\eta'$ in WtmLQCD

We work in the Wilson twisted mass $N_f = 2+1+1$ (unitary) setup:

$$\mathcal{S}_{F,l}[U,\chi_l,\bar{\chi}_l] = a^4 \sum_x \bar{\chi}_l \left(D_W + m_0 + i\mu_l\gamma_5\tau^3\right)\chi_l\,,$$

Frezzotti et. al., JHEP 0108:058 (2001)

$$\mathcal{S}_{F,h}[U,\chi_h,\bar{\chi}_h] = a^4 \sum_x \bar{\chi}_h \left(D_W + m_0 + i\mu_\sigma\gamma_5\tau^1 + \mu_\delta\tau^3\right)\chi_h\,.$$

R. Frezzotti and G.C. Rossi, Nucl.Phys.Proc.Suppl.128 (2004)

- Automatic $\mathcal{O}(a)$ improvement $\rightarrow \not{\mathcal{P}}$ and $\not{\mathcal{F}}$ at finite $a$
- Heavy sector not flavor-diagonal $\rightarrow$ additional propagators $G^{xy}_{cs}$, $G^{xy}_{sc}$

In the physical basis 2 $\gamma$-combinations ($i\gamma_5$, $i\gamma_0\gamma_5$) available; consider only $i\gamma_5$:

phys basis: $$\eta_l^{phys} = \frac{1}{\sqrt{2}}\bar{\psi}_l i\gamma_5\psi_l\,, \qquad \eta_{c,s}^{phys} = \bar{\psi}_h\left(\frac{1\pm\tau^3}{2}i\gamma_5\right)\psi_h = \begin{cases} \bar{c}i\gamma_5 c \\ \bar{s}i\gamma_5 s \end{cases},$$

tm basis: $$\eta_l^{tm} = \frac{1}{\sqrt{2}}\bar{\chi}_l\left(-\tau^3\right)\chi_l \qquad \eta_{c,s}^{tm} = \frac{1}{2}\bar{\chi}_h\left(-\tau^1 \pm i\gamma_5\tau^3\right)\chi_h\,.$$

$\Rightarrow$ Heavy operators are a sum of scalars and pseudoscalars

Considering renormalization we have (up to a global factor)

$$\eta^{tm}_{c,renormalized} = Z\left(\bar{\chi}_c i\gamma_5\chi_c - \bar{\chi}_s i\gamma_5\chi_s\right)/2 + \left(\bar{\chi}_s\chi_c + \bar{\chi}_c\chi_s\right)/2$$
$$\eta^{tm}_{s,renormalized} = Z\left(\bar{\chi}_s i\gamma_5\chi_s - \bar{\chi}_c i\gamma_5\chi_c\right)/2 - \left(\bar{\chi}_s\chi_c + \bar{\chi}_c\chi_s\right)/2 \,.$$

$\rightarrow$ Need (non-singlet) $Z = \frac{Z_P}{Z_S}$; can avoid this for masses ...

Additional rotation of basis to disentangle „heavy" operators

$$\eta_{S,P} = \eta^{tm}_c \pm \eta^{tm}_s = \begin{cases} \frac{1}{\sqrt{2}}\left(\bar{\chi}_c\chi_s + \bar{\chi}_s\chi_c\right) \\ \frac{1}{\sqrt{2}}\left(\bar{\chi}_c i\gamma_5\chi_c - \bar{\chi}_s i\gamma_5\chi_s\right) \end{cases} .$$

$$\Rightarrow \mathcal{C}^{\eta}(t) = \begin{pmatrix} \eta_l(t)\eta_l(0) & \eta_l(t)\eta_S(0) & \eta_l(t)\eta_P(0) \\ \eta_S(t)\eta_l(0) & \eta_S(t)\eta_S(0) & \eta_S(t)\eta_P(0) \\ \eta_P(t)\eta_l(0) & \eta_P(t)\eta_S(0) & \eta_P(t)\eta_P(0) \end{pmatrix} .$$

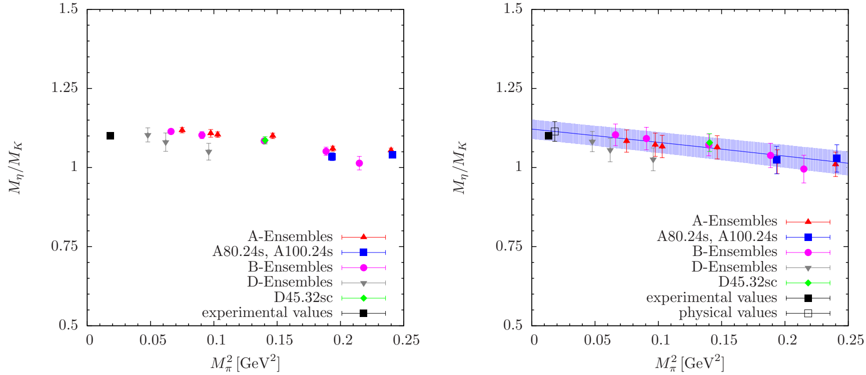
Advantage: Number of contractions per matrix element reduced by a factor 4

Putting in $Z$ and rotating back before solving GEVP:

$\Rightarrow$ Eigenvalues of $\mathcal{C}^{\eta}(t)$ give masses $M_{\eta}$, $M_{\eta'}$

$\Rightarrow$ Eigenvectors of $\mathcal{C}^{\eta}(t)$ give physical amplitudes $\rightarrow$ mixing parameters

# Alternative ratio extrapolation: $M_\eta / M_K$

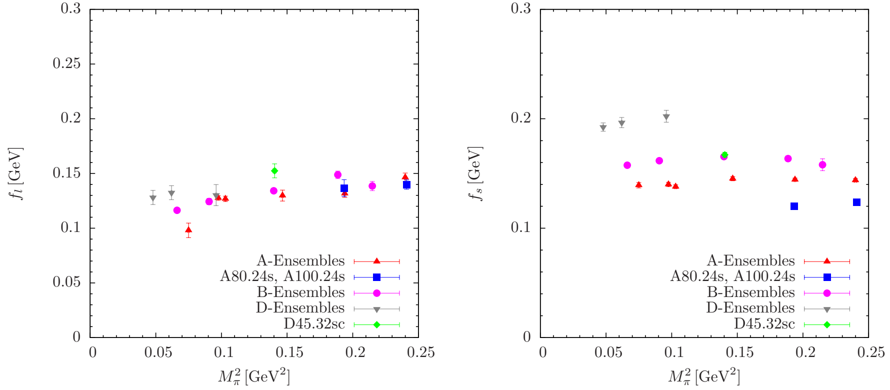



- $M_\eta$ and $M_K$ have very similar dependence on $m_s$.
- Ratio $M_\eta / M_K$ cancels most $m_s$-dependence.

$$(M_\eta / M_K)_{\text{phys}} = 1.114(31)_{\text{stat}} \quad \rightarrow \quad M_{\eta,\text{phys}} = 0.554(15)_{\text{stat}}\,\text{GeV}$$

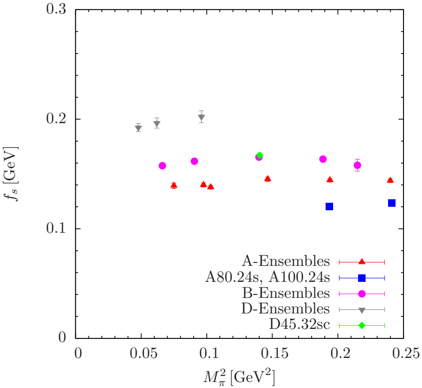
$\rightarrow$ **Confirms the results from the direct chiral + continuum fit**

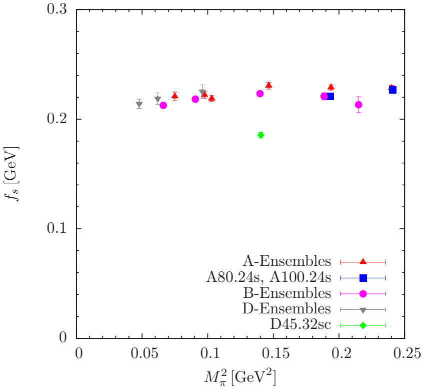
# Decay constants - $f_l$ and $f_s$




- $f_l$ has some $m_l$-dependence, lattice artifacts
- $f_s$ has very strong $m_s$-dependence; huge artifacts
- What about the influence of $Z$?

# Decay constants - Renormalization

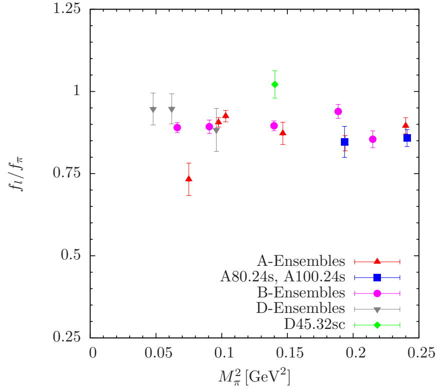


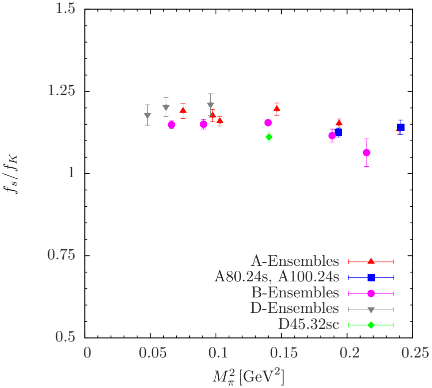




- Renormalization on the lattice not unambiguous $\rightarrow$ different lattice artifacts
- Values for $Z$ from two different methods M1 (left) and M2 (right)
- Huge impact on $f_s$ (effect much smaller for $f_l$)
- $Z$ enters also $\mu_s = (\mu_\sigma - Z\mu_\delta)$ and hence $f_K = (\mu_l + \mu_s)\frac{\langle 0|\mathcal{P}^{+,tm}_{neutral}|K\rangle}{M_K^2}$

**Idea:** Form ratios to cancel $m_s$–dependence, lattice artifacts

# Ratios $f_l/f_\pi$ and $f_s/f_{\mathrm{K}}$




- $f_s/f_{\mathrm{K}}$ cancels most $m_s$,$a$-dependence.
- Rather mild $m_l$-dependence.
- M2 has generally smaller artifacts.

$\rightarrow$ **Use results from M2 for final analysis.**