Properties of the η and η' mesons from lattice QCD

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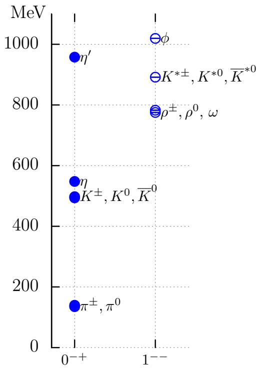
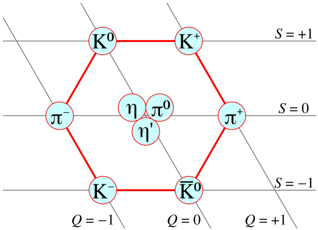
EMMI Workshop, "Meson and hyperon interactions with nuclei", Kitzbühel, Sept. 16th, 2022

Outline

- ⋆ Preliminaries.
- ⋆ Previous work on the lattice.
- ⋆ Simulation details: Extracting η, η' masses and decay constants.
- ⋆ Large-N_c ChPT.
- ⋆ Results: continuum limit at physical quark masses, LECs.
- ⋆ Singlet axial Ward identity, gluonic decay constants.
- ⋆ Summary.

Pseudoscalar meson nonet





If $\exists$ SU(3) flavour symmetry (u,d,s) then for $\bar{q}q$ we have $\bar{3} \otimes 3 = 8 \oplus 1$.

octet: $\pi^0, \pi^{\pm}, K^{\pm}, K^0, \overline{K}^0, \eta$, singlet: η'.

$$\eta = \eta_8 \sim \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \,, \qquad \eta' = \eta_0 \sim \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \,.$$

Pseudoscalar meson nonet

Classical global symmetries of $\mathscr{L}_{QCD}$ for $m_u = m_d = m_s = 0$:

$$\mathrm{SU}_A(3) \times \mathrm{SU}_V(3) \times \mathrm{U}_A(1) \times \mathrm{U}_V(1) \longrightarrow \mathrm{SU}_V(3) \times \mathrm{U}_V(1)$$

$\mathrm{SU}_A(3)$ chiral symmetry spontaneously broken at $T < T_c$,

8 Nambu-Goldstone bosons: $\pi^0, \pi^\pm, K^\pm, K^0, \overline{K}^0, \eta_8$.

$\mathrm{U}_A(1)$ symmetry broken due to quantum corrections (axial anomaly).

η_0 is heavier than octet mesons.

Physical ($m_s > m_u \approx m_d > 0$) η and η' are no flavour eigenstates.

$\rightsquigarrow$ state mixing picture between η_8 and η_0 based on an effective Lagrangian.

Axial-Ward Identity

$m_q = 0$

$$\partial_\mu \widehat{A}^8_\mu = 0, \qquad \partial_\mu \widehat{A}^0_\mu = \sqrt{6}\,\widehat{\omega}.$$

Finite m_q,

$$\partial_\mu \widehat{A}^8_\mu = \frac{2}{3}\left(\widehat{m}_\ell + 2\widehat{m}_s\right)\widehat{P}^8 - \frac{2\sqrt{2}}{3}\delta\widehat{m}\,\widehat{P}^0,$$
$$\partial_\mu \widehat{A}^0_\mu = \frac{2}{3}\left(2\widehat{m}_\ell + \widehat{m}_s\right)\widehat{P}^0 - \frac{2\sqrt{2}}{3}\delta\widehat{m}\,\widehat{P}^8 + \sqrt{6}\,\widehat{\omega}.$$

with e.g. $\widehat{A}^{8\mu} = \overline{\psi}\gamma_\mu\gamma_5 t^8\psi$ and $\widehat{P}^8 = \overline{\psi}\gamma_5 t^8\psi$, $\psi = (u, d, s)$ and topological charge density

$$\omega(x) = -\frac{1}{16\pi^2}\,\mathrm{tr}\left[F_{\mu\nu}(x)\widetilde{F}_{\mu\nu}(x)\right], \qquad Q_{\rm t} = \int \mathrm{d}^4x\,\omega(x)$$

Witten and Veneziano relation (large N_c or t'Hooft limit):

$$\frac{F_\pi^2}{2N_f}\left(M_\eta^2 + M_{\eta'}^2 - 2M_K^2\right) = \chi_{\rm top}, \qquad \chi_{\rm top} = \langle \hat{Q}_{\rm t}^2\rangle / V,$$

where $\chi_{\rm top}$ is the quenched chiral susceptibility.

Axial decay constants

Local axial-vector currents:

$$\langle 0|\widehat{A}^{a\mu}|\mathcal{M}\rangle = i\,F^a_{\mathcal{M}} p^\mu, \qquad \mathcal{M} = \eta, \eta'$$

Cf. $\langle 0|\widehat{A}^{3\mu}|\pi^0\rangle = i\,F_\pi p^\mu$ (normalized so that $F_\pi = 92$ MeV) where F_π appears in the decay rate for $\pi \to \ell\nu$.

Four independent decay constants:

$$\begin{pmatrix} F^8_\eta & F^0_\eta \\ F^8_{\eta'} & F^0_{\eta'} \end{pmatrix} = \begin{pmatrix} F_8 \cos\theta_8 & -F^0 \sin\theta_0 \\ F^8 \sin\theta_8 & F^0 \cos\theta_0 \end{pmatrix} = \begin{pmatrix} \cos\theta_8 & -\sin\theta_0 \\ \sin\theta_8 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} F^8 & 0 \\ 0 & F^0 \end{pmatrix}$$
$$= \Xi(\theta_8, \theta_0)\,\mathrm{diag}(F^8, F^0)$$

In SU(3) limit $(m_u = m_d = m_s)$: $\theta_8 = \theta_0 = 0$, $F^0_\eta = F^8_{\eta'} = 0$.

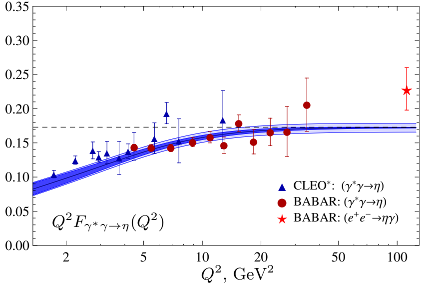
One may also use the "**flavour basis**": $\bar{\ell}\ell = (\bar{u}u + \bar{d}d)/\sqrt{2}$ and $\bar{s}s$:

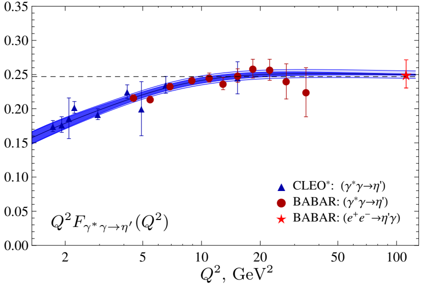
$$\begin{pmatrix} F^\ell_\eta & F^s_\eta \\ F^\ell_{\eta'} & F^s_{\eta'} \end{pmatrix} = \Xi(\theta_\ell, \theta_s)\mathrm{diag}(F^\ell, F^s) = \frac{1}{\sqrt{3}} \begin{pmatrix} F^8_\eta + \sqrt{2}F^0_\eta & -\sqrt{2}F^8_\eta + F^0_\eta \\ \sqrt{2}F^0_{\eta'} + F^8_{\eta'} & F^0_{\eta'} - \sqrt{2}F^8_{\eta'} \end{pmatrix}$$

Note: F^0, F^ℓ, F^s, ϕ_ℓ, ϕ_s depend on the renormalisation scale μ, i.e. $F^0 = F^0(\mu)$ etc.. Flavour basis, at low scales $\phi_\ell \approx \phi_s$ replaced by single ϕ.

$\gamma\gamma^\star \to \eta/\eta'$ form factors

Based on [Agaev,1409.4311] Expt: [BABAR,1101.1142], [CLEO,hep-ex/9707031], not shown [BABAR,1808.08038].





For $\mathcal{M} \in \{\eta, \eta'\}$: **Collinear factorization at large Q^2.**

$$\mathbf{F}_{\gamma^\star\gamma\to\mathcal{M}}(\mathbf{Q^2}) = \frac{\sqrt{2}F^8_{\mathcal{M}}}{3\sqrt{6}} \int_0^1 dx \, \underbrace{T^8_H(x,\mu,Q^2)}_{\text{hard}} \underbrace{\phi^8_{\mathcal{M}}((x,\mu)}_{\text{soft}} +$$

$$\frac{2\sqrt{2}F^0_{\mathcal{M}}}{3\sqrt{3}} \int_0^1 dx \, T^0_H(x,\mu,Q^2)\phi^0_{\mathcal{M}}(x,\mu) + \frac{\sqrt{2}F^0_{\mathcal{M}}}{3\sqrt{3}} \int_0^1 dx \, T^g_H(x,\mu,Q^2)\phi^g_{\mathcal{M}}(x,\mu) \, .$$

$$\lim_{Q^2\to\infty} Q^2 F_{\gamma^*\gamma\to\mathcal{M}}(Q^2) = \frac{2}{\sqrt{3}}\left[\mathbf{F^8_{\mathcal{M}}} + 2\sqrt{2}\mathbf{F^0_{\mathcal{M}}}(\mu_0)\left(1 - \frac{2N_f}{\pi\beta_0}\alpha_s(\mu_0)\right)\right].$$

Gluonic matrix elements

$$\boldsymbol{a_\eta(\mu)} := \langle \Omega | 2\widehat{\omega} | \eta \rangle \qquad \boldsymbol{a_{\eta'}(\mu)} := \langle \Omega | 2\widehat{\omega} | \eta' \rangle$$

Extract $a_\eta(\mu)$ and $a_{\eta'}(\mu)$ via the singlet axial-ward identity.

Relevant for $J/\psi \to \eta(\eta')\gamma$ decays.

If assume $c\bar{c} \to gg\gamma$ dominates the decay [Novikov et al.,1980], [Goldberg,1980].

$$R(J/\psi) = \frac{\Gamma[J/\psi \to \eta'\gamma]}{\Gamma[J/\psi \to \eta\gamma]} \approx \frac{a_{\eta'}^2}{a_\eta^2} \left(\frac{k_{\eta'}}{k_\eta} \right)^3$$

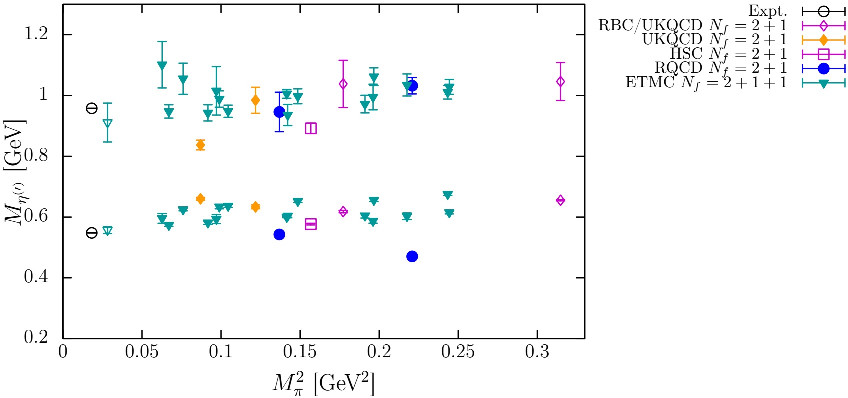
with k_n: momentum of the meson in the rest frame of J/ψ.

A mixing angle can also be defined: ($a_\eta(\mu) = 0$ if $m_s = m_\ell$)

$$\theta_y = -\arctan\left(\frac{a_\eta}{a_{\eta'}} \right)$$

Previous lattice work

Masses:



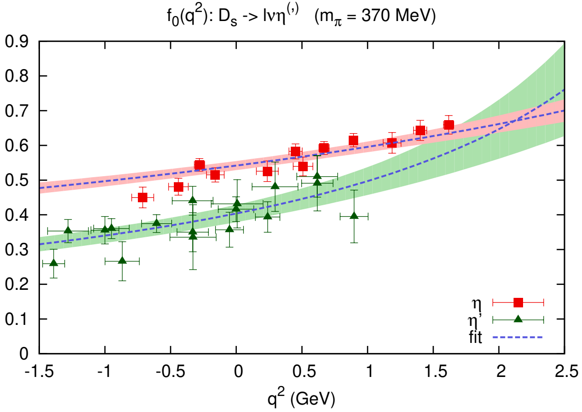
[ETMC,1710.07986]: continuum, chiral extrapolation. Agreement with experiment.

$$M_\eta = 557(11)_{\text{stat}}(03)_{\text{ChPT}}\ \text{MeV} \qquad M_{\eta'} = 911(64)_{\text{stat}}(03)_{\text{ChPT}}\ \text{MeV}$$

Also results for the **decay constants**: use the flavour basis and ChPT with Feldmann-Kroll-Stech scheme [hep-ph/9802409]. **Indirect determination** via pseudoscalar amplitudes [Feldmann,hep-ph/9907491].

Previous lattice work: $D_s \to \eta/\eta' \ell\nu$

[Bali,1406.5449]



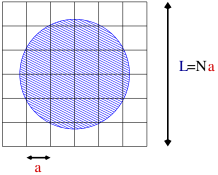
$$|f_0^{D_s\to\eta}| = 0.542(13)_{\text{stat}}, \qquad |f_0^{D_s\to\eta'}| = 0.404(25)_{\text{stat}} \text{ at } M_\pi \approx 370\,\text{MeV}$$

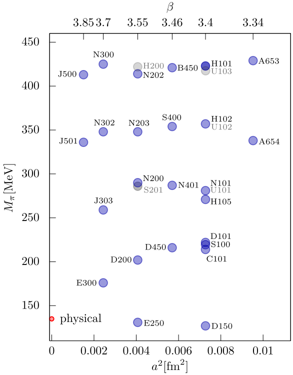
[BESIII,1901.02133]: $f_+^{\eta}(0) = 0.458(7)$, $f_+^{\eta'}(0) = 0.490(51)$

Lattice QCD simulations

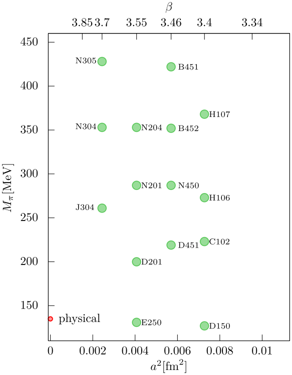
Simulate QCD in **Euclidean space** (imaginary time) on a 4-D lattice,
$S^{cont} = \int d^4x\, \mathcal{L} \to S^{latt} = a^4 \sum_x \mathcal{L}^{latt}$.

Input: $\mathcal{L}^{latt} = \frac{1}{4g^2} FF + \bar{q}_f (\not{D} + m_f) q_f$





$2m_\ell + m_s = \text{const.}$



$m_s = \text{const.}$

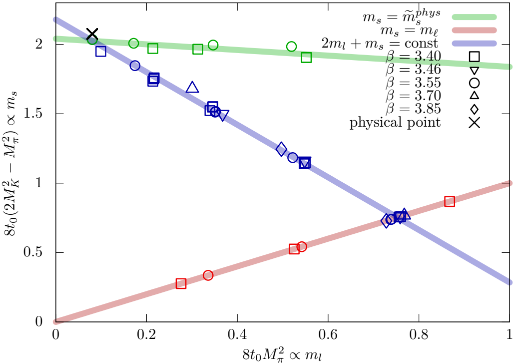
Explore systematics:

Finite lattice spacing a.

(Unphysical) quark mass dependence.

Finite Volume.

CLS $N_f = 2+1$ ensembles: m_ℓ-m_s plane



Simulate in the isospin limit: $m_u = m_d = m_\ell$.

Two trajectories: good control over the quark mass dependence.

$2m_\ell + m_s = \text{const.}$: investigate SU(3) flavour breaking (flavour average quantities roughly constant), approach to physical point involves $m_\pi \downarrow$ and $m_K \uparrow$.

Ensembles on red trajectory ($m_s = m_\ell$) not used here.

Extracting the masses and decay constants

Construct two-point correlation functions:

$$C_{ij}(t) = \frac{1}{N_t}\sum_{t_i=0}^{N_t-1}\langle b_i(t+t_i)b_j^\dagger(t_i)\rangle \longrightarrow$$

$$\boldsymbol{\langle 0|b_i|\eta\rangle}\langle\eta|b_j^\dagger|0\rangle\frac{e^{-\boldsymbol{E_\eta}t}}{2E_\eta V_3} + \boldsymbol{\langle 0|b_i|\eta'\rangle}\langle\eta'|b_j^\dagger|0\rangle\frac{e^{-\boldsymbol{E_{\eta'}}t}}{2E_{\eta'}V_3} + \ldots$$

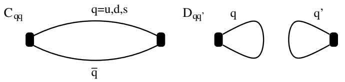
where the interpolators b_j are chosen to have the right QNs: e.g.

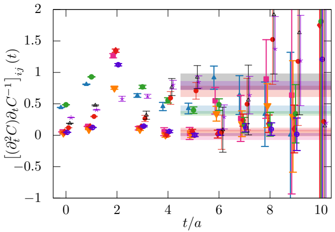
$$b_8 = \frac{1}{\sqrt{6}}(u\gamma_5\bar{u} + d\gamma_5\bar{d} - 2s\gamma_5\bar{s}), \qquad b_0 = \frac{1}{\sqrt{3}}(u\gamma_5\bar{u} + d\gamma_5\bar{d} + s\gamma_5\bar{s})$$

and the axial vector equivalents. Note: could also use $b^\ell \propto (u\gamma_5\bar{u} + d\gamma_5\bar{d})$ and $b^s \propto s\gamma_5\bar{s}$.

[RQCD,2106.05398]

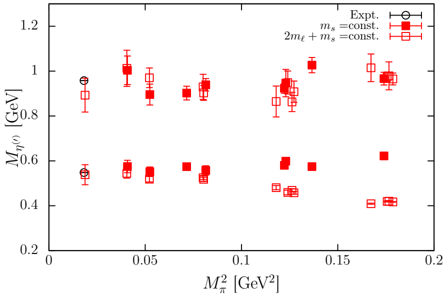
Signal for $C_{ij}(t)$ rapidly falls below the noise as t increases due to disconnected quark line diagrams.

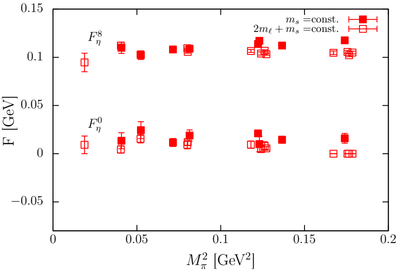


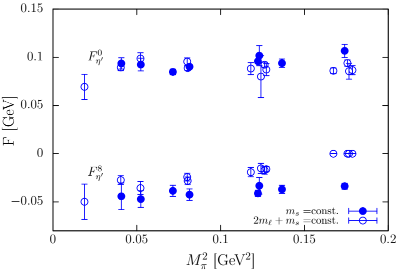


Mass spectrum and decay constants

22 ensembles, $M_\pi = 420 - 135$ MeV, $a = 0.086$, 0.076, 0.064 fm (and 0.050 fm), $M_\pi L \gtrsim 4$







Physical point extrapolation

Results depend on the quark masses (equivalently M_π and M_K) and the lattice spacing.

Fit to the masses and decay constants

$$f_O(a, \overline{M}^2, \delta M^2) = f_O^{\mathrm{cont}}(\overline{M}^2, \delta M^2) \qquad \text{continuum}$$
$$\times\, h_O(a, am_\ell, am_s, a^2/t_0^*, a^2\overline{M}^2, a^2\delta M^2) \qquad O(a), O(a^2)$$

where $O \in \{M_\eta, M_{\eta'}, F_\eta^8, F_\eta^0, F_{\eta'}^8, F_{\eta'}^0\}$ and

$$\overline{M}^2 := \frac{1}{3}(2M_K^2 + M_\pi^2) \approx 2B_0\overline{m}, \quad \delta M^2 := 2(M_K^2 - M_\pi^2) \approx 2B_0\delta m,$$

where $\overline{m}$ is the average quark mass, $\delta m = m_s - m_\ell$ and $B_0 = -\langle \bar{u}u \rangle / F^2 > 0$.

- Parametrize quark mass dependence using large-N_c ChPT.
- Same parameters (low energy constants) appear in large-N_c ChPT expressions for the masses and decay constants $\rightarrow$ perform a simultaneous fit.
- Test of large-N_c ChPT.

Large-N_c ChPT

U(3) EFT, η' becomes a pseudo-Goldstone boson in the t'Hooft limit.
Expansion: $p^2 = O(\delta)$, $m = O(\delta)$, $1/N_c = O(\delta)$.

Known up to NNLO, e.g. [Guo,1503.02248] [Bickert,1612.05473], use NLO.
Contribution of the η_8/η_0 sector to the leading order Large-N_c ChPT Lagrangian ($\eta^\intercal = (\eta_8, \eta_0)$):

$$\mathscr{L} = \ldots + \frac{1}{2}\partial_\mu \eta^\intercal \partial^\mu \eta - \frac{1}{2}\eta^\intercal \mu^2 \eta, \quad \mu^2 = \begin{pmatrix} \mu_8^2 & \mu_{80}^2 \\ \mu_{80}^2 & \mu_0^2 \end{pmatrix}.$$

$$R\mu^2 R^\intercal = \begin{pmatrix} M_\eta^2 & 0 \\ 0 & M_{\eta'}^2 \end{pmatrix}, \quad R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

At leading order:

$$\mu_8^2 = 2B_0(m_\ell + 2m_s) = \overline{M}^2 + \frac{1}{3}\delta M^2, \quad \mu_0^2 = 2B_0(2m_\ell + m_s) + M_0^2 = \overline{M}^2 + M_0^2,$$

$$\mu_{80}^2 = -\frac{2\sqrt{2}}{3}B_0(m_s - m_\ell) = -\frac{\sqrt{2}}{3}\delta M^2, \quad \tan(2\theta) = -2\sqrt{2}\frac{\delta M^2}{3M_0^2 - \delta M^2}.$$

and $M_0^2 = 2N_f \chi_{\text{top}}/F_\pi^2$.

NLO Large-N_c ChPT

$$(\mu_8^{\rm NLO})^2 = \overline{M}^2 + \frac{1}{3}\delta M^2 + \frac{8}{3F^2}\left(2\mathbf{L_8} - \mathbf{L_5}\right)\delta M^4,$$

$$(\mu_0^{\rm NLO})^2 = \overline{M}^2 + M_0^2 + \frac{4}{3F^2}\left(2\mathbf{L_8} - \mathbf{L_5}\right)\delta M^4 - \frac{8}{F^2}\mathbf{L_5}\overline{M}^2 M_0^2 - \tilde{\mathbf{\Lambda}}\overline{M}^2 - \mathbf{\Lambda_1} M_0^2,$$

$$(\mu_{80}^{\rm NLO})^2 = -\frac{\sqrt{2}}{3}\delta M^2 - \frac{4\sqrt{2}}{3F^2}\left(2\mathbf{L_8} - \mathbf{L_5}\right)\delta M^4 + \frac{4\sqrt{2}}{3F^2}\mathbf{L_5} M_0^2 \delta M^2 + \frac{\sqrt{2}}{6}\tilde{\mathbf{\Lambda}}\delta M^2.$$

where $\tilde{\Lambda} = \Lambda_1(\mu) - 2\Lambda_2(\mu)$ is scale-independent and $M_0 = M_0(\mu)$. No chiral logs/ChPT renormalization scale at this order!

$$F_\eta^8 = F\left[\cos\theta + \frac{4\mathbf{L_5}}{3F^2}\left(3\cos\theta\overline{M}^2 + (\sqrt{2}\sin\theta + \cos\theta)\delta M^2\right)\right],$$

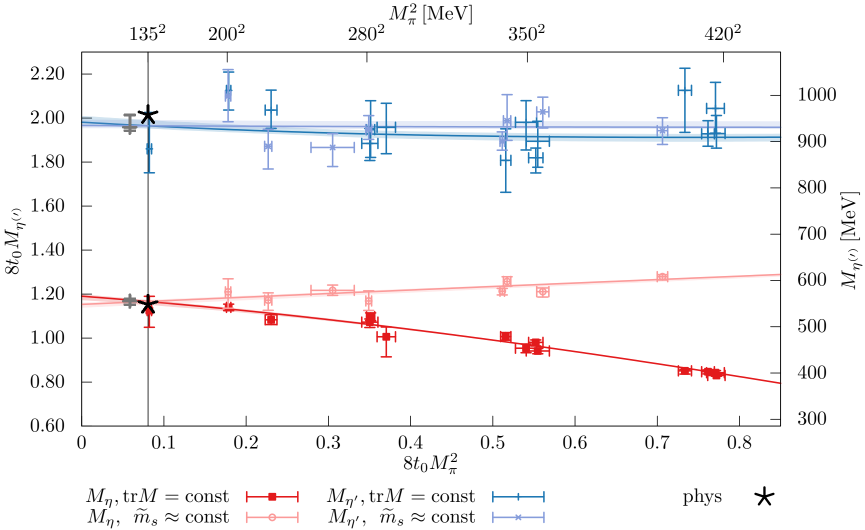
$$F_{\eta'}^8 = F\left[\sin\theta + \frac{4\mathbf{L_5}}{3F^2}\left(3\sin\theta\overline{M}^2 + (\sin\theta - \sqrt{2}\cos\theta)\delta M^2\right)\right],$$

$$F_\eta^0 = -F\left[\sin\theta\left(1 + \frac{\mathbf{\Lambda_1}}{2}\right) + \frac{4\mathbf{L_5}}{3F^2}\left(3\sin\theta\overline{M}^2 + \sqrt{2}\cos\theta\delta M^2\right)\right],$$

$$F_{\eta'}^0 = F\left[\cos\theta\left(1 + \frac{\mathbf{\Lambda_1}}{2}\right) + \frac{4\mathbf{L_5}}{3F^2}\left(3\cos\theta\overline{M}^2 - \sqrt{2}\sin\theta\delta M^2\right)\right].$$

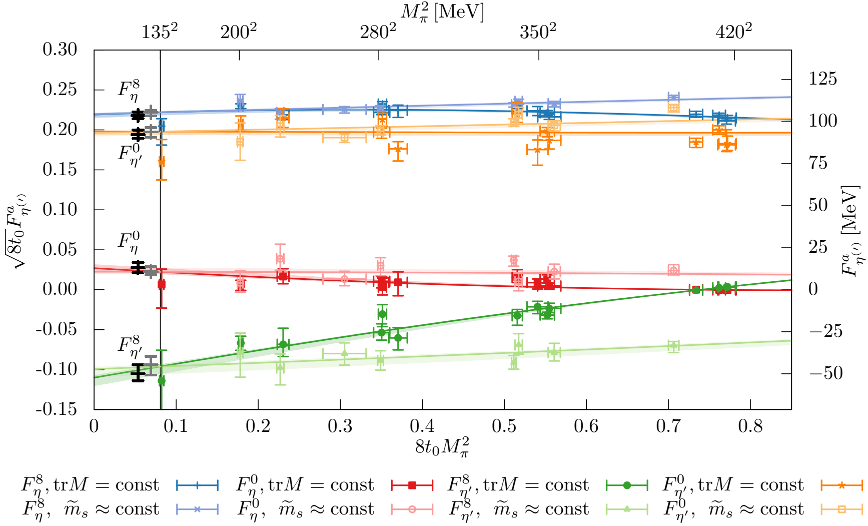
LECs common to masses and decay constants.

Physical point results: masses



Data points are shifted to remove discretisation effects.

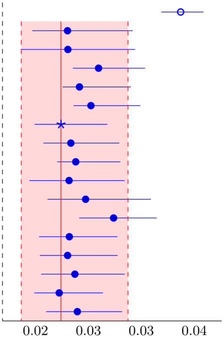
Physical point results: decay constants

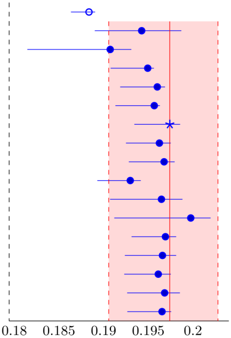


Combined, fully correlated fit gives $\chi^2/N_{\mathrm{df}} \approx 179/122 \approx 1.47$

Systematics

- Volume: only large volumes: $L_s^3 > (2.2\,\mathrm{fm})^3 \gg R_\eta^3 \approx R_\pi^3$ [Bernstein,1511.03242] and typically $L_s M_\pi > 4$.
- Lattice spacing: vary parametrization of discretization effects.
- NLO Large-N_c ChPT: impose cutoffs on the average (non-singlet) pseudoscalar mass: $\overline{M}^2 \leq \overline{M}^2_{\mathrm{max}}$, $12t_0\overline{M}^2_{\mathrm{max}} \in \{1.2, 1.4, 1.6\}$.
- Renormalization: matching to PT done at $\mu \in \{a^{-1}/2, a^{-1}, 2a^{-1}\}$.

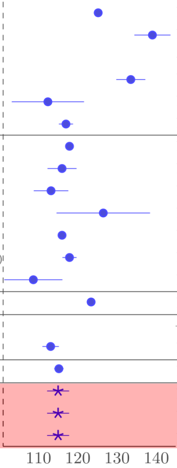


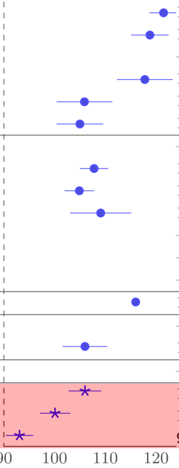


id	$\sqrt{8t_0^{\mathrm{ph}}}F_\eta^0$	$\sqrt{8t_0^{\mathrm{ph}}}F_{\eta'}^0$
1	$0.0337\,(^{22}_{18})$	$0.1884\,(^{7}_{20})$
2	$0.0230\,(^{62}_{33})$	$0.1943\,(^{44}_{53})$
3	$0.0231\,(^{64}_{45})$	$0.1907\,(^{24}_{93})$
4	$0.0260\,(^{44}_{24})$	$0.1949\,(^{7}_{42})$
5	$0.0242\,(^{49}_{16})$	$0.1960\,(^{9}_{42})$
6	$0.0252\,(^{47}_{16})$	$0.1957\,(^{6}_{44})$
7	$0.0224\,(^{44}_{25})$	$0.1974\,(^{12}_{40})$
8	$0.0233\,(^{46}_{26})$	$0.1962\,(^{13}_{37})$
9	$0.0238\,(^{42}_{17})$	$0.1968\,(^{12}_{39})$
10	$0.0232\,(^{53}_{38})$	$0.1930\,(^{12}_{37})$
11	$0.0247\,(^{62}_{36})$	$0.1965\,(^{24}_{58})$
12	$0.0274\,(^{41}_{32})$	$0.1998\,(^{22}_{86})$
13	$0.0232\,(^{45}_{29})$	$0.1969\,(^{12}_{38})$
14	$0.0230\,(^{47}_{26})$	$0.1966\,(^{15}_{42})$
15	$0.0237\,(^{47}_{32})$	$0.1961\,(^{14}_{38})$
16	$0.0222\,(^{41}_{24})$	$0.1968\,(^{17}_{42})$
17	$0.0240\,(^{42}_{29})$	$0.1965\,(^{10}_{39})$

Physical point results

ref	F^8/MeV	F^0/MeV
Benayoun et al. [101]	125.2(9)	121.5(2.8)
Escribano and Frere [102]	139.0(4.6)	118.8(3.7)
Escribano et al. [103]	– – –	—
Chen et al. [104]	133.5(3.7)	117.8(5.5)
Escribano et al. [105]	112.4(9.2)	105.9(5.5)
Escribano et al. [98]	117.0(1.8)	105.0(4.6)
Leutwyler [6]	118	—
Feldmann [97]	116.0(3.7)	107.8(2.8)
Guo et al. [81] NLO-A	113.2(4.4)	104.9(2.9)
Guo et al. [81] NNLO-B	126(12)	109.1(6.0)
Bickert et al. [42] NLO-I	116.0(9)	—
[42] NNLO w/o Ci ($\mu_{\rm EFT} = 1$GeV)	117.9(1.8)	—
[42] NNLO w/ Ci ($\mu_{\rm EFT} = 1$GeV)	109(7)	—
Ding et al. [106]	123.4	116.0
ETMC [19]	—	—
Gu et al. [107] NNLO-A9p(F_π)	113.1(2.1)	106.0(4.4)
eq. (7.16)	115.2(1.2)	—
this work ($\mu = 1$GeV)	115.0(2.8)	106.0(3.2)
this work ($\mu = 2$ GeV)	115.0(2.8)	100.1(3.0)
this work ($\mu = \infty$)	115.0(2.8)	93.1(2.7)

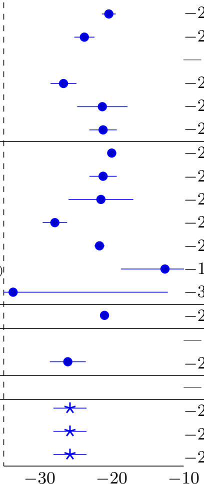


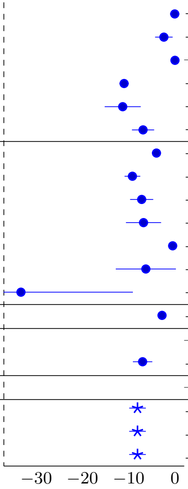


$$F^8 = \sqrt{(F^8_\eta)^2 + (F^8_{\eta'})^2}, \quad F^0 = \sqrt{(F^0_\eta)^2 + (F^0_{\eta'})^2}$$

Physical point results

ref	θ_8	θ_0
Benayoun et al. [101]	$-20.4(1.0)^\circ$	$-0.1(1.0)^\circ$
Escribano and Frere [102]	$-23.8(1.4)^\circ$	$-2.4(1.9)^\circ$
Escribano et al. [103]	—	—
Chen et al. [104]	$-26.7(1.8)^\circ$	$-11.0(1.0)^\circ$
Escribano et al. [105]	$-21.3(3.5)^\circ$	$-11.3(3.9)^\circ$
Escribano et al. [98]	$-21.2(1.9)^\circ$	$-6.9(2.4)^\circ$
Leutwyler [6]	-20°	-4°
Feldmann [97]	$-21.2(1.9)^\circ$	$-9.2(1.7)^\circ$
Guo et al. [81] NLO-A	$-21.5(4.5)^\circ$	$-7.2(2.5)^\circ$
Guo et al. [81] NNLO-B	$-27.9(1.7)^\circ$	$-6.8(3.8)^\circ$
Bickert et al. [42] NLO-I	$-21.7(7)^\circ$	$-0.5(7)^\circ$
[42] NNLO w/o Ci ($\mu_{\mathrm{EFT}} = 1\mathrm{GeV}$)	$-12.6(6.1)^\circ$	$-6.3(6.5)^\circ$
[42] NNLO w/ Ci ($\mu_{\mathrm{EFT}} = 1\mathrm{GeV}$)	$-34(22)^\circ$	$-33(24)^\circ$
Ding et al. [106]	-21°	-2.8°
ETMC [19]	—	—
Gu et al. [107] NNLO-A9p(F_π)	$-26.1(2.5)^\circ$	$-7.0(2.1)^\circ$
eq. (7.16)	—	—
this work ($\mu = 1\mathrm{GeV}$)	$-25.8(2.3)^\circ$	$-8.1(1.8)^\circ$
this work ($\mu = 2\,\mathrm{GeV}$)	$-25.8(2.3)^\circ$	$-8.1(1.8)^\circ$
this work ($\mu = \infty$)	$-25.8(2.3)^\circ$	$-8.1(1.8)^\circ$

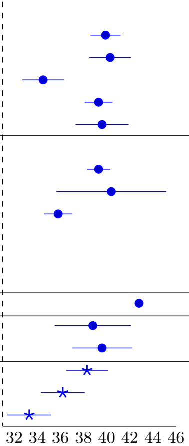


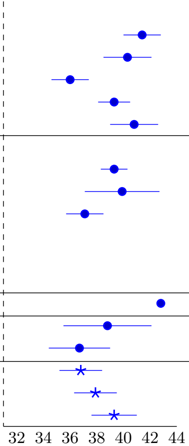


$$\tan\theta_8 = F^8_{\eta'}/F^8_{\eta}, \quad \tan\theta_0 = -F^0_{\eta}/F^8_{\eta'}.$$

Physical point results

ref	ϕ_ℓ	ϕ_s
Benayoun et al. [101]	—	—
Escribano and Frere [102]	$39.9(1.3)^\circ$	$41.4(1.4)^\circ$
Escribano et al. [103]	$40.3(1.8)^\circ$	$40.3(1.8)^\circ$
Chen et al. [104]	$34.5(1.8)^\circ$	$36.0(1.4)^\circ$
Escribano et al. [105]	$39.3(1.2)^\circ$	$39.2(1.2)^\circ$
Escribano et al. [98]	$39.6(2.3)^\circ$	$40.8(1.8)^\circ$
Leutwyler [6]	—	—
Feldmann [97]	$39.3(1.0)^\circ$	$39.3(1.0)^\circ$
Guo et al. [81] NLO-A	$40.4(4.8)^\circ$	$39.9(2.8)^\circ$
Guo et al. [81] NNLO-B	$35.8(1.2)^\circ$	$37.1(1.4)^\circ$
Bickert et al. [42] NLO-I	—	—
[42] NNLO w/o Ci ($\mu_{\rm EFT} = 1$GeV)	—	—
[42] NNLO w/ Ci ($\mu_{\rm EFT} = 1$GeV)	—	—
Ding et al. [106]	42.8°	42.8°
ETMC [19]	$38.8(3.3)^\circ$	$38.8(3.3)^\circ$
Gu et al. [107] NNLO-A9p(F_π)	$39.6(2.6)^\circ$	$36.7(2.3)^\circ$
this work ($\mu = 1$GeV)	$38.3(1.8)^\circ$	$36.8(1.6)^\circ$
this work ($\mu = 2$ GeV)	$36.2(1.9)^\circ$	$37.9(1.6)^\circ$
this work ($\mu = \infty$)	$33.3(2.0)^\circ$	$39.3(1.7)^\circ$





$$\tan\theta_\ell = F^\ell_{\eta'}/F^\ell_\eta, \quad \tan\theta_s = -F^s_\eta/F^s_{\eta'}.$$

At low scales $\theta_\ell \approx \theta_s$. Agreement with ETMC results also for $F^{\ell,s}$.

Large-N_c ChPT LECs

Our LECs in the continuum limit, all errors added in quadrature:

$$
\begin{aligned}
M_0(\mu = 2\,\mathrm{GeV}) &= 818(27)\,\mathrm{MeV}, & F &= 87.7(2.8)\,\mathrm{MeV},\\
\Lambda_1(\mu = 2\,\mathrm{GeV}) &= -0.13(5), & L_5 &= 1.66(23)\cdot 10^{-3},\\
\Lambda_2(\mu = 2\,\mathrm{GeV}) &= 0.19(10), & L_8 &= 1.08(13)\cdot 10^{-3}.
\end{aligned}
$$

Scale-independent combinations:

$$
M_0/\sqrt{1+\Lambda_1} = 877(22)\,\mathrm{MeV}, \quad \tilde{\Lambda} = \Lambda_1 - 2\Lambda_2 = -0.46(19).
$$

Scale-independent: F^8, θ_8, θ_0. Scale-dependent: F^0, F^ℓ, F^s, ϕ_ℓ, ϕ_s.

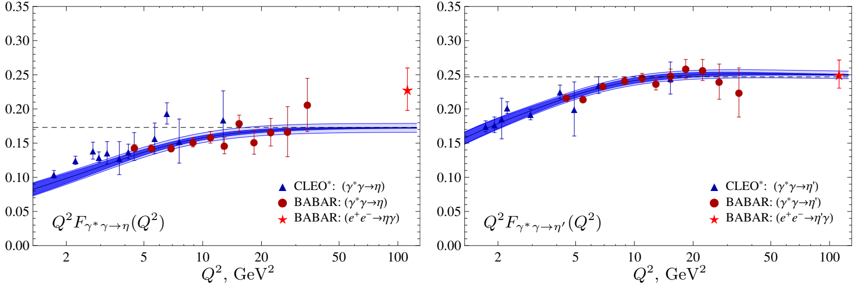
In the Feldmann-Kroll-Stech model [Feldmann,hep-ph/9907491], NLO LEC $\Lambda_1(\mu) = 0$ and any scale dependence is neglected. Then $\phi = \phi_\ell = \phi_s$ and

$$
\sin^2\phi = \frac{\left(M_{\eta'}^2 - (2M_K^2 - M_\pi^2)\right)(M_\eta^2 - M_\pi)}{2(M_{\eta'}^2 - M_\eta^2)(M_K^2 - M_\pi^2)}, \quad F^\ell = F_\pi, \quad F^s = \sqrt{2F_K^2 - F_\pi^2}.
$$

We find: $F^\ell(\mu = 2\mathrm{GeV}) = 88(^5_3)$ MeV cf. $F_\pi = 92.1$ MeV,
$F^s(\mu = 2\mathrm{GeV}) = 124(^4_5)$ MeV cf. $\sqrt{2F_K^2 - F_\pi^2} \sim 125$ MeV.

$\gamma\gamma^\star \to \eta/\eta'$ form factors

Based on [Agaev,1409.4311] Expt: [BABAR,1101.1142], [CLEO,hep-ex/9707031], not shown [BABAR,1808.08038].



Use $F^{0,8}_{\eta,\eta'}$ to obtain dashed lines:

$$\lim_{Q^2\to\infty} Q^2 F_{\gamma\gamma^*\to\eta}(Q^2) = 160.5(10.0)\,\mathrm{MeV},$$

$$\lim_{Q^2\to\infty} Q^2 F_{\gamma\gamma^*\to\eta'}(Q^2) = 230.5(10.1)\,\mathrm{MeV}.$$

Gluonic matrix elements from fermions

All factors needed to renormalise $\widehat{\omega}$ are not known $\rightarrow$ obtain renormalized gluonic matrix elements through the singlet AWI.

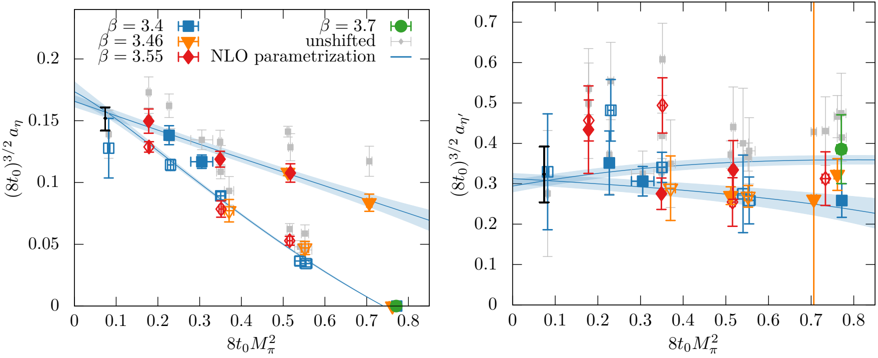
$$\partial_\mu \widehat{A}^0_\mu = \frac{2}{3}\left(2\widehat{m}_\ell + \widehat{m}_s\right)\widehat{P}^0 - \frac{2\sqrt{2}}{3}\delta\widehat{m}\,\widehat{P}^8 + \sqrt{6}\,\widehat{\omega}.$$

$\delta\widehat{m} = \widehat{m}_s - \widehat{m}_\ell$. Use the singlet decay constants F^0_n ($n \in \{\eta, \eta'\}$) and pseudoscalar matrix elements H^0_n and H^8_n

$$\begin{aligned} a_n(\mu) &:= \langle\Omega|2\widehat{\omega}|n\rangle \\ &= \sqrt{\frac{2}{3}} M_n^2 F^0_n(\mu) + \frac{4}{3\sqrt{3}}\delta\widehat{m}\, H^8_n - \frac{2}{3}\sqrt{\frac{2}{3}}(2\widehat{m}_\ell + \widehat{m}_s) H^0_n. \end{aligned}$$

Note that $\widehat{m}H^8_n = Z_A\widetilde{m}\langle\Omega|P^8|n\rangle$, $\widehat{m}H^0_n = Z_A r_P \widetilde{m}\langle\Omega|P^0|n\rangle$, $r_P = 1 + \mathcal{O}(g^6)$.

Gluonic matrix elements from the singlet AWI



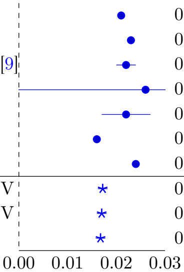
Parametrization is NLO U(3) Large-N_c ChPT. 6 LECs (with priors from analysis of decay constants) plus 3 parameters to account for $\mathcal{O}(a)$ effects. $\chi^2/N_{\mathrm{df}} \approx 34/31$. Shown: $\mu = \infty$.

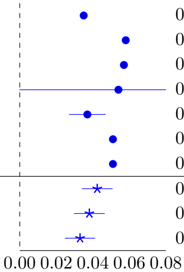
$$a_\eta(\mu = 2\,\mathrm{GeV}) = 0.01700\,\binom{40}{69}_{\mathrm{stat}}\,(48)_{\mathrm{syst}}\,(66)_{t_0}\ \mathrm{GeV}^3,$$
$$a_{\eta'}(\mu = 2\,\mathrm{GeV}) = 0.0381\,\binom{18}{17}_{\mathrm{stat}}\,(80)_{\mathrm{syst}}\,(17)_{t_0}\ \mathrm{GeV}^3.$$

Systematic error due to difference with prediction obtained using NLO Large-N_c ChPT LECs. Also $\theta_y = -\arctan a_\eta / a_{\eta'} = -24.0(3.3)^\circ$.

Comparison with the literature

ref	a_η/GeV^3	$a_{\eta'}/\mathrm{GeV}^3$
Novikov et al. [116]	0.021	0.035
Feldmann [97]	0.023	0.058
Beneke and Neubert [9]	0.022(2)	0.057(2)
Cheng et al. [118]	0.026(28)	0.054(57)
Singh [117]	0.0220(50)	0.037(10)
Qin et al. [119]	0.016	0.051
Ding et al. [106]	0.024	0.051
this work at $\mu = 1\mathrm{GeV}$	0.0172(10)	0.0424(84)
this work at $\mu = 2\mathrm{GeV}$	0.0170(10)	0.0381(84)
this work at $\mu = \infty$	0.0168(10)	0.0330(83)





Systematics from parametrization, renormalization and scale setting included.

If anomaly dominates [Novikov et al., NPB165(80)55]:

$$R(J/\psi) = \frac{\Gamma[J/\psi \to \eta'\gamma]}{\Gamma[J/\psi \to \eta\gamma]} \approx \frac{a_{\eta'}^2}{a_\eta^2} \left(\frac{k_{\eta'}}{k_\eta}\right)^3$$

with k_n: momentum of the meson in the rest frame of J/ψ. From this:

$$R(J/\psi, \mu = 2\,\mathrm{GeV}) = 5.03\,(^{19}_{45})_{\mathrm{stat}}\,(1.94)_{\mathrm{syst}}\,, \quad \text{PDG:} \quad R(J/\psi) = 4.74(13).$$

Summary

- Lattice studies of the η and η' mesons are challenging. Enormous progress has been made and results are now extracted at the physical point in the continuum limit.
- Meson masses determined with 1.7% error on M_η and 2.3% error on $M_{\eta'}$. Agreement with the experimental masses.
- First direct lattice determination of the η and η' decay constants.
- Determination of gluonic matrix elements via the singlet axial Ward identity.
- NLO Large-N_c ChPT describes all data (two meson masses, four decay constants, two gluonic matric elements) reasonably well with just six LECs, but there are some tensions. NNLO?
- The Feldmann-Kroll-Stech model works OK where $\Lambda_1(\mu)$ is small ($\mu \in [0.8, 1.5]$ GeV).
- Other properties will be computed in the future.