The over-populated Quark-Gluon Plasma on the Lattice

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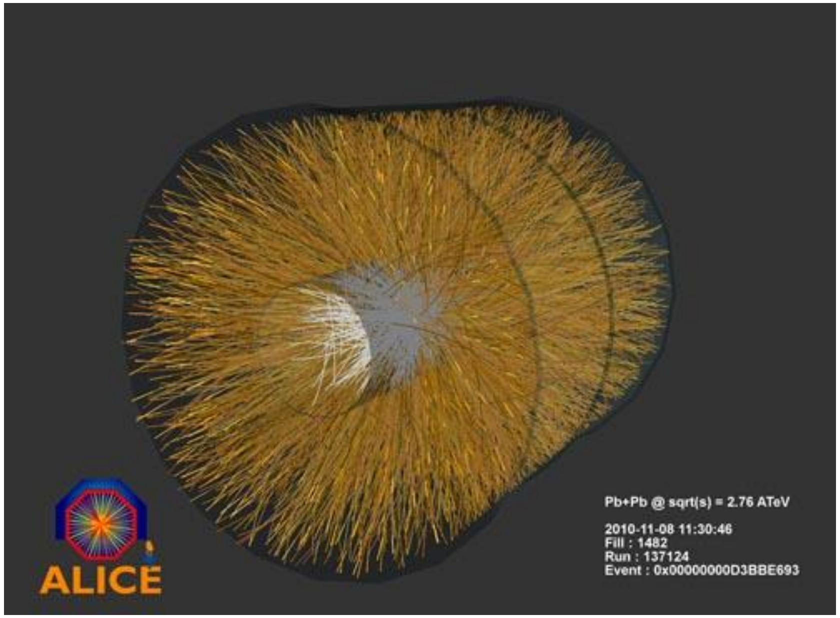
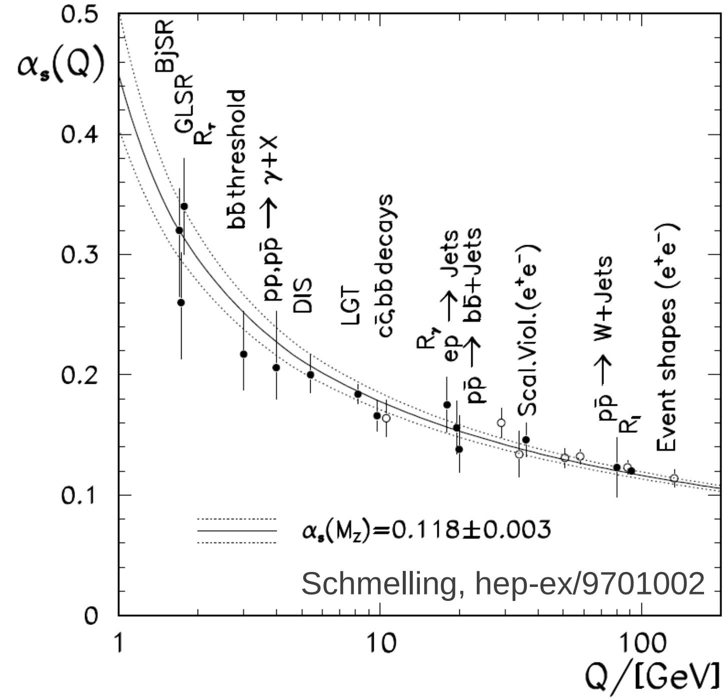
Relativistic heavy-ion collisions





CGC: Energy density of gluons with typical momentum Q_s (at time $1/Q_s$)

$$\epsilon \sim \frac{Q_s^4}{\alpha_s} \quad \text{i.e. occupation numbers} \quad n(p \lesssim Q_s) \sim \frac{1}{\alpha_s}$$

Nonperturbative even for weak coupling $\alpha_s \ll 1$

Real-time lattice gauge theory

Far-from-equilibrium phenomena (plasma instabilities, turbulence,…)
cannot be described in Euclidean space-time

Wilson action in Minkowski space-time:

$$S = -\beta_0 \sum_x \sum_i \left\{ \frac{1}{2\mathrm{Tr}\,\mathbf{1}} \left(\mathrm{Tr}\, U_{x,0i} + \mathrm{Tr}\, U_{x,0i}^{-1} \right) - 1 \right\} + \beta_s \sum_x \sum_{\substack{i,j \\ i<j}} \left\{ \frac{1}{2\mathrm{Tr}\,\mathbf{1}} \left(\mathrm{Tr}\, U_{x,ij} + \mathrm{Tr}\, U_{x,ij}^{-1} \right) - 1 \right\} + S_{\text{quarks}}$$

Plaquette variables

$$U_{x,\mu\nu} \equiv U_{x,\mu} U_{x+\hat{\mu},\nu} U^{\dagger}_{x+\hat{\nu},\mu} U^{\dagger}_{x,\nu} \approx \exp\left[-\mathrm{i}ga^2 F_{\mu\nu}(x)\right], \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

Real time:

$$\sim e^{iS}$$

non-positive definite, i.e.
no importance sampling!

Classical-statistical lattice gauge theory

Accurate for sufficiently large fields/high gluon occupation numbers:

anti-commutators $\langle\{A, A\}\rangle \gg \langle[A, A]\rangle$ commutators

'working horse' for instability dynamics/turbulence: Romatschke, Venugopalan; Berges, Gelfand, Sexty, Scheffler, Schlichting; Kunihiro, Müller, Ohnishi, Schäfer, Takahashi, Yamamoto; Fukushima, Gelis; …

starting with scalars in mid 90s: Khlebnikov, Tkachev; Prokopec, Roos ; Tkachev, Khlebnikov, Kofman, Linde; Son; …

tested against quantum (2PI) over last decade: Aarts, Berges; Arrizabalaga, Smit, Tranberg; Tranberg, Rajantie; Berges, Rothkopf, Schmidt; Berges, Gelfand, Pruschke; …

$n(p) \sim 1/\alpha_s$	$1/\alpha_s \gg n(p) \gg 1$	$n(p) \lesssim 1$
('over-populated')	(classical particle)	(quantum)

$\underbrace{\text{(`over-populated') (classical particle)}}_{\text{classical-statistical field theory}}$

classical-statistical field theory

kinetic theory

($\alpha_s \ll 1$)

From instabilities to (wave) turbulence: some general remarks

en.wikipedia.org/wiki/Capillary_wave

Experimental example: Modulation *instability* and capillary wave *turbulence*

Instability leads to breaking of waves and development of wave turbulence

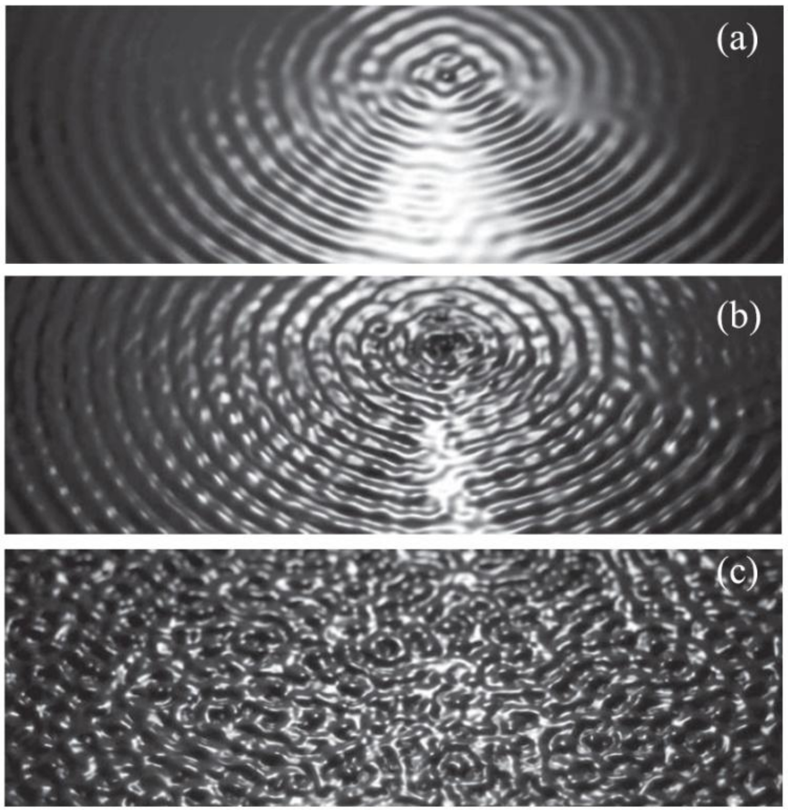


Fig. 1: (Colour on-line) Snapshots of the wave field evolution during the startup: (a) $t = 0.125\,\text{s}$, (b) $t = 0.25\,\text{s}$, (c) $t = 0.625\,\text{s}$.

Xia, Shats, Punzmann, EPL91 (2010) 14002

Energy injection limited by droplet formation!

Increasing forcing (higher wave amplitudes)

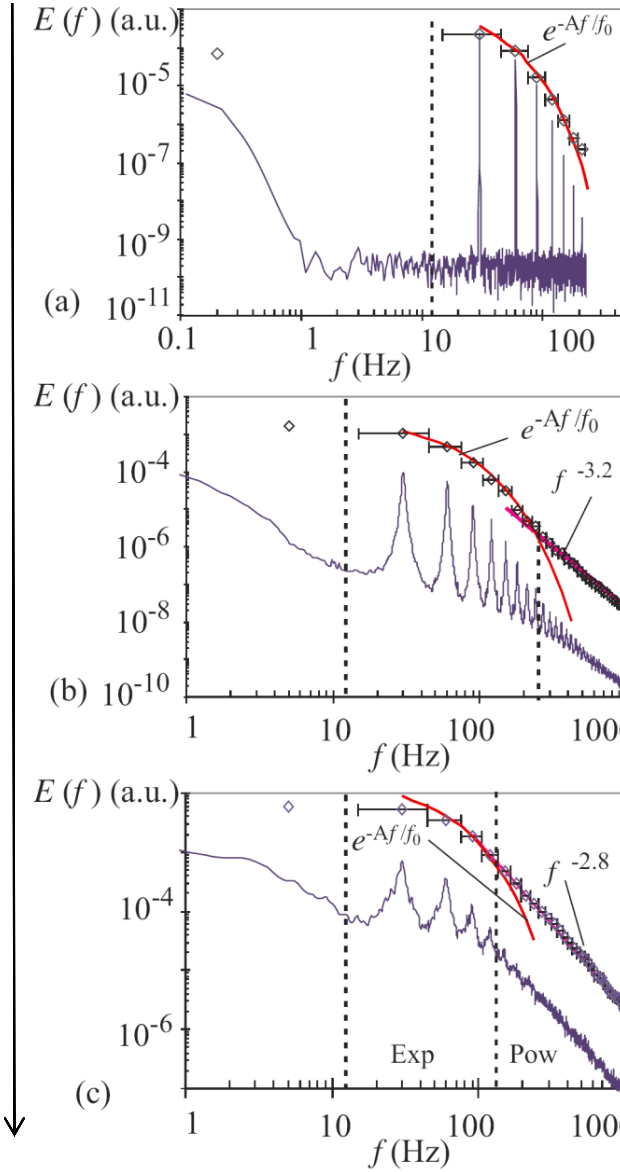


Fig. 2: (Colour on-line) Frequency spectra of capillary waves at different accelerations. (a) $\Delta a = 0.5g$, (b) $\Delta a = 1.4g$ and (c) $\Delta a = 2.1g$. Open diamonds show the spectral powers E_f of each harmonics.

Digression: wave turbulence

Boltzmann equation for *number conserving* 2↔2 scattering, $n_1 \equiv n(t,p_1)$:

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = \int \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} \int \frac{\mathrm{d}^3 p_3}{(2\pi)^3 2E_3} \int \frac{\mathrm{d}^3 p_4}{(2\pi)^3 2E_4}$$

$$\times \underbrace{\delta^3(p_1 + p_2 - p_3 - p_4)}_{\text{momentum conservation}} \underbrace{\delta(E_1 + E_2 - E_3 - E_4)}_{\text{energy conservation}} (2\pi)^4 \underbrace{|M|^2}_{\text{scattering}}$$

$$\times \Big(\underbrace{n_3 n_4 (1 + n_1)(1 + n_2)}_{\text{“gain“ term}} - \underbrace{n_1 n_2 (1 + n_3)(1 + n_4)}_{\text{“loss“ term}} \Big)$$

http://www.aeiou.at/aeiou.encyclop.b/b638771.htm

has different stationary solutions, $\mathrm{d}n_1/\mathrm{d}t=0$, in the (classical) regime $n(p) \gg 1$:

1. $n(p) = 1/(e^{\beta\omega(p)} - 1)$ thermal equilibrium

2. $n(p) \sim 1/p^{4/3}$ turbulent *particle* cascade

3. $n(p) \sim 1/p^{5/3}$ *energy* cascade

Kolmogorov -Zakharov spectrum

…associated to stationary transport of conserved charges

Range of validity of Kolmogorov-Zakharov

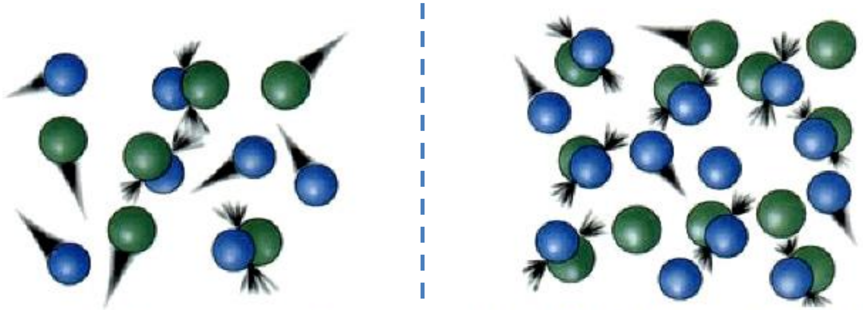
E.g. self-interacting scalars with quartic coupling: $|M|^2 \sim \lambda^2 \ll 1$

$n(p) \lesssim 1$

$1 \ll n(p) \ll 1/\lambda$

$n(p) \sim 1/\lambda$

nonperturbative



Low concentration = Few collisions

High concentration = More collisions

Very high concentration = ?

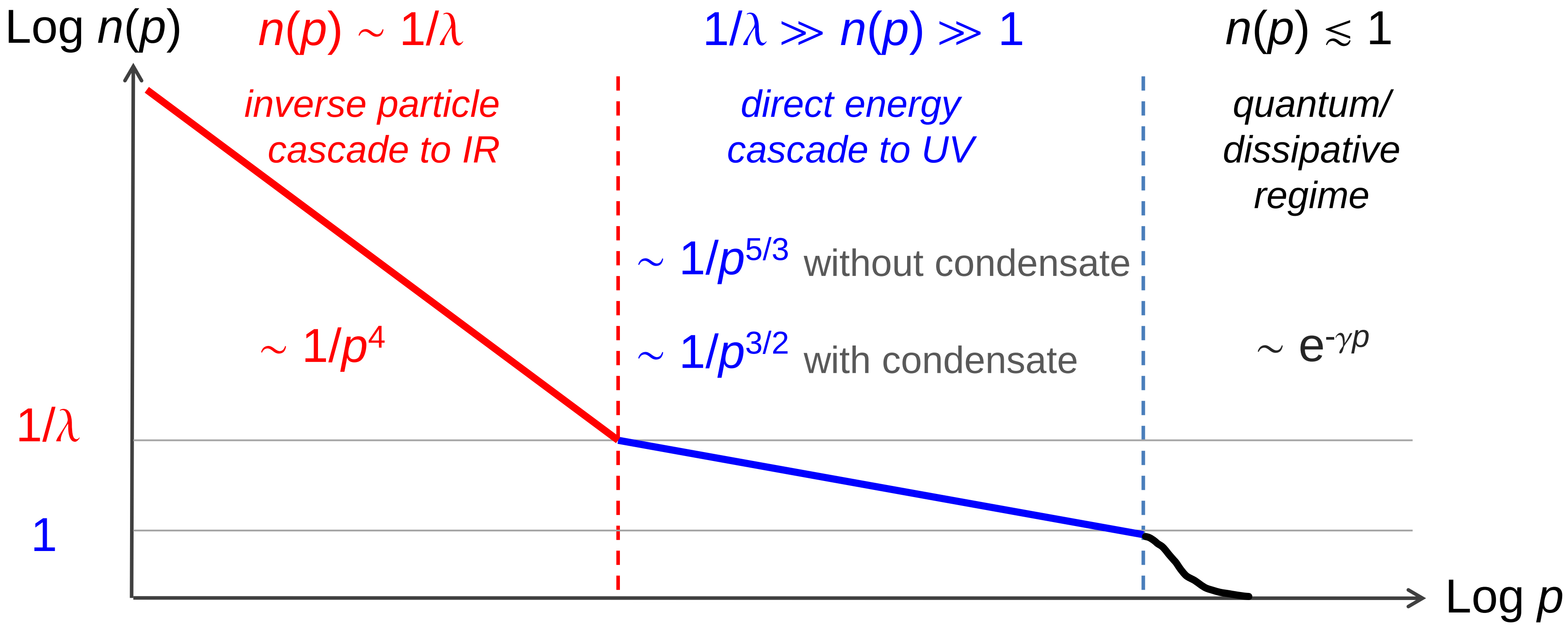
http://upload.wikimedia.org/wikipedia/commons/4/41/Molecular-collisions.jpg

Kolmogorov-Zakharov solutions are limited to the “window“

$1 \ll n(p) \ll 1/\lambda$, since for

$n(p) \sim 1/\lambda$ the n↔m scatterings for n,m=1,..,∞ are as important as 2↔2 **!**

Dual cascade



Infrared particle cascade: $n(p) \sim 1/p^{d+z-\eta}$

Berges, Rothkopf, Schmidt
PRL 101 (2008) 041603

(d=3 space dimensions, rel. dispersion $\omega(p) \sim p^z$ with z=1, scalar anomalous dimension $\eta \approx 0$)

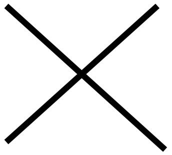
Berges, Hoffmeister, NPB 813 (2009) 383, see also Carrington, Rebhan, EPJ C 71 (2011) 1787, …

Analytical method: Resummed (2PI) 1/N expansion to NLO in scalar QFT

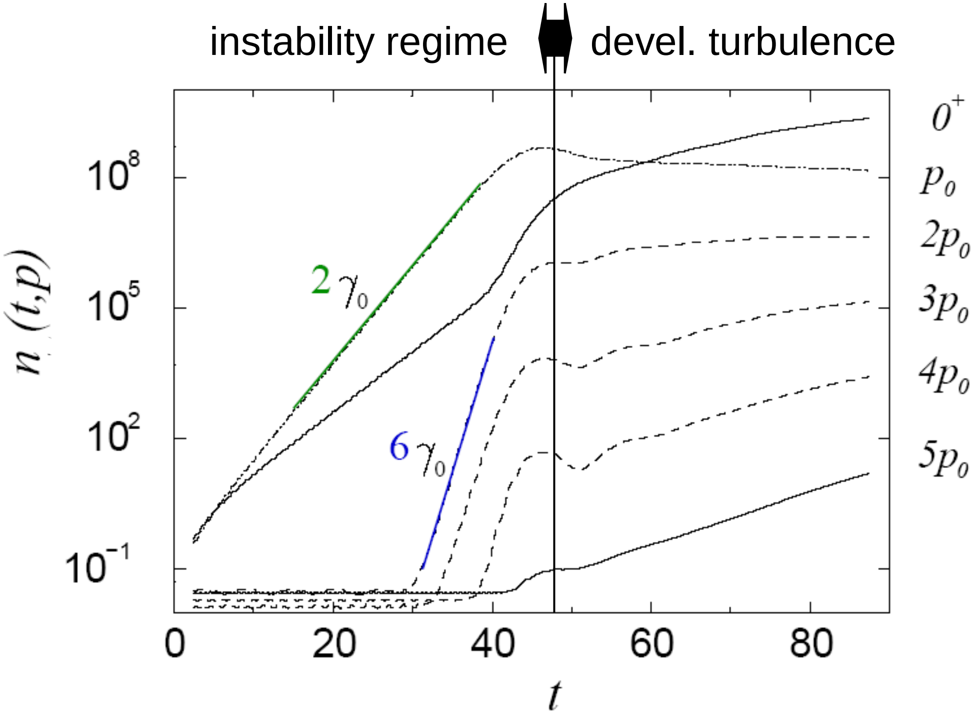
Berges, NPA 699 (2002) 847; Aarts, Ahrensmeier, Baier, Berges, Serreau PRD 66 (2002) 045008

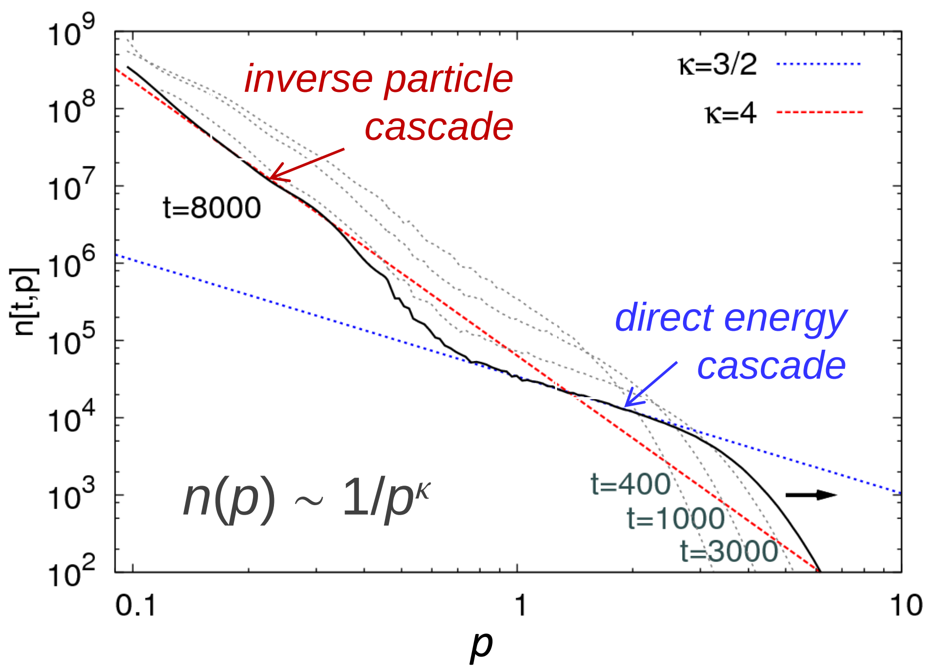
A simple (quantum) example

N=4 component linear sigma-model with quartic self-interaction λ



E.g. parametric resonance instability followed by dual cacade:



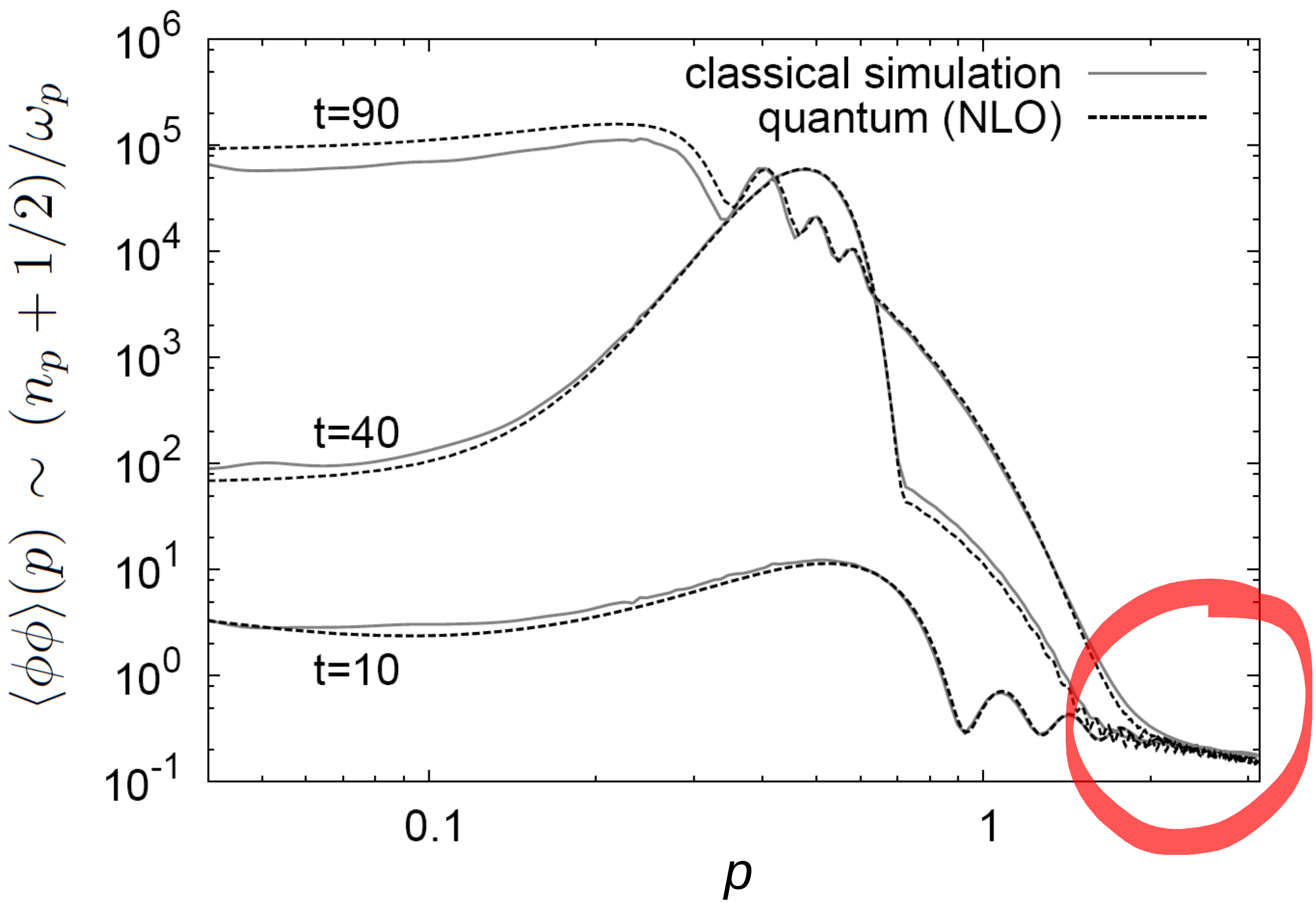


$\lambda \sim 10^{-4}$, $\phi\ (t) = \sigma(t)\sqrt{6N/\lambda}$ in units of $\sigma(t{=}0)$

Berges, Rothkopf, Schmidt, PRL 101 (2008) 041603

Direct cascade: Micha, Tkachev, PRL 90 (2003) 121301, …

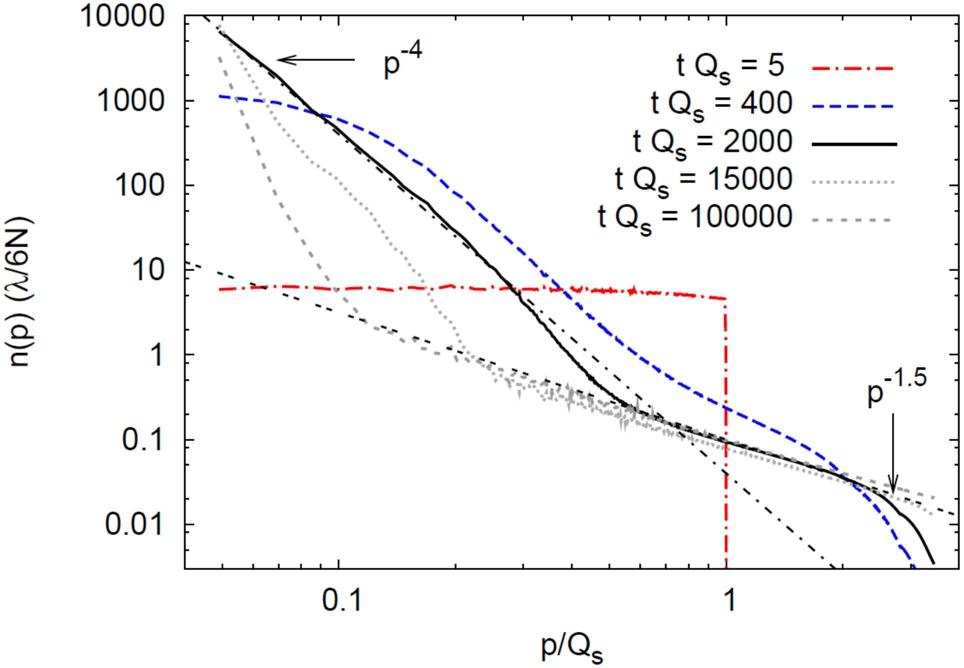
Comparing quantum to classical-statistical



Berges, Rothkopf, Schmidt, PRL 101 (2008) 041603

Classical describes well quantum (NLO 2PI-1/N) at early/intermediate times

Bose condensation from 'over-population'



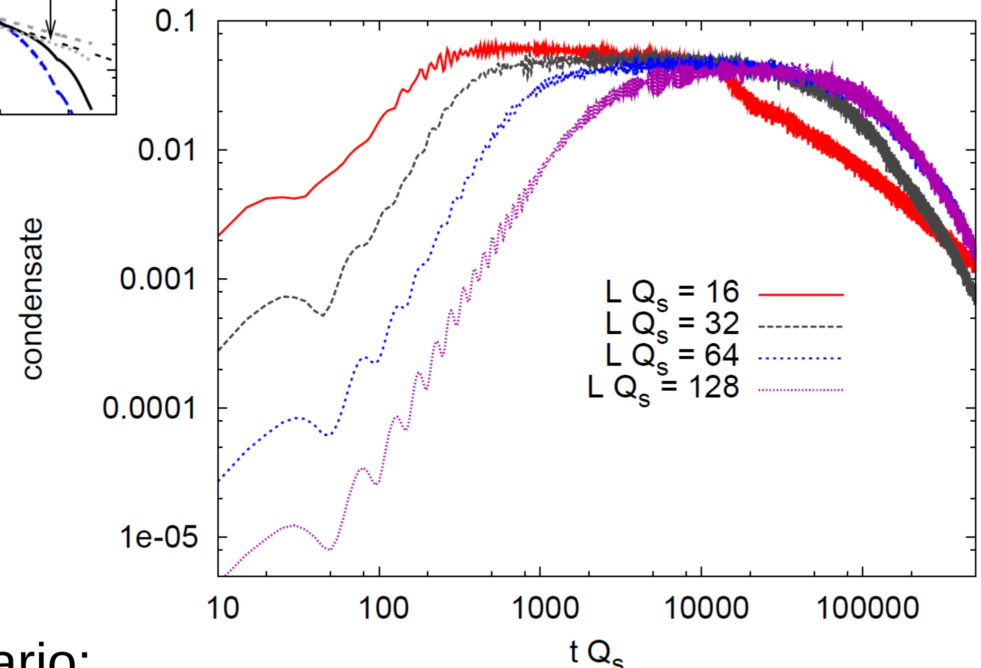
Berges, Schlichting, Sexty

$\epsilon \sim \frac{Q_s^4}{\lambda}$ i.e. initial occupation

$$n(p \lesssim Q_s) \sim \lambda^{-1}$$

Condensate from inverse particle cascade ($V = L^3$):

$$\langle \, (\int d^3x \, \phi(t,\vec{x})/V)^2 \, \rangle$$
$$= \langle \, (\phi(t,\vec{p}=0)/V)^2 \, \rangle$$



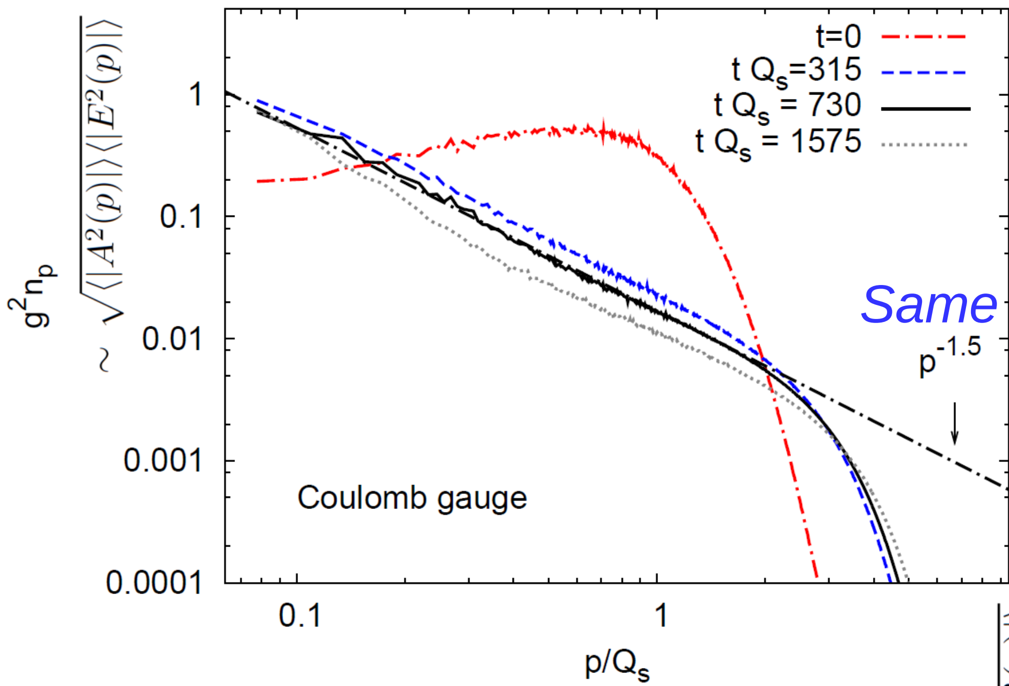
Compare Bose condensation scenario:

Blaizot, Gelis, Liao, McLerran, Venugopalan, arXiv:1107.5296 [hep-ph], Epelbaum, Gelis, NPA 872 (2011) 210

Nonabelian lattice gauge theory

Occupancy:

Berges, Schlichting, Sexty

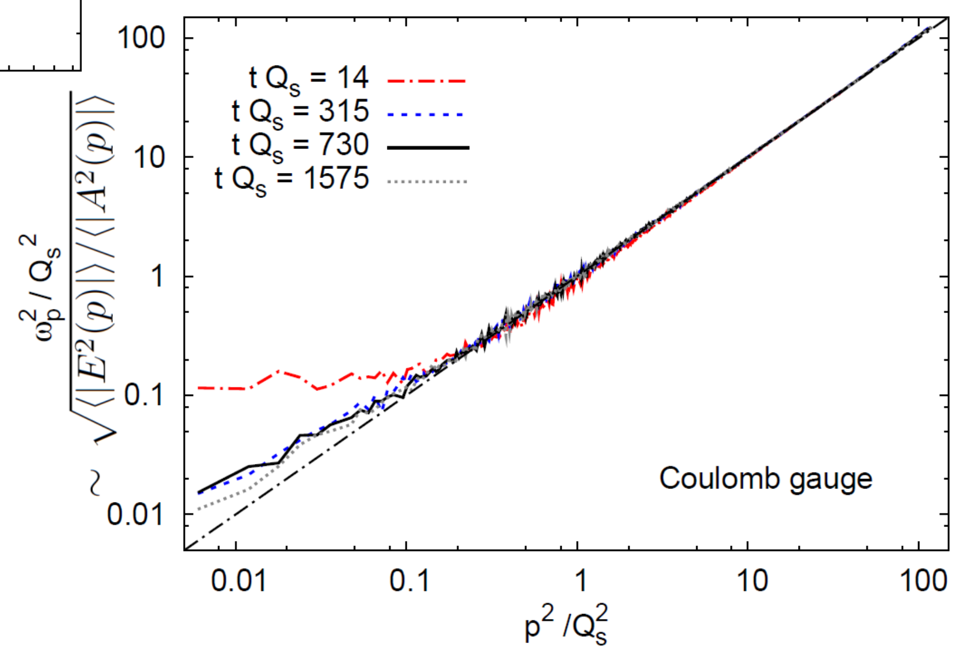


Classical-statistical *SU*(2) from

$$\epsilon \sim \frac{Q_s^4}{g^2} \quad \text{i.e.} \quad n(p \simeq Q_s) \sim \frac{1}{g^2}$$

Same Kolmogorov-3/2 as for scalars!

Dispersion:



- Notion of occupation numbers exceeding $g^2 n_p \sim 1$ unclear

- Any gauge-invariant ‘Bose-condensate equivalent‘?

Digression: Turbulence in QCD

Berges, Scheffler, Sexty, PLB 681 (2009) 362

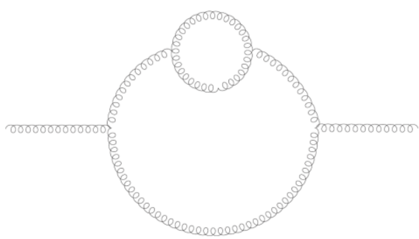
Turbulent scaling exponents are encoded in correlation functions:

$$F^{ab}_{\mu\nu}(p) = \frac{1}{2}\int d^4x\, e^{ipx}\langle\{A^a_\mu(x), A^b_\nu(0)\}\rangle \quad , \quad \rho^{ab}_{\mu\nu}(p) = \int d^4x\, e^{ipx}\langle[A^a_\mu(x), A^b_\nu(0)]\rangle$$

anti-commutator ⟵ *no fluctuation dissipation relation!* ⟶ commutator

Resummed (2PI) loop expansion, e.g.

$\Sigma \sim$



Scaling ansatz:

$$\bar{F}_{\mu\nu}(sp) = |s|^{-(2+\kappa)}\bar{F}_{\mu\nu}(p)$$
$$\bar{\rho}_{\mu\nu}(sp) = |s|^{-2}\,\mathrm{sgn}(s)\bar{\rho}_{\mu\nu}(p)$$

Translation invariance ($\Sigma_\rho F = \Sigma_F \rho$):

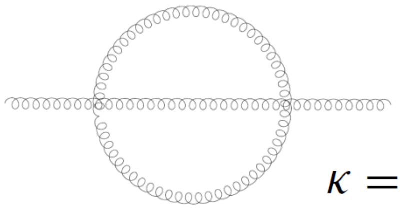
$$0 = \int_{\mathbf{p}qklr} \delta^{(4)}(p+k+q)\delta^{(4)}(k+l+r)V_{\mu\delta\gamma,\nu\tau\lambda}(p,k,q)$$
$$\times V_{\alpha\delta'\gamma',\beta\tau'\lambda'}(k,l,r)\bar{G}^{\tau\alpha}_{(R)}(k)\bar{G}^{\beta\delta}_{(A)}(k)\bar{\rho}^{\lambda\gamma}(q)\bar{F}^{\nu\mu}(p)$$
$$\times\left\{\bar{F}^{\tau'\delta'}(l)\bar{F}^{\lambda'\gamma'}(r)\left[\left|\frac{p^0}{r^0}\right|^{\Delta}\mathrm{sgn}\left(\frac{p^0}{r^0}\right) + \left|\frac{p^0}{l^0}\right|^{\Delta}\mathrm{sgn}\left(\frac{p^0}{l^0}\right)\right.\right.$$
$$\left.\left. - \left|\frac{p^0}{q^0}\right|^{\Delta}\mathrm{sgn}\left(\frac{p^0}{q^0}\right) - 1\right] + \mathcal{O}(\rho^2)\right\},\ \Delta = 4 - 3\kappa$$

Solutions:

$$\left.\begin{array}{ll}\text{a)} & \Delta = 0 \\ \text{b)} & \Delta = -1\end{array}\right\}$$

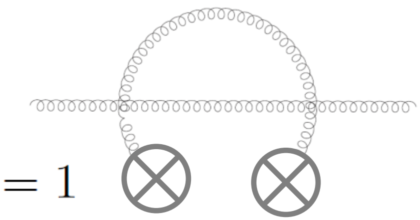
$$\boxed{\kappa = \frac{5}{3}, \quad \text{or} \quad \kappa = \frac{4}{3}}$$

Discussion



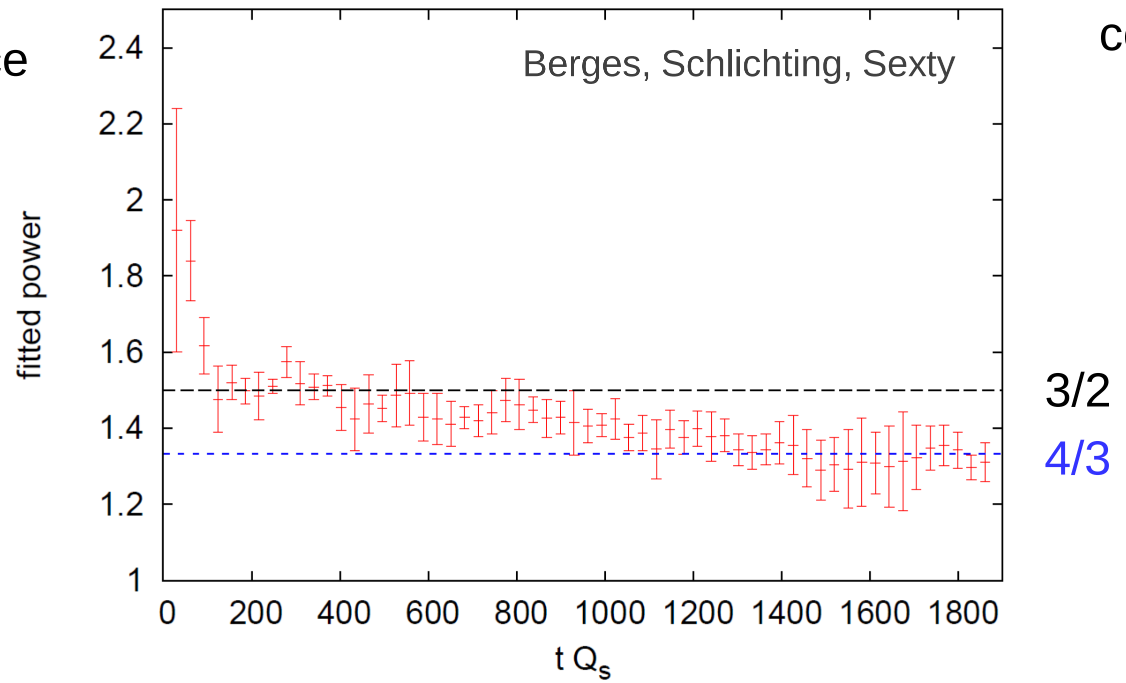
$\kappa = \frac{5}{3}$, or $\kappa = \frac{4}{3}$

$\kappa = \frac{3}{2}$, or $\kappa = 1$



'time-dependent condensate'

Classical lattice gauge theory:

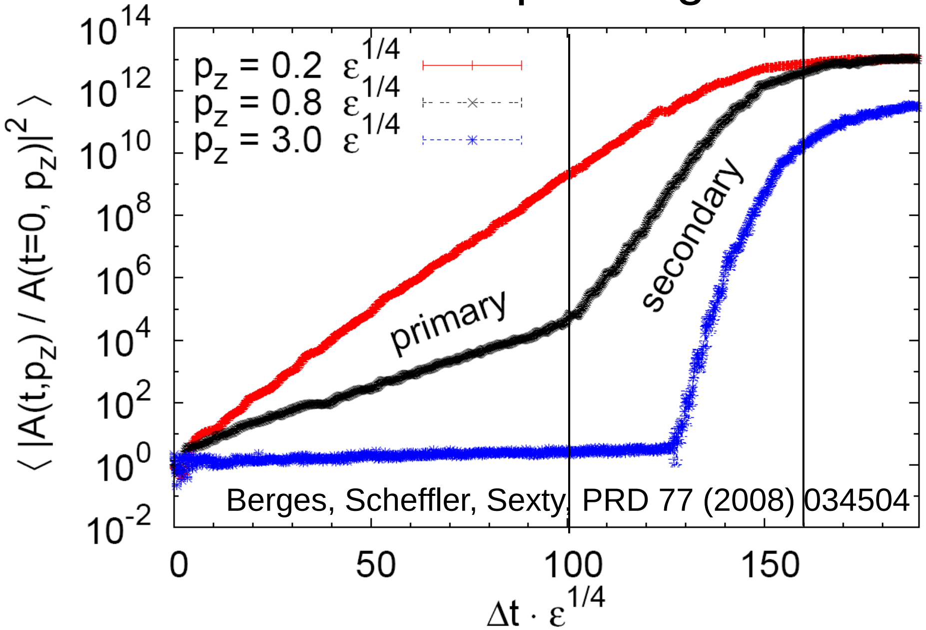


See, however, e.g. Arnold, Moore PRD 73 (2006) 025006; Mueller, Shoshi, Wong, NPB 760 (2007) 145; Ipp, Rebhan, Strickland, PRD 84 (2011) 056003; …

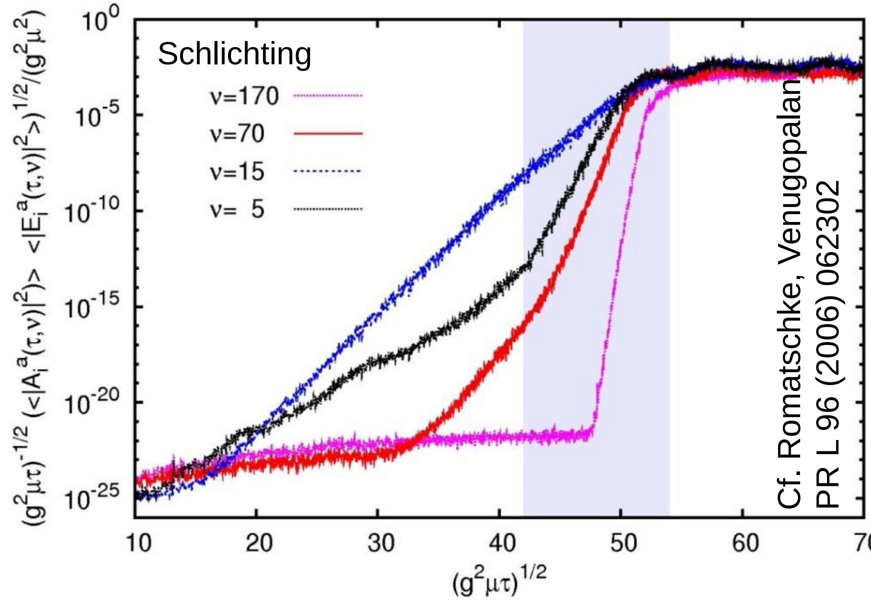
Hard-loop approach: $1.8 \lesssim \kappa \lesssim 2.4$

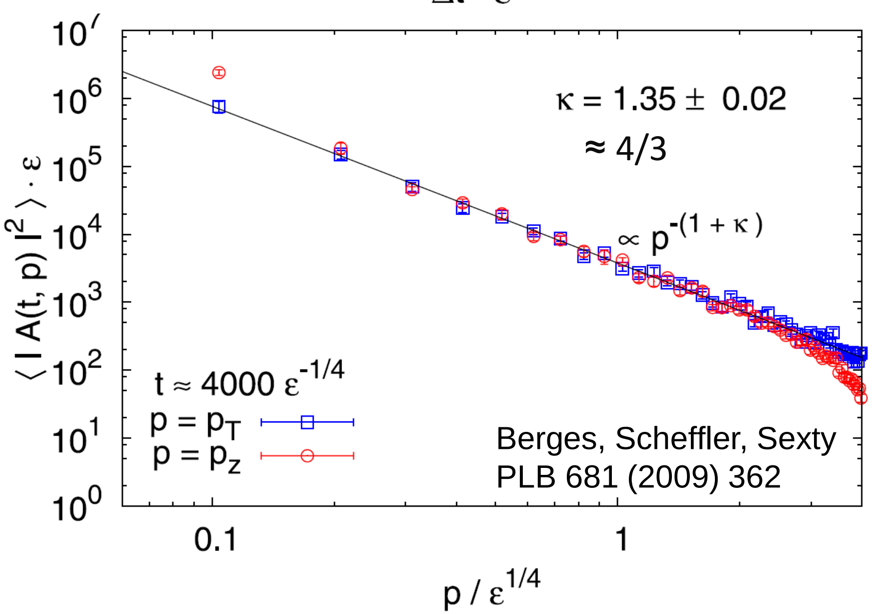
Turbulence from plasma instabilities: Lattice

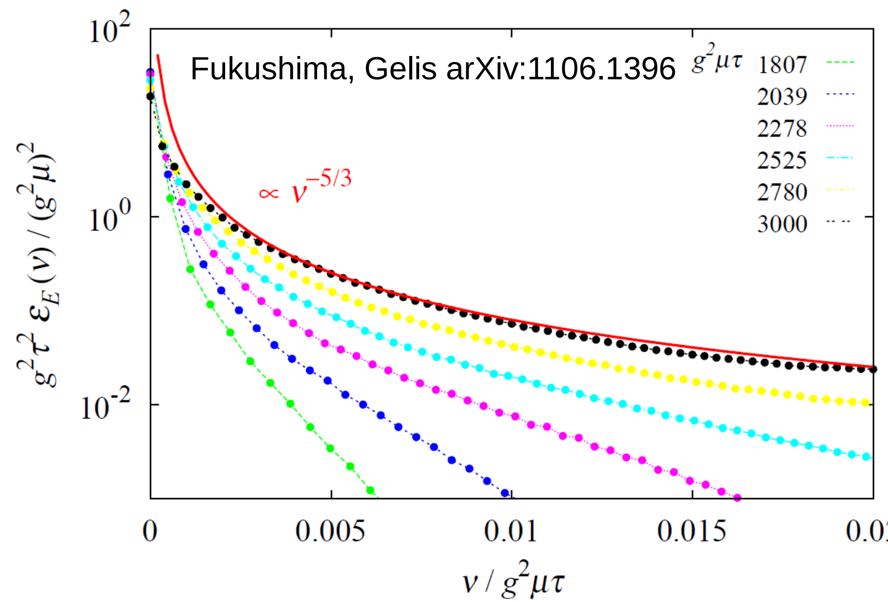
Non-expanding



With longitudinal expansion







Lattice simulations with dynamical fermions

Consider general class of models including lattice gauge theories

$$\mathcal{L} = \frac{1}{2}\partial\Phi^*\partial\Phi - V(\Phi) + \sum_k^{N_f}\left[i\bar\Psi_k\gamma^\mu\partial_\mu\Psi_k - \bar\Psi_k(\overset{m - g\Phi(x)}{\overset{\downarrow}{M}}\underset{\frac{1}{2}(1-\gamma^5)}{\underset{\nearrow}{P_L}} + M^*\underset{\frac{1}{2}(1+\gamma^5)}{\underset{\nwarrow}{P_R}})\Psi_k\right]$$

$$\int\prod_k D\Psi_k^+ D\Psi_k e^{i\int\mathcal{L}(\Phi,\Psi^+,\Psi)} \Longrightarrow \boxed{\partial_x^2\Phi(x) + V'(\Phi(x)) + N_f J(x) = 0}$$

$$J(x) = J^{\mathrm{S}}(x) + J^{\mathrm{PS}}(x)$$

$$J^{\mathrm{S}}(x) = -g\left\langle\bar\Psi(x)\Psi(x)\right\rangle = g\mathrm{Tr}\, D(x,x)\,,$$

$$J^{\mathrm{PS}}(x) = -g\left\langle\bar\Psi(x)\gamma^5\Psi(x)\right\rangle = g\mathrm{Tr}\, D(x,x)\gamma^5$$

For classical $\Phi(x)$ the exact equation for the fermion $D(x,y)$ reads:

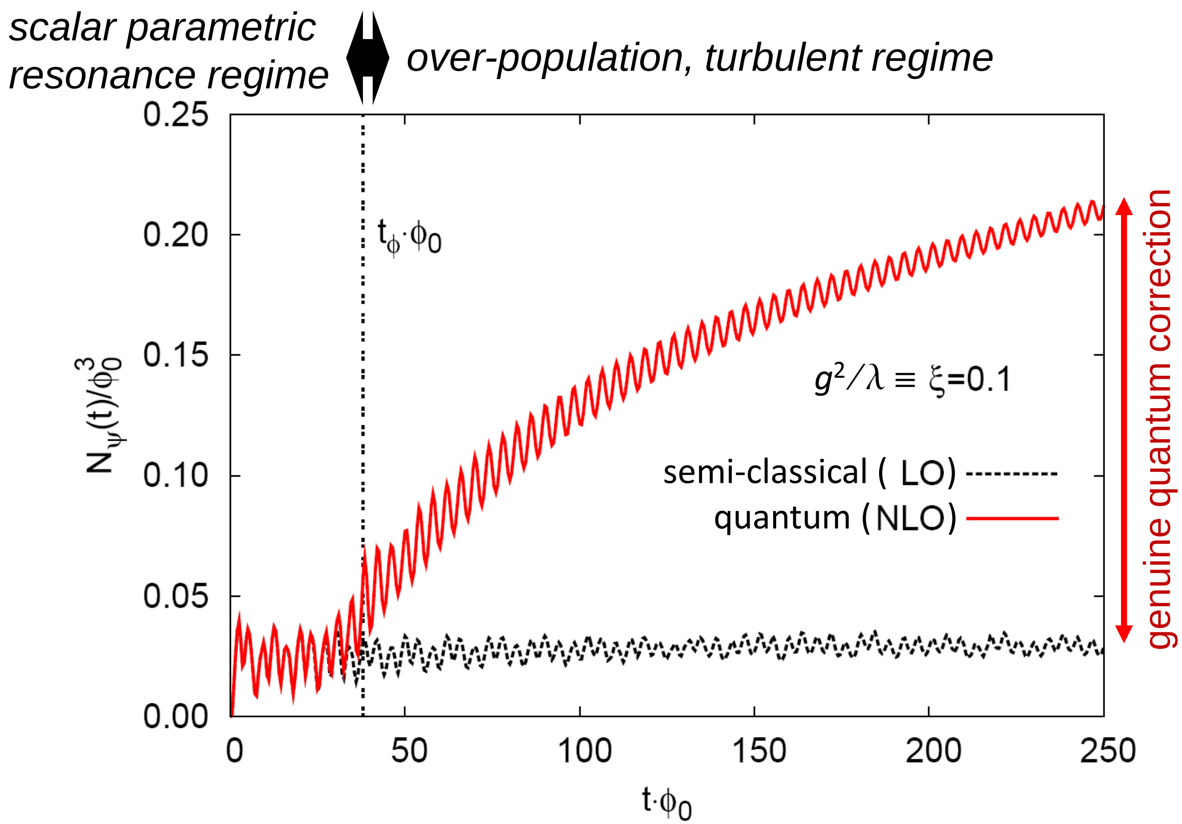
$$\boxed{(i\gamma^\mu\partial_{x,\mu} - m + g\mathrm{Re}\,\Phi(x) - ig\mathrm{Im}\,\Phi(x)\gamma^5)D(x,y) = 0}$$

Very costly ($4\ 4\ N^3\ N^3$)! Use low-cost fermions of Borsanyi & Hindmarsh

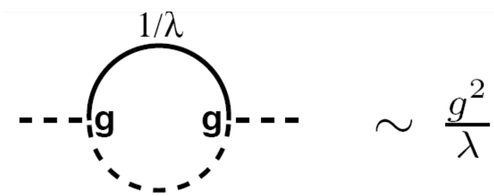
Aarts, Smit, NPB 555 (1999) 355 PRD 79 (2009) 065010

Fermion production from over-populated bosons

N=4 component linear sigma-model with quartic self-interaction coupled to N_f =2 massless Dirac fermions:



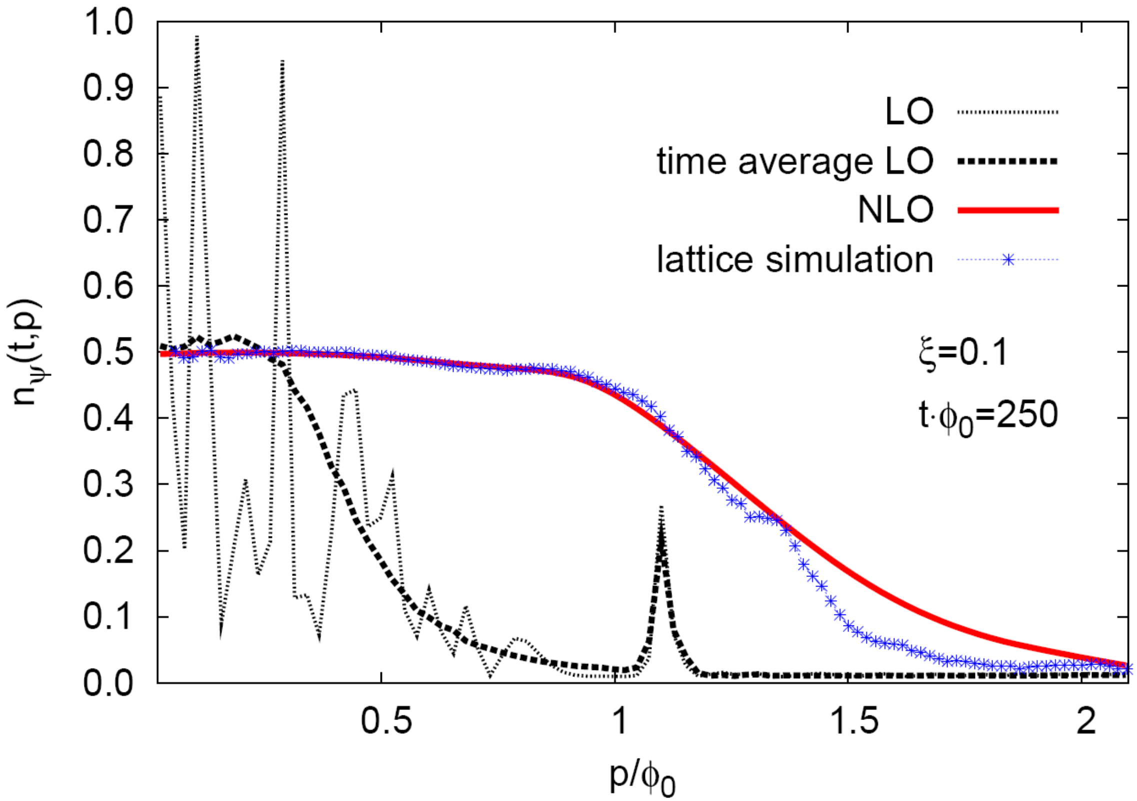
2PI-NLO:



Berges, Gelfand, Pruschke, PRL 107 (2011) 061301

fermion production rate (NLO): $\sim (g^2/\lambda)\ \phi_0$ **!**

Real-time dynamical fermions in 3+1 dimensions!



- Wilson fermions on a 64^3 lattice

Berges, Gelfand, Pruschke, PRL 107 (2011) 061301

- Very good agreement with NLO quantum result (2PI) for $\xi \ll 1$
  (differences at larger p depend on Wilson term $\rightarrow$ larger lattices)

- Lattice simulation can be applied to $\xi \sim 1$ relevant for QCD

Conclusions

- Classical-statistical lattice gauge theory: describes both classical field and classical particle regime (no separation of scales required for $\alpha_s \ll 1$)

→ Plasma instabilities at early times, followed by non-linear amplification

→ Kolmogorov turbulence at intermediate times (massless dispersion!)

→ Indications of 'Bose-condensation equivalent' physics (!?)

- Similar far-from-equilibrium phenomena in scalar/fermion field theories both analytically (2PI) and numerically (2PI or classical) very well understood

→ Parametric resonance instability → dual cascade → Bose condensation

→ Strongly enhanced fermion production from over-populated bosons

→ Very good agreement between classical-statistical and 2PI *excluding* late-time approach to thermal equilibrium (n ~ 1: 2PI or kinetic theory)

- Question:

→ Hard-loop effective theories seem to qualitatively agree but *quantitatively?*