Vortices, Superfluid turbulence
& Condensate Formation in Bose Gases

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Center for Quantum Dynamics

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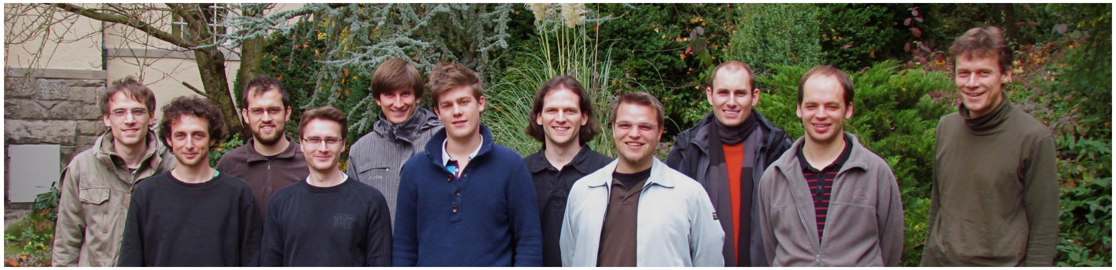
Thanks & credits to...



...my work group in Heidelberg:

Sebastian Bock
Sebastian Erne
Martin Gärttner
Roman Hennig
Markus Karl
Steven Mathey
Boris Nowak
Nikolai Philipp
Maximilian Schmidt
Jan Schole
Dénes Sexty
Martin Trappe
Jan Zill

...my former students:

Cédric Bodet (→ NEC), Alexander Branschädel (→ KIT Karlsruhe), Stefan Keßler (→ U Erlangen), Matthias Kronenwett (→ R. Berger), **Christian Scheppach** (→ Cambridge, UK), Philipp Struck (→ Konstanz), Kristan Temme (→ Vienna)

€€€...

Alexander von Humboldt
Stiftung / Foundation

DFG
Heisenberg-Programm
Deutsche Forschungsgemeinschaft

RUPRECHT-KARLS-
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HEIDELBERG

LGFG BaWue

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HELMHOLTZ
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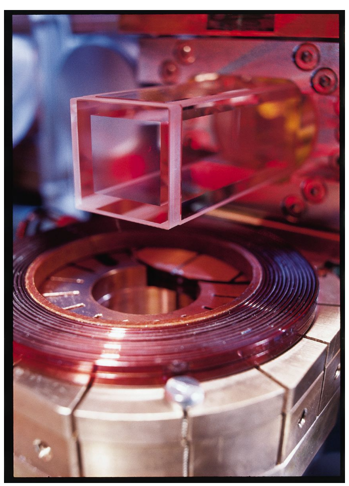
ULTRAcool...

... atoms @ nanokelvins –

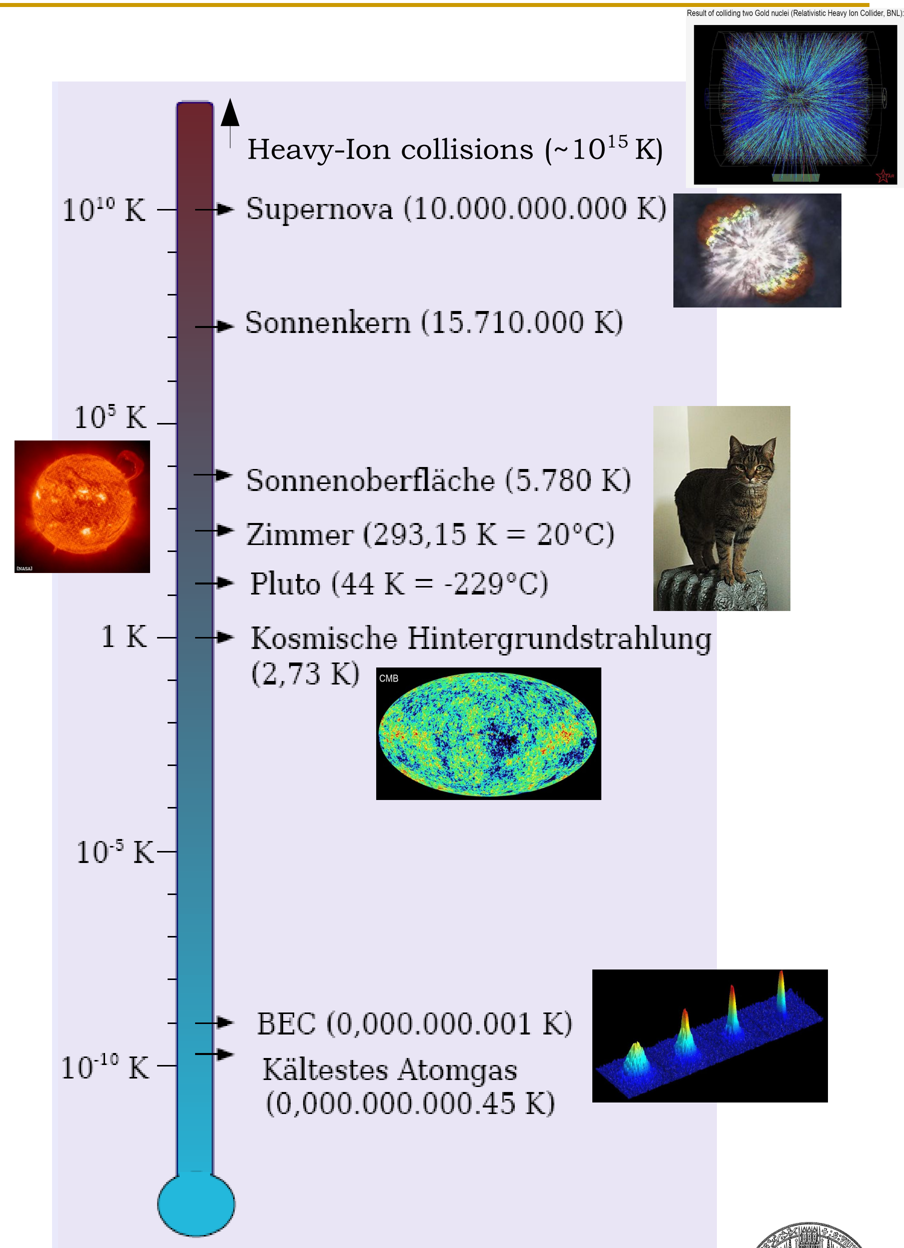
trapped only a few mm away from

glass cell @ room temperature



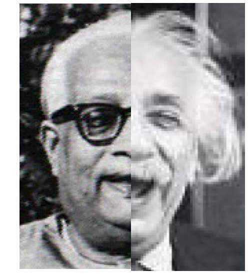
(vacuum of 10^{-12} Torr,
i.e. 10^{-15} bar,
or 10^{-10} Pa,

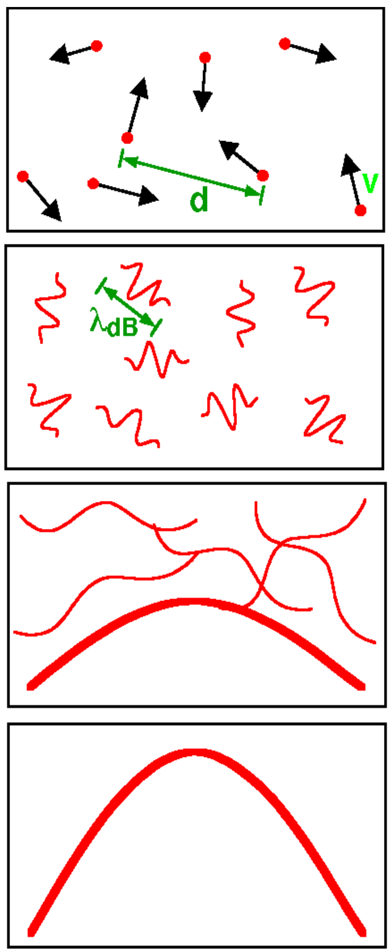
≈ atmospheric pressure on the moon)



Condensate formation in an ultracold Bose gas

Bose-Einstein condensation





High Temperature T:
thermal velocity v
density d^{-3}
"Billiard balls"

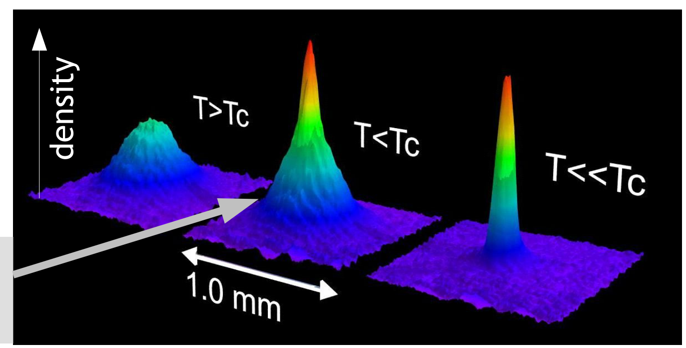
Low Temperature T:
De Broglie wavelength
$\lambda_{dB} = h/mv \propto T^{-1/2}$
"Wave packets"

$T = T_{crit}$:
Bose-Einstein Condensation
$\lambda_{dB} \approx d$
"Matter wave overlap"

T=0:
Pure Bose condensate
"Giant matter wave"

Experimental picture after free expansion of the trapped cloud:

Bose-Einstein condensation (BEC)



bimodal distribution

Equilibration

Transient, metastable state
e.g. Turbulence
Non-thermal fixed point

Superfluid hydro of Bose-condensed Gas

The Gross-Pitaevskii Equation, ($g = 4\pi a_0/m$)

$$i\frac{\partial\Psi(\boldsymbol{\rho},t)}{\partial t} = \left(-\frac{\nabla^2}{2} + g|\Psi(\boldsymbol{\rho},t)|^2\right)\Psi(\boldsymbol{\rho},t)$$

using defs.

$$\Psi(\boldsymbol{\rho},t) = \sqrt{n(\boldsymbol{\rho},t)}\exp[i\varphi(\boldsymbol{\rho},t)]$$

$$Q = gn \qquad \mathbf{u}(\boldsymbol{\rho},t) = \nabla\varphi(\boldsymbol{\rho},t)$$

can be written as

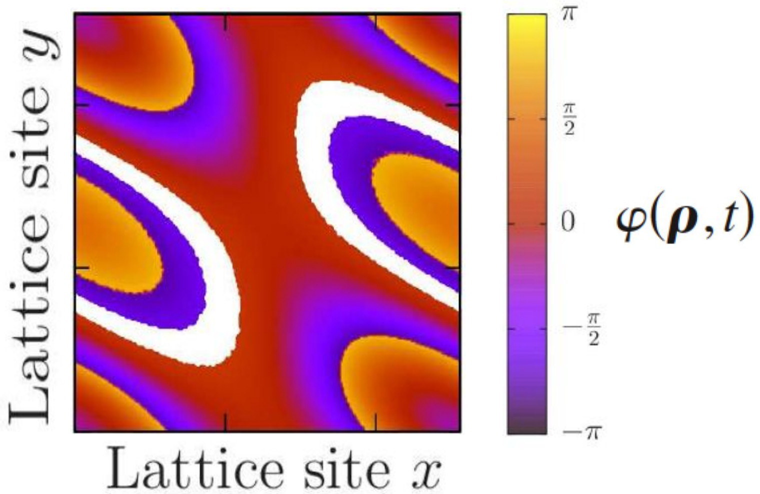
$$\frac{\partial}{\partial t}n + \nabla\cdot(n\mathbf{u}) = 0 \qquad \frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla Q$$

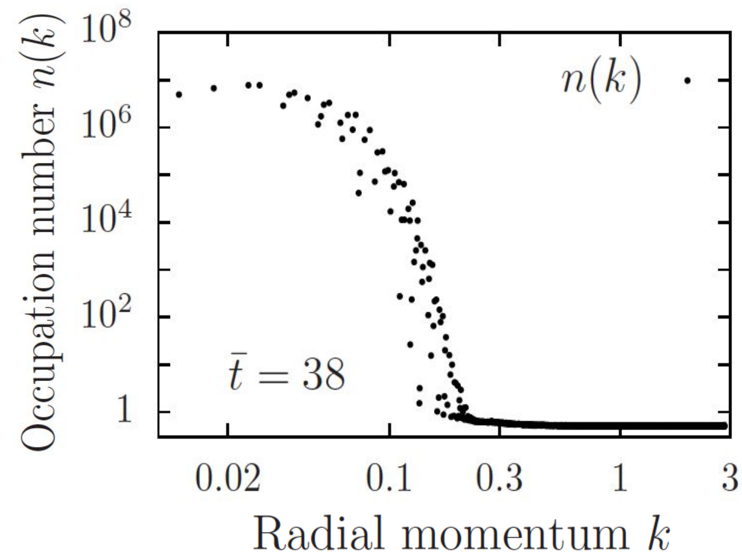
Euler equation

Movie 1: Phase evolution & Spectrum

$$\Psi(\boldsymbol{\rho},t) = \sqrt{n(\boldsymbol{\rho},t)}\exp[i\varphi(\boldsymbol{\rho},t)]$$

$$n(k) = \langle \Psi^*(\mathbf{k})\Psi(\mathbf{k})\rangle\big|_{\text{angle average}}$$





http://www.thphys.uni-heidelberg.de/~smp/gasenzer/videos/boseqt.html

Movie by Jan Schole

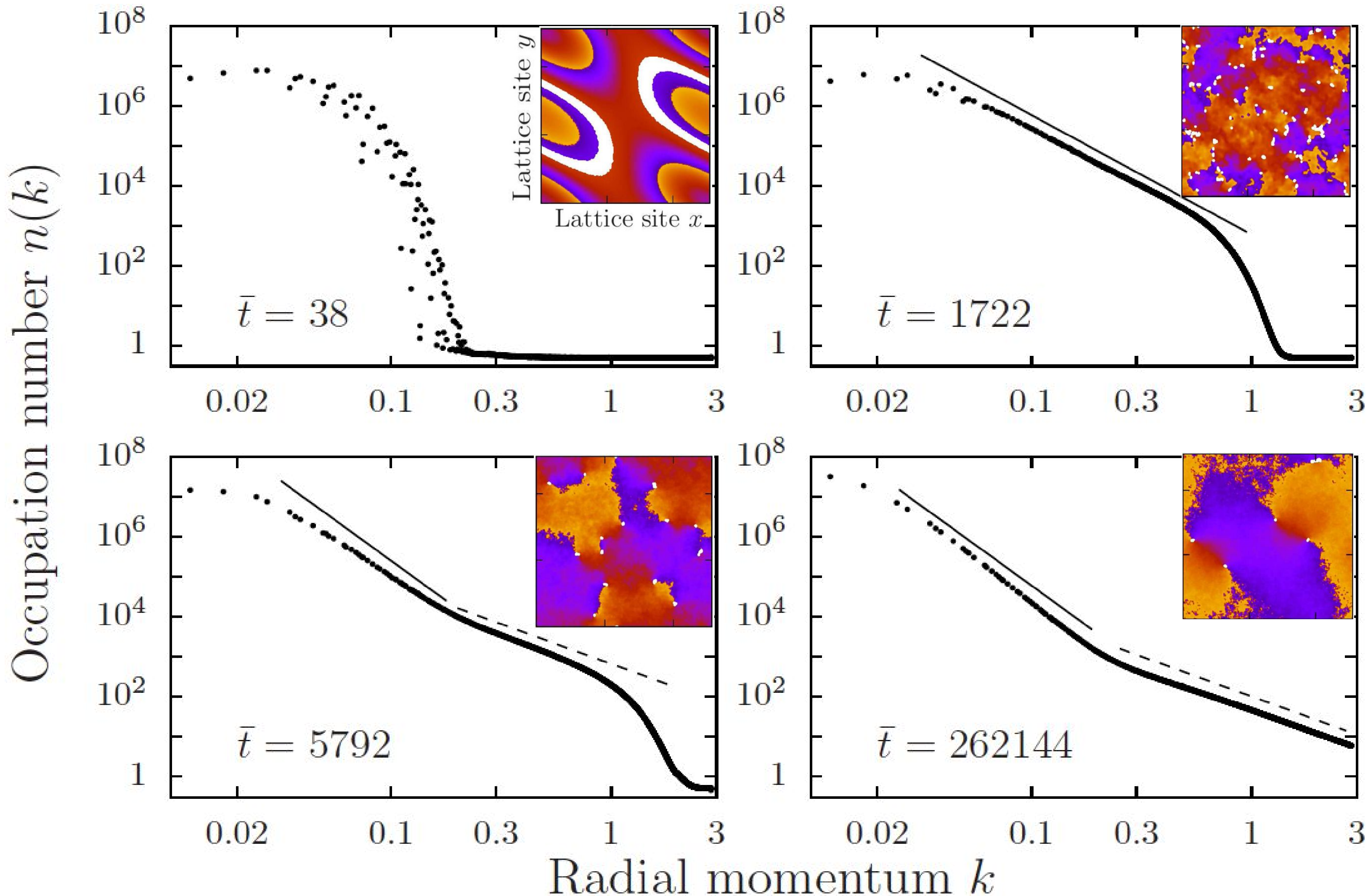
Movie 2: Vortex “gas” & Spectrum

$$n(k) = \left\langle \Psi^*(\mathbf{k})\Psi(\mathbf{k}) \right\rangle\Big|_{\text{angle average}}$$

http://www.thphys.uni-heidelberg.de/~smp/gasenzer/videos/boseqt.html

B. Nowak, D. Sexty, TG, PRB **84**: 020506(R), 2011

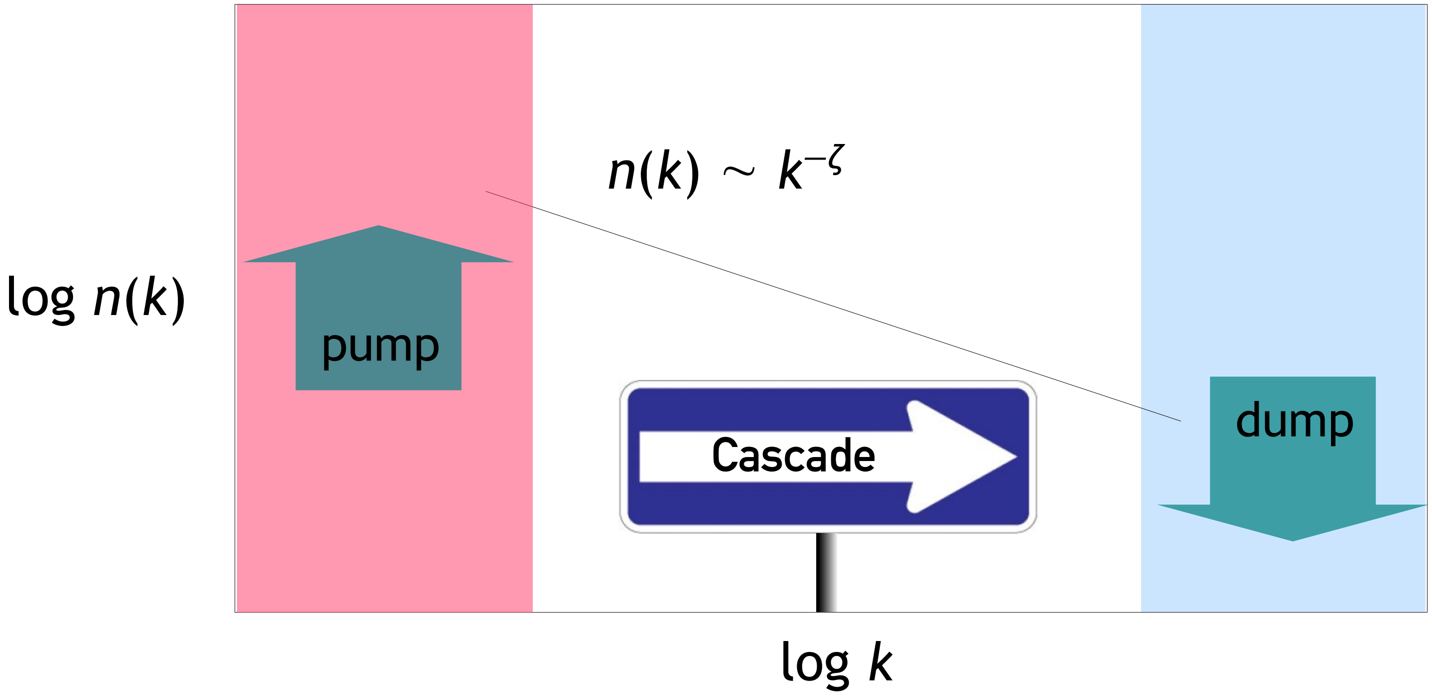
Spectrum in 2+1 D



Wave turbulence in an ultracold Bose gas

Wave turbulence

Stationary scaling $n(k)$ within inertial region:



Dilute ultracold Bose Gas

Gross-Pitaevskii Equation: ($g = 4\pi a_0/m$)

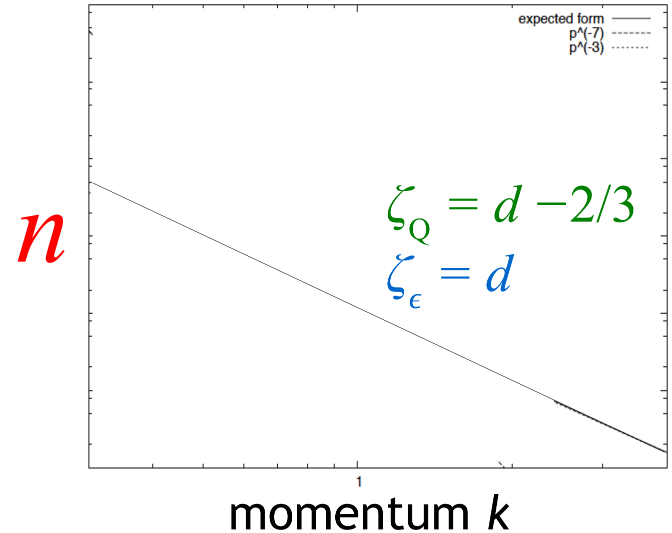
$$i\frac{\partial\Psi(\boldsymbol{\rho},t)}{\partial t} = \left(-\frac{\nabla^2}{2} + g|\Psi(\boldsymbol{\rho},t)|^2\right)\Psi(\boldsymbol{\rho},t)$$

Momentum spectrum:

$$n(\mathbf{k}) = \langle \Psi^*(\mathbf{k})\Psi(\mathbf{k})\rangle$$

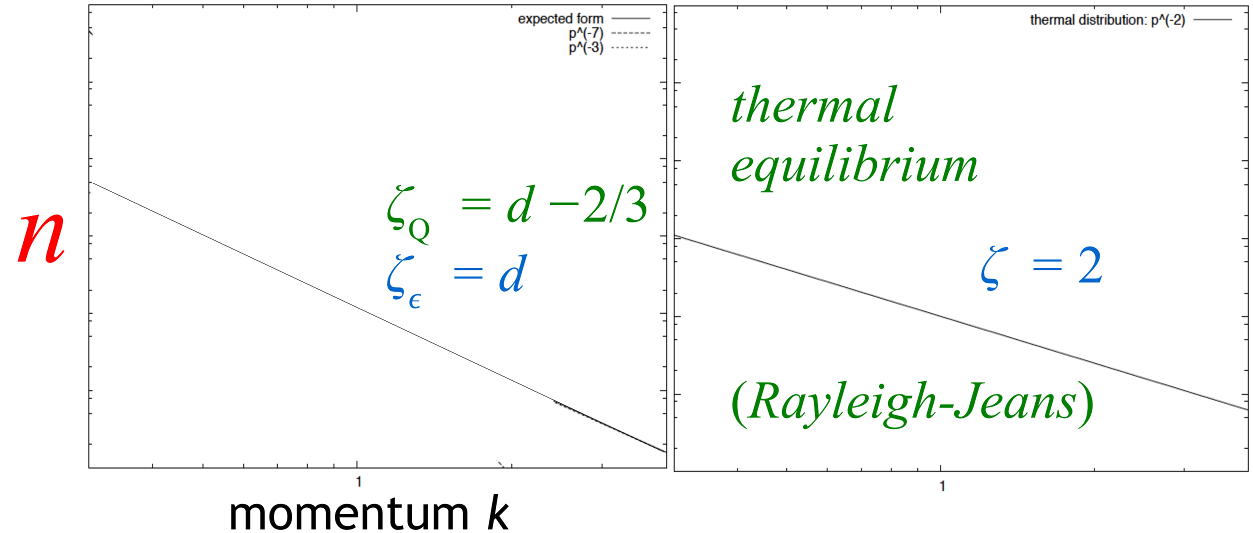
Bose gas in d spatial dimensions

$$n \sim k^{-\zeta}$$



J. Berges, A. Rothkopf, J. Schmidt, PRL **101** (08) 041603
C. Scheppach, J. Berges, TG PRA **81** (10) 033611

Bose gas in d spatial dimensions $n \sim k^{-\zeta}$



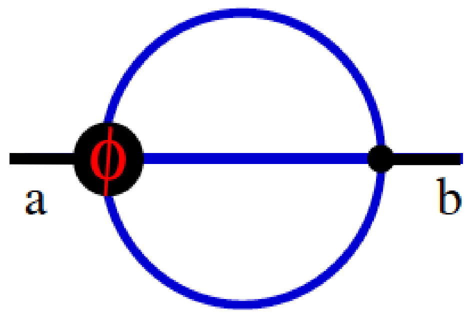
J. Berges, A. Rothkopf, J. Schmidt, PRL **101** (08) 041603
C. Scheppach, J. Berges, TG PRA **81** (10) 033611

Dyn. QFT: Resummed Vertex

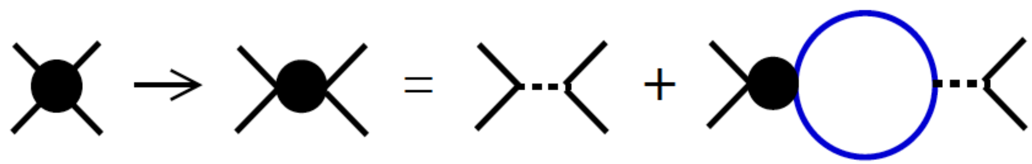
$p = (p_0, \mathbf{p})$:

$$J(p) := \Sigma^{\rho}_{ab}(p)F_{ba}(p) - \Sigma^{F}_{ab}(p)\rho_{ba}(p) \stackrel{!}{=} 0$$

$\Sigma_{ab}(x,y)$ =



Vertex bubble resummation:
(e.g. 2PI to NLO in $1/N$)

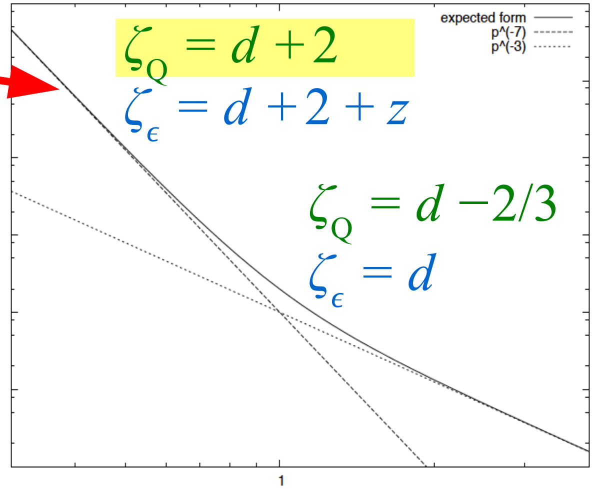


[*Dynamics:* J. Berges, (02); G. Aarts et al., (02);
Nonthermal fixed points: J. Berges, A. Rothkopf, J. Schmidt, PRL (08)]

Bose gas in d spatial dimensions $n \sim k^{-\zeta}$

New exponent
beyond
Quantum Boltzmann!

n



momentum k

J. Berges, A. Rothkopf, J. Schmidt, PRL **101** (08) 041603; J. Berges, G. Hoffmeister, NPB 813, 383 (2009)
C. Scheppach, J. Berges, TG PRA **81** (10) 033611

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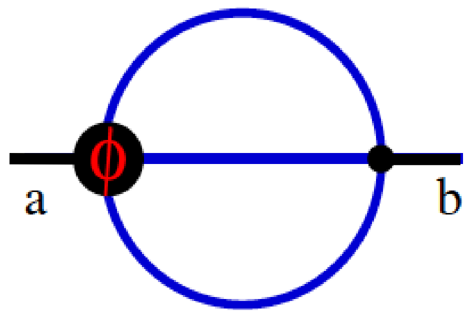
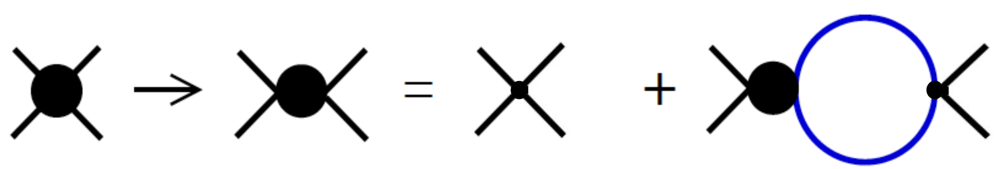
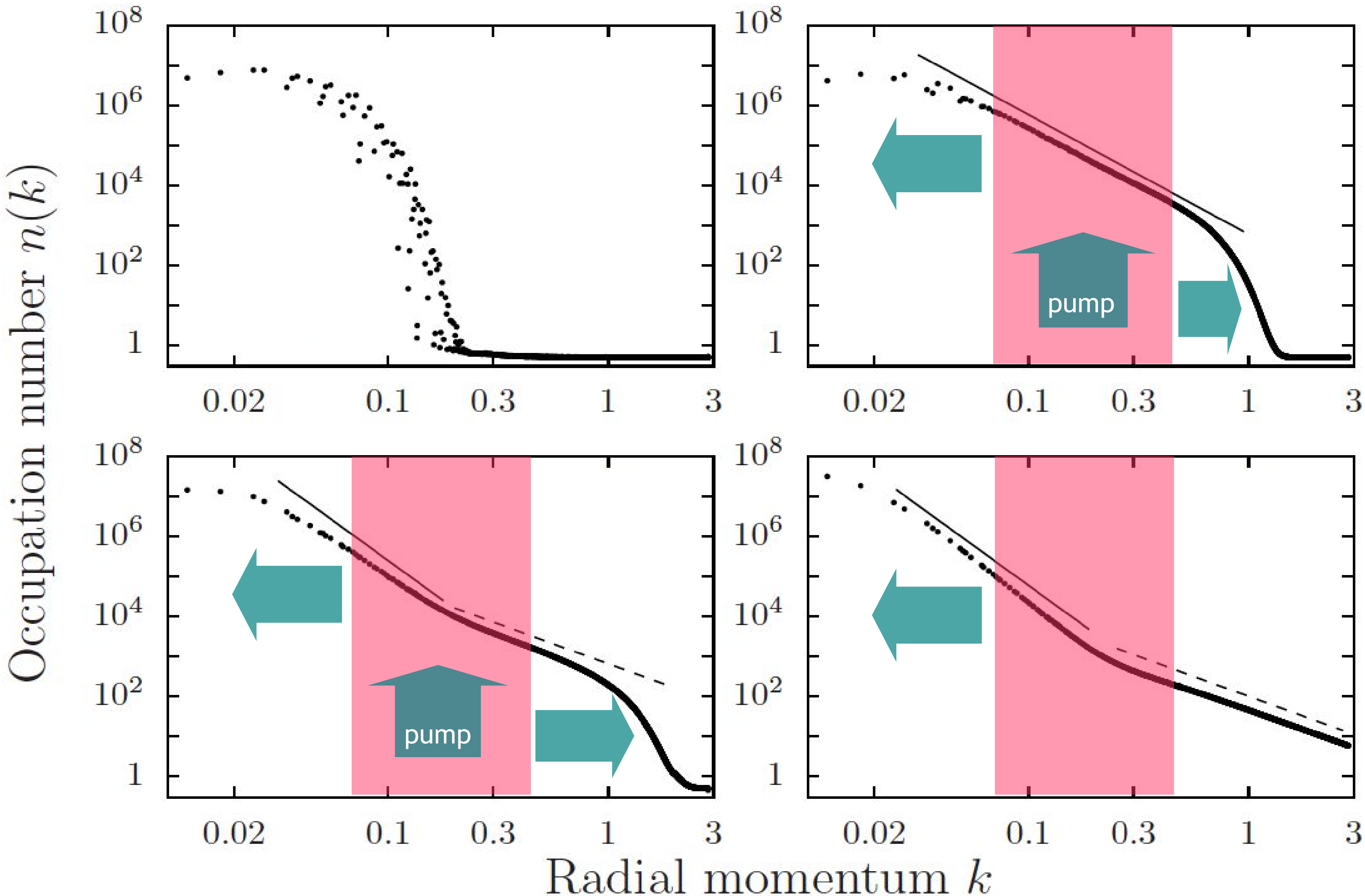


Vertex bubble resummation:



Cascades in 2+1 D



Decomposition of Energy

$$E_{tot} = \int \left(\frac{1}{2} |\nabla \sqrt{n} e^{-i\varphi}|^2 + \frac{1}{2} g n^2 \right) d\boldsymbol{\rho}$$

$$= E_{kin} + E_q + E_{int}$$

$$\mathbf{u}(\boldsymbol{\rho},t) = \nabla \varphi(\boldsymbol{\rho},t)$$

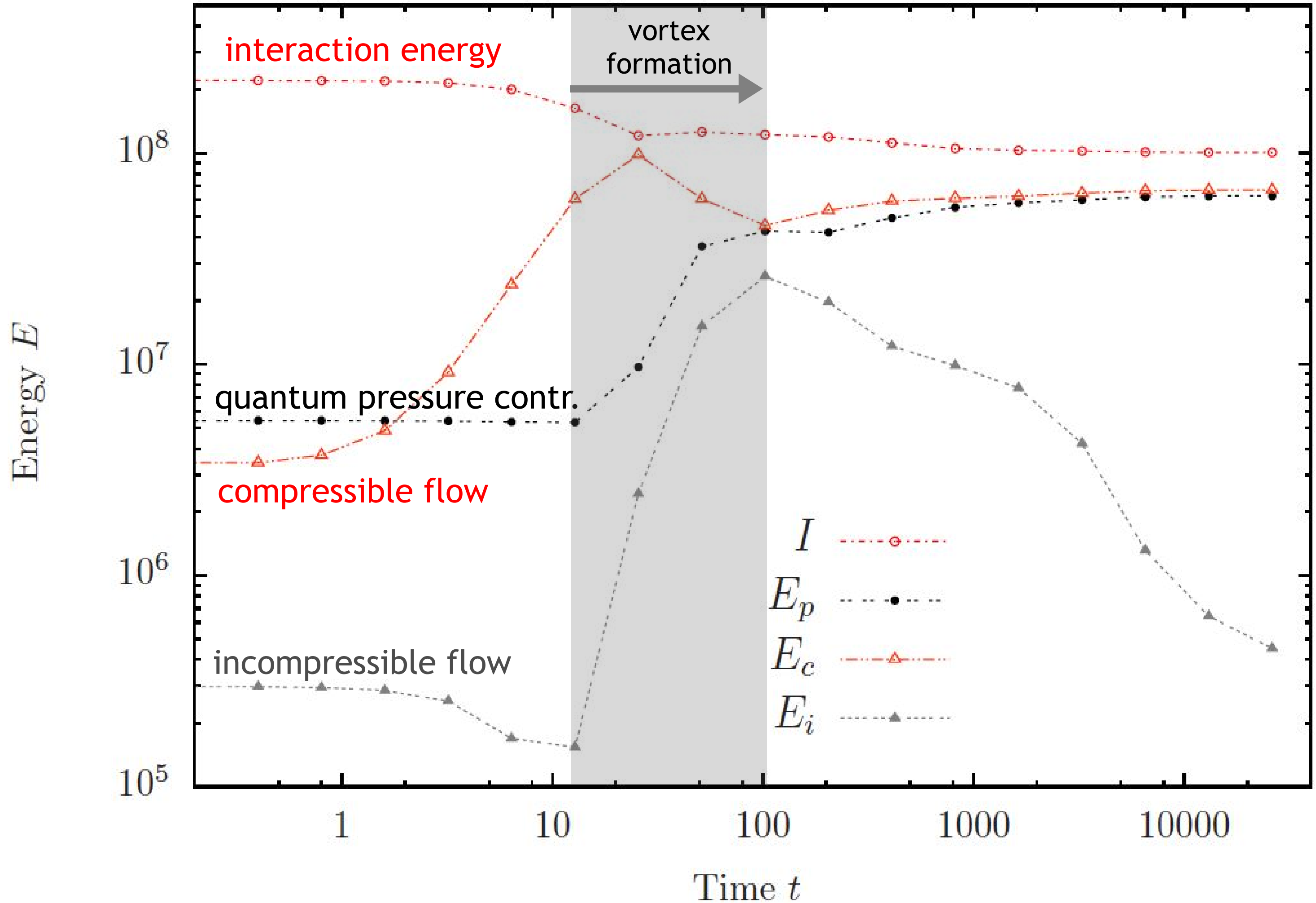
$$\nabla \times (\sqrt{n}\mathbf{u})^c = 0$$

$$E_{kin} = \frac{1}{2} \int |\sqrt{n}\mathbf{u}|^2 d\boldsymbol{\rho} \quad = E^i_{kin} + E^c_{kin}$$

$$\nabla \cdot (\sqrt{n}\mathbf{u})^i = 0$$

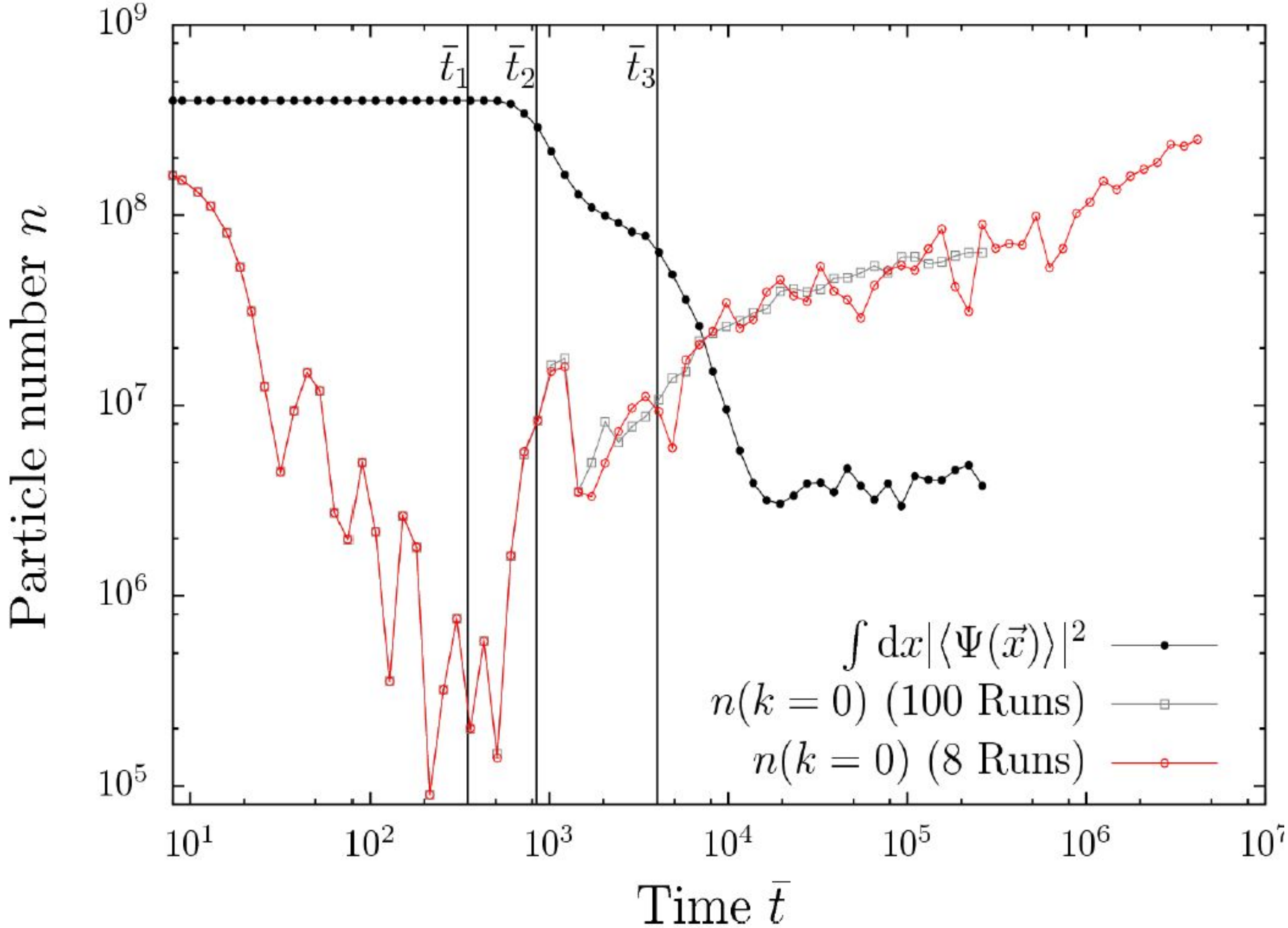
$$E_q = \frac{1}{2} \int (\nabla \sqrt{n})^2 d\boldsymbol{\rho}$$

Time evolution of Energy Components (3+1 D)



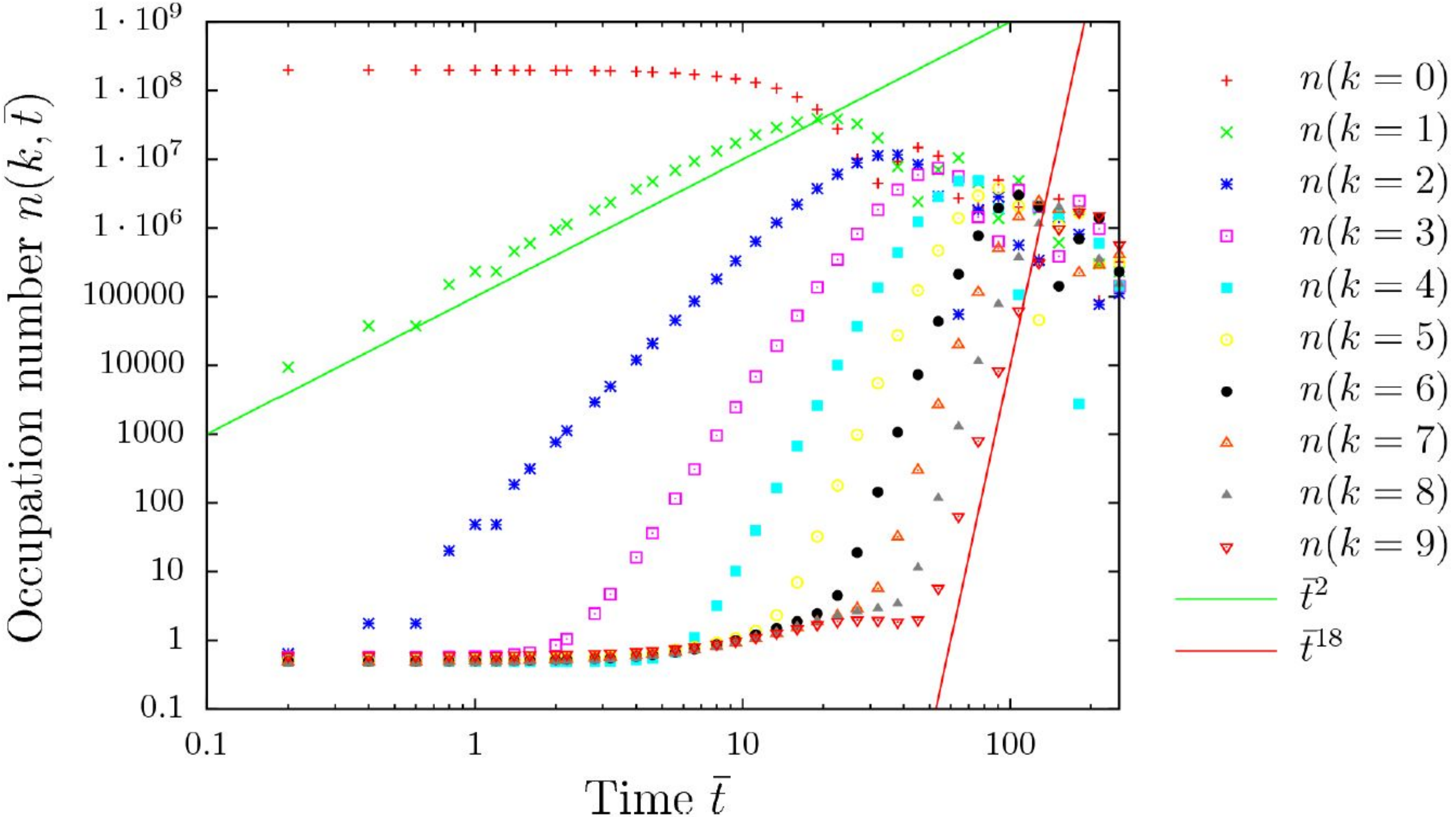
J. Schole, B. Nowak, D. Sexty, TG (unpublished)

Evolution of Zero-Mode



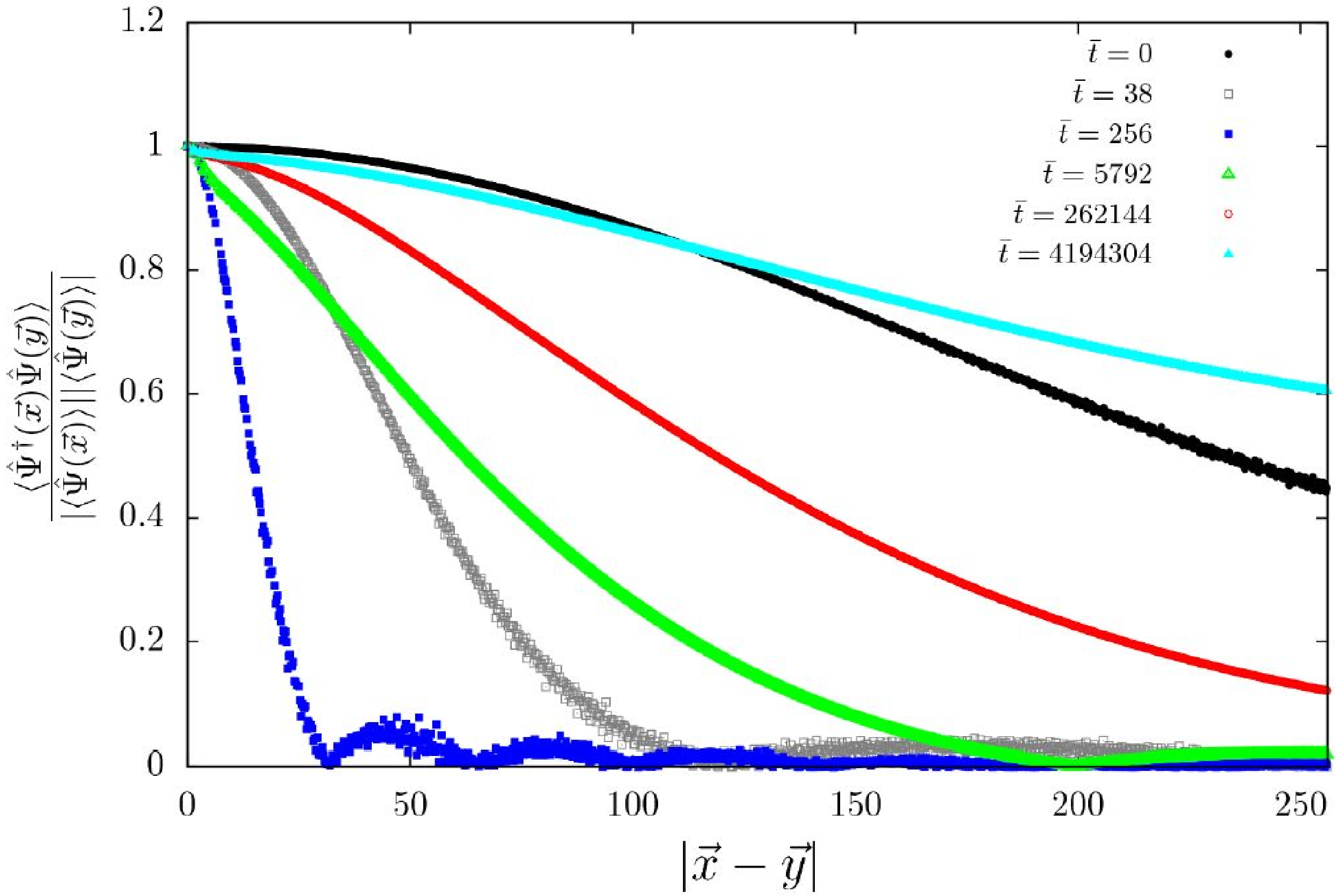
J. Schole, B. Nowak, D. Sexty, TG (unpublished)

Mode Occupations



J. Schole, B. Nowak, D. Sexty, TG (unpublished)

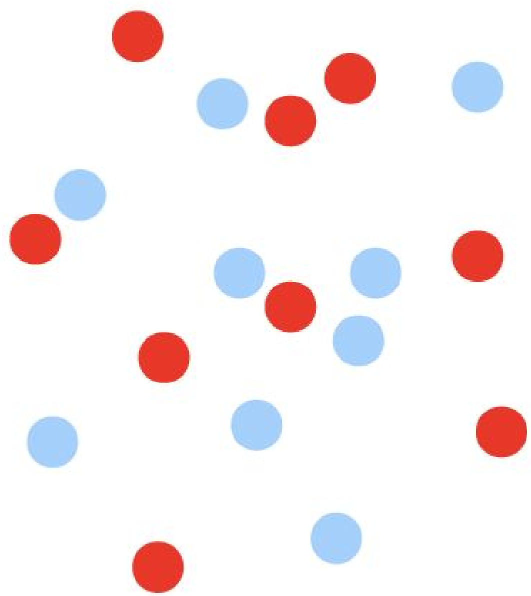
1st-order Coherence



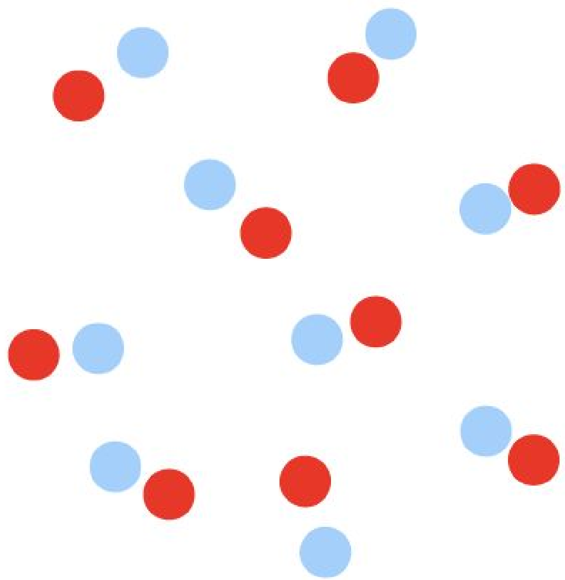
J. Schole, B. Nowak, D. Sexty, TG (unpublished)

Random vortex distributions

Point vortex model



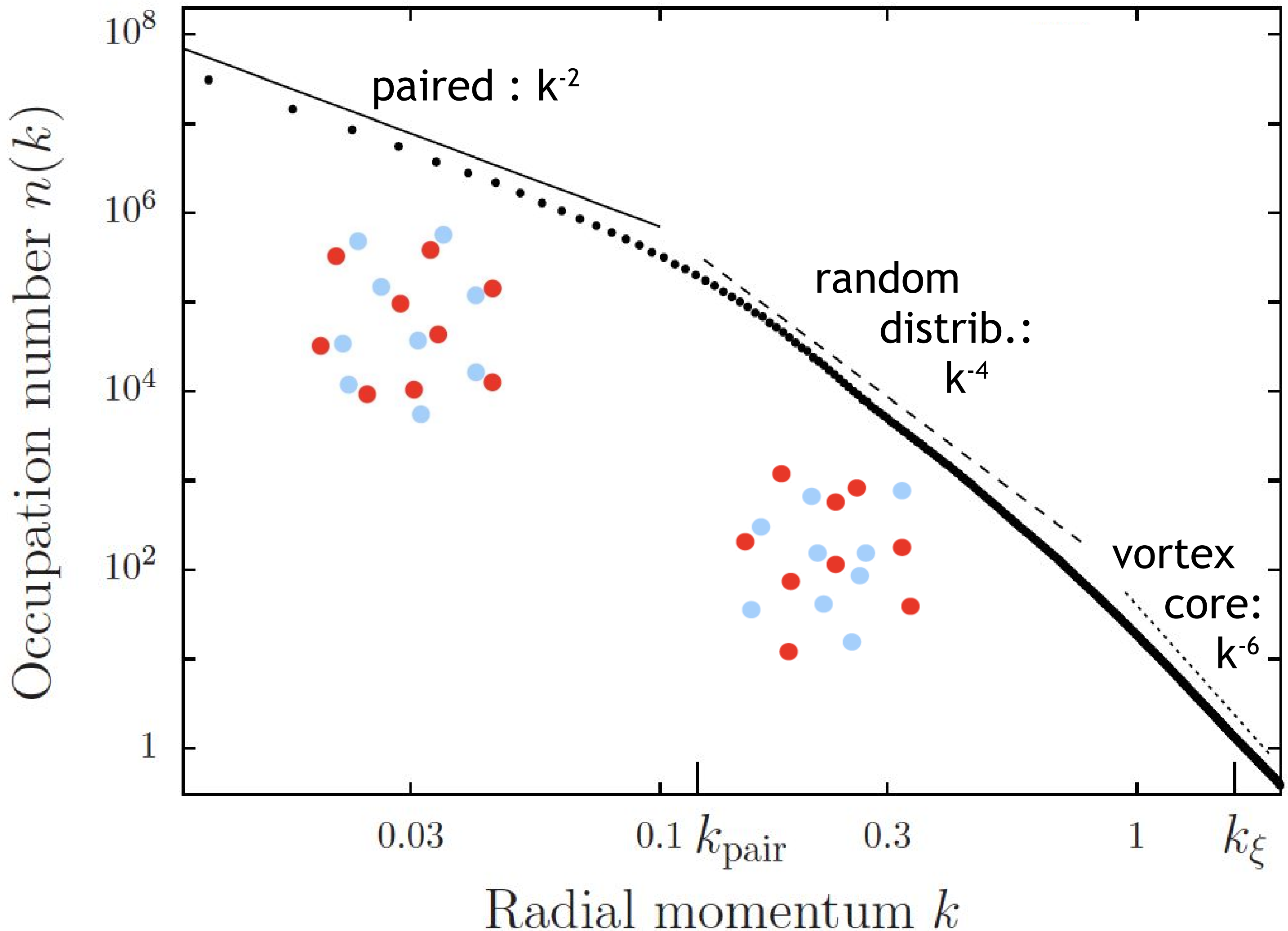
$$n_k \sim k^{-4}$$



$$n_k \sim k^{-2}, \;\; k < k_{\mathrm{pair}}$$
$$n_k \sim k^{-4}, \;\; k > k_{\mathrm{pair}}$$

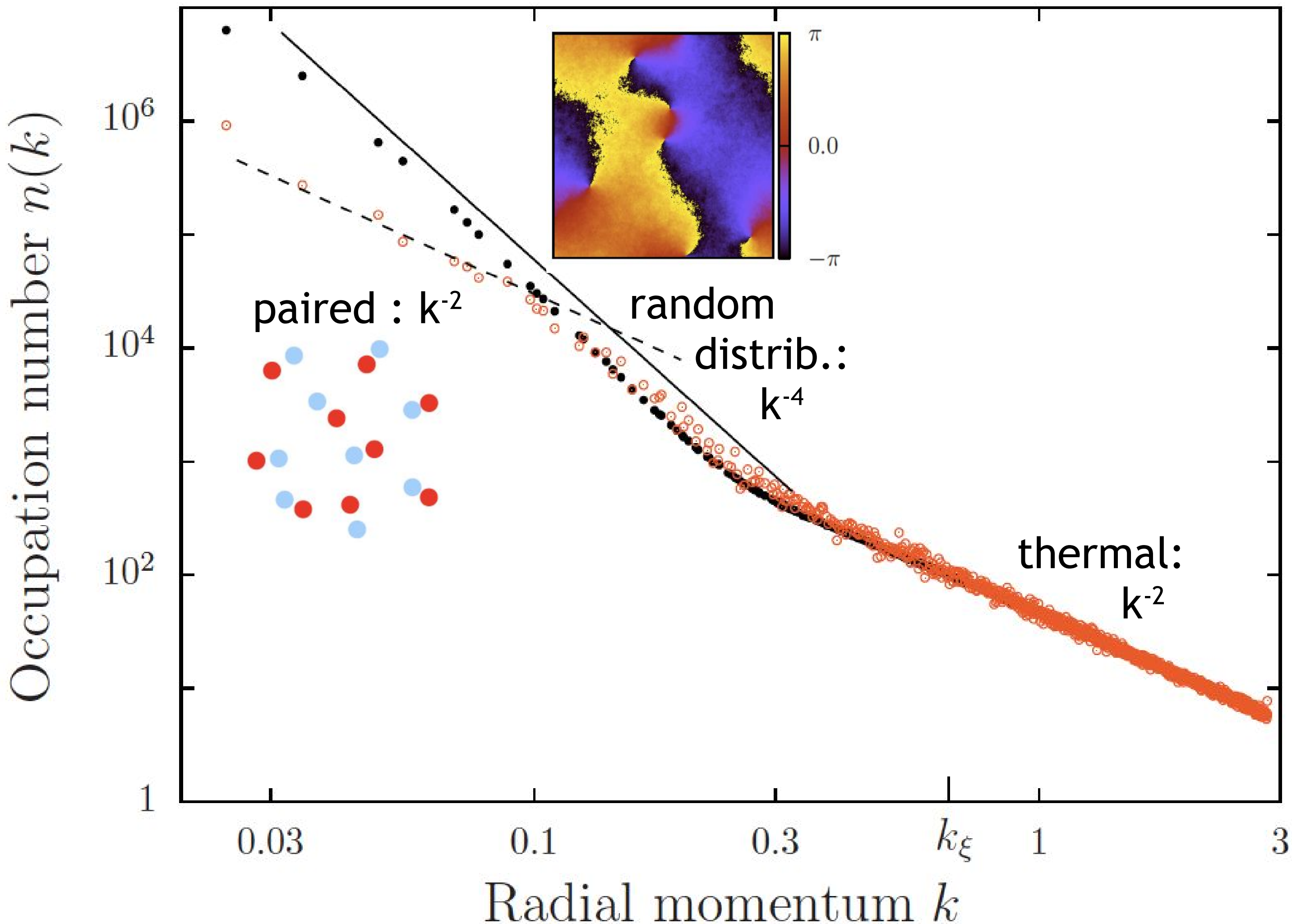
B. Nowak, J. Schole, D. Sexty, TG, arXiv:1111.61XX [cond-mat.quant-gas]

Point vortex model in 2+1 D



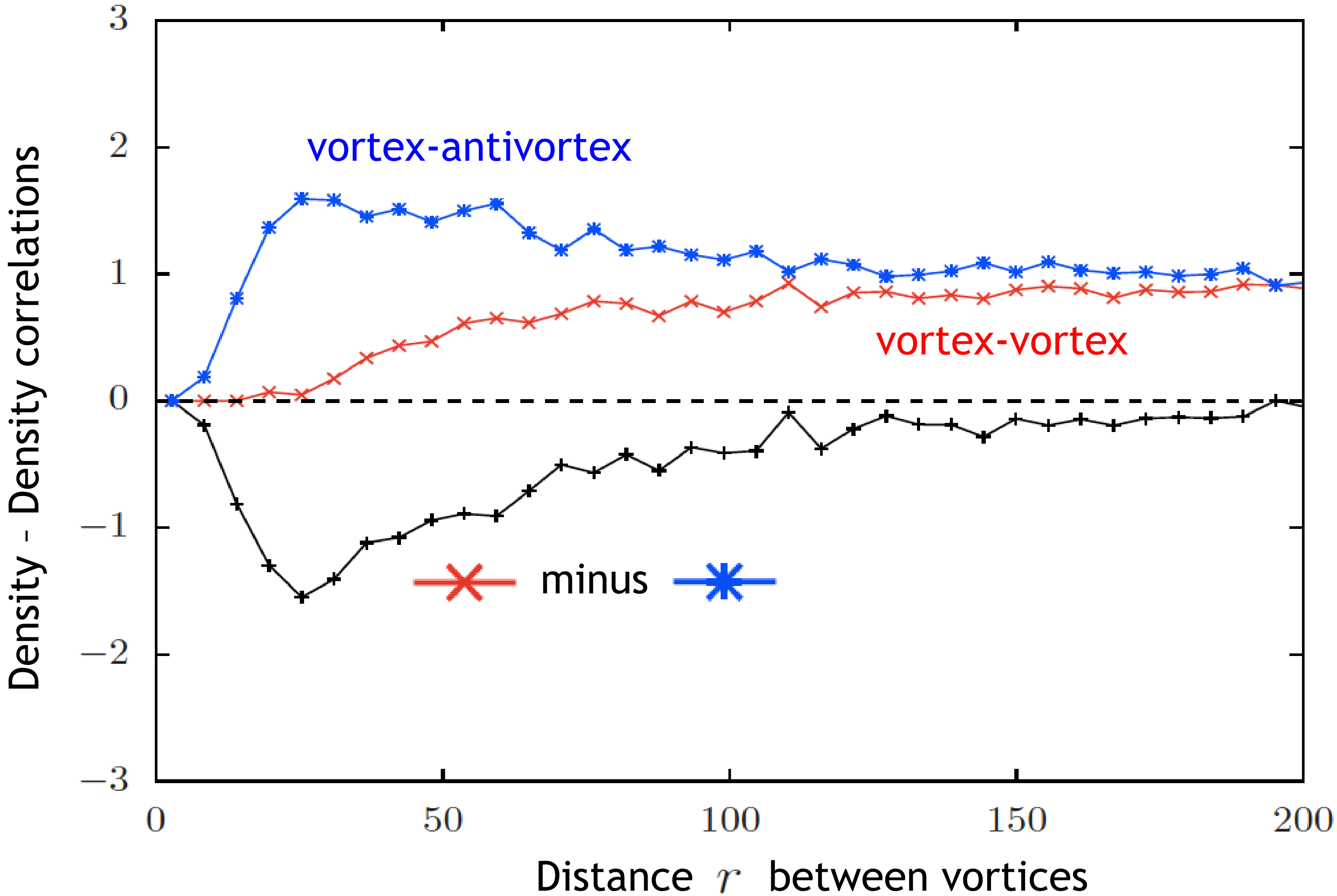
B. Nowak, J. Schole, D. Sexty, TG, arXiv:1111.61XX [cond-mat.quant-gas]

Simulations in 2+1 D



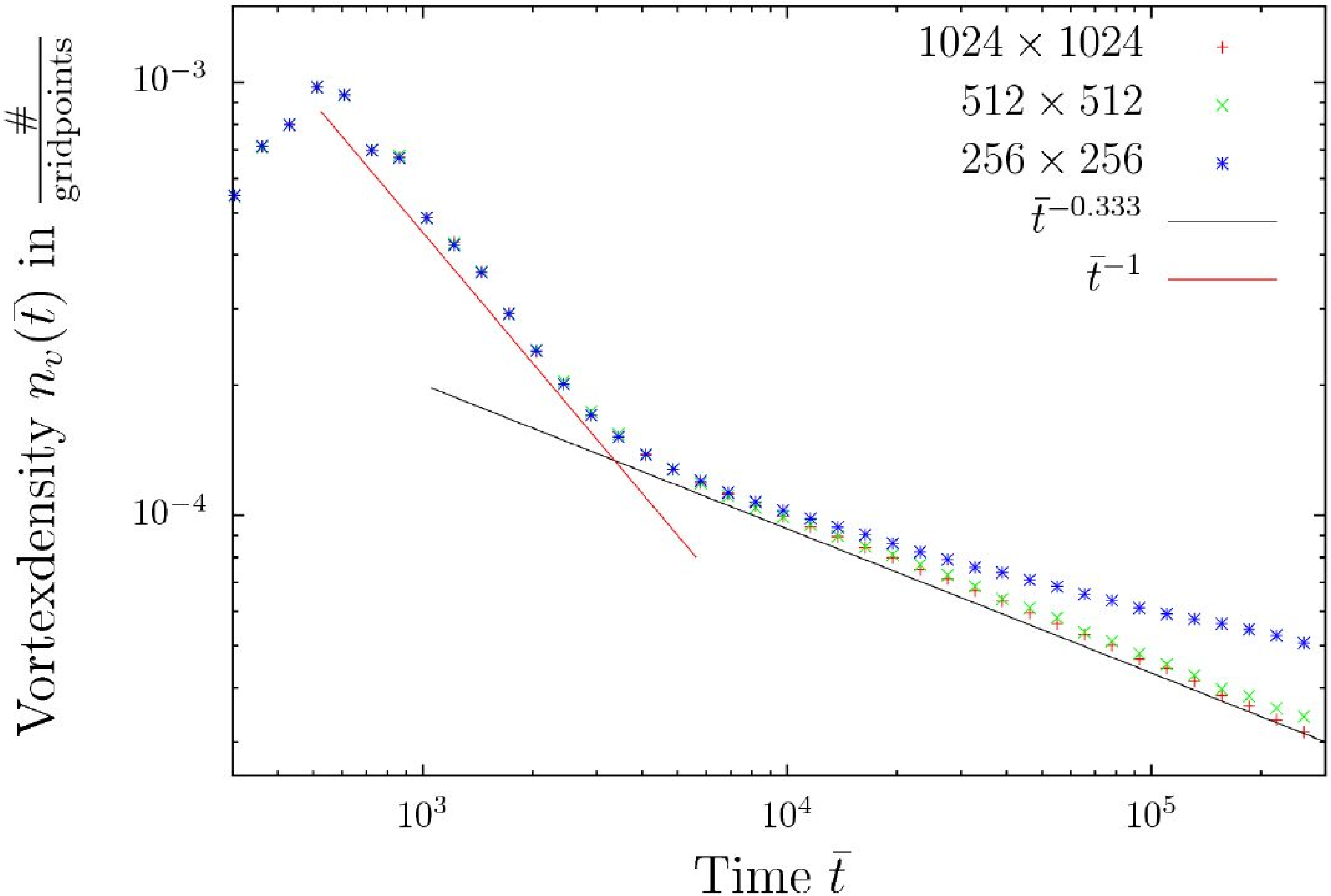
B. Nowak, J. Schole, D. Sexty, TG, arXiv:1111.61XX [cond-mat.quant-gas]

Vortex position correlations



B. Nowak, J. Schole, D. Sexty, TG, arXiv:1111.61XX [cond-mat.quant-gas]

Vortex-Density Decay in 2d



J. S., B. Nowak, D. Sexty, T. Gasenzer (unpublished)

Vortices in a Na condensate

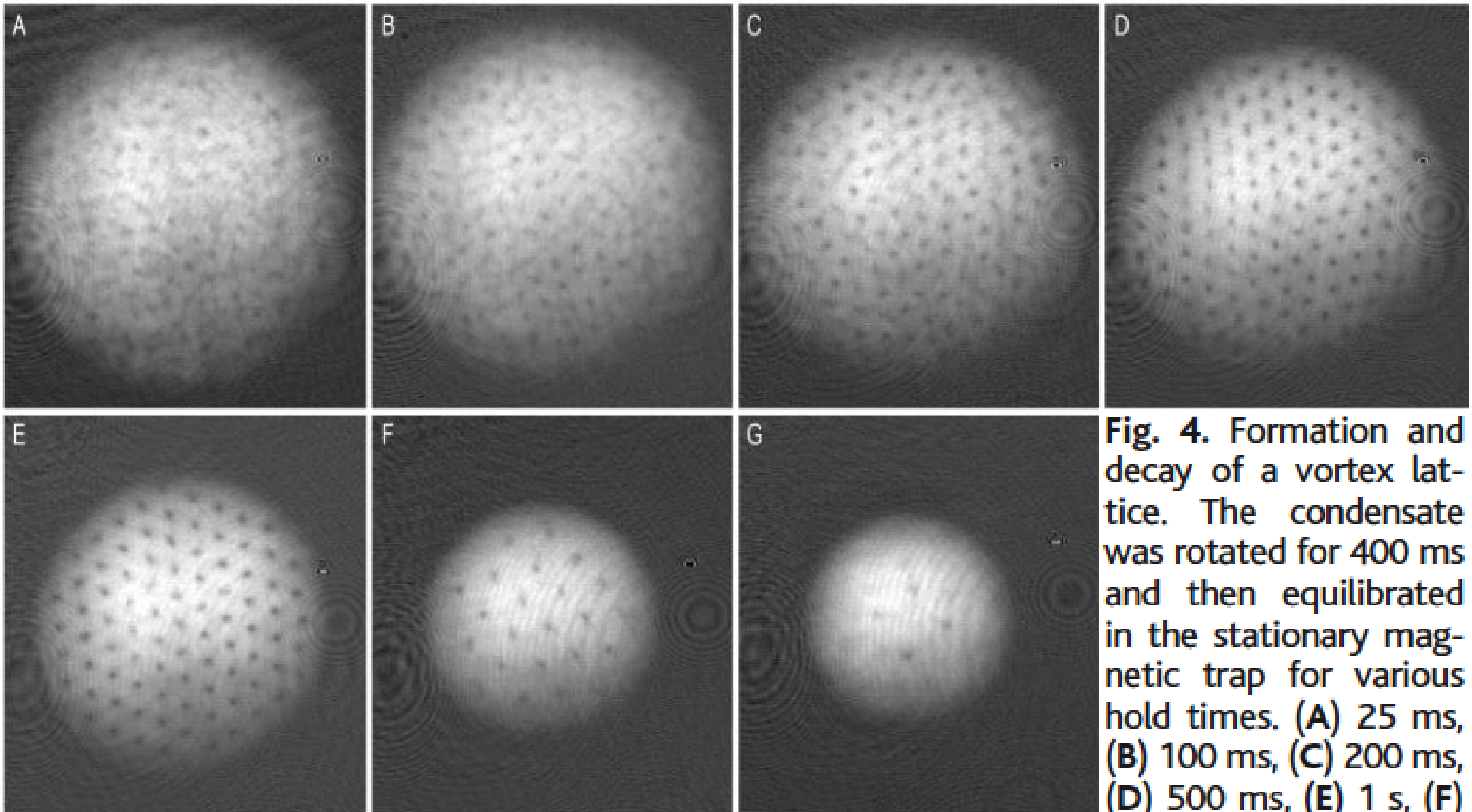


Fig. 4. Formation and decay of a vortex lattice. The condensate was rotated for 400 ms and then equilibrated in the stationary magnetic trap for various hold times. (A) 25 ms, (B) 100 ms, (C) 200 ms, (D) 500 ms, (E) 1 s, (F) 5 s, (G) 10 s,

J. R. Abo-Shaeer, C. Raman, J. M. Vogels, W. Ketterle
20 APRIL 2001 VOL 292 SCIENCE

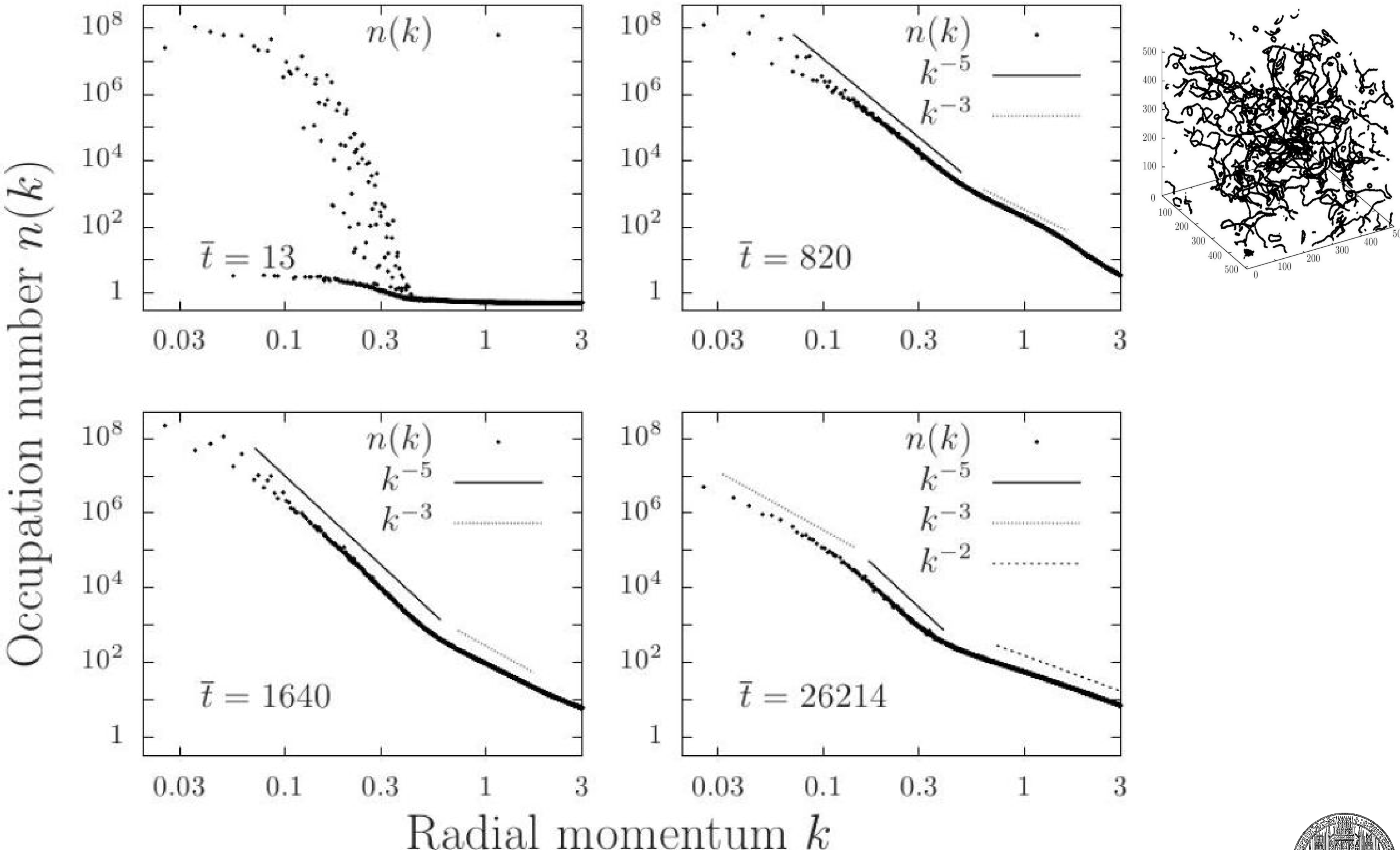
Movie 3: Vortex Lines in 3+1 D

$$n(k) = \left\langle \Psi^*(\mathbf{k})\Psi(\mathbf{k}) \right\rangle\Big|_{\text{angle average}}$$

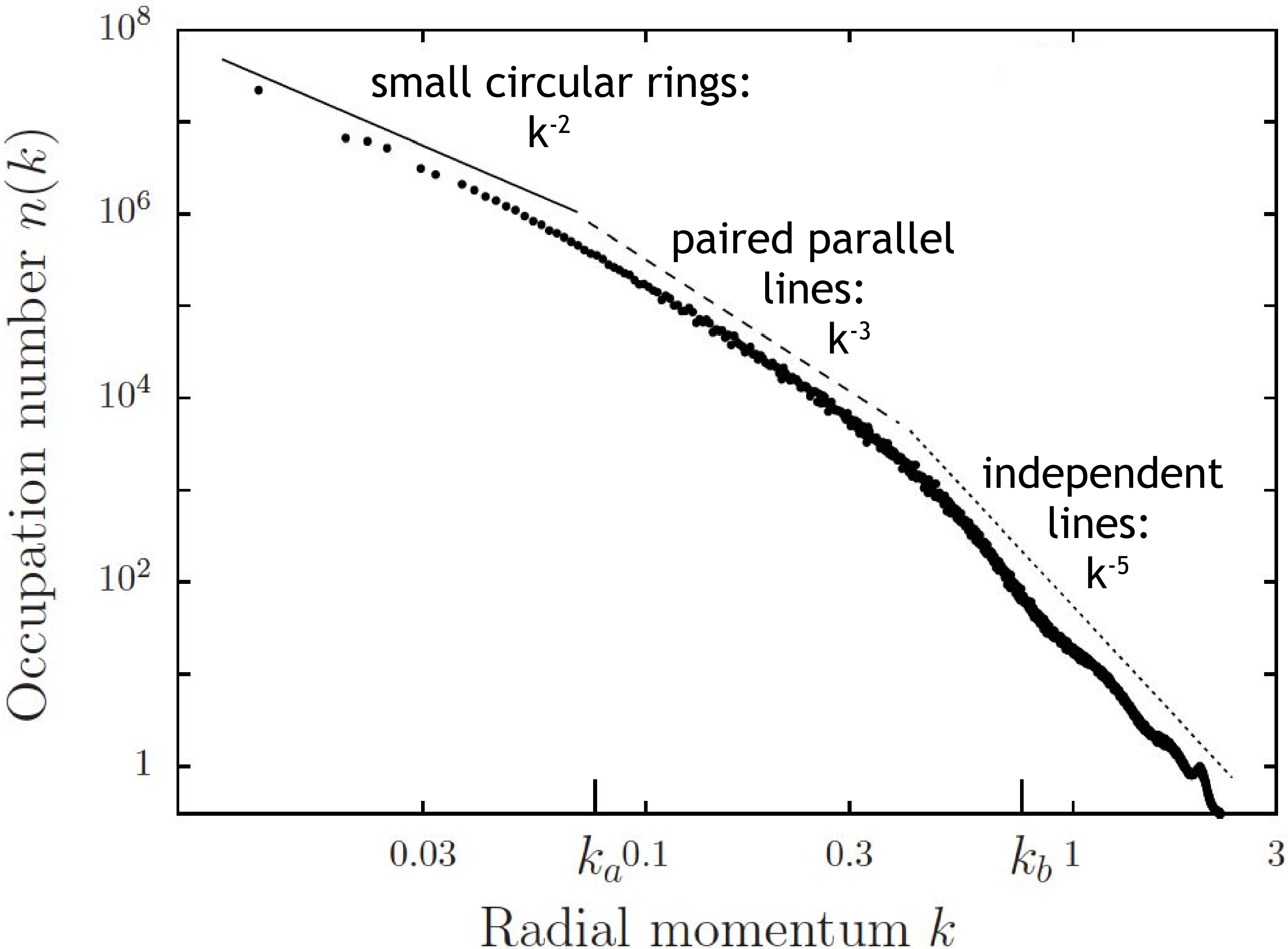
http://www.thphys.uni-heidelberg.de/~smp/gasenzer/videos/boseqt.html

B. Nowak, D. Sexty, TG, PRB **84**: 020506(R), 2011

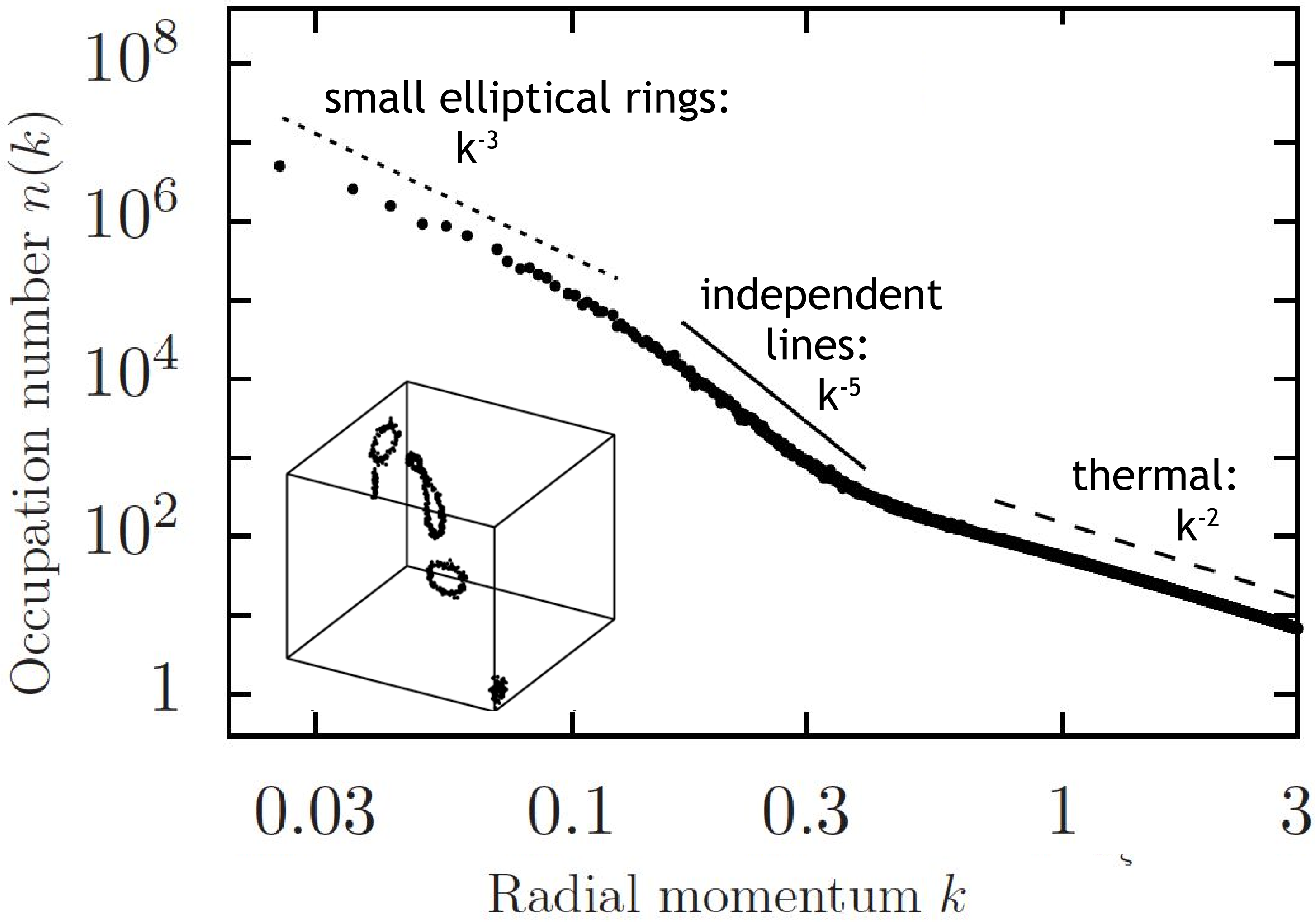
3+1 D simulations



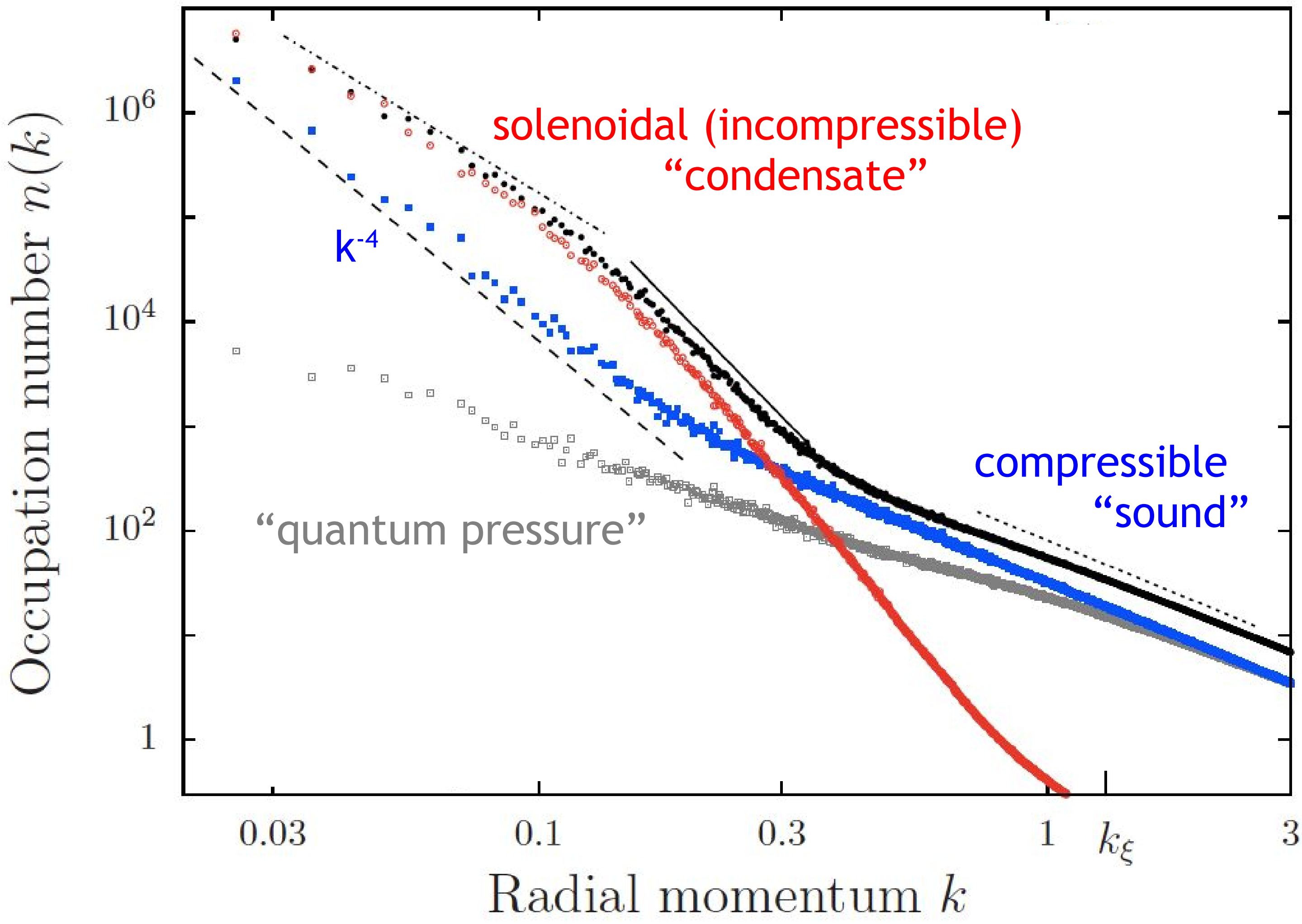
Line vortex model in 3+1 D



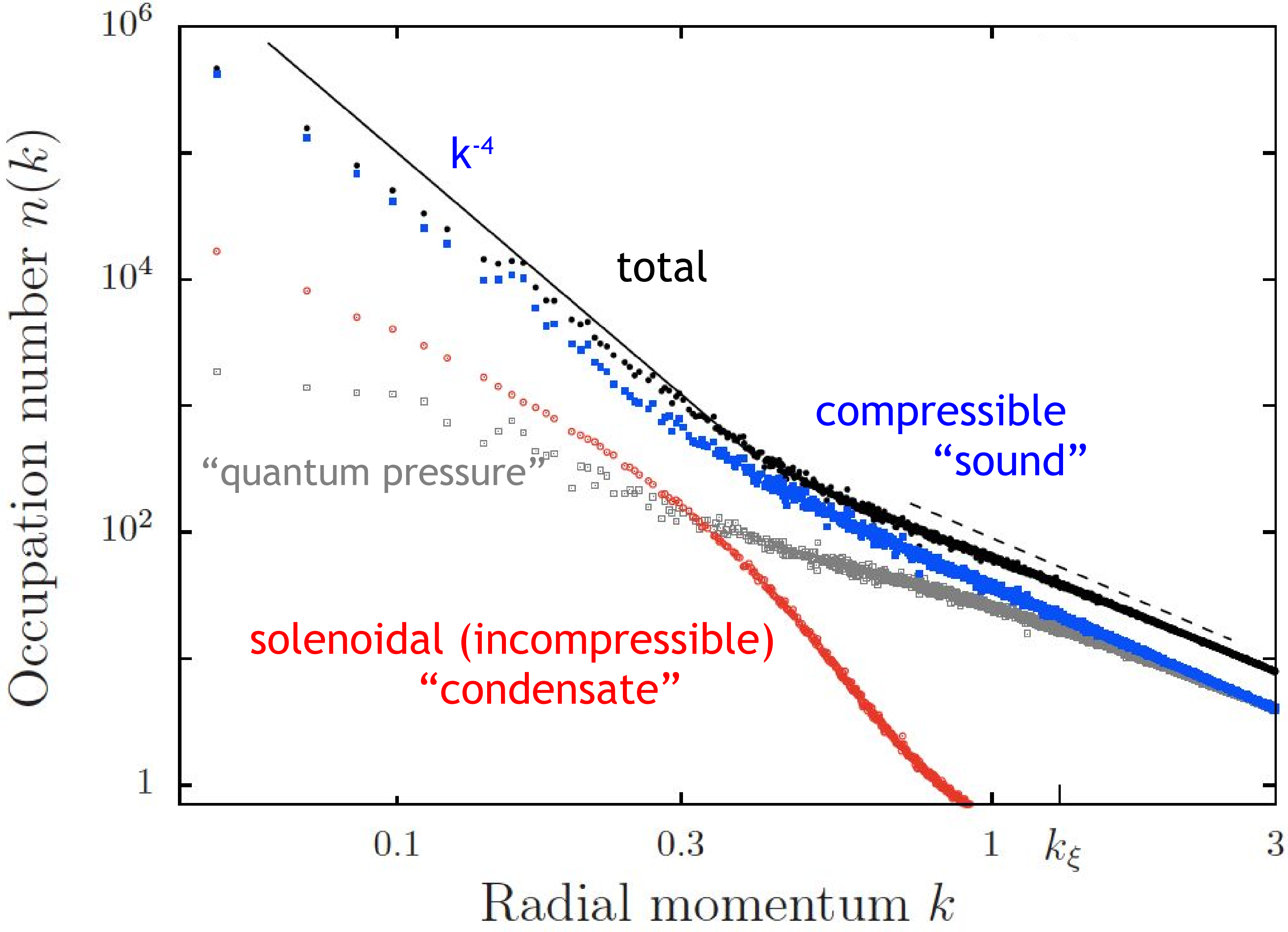
Simulations in 3+1 D



Decomposition of flow

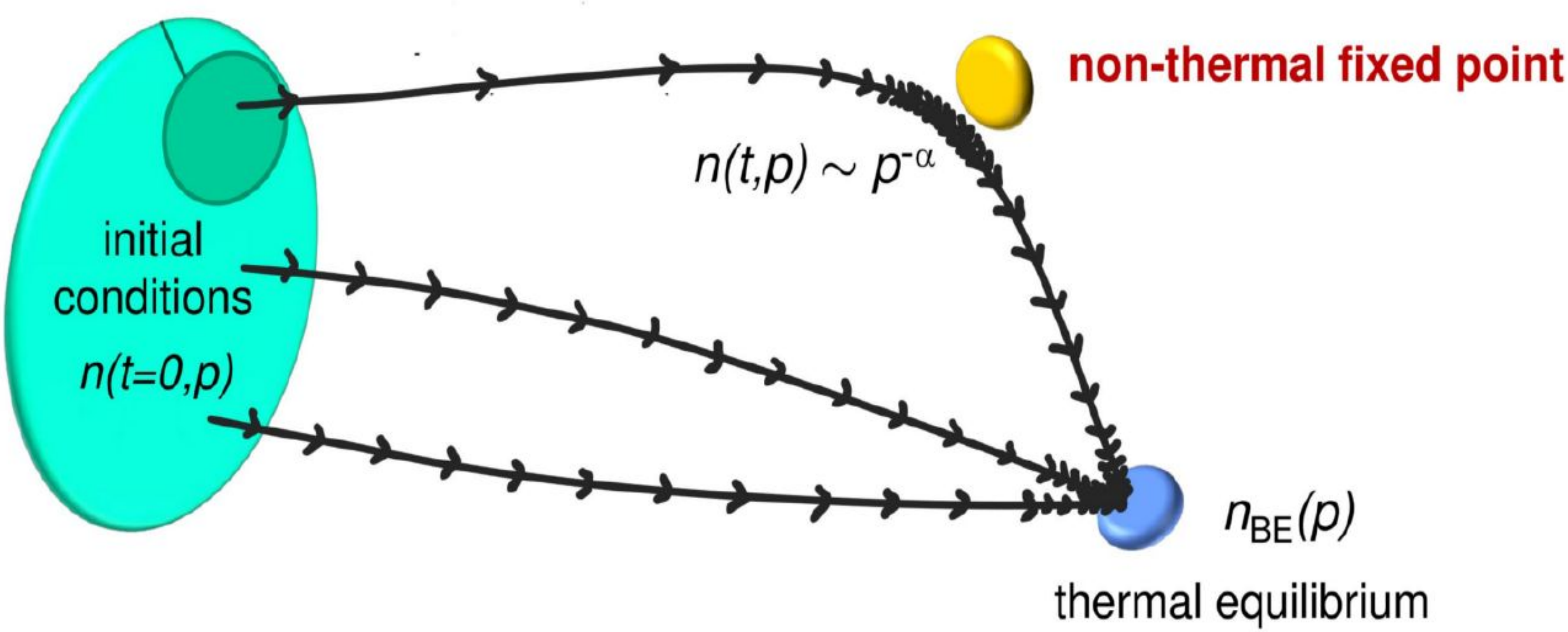


Acoustic turbulence



B. Nowak, J. Schole, D. Sexty, TG, arXiv:1111.61XX [cond-mat.quant-gas]

Non-thermal fixed point



[Fig. courtesy: J. Berges '08]

Relativistic scalar field

Strong Turbulence

Simulations of the non-linear Klein-Gordon equation, O(2) symmetry

$$(\partial_t^2 - \partial_x^2)\varphi(x,t) + \lambda\varphi^3(x,t) = 0$$

Initial condition: Highly occupied zero mode, Unoccupied modes with k>0

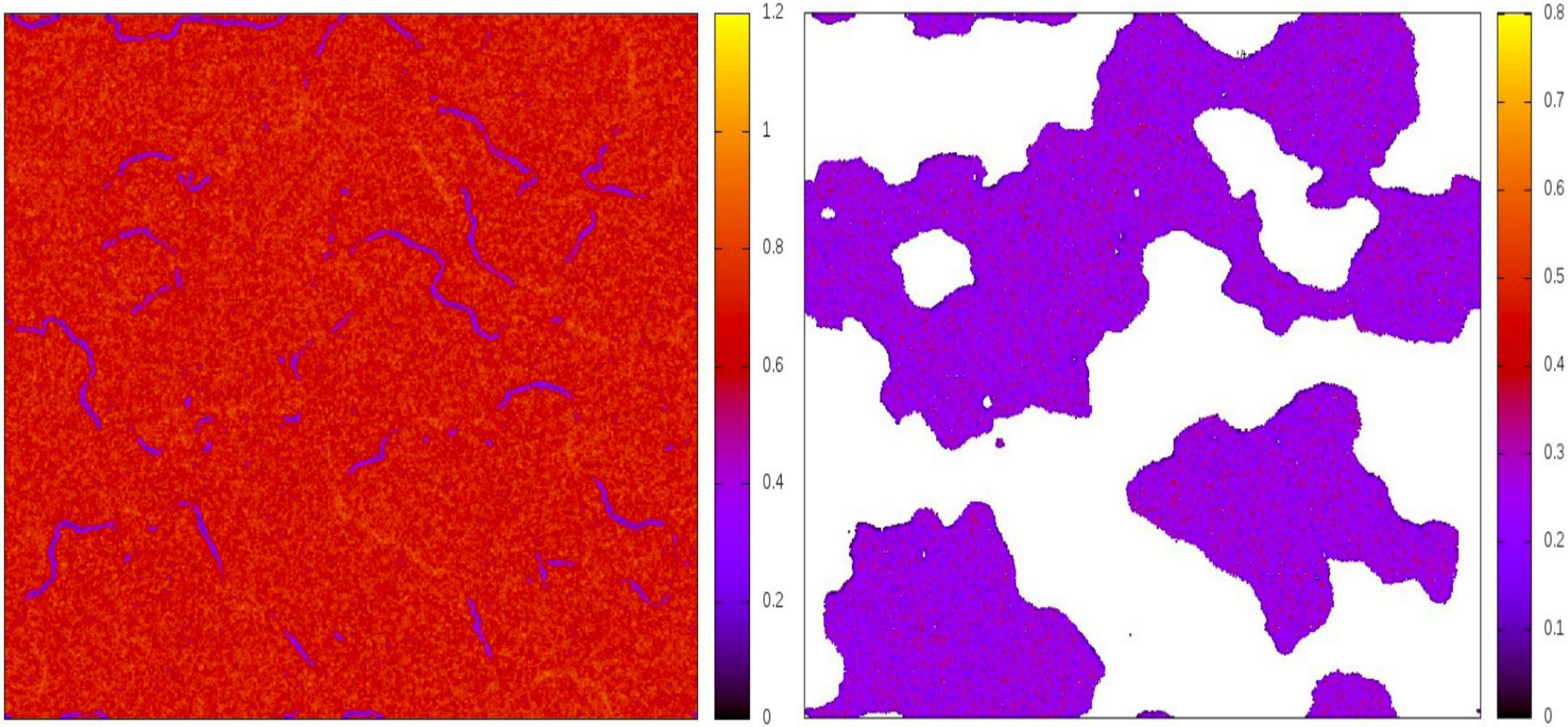
(video)

See also: http://www.thphys.uni-heidelberg.de/~sexty/videos

TG, B. Nowak, D. Sexty, arXiv:1108.0541 [hep-ph]

Strong Turbulence = Charge Separation

Modulus of complex field $|\varphi|$ vs. mean charge distribution

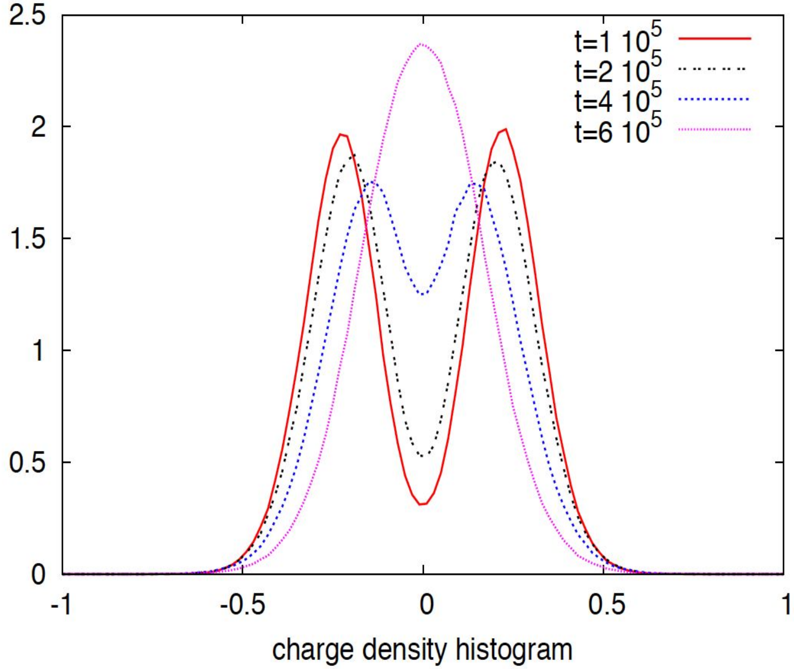


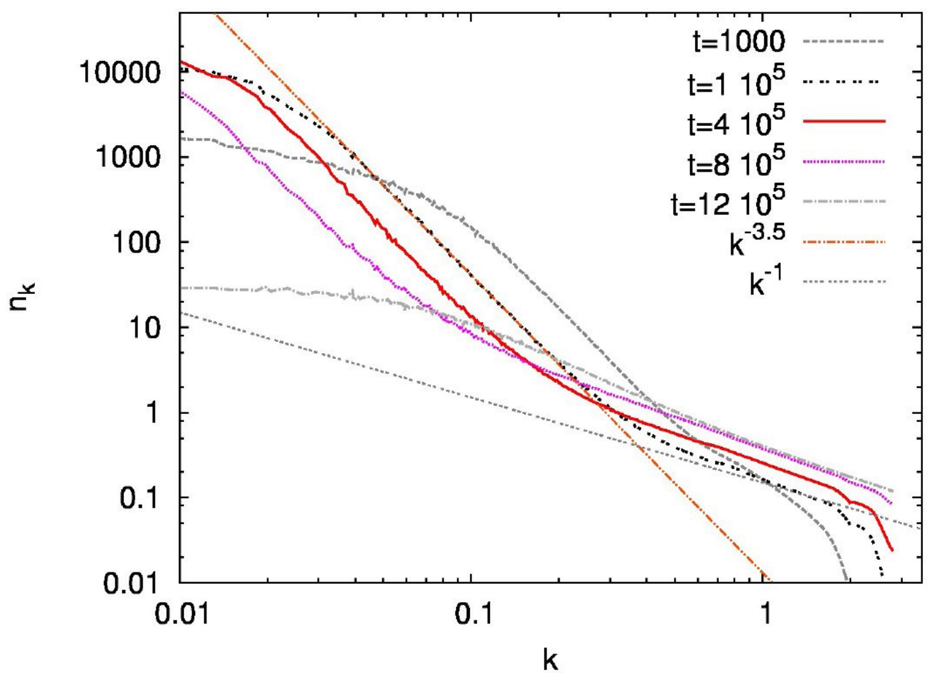
TG, B. Nowak, D. Sexty, arXiv:1108.0541 [hep-ph]
cf. also Tkachev, Kofman, Starobinsky, Linde (1998)

Strong Turbulence = Charge Separation

Charge density distribution vs. power spectrum

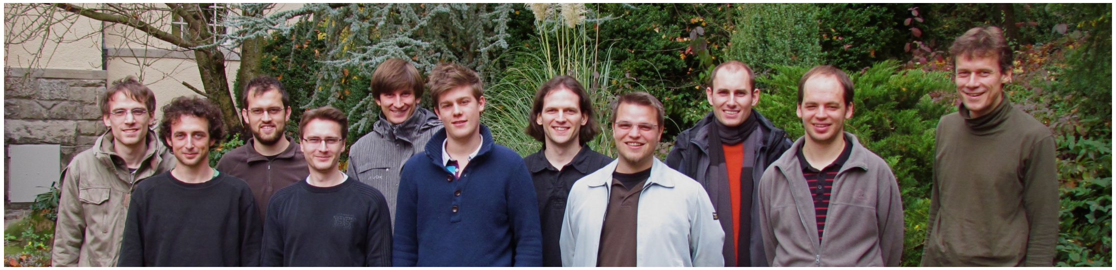
(d = 2, N = 2)





TG, B. Nowak, D. Sexty, arXiv:1108.0541 [hep-ph]

Thanks & credits to...



...my work group in Heidelberg:

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Steven Mathey
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Martin Trappe
Jan Zill
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€€€...

Heisenberg-Programm
DFG Deutsche Forschungsgemeinschaft

RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

LGFG BaWue

DAAD Deutscher Akademischer Austausch Dienst German Academic Exchange Service

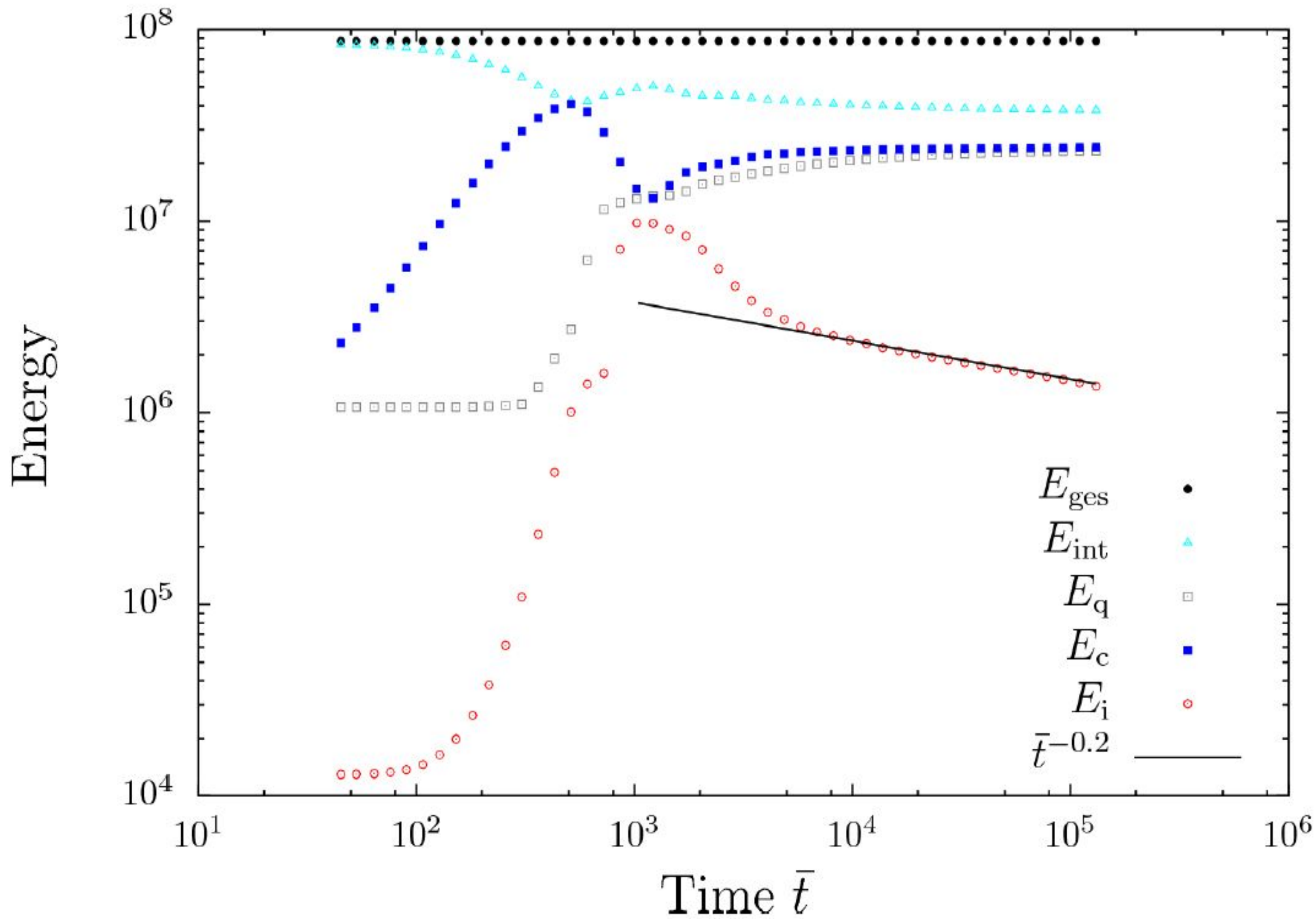
Center for Quantum Dynamics

EMMI

HELMHOLTZ | GEMEINSCHAFT

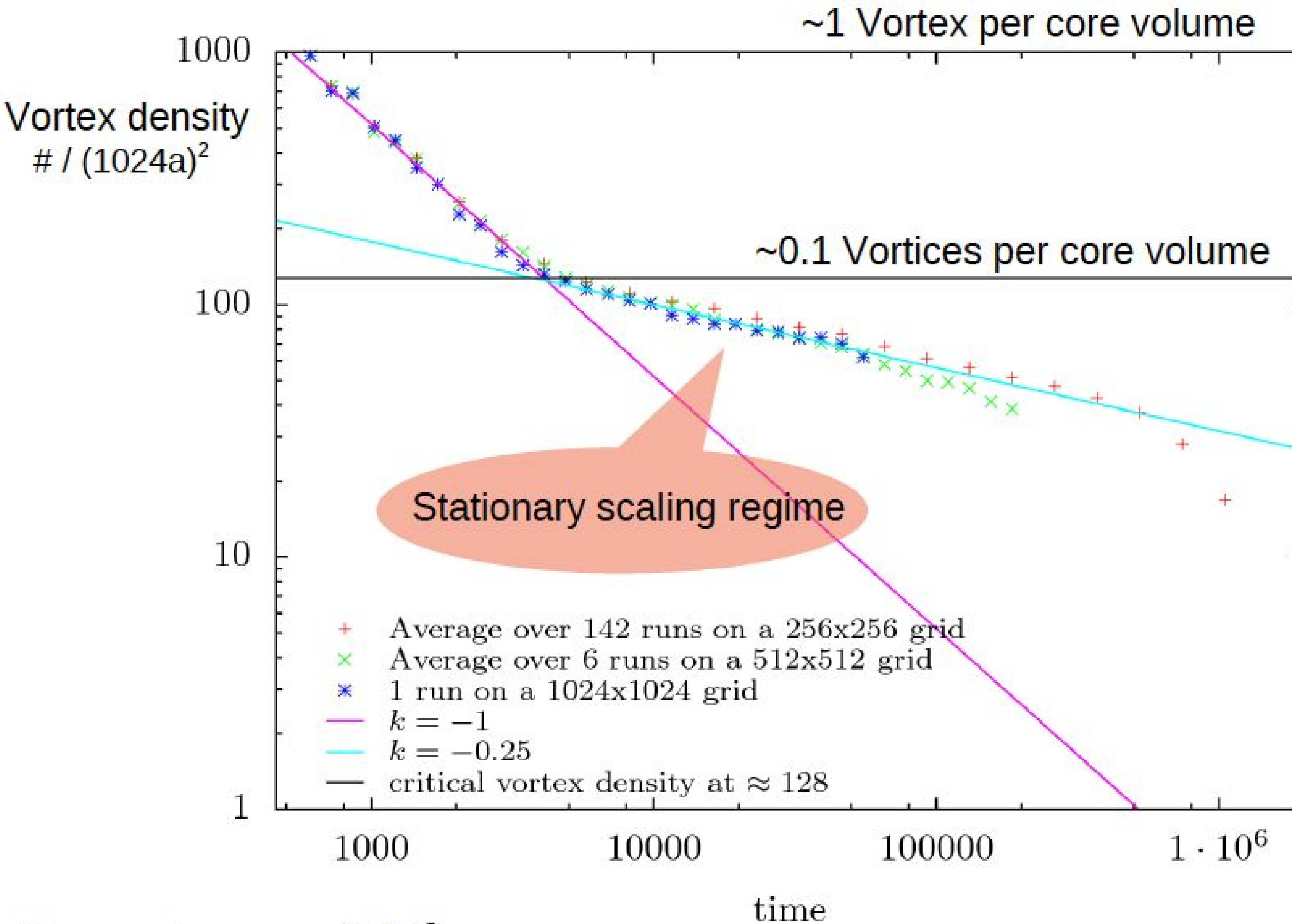
Supplementary slides

Time-Evolution of Energy-components (2+1 D)



J. S., B. Nowak, D. Sexty, T. Gasenzer (unpublished)

Time evolution of vortex density



Core volume ~ $\pi(3\xi)^2$

J. Schole, B. Nowak, D. Sexty, TG (unpublished)

Enstrophy in classical turbulence

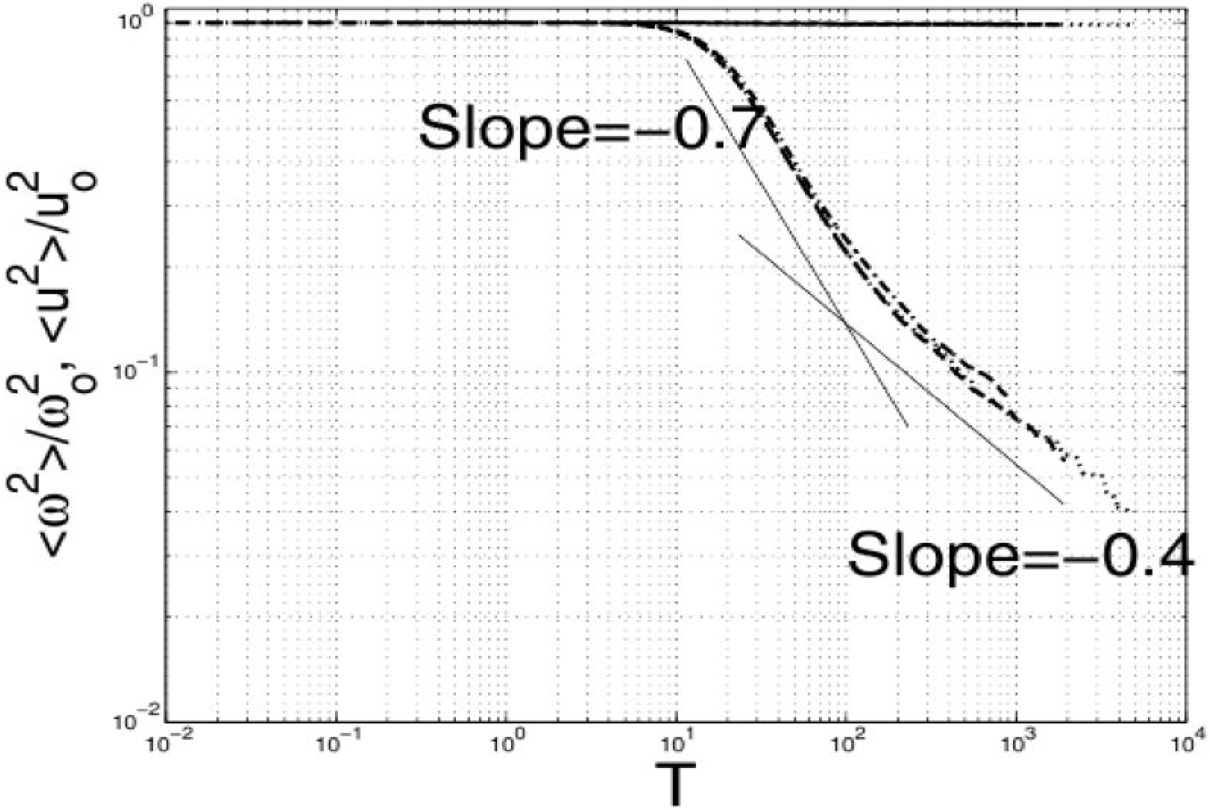
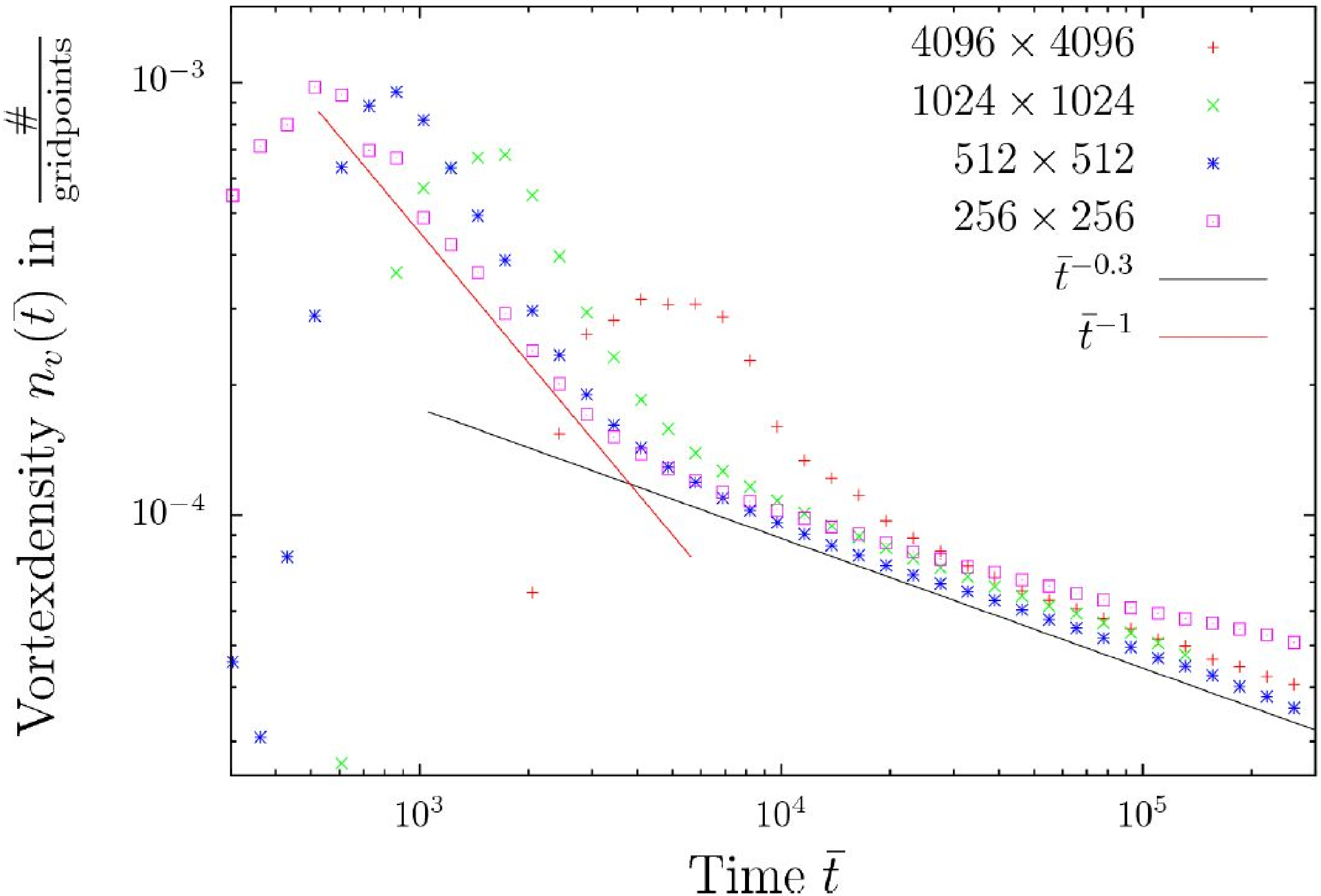


FIG. 1. Time evolution of energy (upper horizontal curve) and enstrophy. Resolution: dotted line (512^3); dashed line (1024^3); dash-dotted line (2048^3).

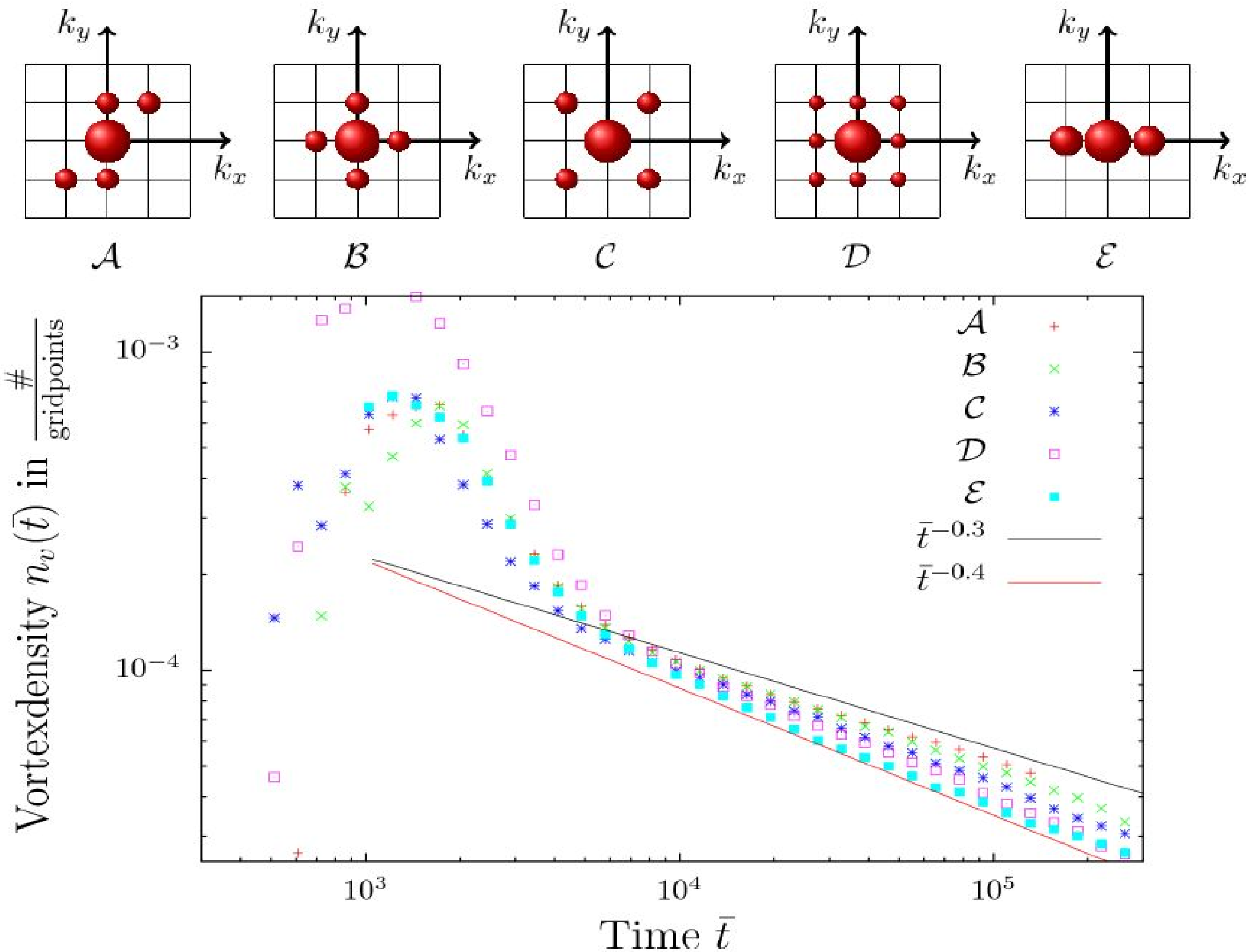
V. Yakhot, J. Wanderer, PRL 93:154502

Vortex-Density Decay in 2d



J. S., B. Nowak, D. Sexty, T. Gasenzer (unpublished)

Vortex-Density Decay in 2d



J. S., B. Nowak, D. Sexty, T. Gasenzer (unpublished)

Kinetic Theory

One of the power laws can be explained by a kinetic theory:

$$\partial_t n_v(t) = -\frac{n_{\text{dip}}}{\tau_{\text{ann}}}$$

$$n_{\text{dip}} \sim n_v \qquad \sigma \sim d$$

$$\tau_{\text{ann}} = \tau_{\text{coll}}\,\alpha \qquad \bar{v} = \frac{1}{d}$$

$$\tau_{\text{coll}} = \frac{l}{\bar{v}} \qquad d = \frac{1}{\sqrt{n_v}}$$

$$l \sim \frac{1}{n_v \sigma}$$

d : average pair distance

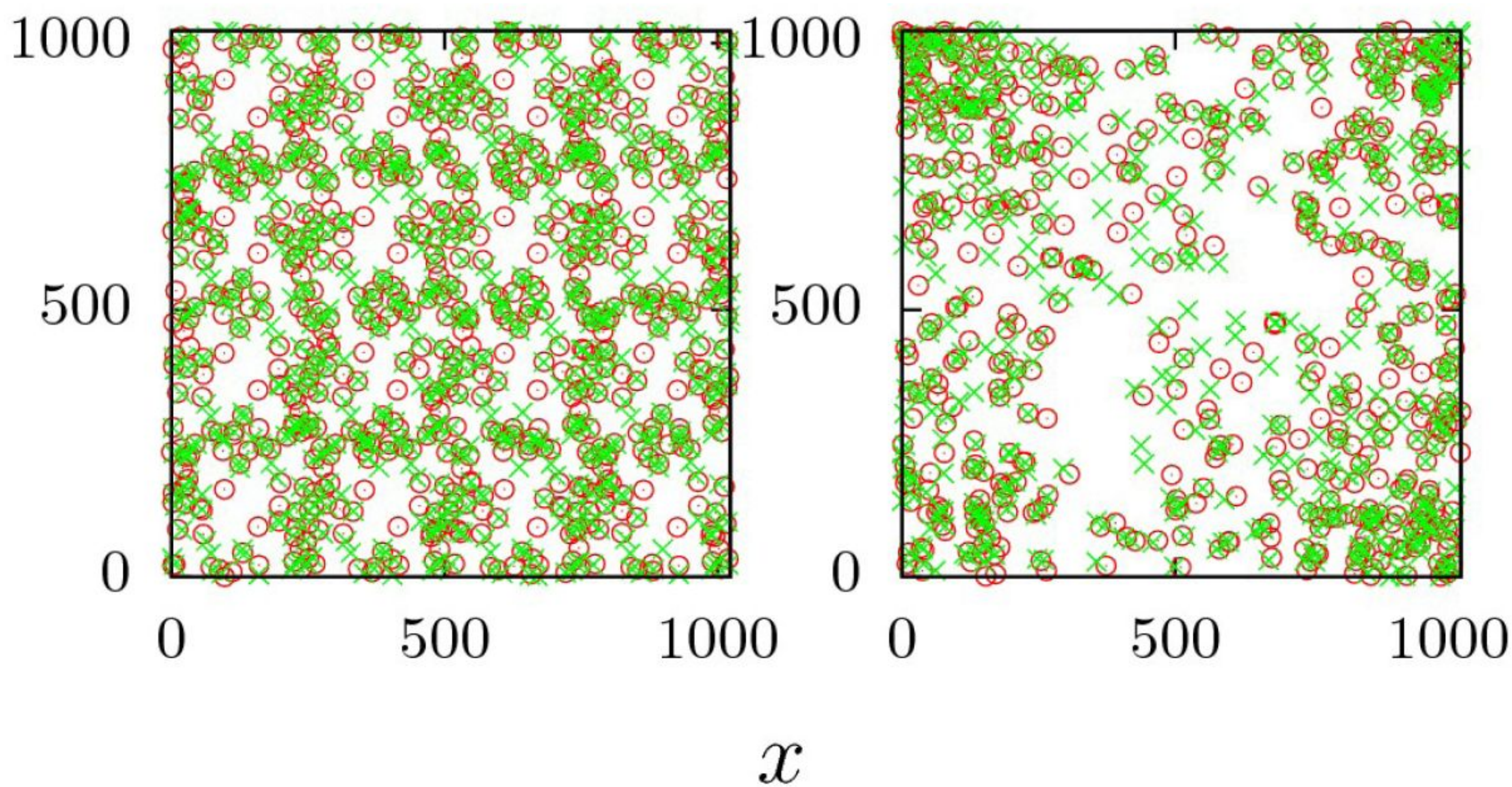
$\bar{v}$: average pair velocity

l : mean free path

$$\Rightarrow \quad \partial_t n_v(t) \sim -n_v^2 \quad \Rightarrow \quad n_v(t) \sim t^{-1}$$

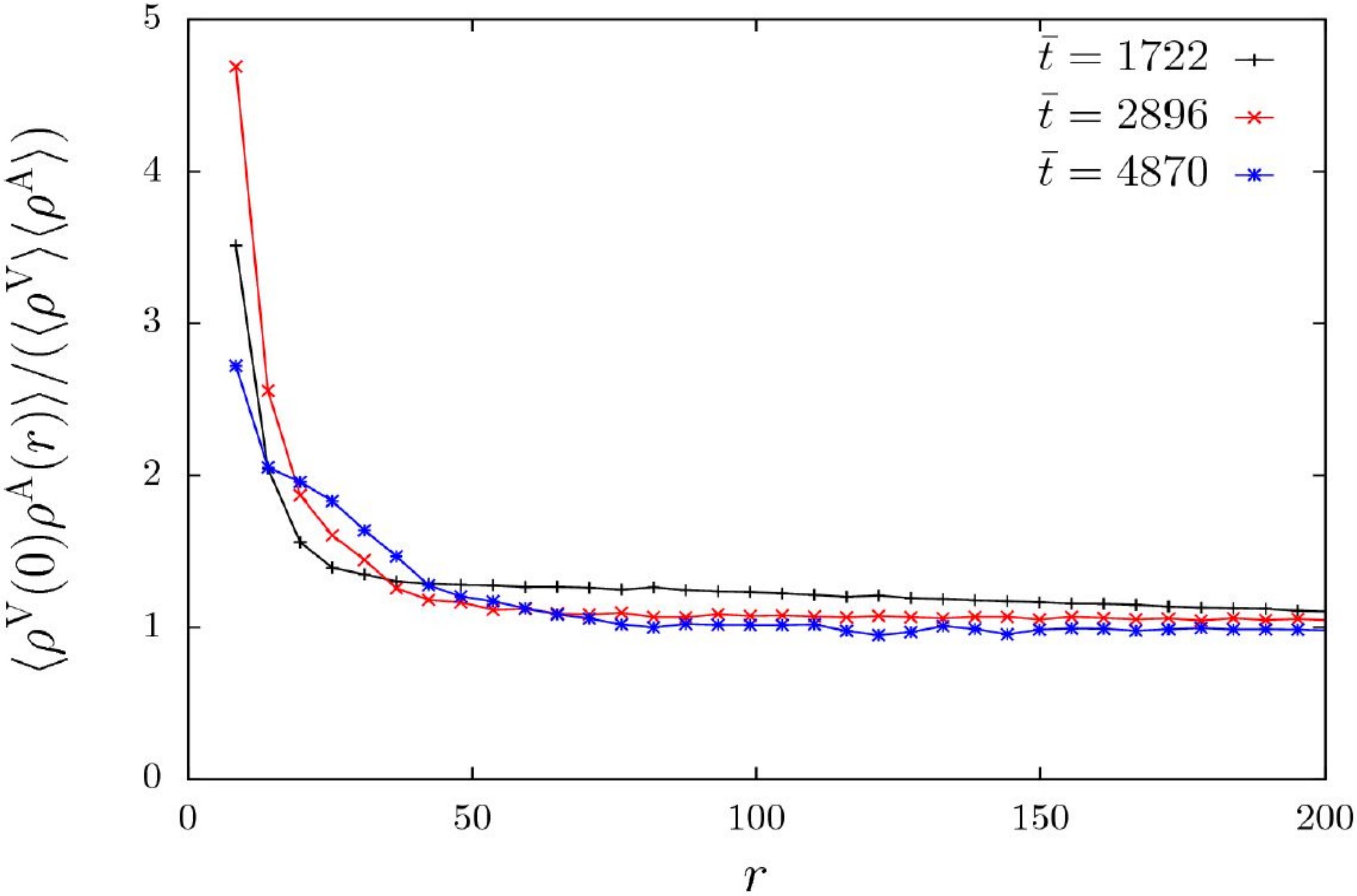
This result is valid under the assumption that the vortices are moving in pairs and that the pairs are homogeneously distributed.

Vortex-Distribution



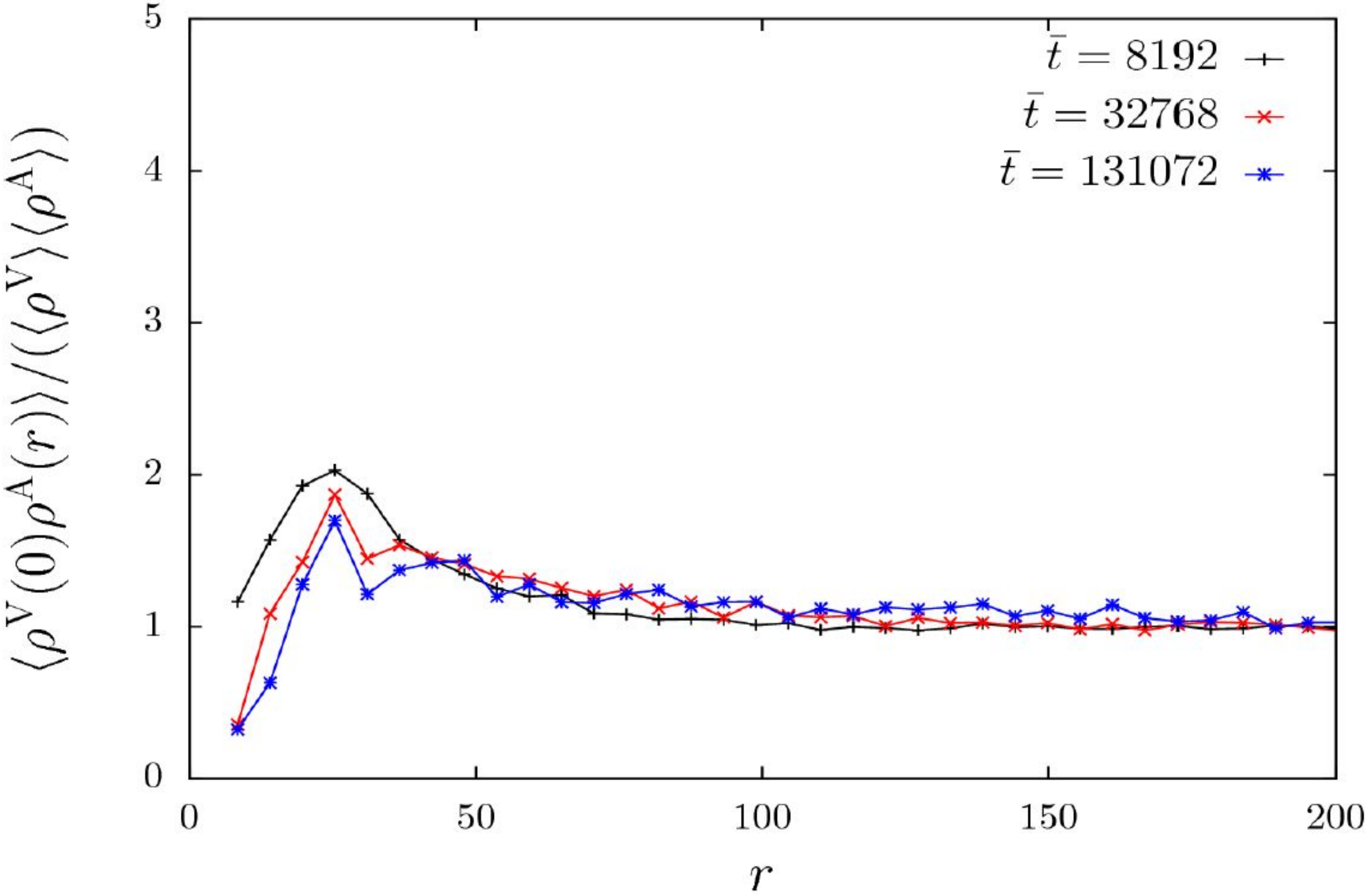
J. S., B. Nowak, D. Sexty, T. Gasenzer (unpublished)

Vortex-Antivortex-Correlations



J. S., B. Nowak, D. Sexty, T. Gasenzer (unpublished)

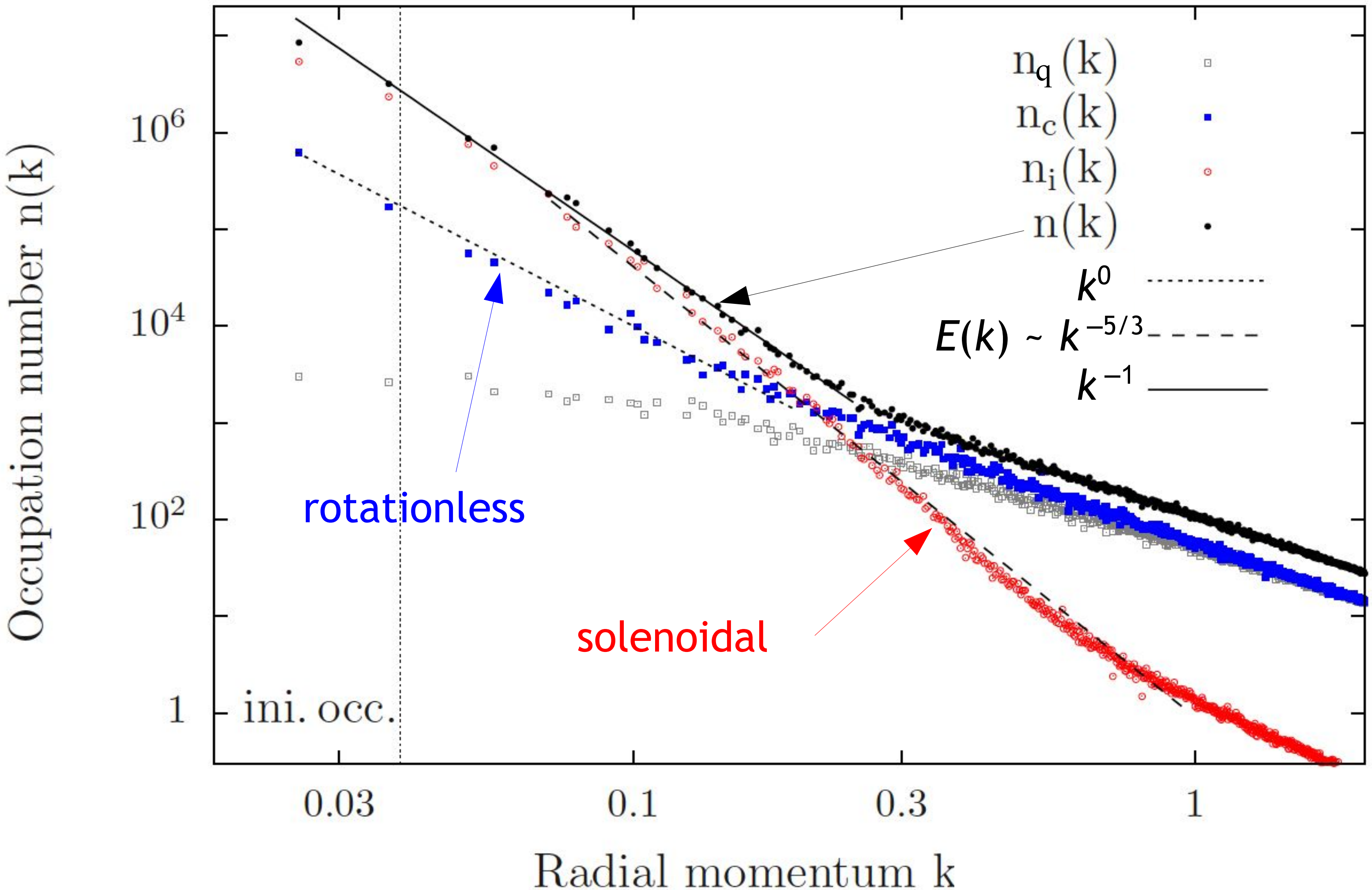
Vortex-Antivortex-Correlations



J. S., B. Nowak, D. Sexty, T. Gasenzer (unpublished)

Simulations in 2+1 D

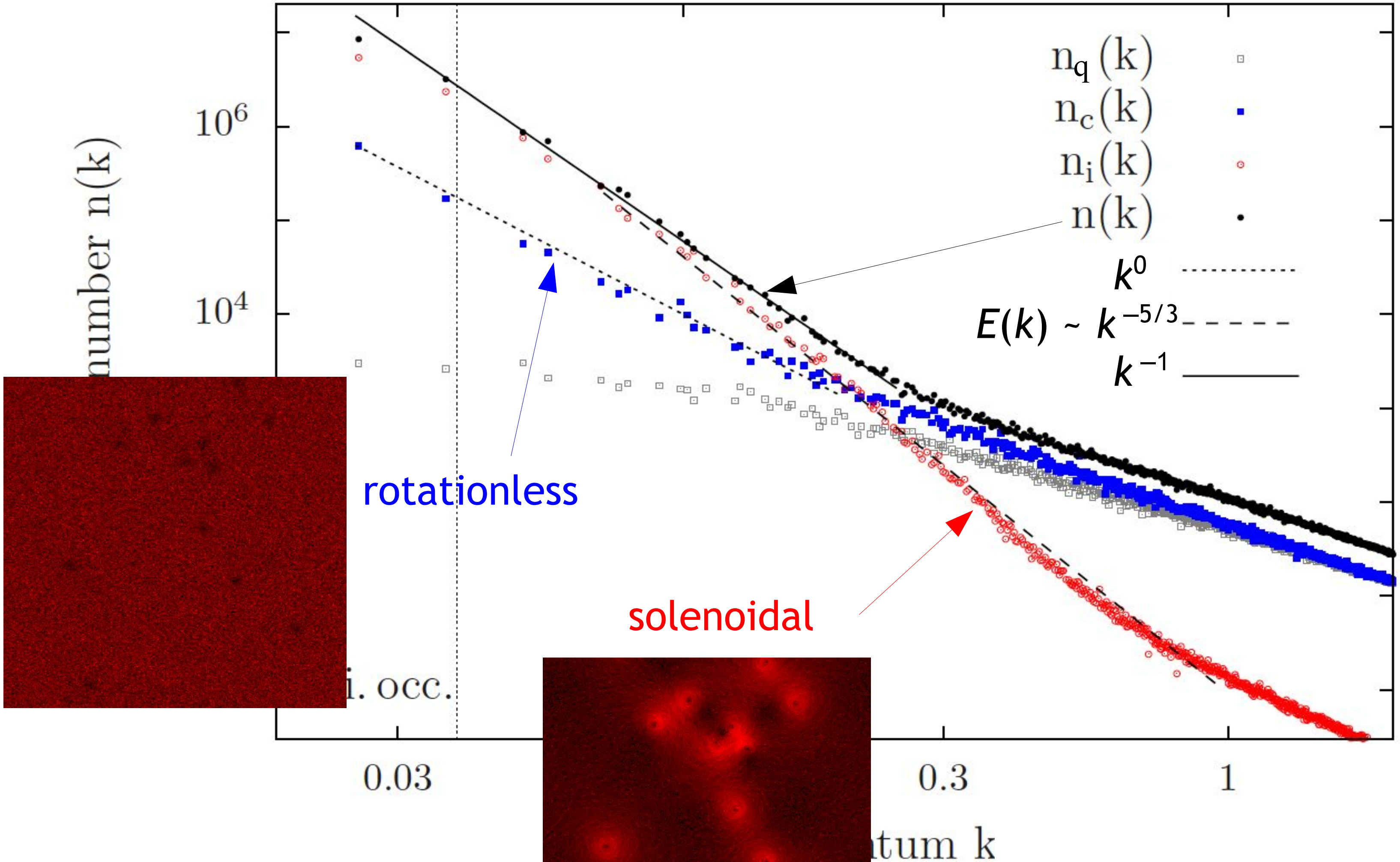
$$E(k) = \omega(k) k^{d-1} n(k)$$



B. Nowak, D. Sexty, TG (arXiv:1012.4437), PRB tbp.

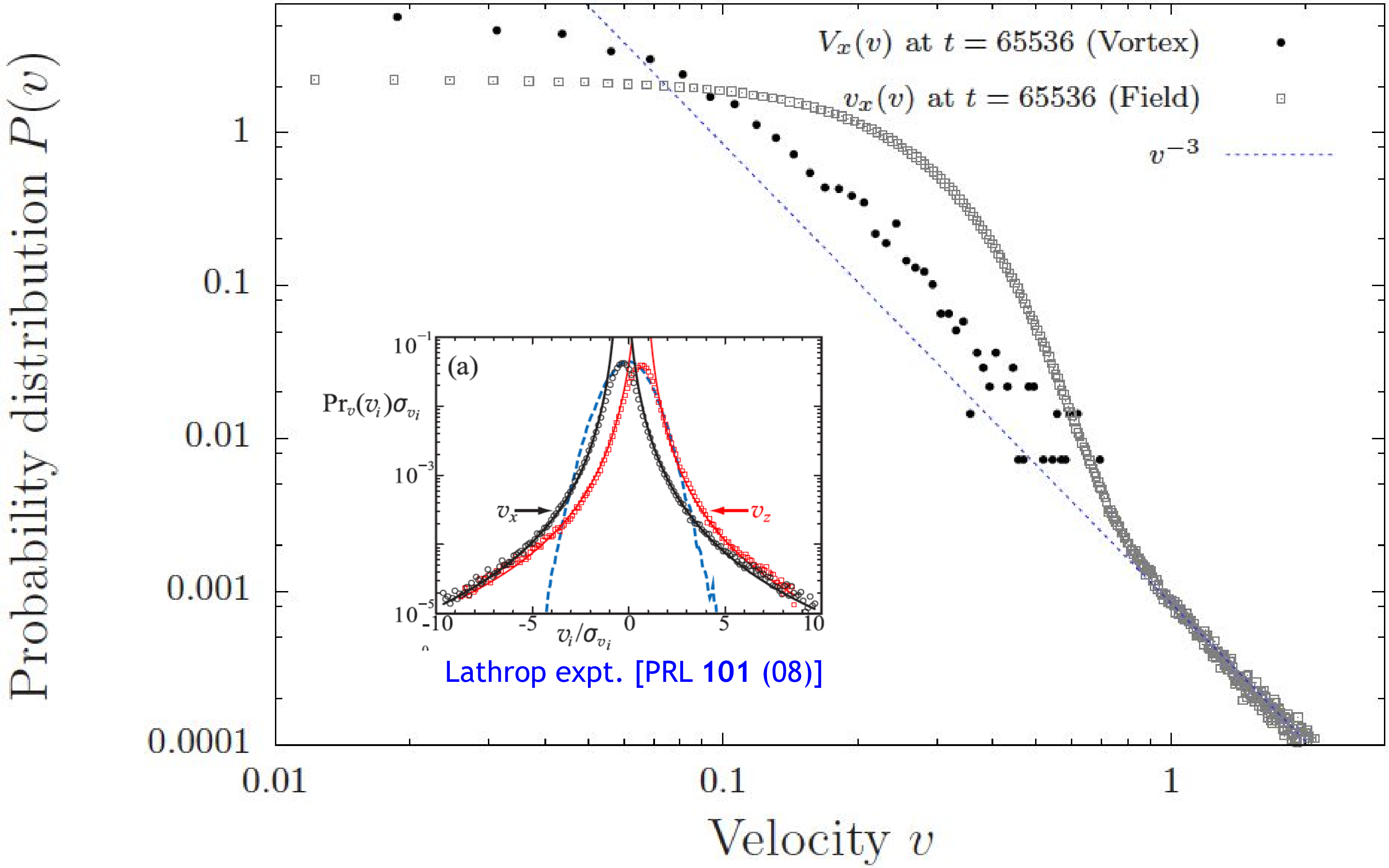
Simulations in 2+1 D

$$E(\mathrm{k}) = \omega(\mathrm{k})\mathrm{k}^{d-1}\mathrm{n}(\mathrm{k})$$



, D. Sexty, TG (arXiv:1012.4437), PRB tbp.

Vortex velocity distribution



Lathrop expt. [PRL **101** (08)]

J. Schole, B. Nowak, D. Sexty, TG (unpublished)
s. also C.F. White et al., PRL **104** (10); I.A. Min, Phys. Fluids **8** (96)

Velocity distributions

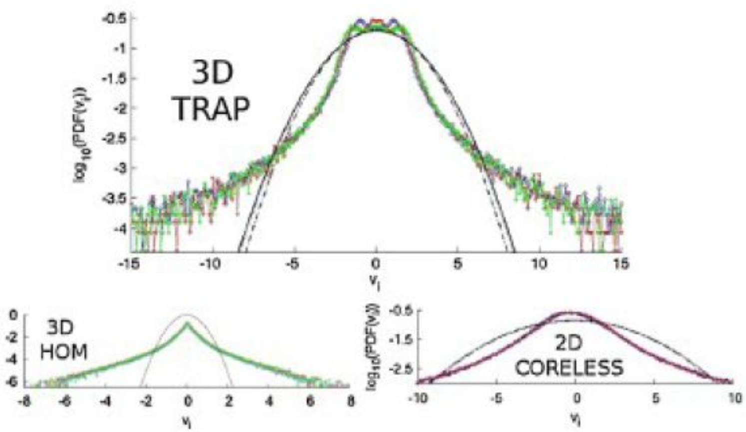
Paoletti et al. PRL 101, 154501 (2008):

Power law tails distinguish classical turbulence from classical turbulence.

Min et al. Phys. Fluids 8, 1169 (1996), White et al. PRL 104, 075301 (2010):

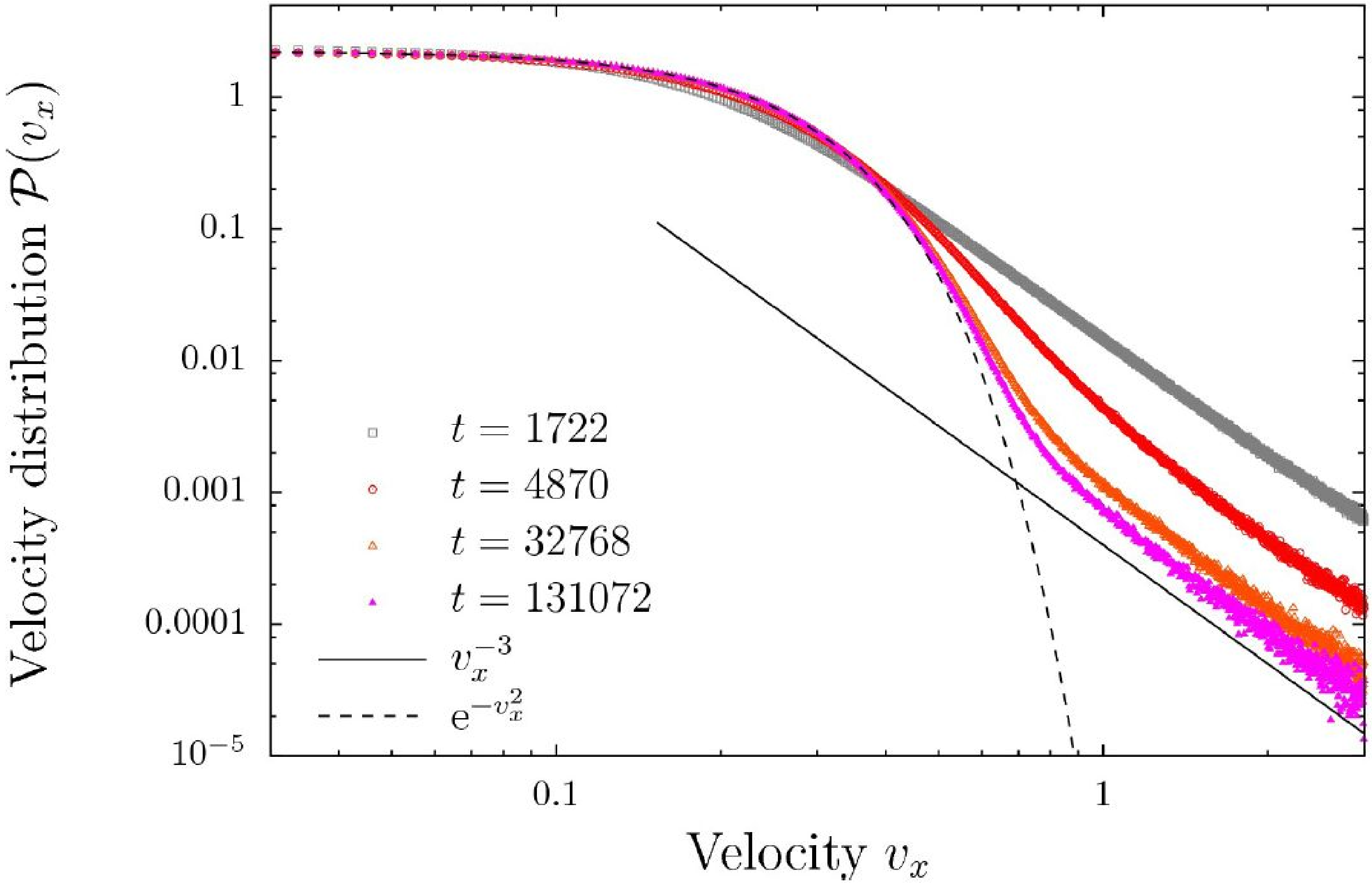
Point vortices: Power law tails

Vorticity patches: Gaussian distributions



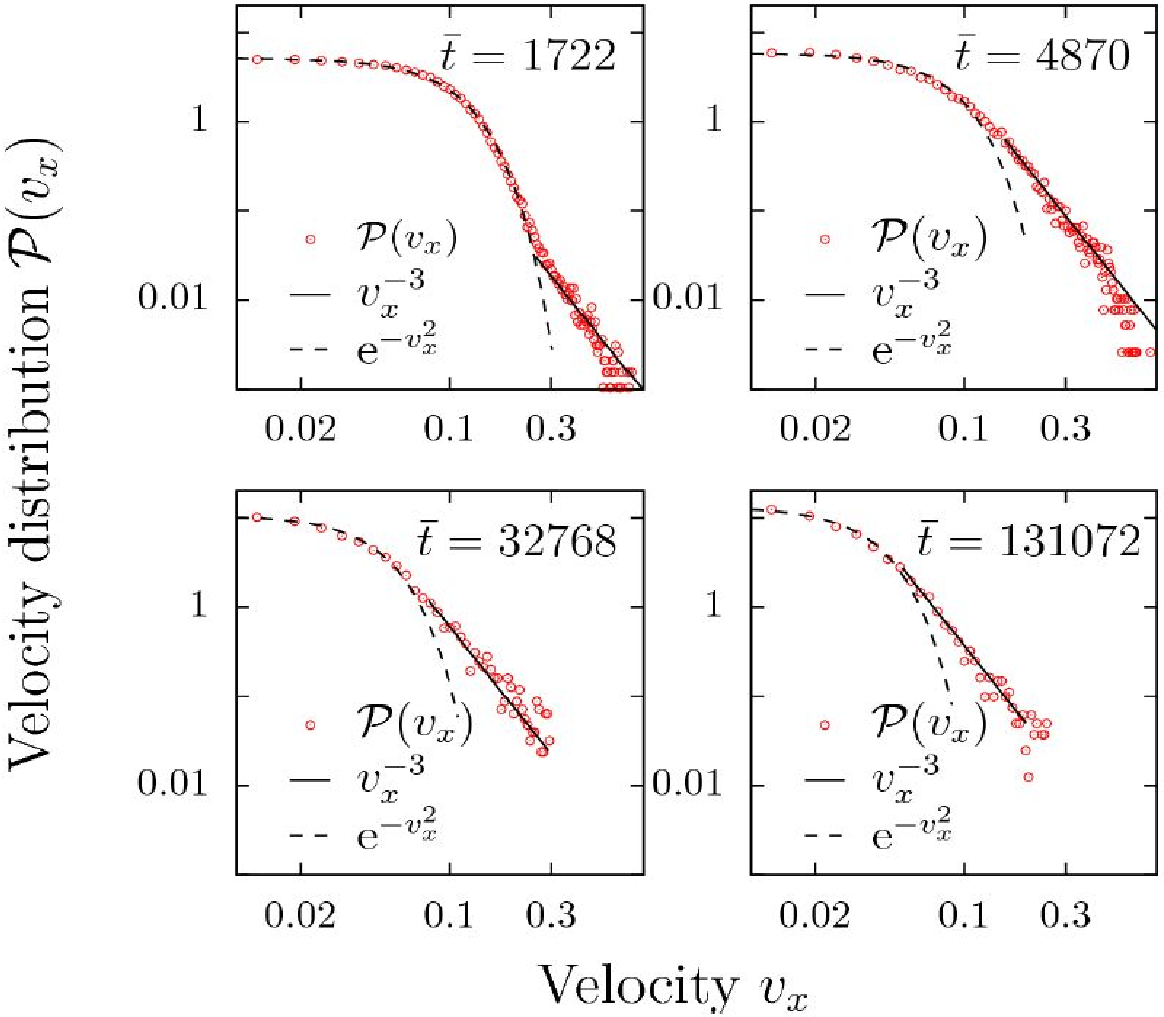
White et al. PRL 104, 075301 (2010)

Velocity distributions (Field)



J. S., B. Nowak, D. Sexty, T. Gasenzer (unpublished)

Velocity distributions (Vortices)



J. S., B. Nowak, D. Sexty, T. Gasenzer (unpublished)

Condensation

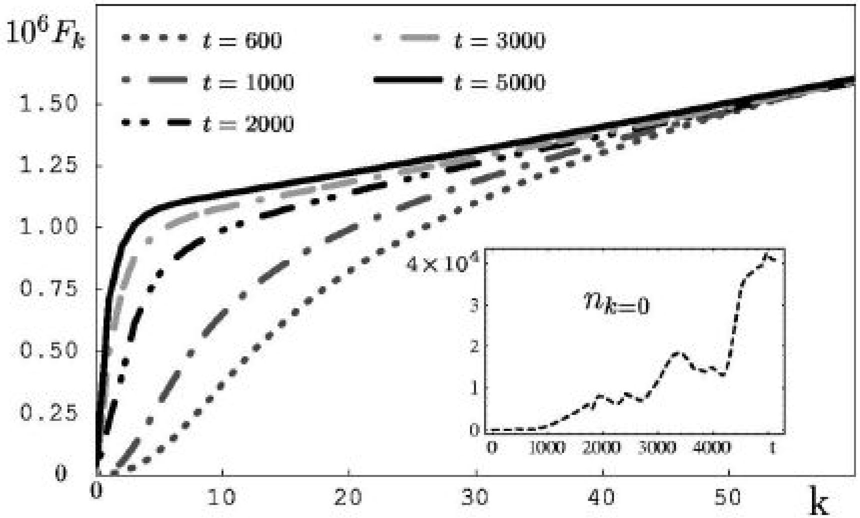


FIG. 2. Evolution of the integral distribution of particles $F_k = \Sigma_{k' \leq k} n_{\mathbf{k}'}$. Notice the appearance of a "shoulder" of F_k indicative of quasicondensate formation. The evolution of $n_{\mathbf{k}} = \mathbf{0}$ is presented in the inset. Note the strong fluctuations typical for the evolution of a single harmonic. The fluctuations are also seen in the graph of the first shell [see inset (a) of Fig. 1].

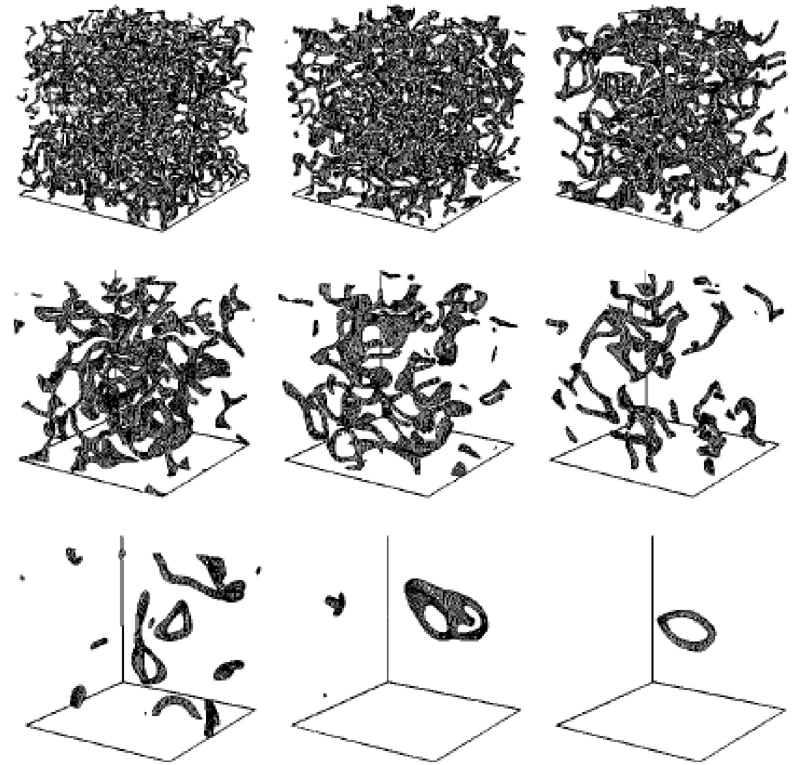
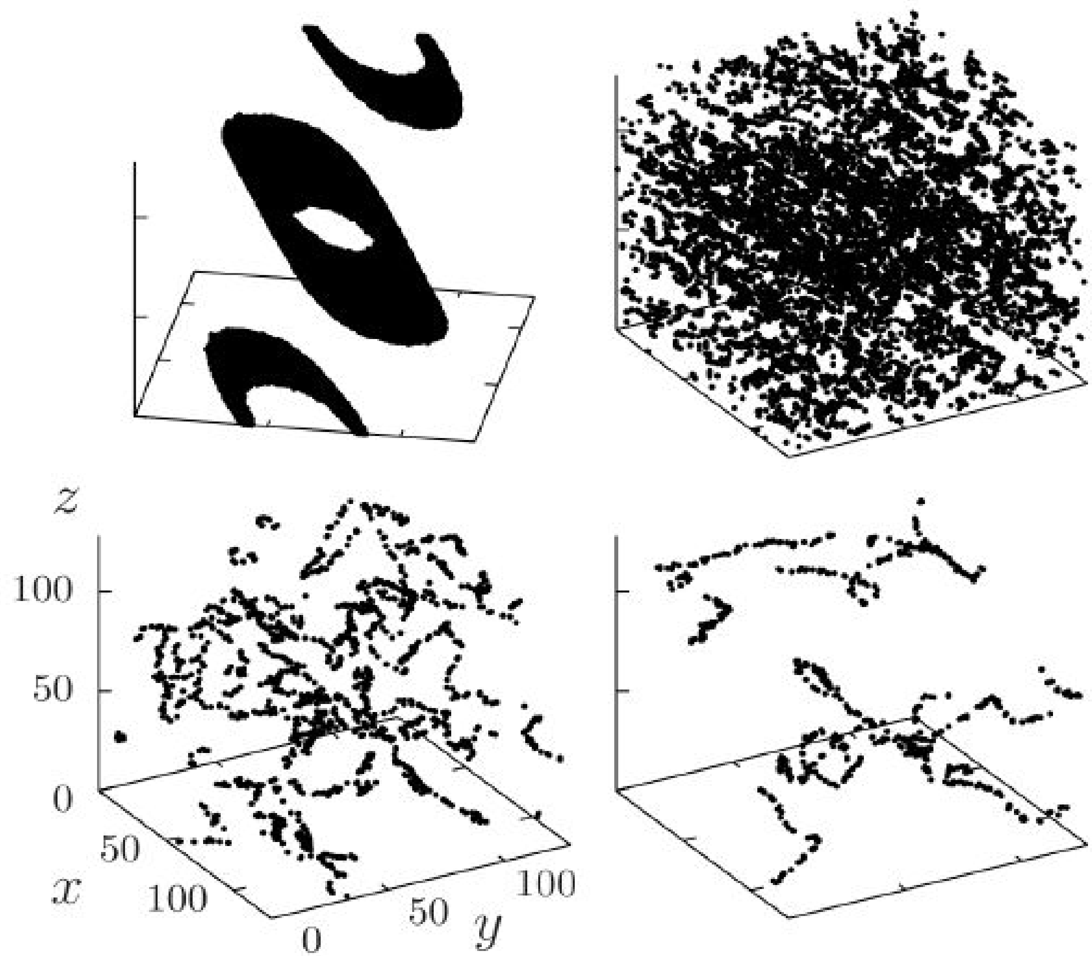


FIG. 4. Evolution of topological defects in the phase of the long-wavelength part $\tilde{\psi}$ of the field ψ in the computational box 128^3. The defects are visualized by isosurfaces $|\tilde{\psi}|^2 = 0.05\langle|\tilde{\psi}|^2\rangle$. High-frequency spatial waves are suppressed by the factor $\max\{1 - k^2/k_c^2, 0\}$, where the cutoff wave number is chosen according to the phenomenological formula $k_c = 9 - t/1000$.

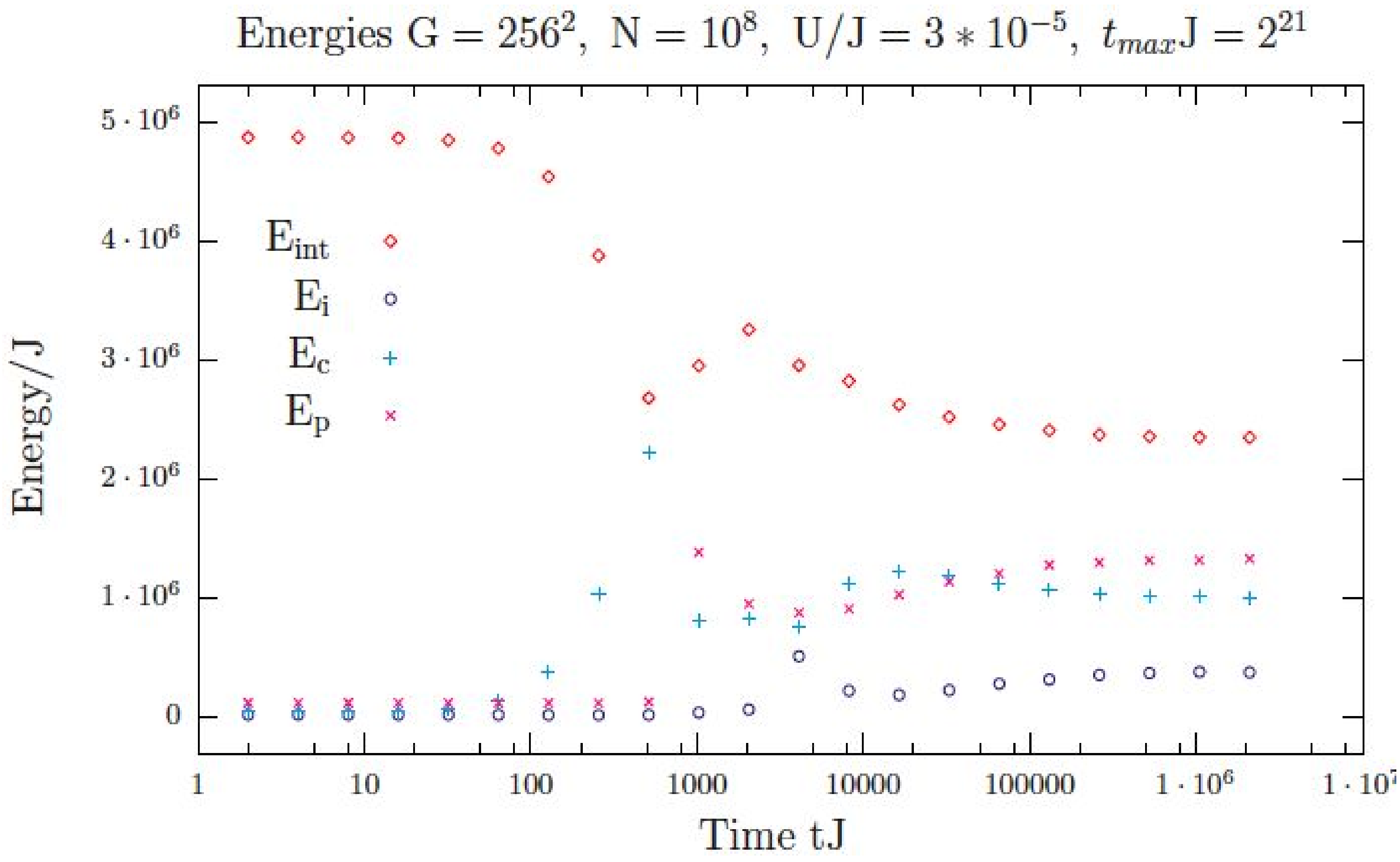
N.G. Berloff, S.V. Svistunov, PRA 66, 013603 (2002)

Simulations in 3+1 D



B. Nowak, D. Sexty, TG (arXiv:1012.4437)

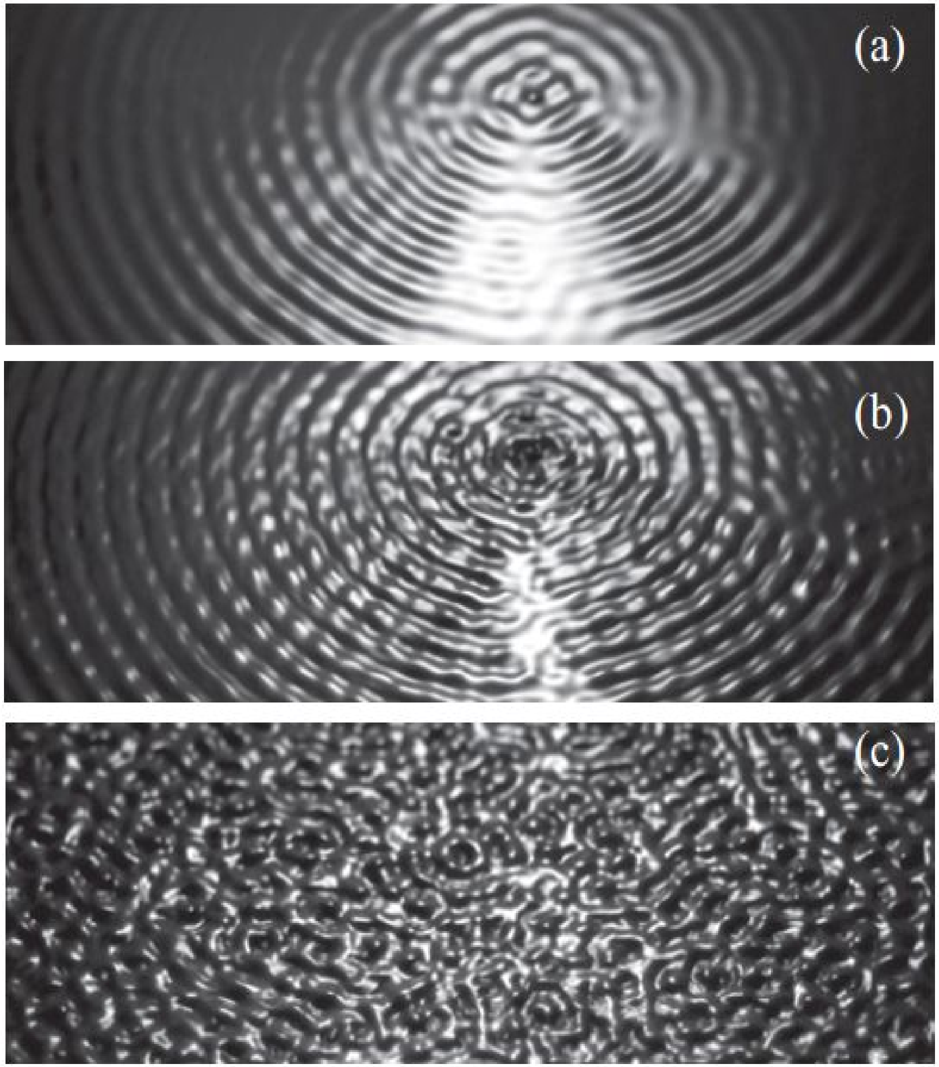
Energies

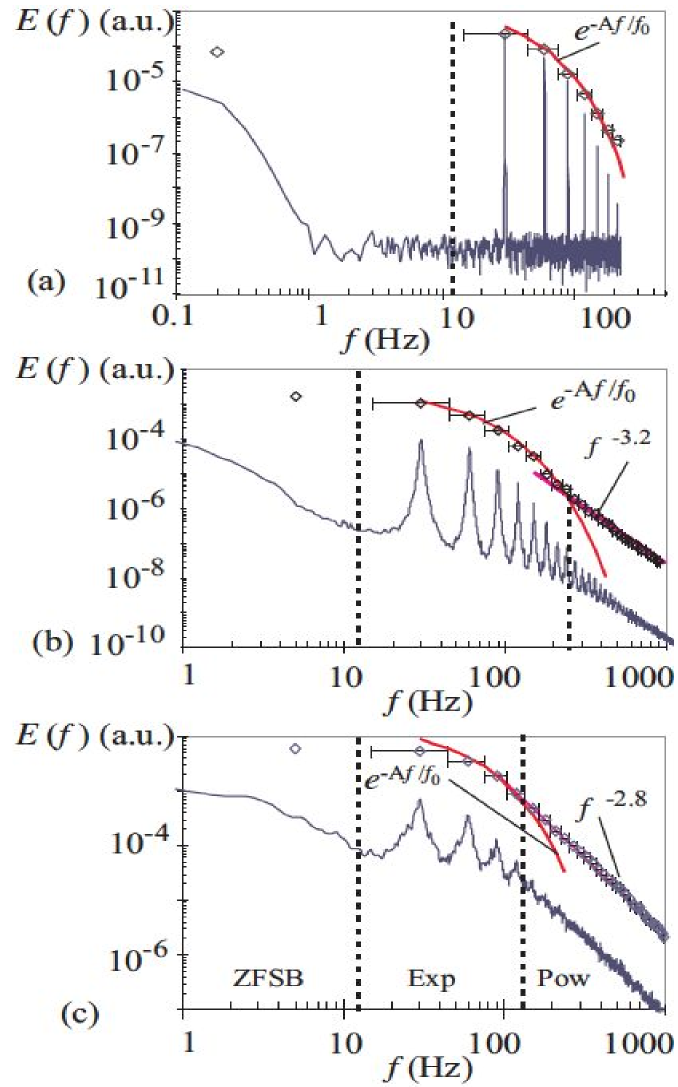


Energies $G = 256^2$, $N = 10^8$, $U/J = 3 * 10^{-5}$, $t_{max}J = 2^{21}$
E_{int}
E_i
E_c
E_p
Energy/J
$5 \cdot 10^6$
$4 \cdot 10^6$
$3 \cdot 10^6$
$2 \cdot 10^6$
$1 \cdot 10^6$
0
1
10
100
1000
10000
100000
$1 \cdot 10^6$
$1 \cdot 10^7$
Time tJ

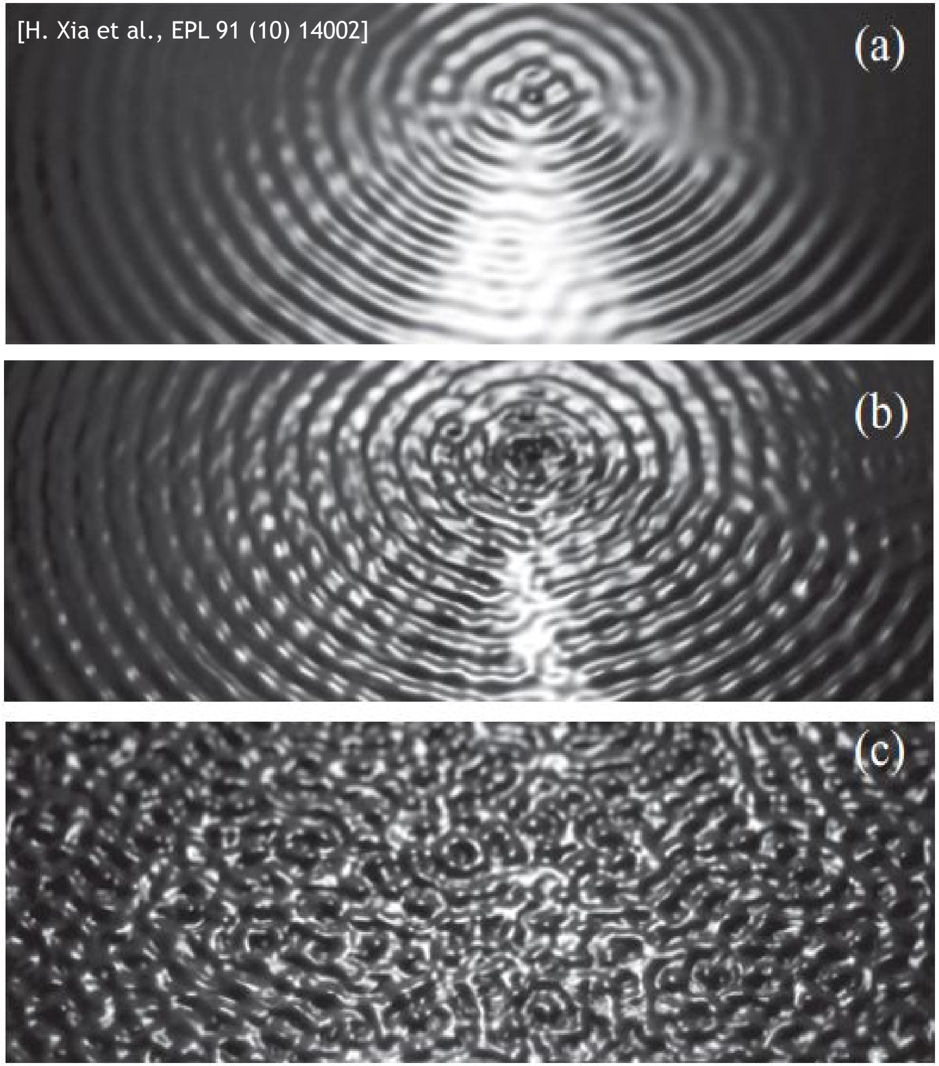
[H. Xia et al., EPL 91 (10) 14002]

Wave Turbulence – e.g. on water





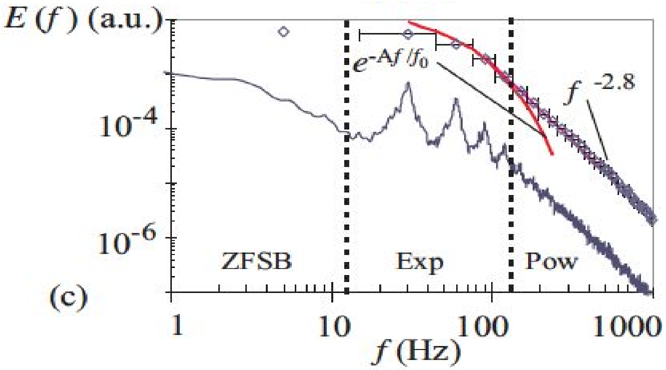
Wave Turbulence – e.g. on water



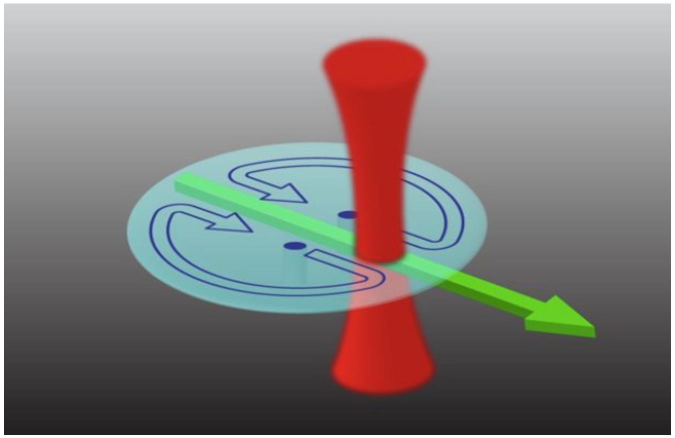
Theory prediction:

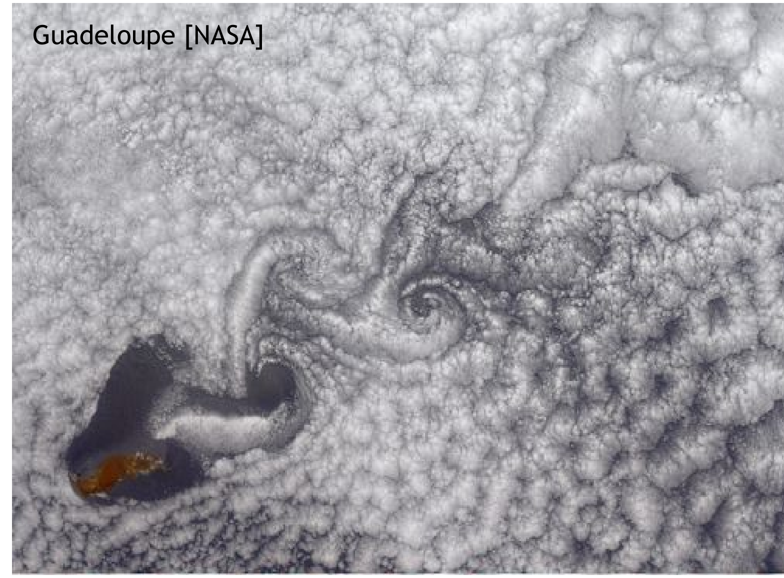
$$E_\omega \sim \omega^{-17/6}.$$

[Zakharov & Filonenko (67)]

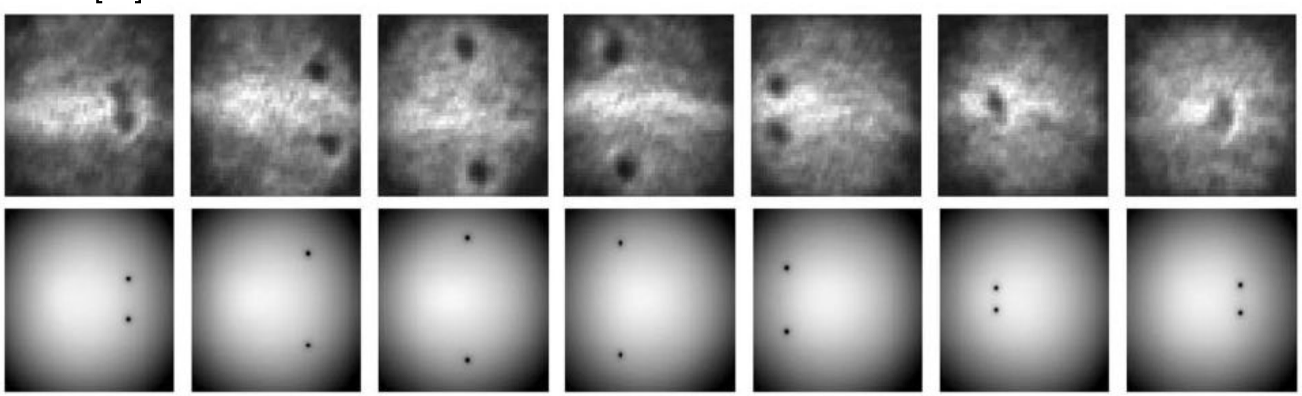


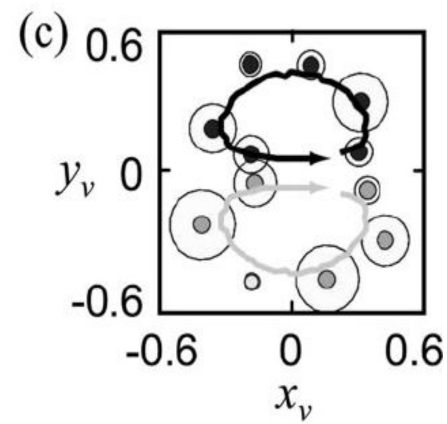
Vortex pairs





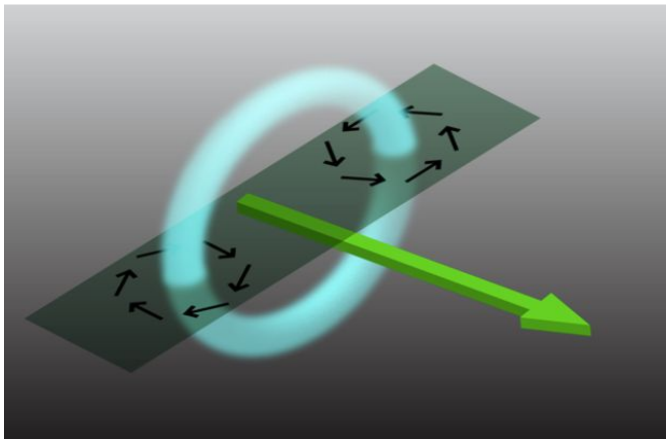
Tucson [AZ]

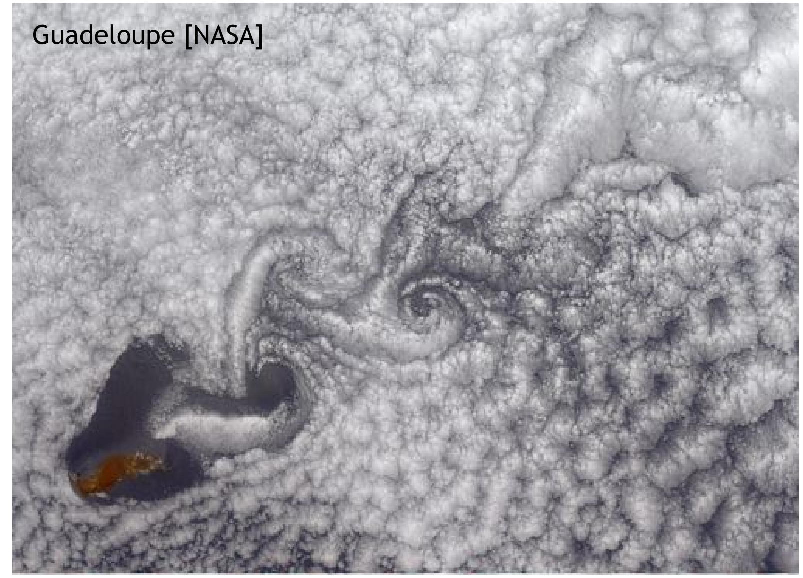




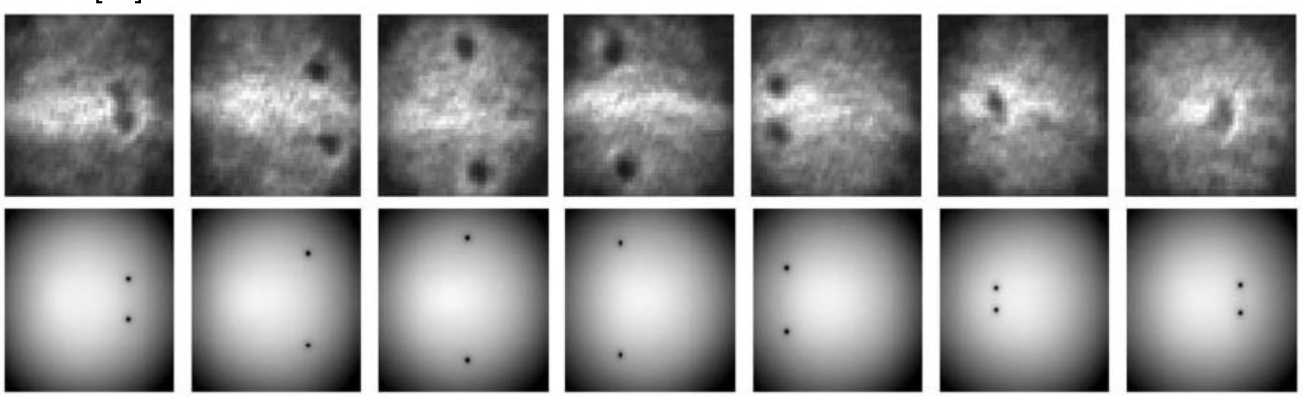
[T.W. Neely et al. PRL **104** (10)]

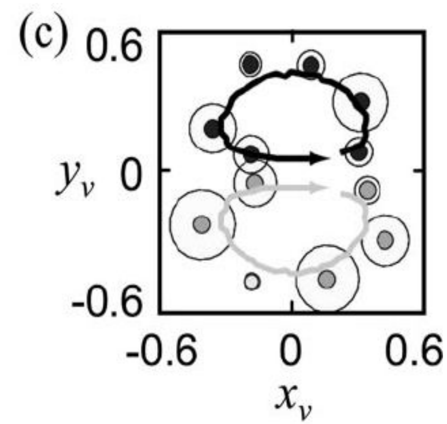
Helmholtz Vortex Law




Guadeloupe [NASA]

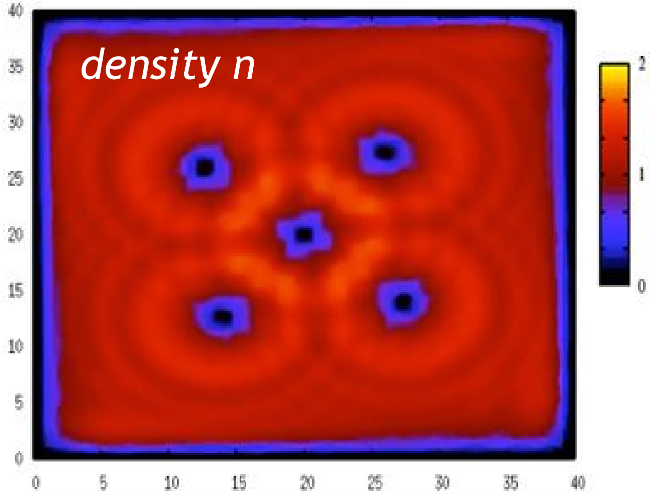
Tucson [AZ]

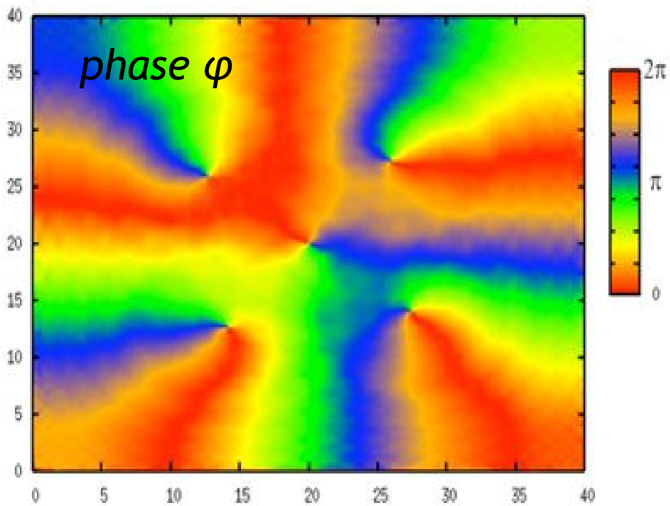




[T.W. Neely et al. PRL **104** (10)]

Quantum Vortices





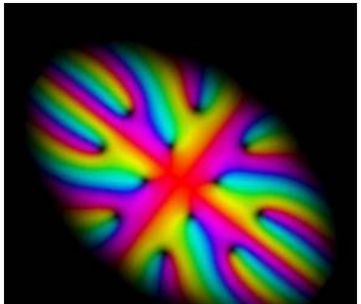
[W. P. Reinhardt, Seattle]

$$\Psi(\boldsymbol{\rho},t)=\sqrt{n(\boldsymbol{\rho},t)}\exp[i\varphi(\boldsymbol{\rho},t)]$$

complex field

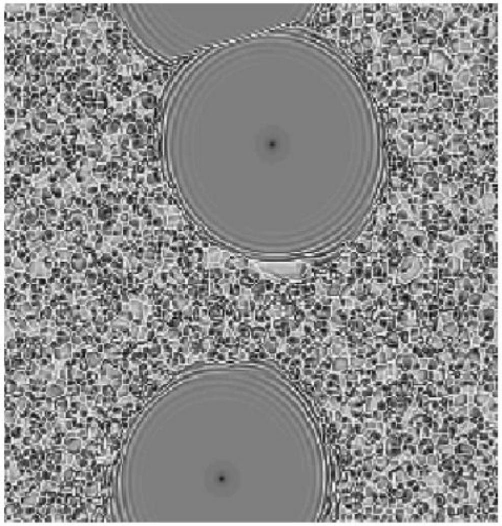
$$\mathbf{u}(\boldsymbol{\rho},t)=\nabla\varphi(\boldsymbol{\rho},t)$$

velocity



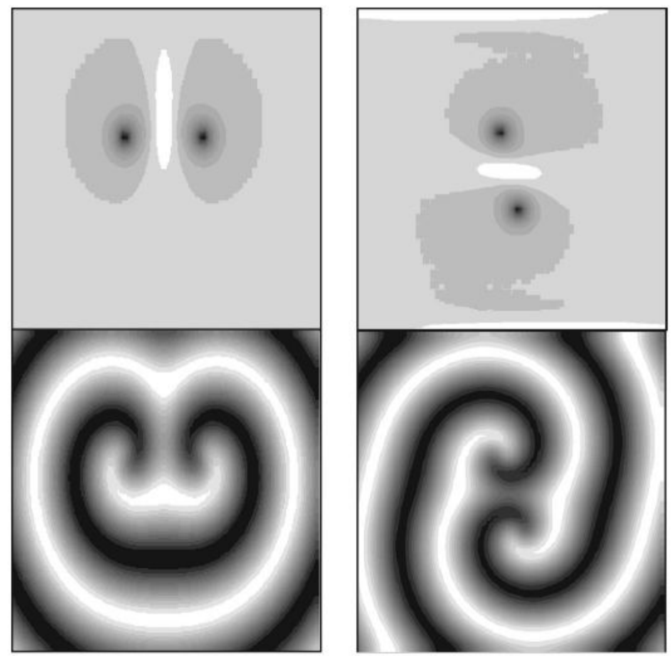
[Wolfgang Ketterle, MIT]

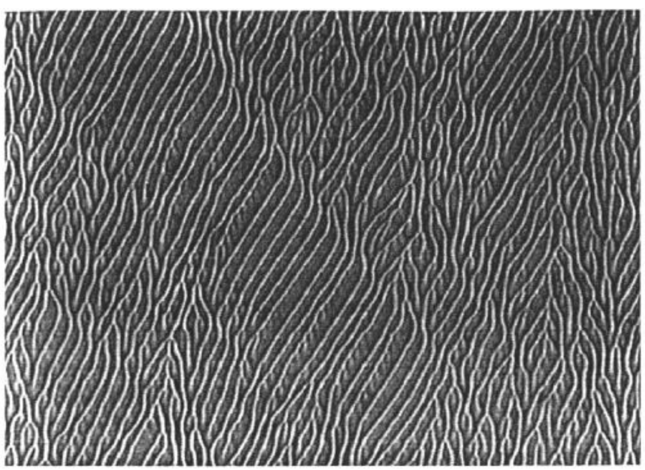
Nonlinear dynamics: Pattern formation



I. S. Aranson and L. Kramer: The complex Ginzburg-Landau equation

REVIEWS OF MODERN PHYSICS, VOLUME 74, JANUARY 2002



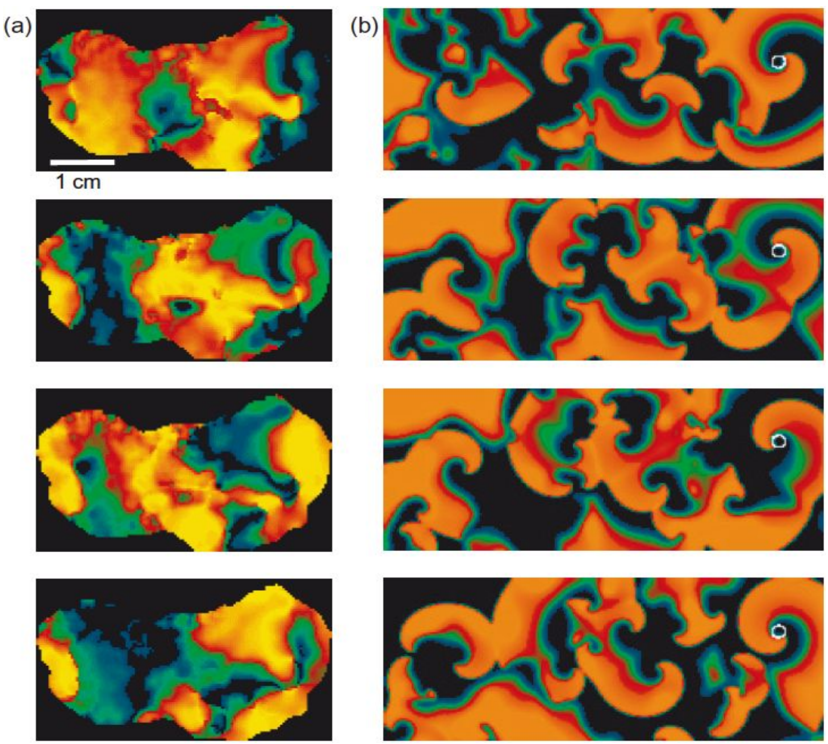


Nonlinear dynamics: Pattern formation

Visualization of spiral and scroll waves in simulated and experimental cardiac tissue

E M Cherry and F H Fenton
Department of Biomedical Sciences, Cornell University, Ithaca, NY 14853, USA
and
Max Planck Institute for Dynamics and Self-organization, Göttingen, Germany
E-mail: elizabeth.m.cherry@cornell.edu and flavio.h.fenton@cornell.edu

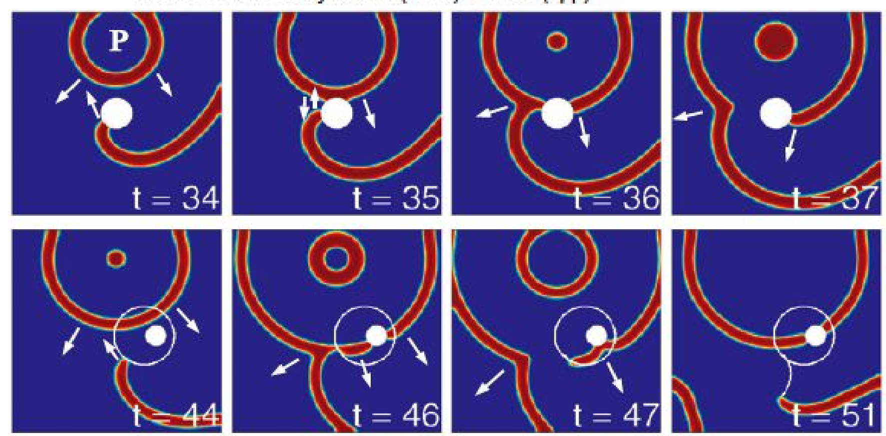
New Journal of Physics **10** (2008) 125016 (43pp)



Far field pacing supersedes anti-tachycardia pacing in a generic model of excitable media

Philip Bittihn[1,2,4], Gisela Luther[2], Eberhard Bodenschatz[2], Valentin Krinsky[2,3], Ulrich Parlitz[1] and Stefan Luther[2]
[1] Drittes Physikalisches Institut, Göttingen University, Friedrich-Hund-Platz 1, 37077 Göttingen, Germany
[2] Max Planck Institute for Dynamics and Self-Organization, Bunsenstraße 10, 37073 Göttingen, Germany
[3] Institut Non Linéaire de Nice, 1361 Rte des Lucioles, 06560 Valbonne/Sophia-Antipolis, France
E-mail: bittihn@physik3.gwdg.de

New Journal of Physics **10** (2008) 103012 (9pp)

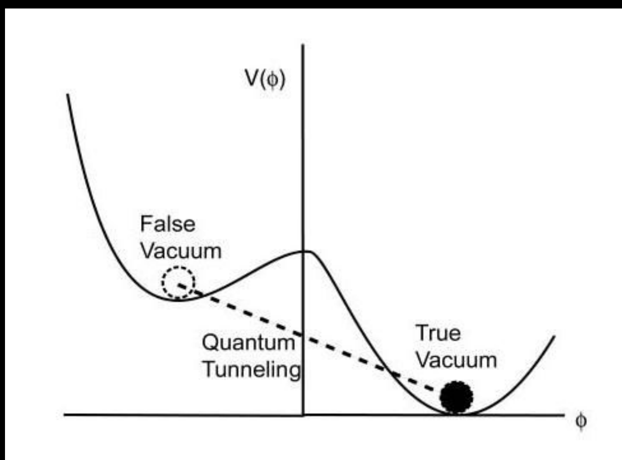


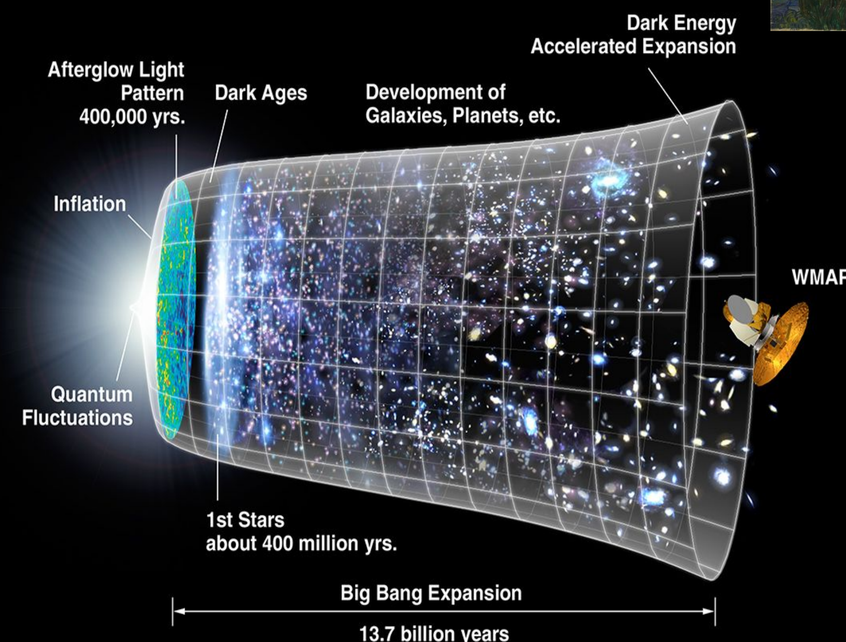
$$\frac{\partial u}{\partial t} = \varepsilon^{-1} u(1-u)\left(u - \frac{v+b}{a}\right) + \nabla^2 u$$

Barkley model

Thermalisation dynamics: Turbulence?

Cosmology:
Reheating after Inflation



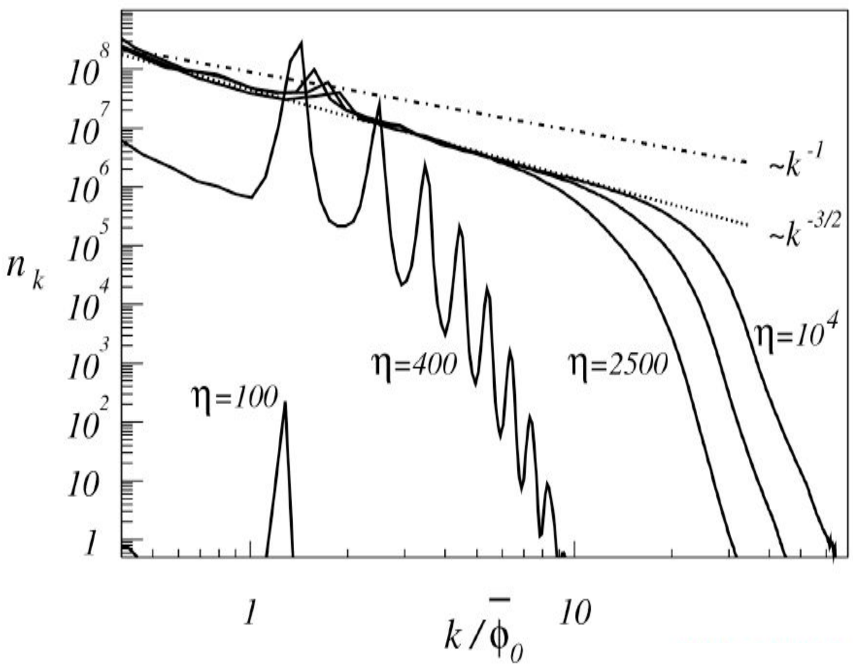


NASA/WMAP Science Team

Turbulence in reheating after inflation

Simulations of the non-linear Klein-Gordon equation,

$$(\partial_t^2 - \partial_x^2)\varphi(x,t) + \lambda\varphi^3(x,t) = 0$$



Initial condition:

Highly occupied zero mode

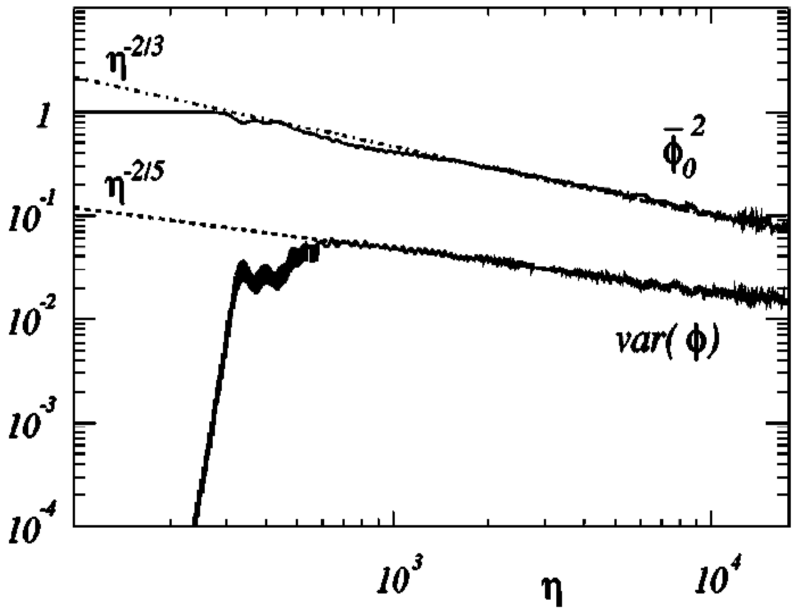
Unoccupied modes with k>0

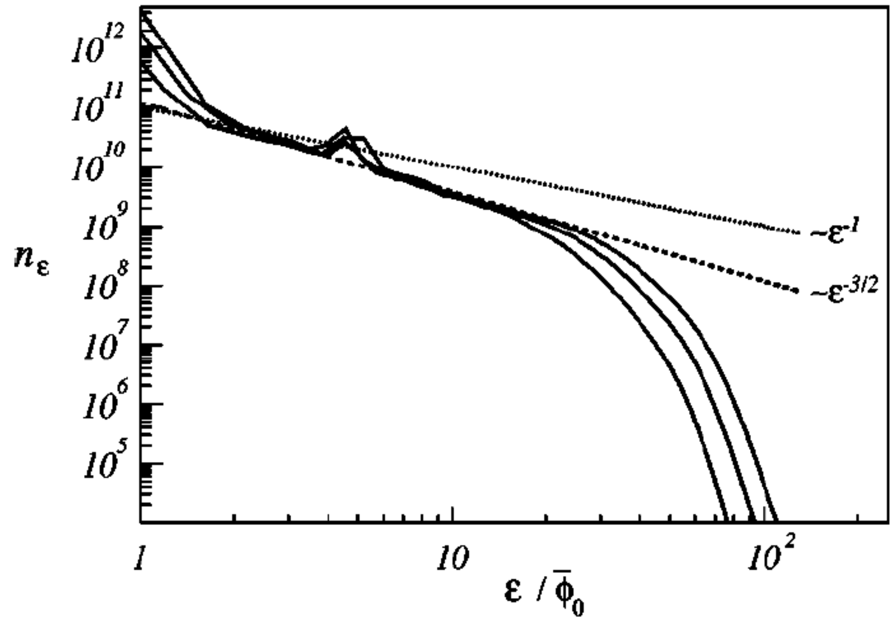
Turbulent spectrum emerges

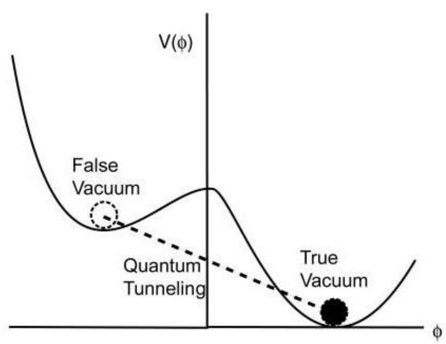
Exponent: weak wave turbulence

Kofmann, Linde, Starobinsky (96)
Micha, Tkachev, PRL & PRD (04)

Wave turbulence







Turbulent thermalisation after universe inflation

[Micha & Tkachev, PRL **90** (03) 121301, PRD **70** (04) 043538]

Lewis Fry Richardson, FRS (1881-1953)

Big whirls have little whirls that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity.

(L.F. Richardson, *The supply of energy from and to Atmospheric Eddies*, 1920)

Great fleas have little fleas upon their backs to bite 'em,
And little fleas have lesser fleas, and so ad infinitum.
And the great fleas themselves, in turn, have greater fleas to go on;
While these again have greater still, and greater still, and so on.

(Augustus de Morgan, *A Budget of Paradoxes, 1872, p. 370*)

So, naturalists observe, a flea
Has smaller fleas that on him prey;
And these have smaller still to bite 'em;
And so proceed ad infinitum.

(Jonathan Swift: *Poetry, a Rhapsody*, 1733)