Lattice Gauge Theory Computation of Heavy Quark Potentials and Prediction of Quarkonium Bound States

FAIR Next Generation Scientists - 7th Edition

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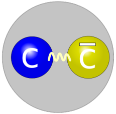
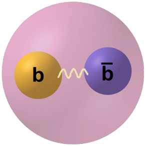
Introduction

- Experiment: Non-relativistic nature for b-quarks, quantum numbers J^{PC} or $^{2S+1}L_J$

  $\rightarrow$ Use Schrödinger equation to predict $b\bar{b}$ bound states (Good approximation and very cheap!)
- Potential models: Construct $V(r)$ such that phenomenological expectations are fulfilled
- Example heavy quarks: Static potential + corrections in m^{-1}





[1]

[1] wikipedia.org/wiki/Quarkonium

Introduction

- Effective theories: Systematic way to derive $V(r)$ from QCD Lagrangian
- QCD $\rightarrow$ NRQCD $\rightarrow$ pNRQCD $\stackrel{\text{LQCD}}{\rightarrow}$ Potential model [Brambilla, 2204.11295]
- Two steps:
  - Compute potentials in LQCD
  - Solve the Schrödinger equation $H|\psi\rangle = E|\psi\rangle$ with $H = \frac{p^2}{m} + V(r)$

$\rightarrow$ Only parameters to fit are quarkmass m and lattice scale r_0

Schrödinger Equation

- Predict bound states for heavy quarkonium by solving Schrödinger equation $H|\psi\rangle = E_\psi|\psi\rangle$ with

$$H = 2m + \frac{\mathrm{p}^2}{m} - \frac{\mathrm{p}^4}{4m^3} + V(r)$$

using $V = V^{(0)} + \frac{1}{m}V^{(1)} + \frac{1}{m^2}V^{(2)} + \dots$

- Define $H = H^{(0)} + \delta H$ [Bali, hep-ph/0001312]

$$H^{(0)} = \frac{\mathrm{p}^2}{m} + V^{(0)}(r) + \text{const.}$$

$$\delta H = -\frac{\mathrm{p}^4}{4m^3} + \frac{1}{m}V^{(1)}(r) + \frac{1}{m^2}V^{(2)}(r, \mathrm{p}, \mathrm{L}, \mathrm{S})$$

Schrödinger Equation

- To solve now:

$$H^{(0)}\psi^{(0)}_{nlm}(r,\theta,\varphi) = E^{(0)}_{nl}\psi^{(0)}_{nlm}(r,\theta,\varphi)$$

- Wave function in position space:

$$\psi^{(0)}_{nlm}(r,\theta,\varphi) = \frac{u^{(0)}_{nl}(r)}{r} Y_{lm}(\theta,\varphi)$$

- Total energy:

$$E^{\text{tot.}} = E^{(0)}_{nl} + \langle \delta H \rangle$$

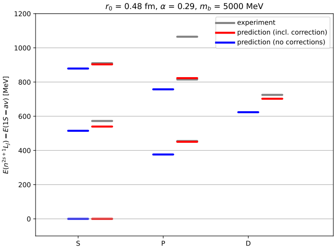


Figure: Comparison between $E^{(0)}_{nl}$ (blue) and $E^{\text{tot.}}$ (red) for bottomonium. Experimental data (grey) can be found here: [Particle Data Group, doi: 10.1093/ptep/ptaa104]

Results for Charmonium and Bottomonium

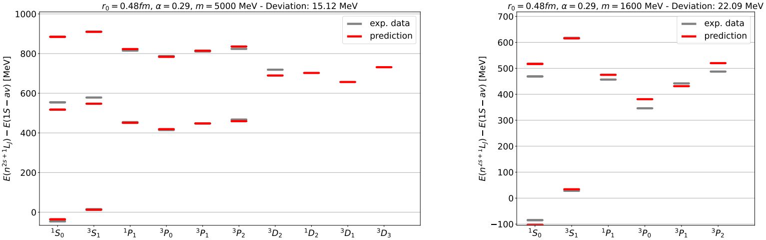


Figure: Bound states for bottomonium (left) and charmonium (right), normed at spin-averaged 1S-mass. Experimental data (grey) can be found here: [Particle Data Group, doi: 10.1093/ptep/ptaa104]

Composition of the Heavy Quark Potential

- Derivation can be found e. g. in [Pineda, Vairo, hep-ph/0009145]
- In the case of $m_1 = m_2 \equiv m$:

$$V(r) = V^{(0)}(r) + \frac{1}{m} V^{(1)}(r) + \frac{1}{m^2} V^{(2)}(r, \mathrm{p}, \mathrm{L}, \mathrm{S}) + \mathcal{O}(m^{-3}) \quad (1)$$

with

$$V^{(2)}(r, \mathrm{p}, \mathrm{L}, \mathrm{S}) = V_{\mathrm{LS}}(r)\mathrm{LS} + \frac{\mathrm{S}_1\mathrm{S}_2}{3} V_{\mathrm{S}^2}(r) + \cdots$$

Colour Field Insertions

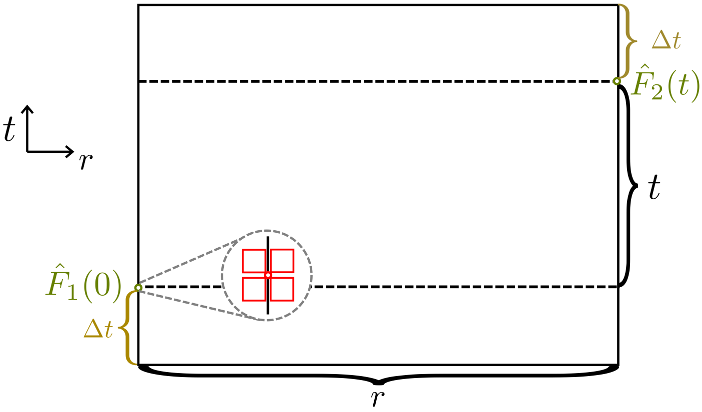


Figure: $\langle F_1(0) F_2(t) \rangle$ with fields as Wilson loop insertions.

Colour Field Insertions

- $V(r)$ computed using colour field correlators
- Define lattice correlators for $F = E, B$ as insertions into a Wilson loop

$$\langle\langle F_i^1(0) F_j^2(t) \rangle\rangle \equiv \frac{\langle F_i^1(0) F_j^2(t) \rangle}{\langle 1 \rangle}$$

$$\langle\langle F_i^1(0) F_j^2(t) \rangle\rangle_c \equiv \langle\langle F_i^1(0) F_j^2(t) \rangle\rangle - \langle\langle F_i^1(0) \rangle\rangle \langle\langle F_j^2(t) \rangle\rangle$$

- Potential value at r is proportional to integral of type $\lim_{T\to\infty} \int_0^T dt\ t^s \langle\langle F_i^1(0) F_j^2(t) \rangle\rangle_{(c)}$, where $s = 0, 1$ [Pineda, Vairo, hep-ph/0009145]

Comparing Analytic and LQCD Results

- Use analytic methods, e. g. weak coupling expansion, area law, to parametrize the potential

  $\rightarrow$ Ansatz which can be fitted to LQCD results
- Example $V^{(0)}_{\text{pert.}}(r) = -\frac{C_f \alpha_s}{r}$: Comparison with LQCD motivates Cornell-ansatz $V^{(0)}(r) = -\frac{\alpha}{r} + \sigma r$

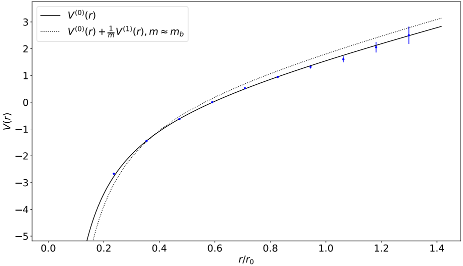


Figure: Static potential $V^{(0)}$ and the correction at $\mathcal{O}(1/m)$ (dashed), plottet against lattice data points for $V^{(0)}$.

Results for Charmonium and Bottomonium

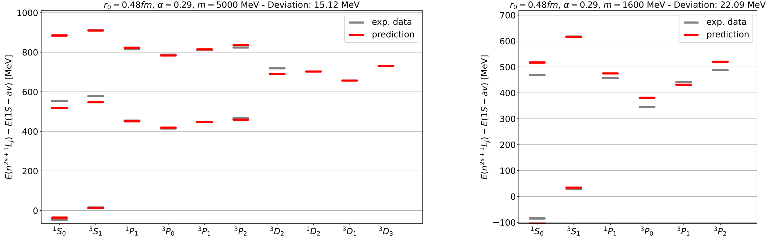


Figure: Bound states for bottomonium (left) and charmonium (right), normed at spin-averaged $1S$-mass. Experimental data (grey) can be found here: [Particle Data Group, doi: 10.1093/ptep/ptaa104]

Summary

- Heavy Quarks can be treated in a non-relativistic way, predict bound states using $V = V^{(0)} + \frac{1}{m} V^{(1)} + \frac{1}{m^2} V^{(2)} + \dots$ in a Schrödinger equation
- $V(r)$ can be computed either in lattice QCD or using analytic methods
- Numerical and analytic results complement each other
- Resulting spectrum reproduces experiment within few MeV

Outlook

- Further applications:
  - Near (above) threshold states (resonances)
  - D and B-mesons
  - Hybrid mesons
  - Heavy-light tetraquarks
- Perform computation in lattice QCD and test state-of-the-art techniques to reduce discretization errors, see e. g. reference [Lüscher, 1006.4518]

The Potential at $\mathcal{O}(m^{-2})$

In the case of $m_1 = m_2 \equiv m$:

$$V(r) = V^{(0)}(r) + \frac{1}{m} V^{(1)}(r) + \frac{1}{m^2} V^{(2)}(r, \mathrm{p}, \mathrm{L}, \mathrm{S}) + \mathcal{O}(m^{-3}) \qquad (2)$$

with

$$\begin{aligned}
V^{(2)}(r, \mathrm{p}, \mathrm{L}, \mathrm{S}) &= V_{\mathrm{SI}}(r, \mathrm{p}, \mathrm{L}) + V_{\mathrm{SD}}(r, \mathrm{L}, \mathrm{S}) \\
V_{\mathrm{SI}}(r, \mathrm{p}, \mathrm{L}) &= V_r(r) + \frac{1}{2}\{\mathrm{p}^2, V_{\mathrm{p}^2}(r)\} + \frac{V_{\mathrm{L}^2}(r)}{r^2}\mathrm{L}^2 \\
V_{\mathrm{SD}}(r, \mathrm{L}, \mathrm{S}) &= V_{\mathrm{LS}}(r)\mathrm{LS} + \mathrm{S}_{12} V_{\mathrm{S}_{12}}(r) + \frac{\mathrm{S}_1\mathrm{S}_2}{3} V_{\mathrm{S}^2}(r) \\
\mathrm{S}_{12} &= \left((\mathrm{S}_1\hat{\mathrm{r}})(\mathrm{S}_2\hat{\mathrm{r}}) - \frac{\mathrm{S}_1\mathrm{S}_2}{3}\right)
\end{aligned}$$

The Potential at $\mathcal{O}(m^{-2})$

- The potentials at $\mathcal{O}(m^{-2})$ are composed of [Pineda, Vairo, hep-ph/0009145]

$$
\begin{aligned}
V_{\mathbf{p}^2}(r) =& \frac{2}{3}V_c(r) - V_b(r) + 2V_d(r) - \frac{4}{3}V_e(r) \\
V_{\mathbf{L}^2}(r) =& - V_c(r) + 2V_e(r) \\
V_{LS}(r) =& \frac{1}{2r}\left((2c_F - 1)\frac{dV^{(0)}(r)}{dr} + 2c_F(V_1'(r) + V_2'(r))\right) \\
V_{S_{12}}(r) =& c_F^2 V_3(r) \\
V_{\mathbf{S}^2}(r) =& {c_F}^2 V_4(r) - 12 d_v \alpha \cdot 4\pi\delta^{(3)}(r) \\
& \vdots
\end{aligned}
$$

- c_F, d_v: NRQCD matching coefficients [Pineda, 1111.0165]

The Potential at $\mathcal{O}(m^{-2})$

Eichten-Feinberg-Gromes formulae:

$$
\begin{aligned}
\frac{dV^{(0)}(r)}{dr} &= V_2'(r) - V_1'(r) \\
\frac{r}{6}\frac{dV^{(0)}(r)}{dr} - \frac{1}{2}V^{(0)}(r) &= V_b(r) + 2V_d(r) \\
-\frac{r}{2}\frac{dV^{(0)}(r)}{dr} &= V_c(r) + 2V_e(r)
\end{aligned}
$$