QCD's equation of state from Dyson–Schwinger equations

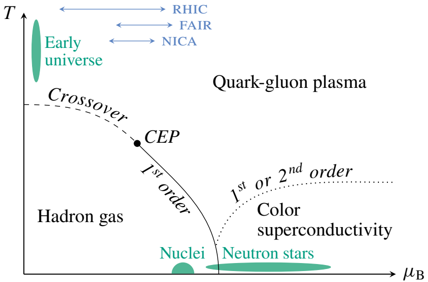
P.I., Fischer, Steinert, PRD 103 (2021) 054012, arXiv:2012.04991

Philipp Isserstedt

Institut für Theoretische Physik
Justus-Liebig-Universität Gießen

FAIR next generation scientists workshop; 7th edition
23–27 May 2022, Paralia (Pieria), Greece

Phase transitions in QCD



Equation of state (EoS)

$p(T,\mu_B)$, $s(T,\mu_B)$, and $\varepsilon(T,\mu_B)$

All encoded in thermodynamic potential Ω

Nonperturbative approaches:

- Lattice QCD . . . sign problem!
- Effective models . . . generalizable?
- Functional methods . . . all QCD degrees of freedom & no sign problem (but truncations are necessary)

Setting the scene: Euclidean QCD

Generating functional

$$\mathcal{Z}_{\mathrm{QCD}} = \int \mathcal{D}[\text{fields}] \exp\Big\{ - \int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^3\vec{x} \, \Big(\bar{\psi} \, (\not{D} + \hat{m} + \gamma_4 \, \hat{\mu}) \, \psi + \frac{1}{4} F^a_{\nu\sigma} F^a_{\nu\sigma} + \text{gauge fixing} + \text{sources} \Big) \Big\}$$

Propagators in Landau gauge and momentum space, $q = (\omega_q, \vec{q}\,)$:

$$S_f(q) = \left[\mathrm{i} \, (\omega_q + \mathrm{i} \, \mu_f) \, \gamma_4 \, C_f(q) + \mathrm{i} \, \vec{\not{q}} \, A_f(q) + B_f(q) \right]^{-1}$$

$$D_{\nu\sigma}(q) = \frac{Z^{\mathrm{T}}(q)}{q^2} \, P^{\mathrm{T}}_{\nu\sigma}(q) + \frac{Z^{\mathrm{L}}(q)}{q^2} \, P^{\mathrm{L}}_{\nu\sigma}(q)$$

Goal: gauge-independent information from gauge-fixed, nonperturbative approach

Dyson–Schwinger equations

- Nonperturbative functional approach
- Equations of motions for correlation functions
- Bound states as composite objects of quarks and gluons (Bethe–Salpeter/Faddeev equations)

Working areas:

Hadron physics

- Meson and baryon properties
- Spectra
- Scattering amplitudes
- Decays
- Form factors
- Exotics (tetraquarks, glueballs, and hybrids)

Nonzero T and μ

- Phase structure of QCD
- Thermodynamics
- In-medium properties of mesons

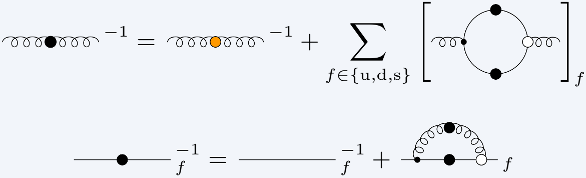
Additionally

- Finite-volume effects
- Muon $g-2$ (HLbL)
- ...

Reviews: Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer, PPNP 91 (2016) 1
Fischer, PPNP 105 (2019) 1

Dyson–Schwinger equations

Truncated DSEs for $(2+1)$-flavor QCD



More details: Fischer, PPNP 105 (2019) 1
P.I., Buballa, Fischer, Gunkel, PRD 100 (2019) 074011
Eichmann, Fischer, Welzbacher, PRD 93 (2016) 034013
Fischer, Luecker, Welzbacher, PRD 90 (2014) 034022

- Quenched lattice gluon propagator as input & unquenching via quark loops
- Nontrivial coupling between different quark flavors
- Vertex ansatz built along STI and perturbation theory

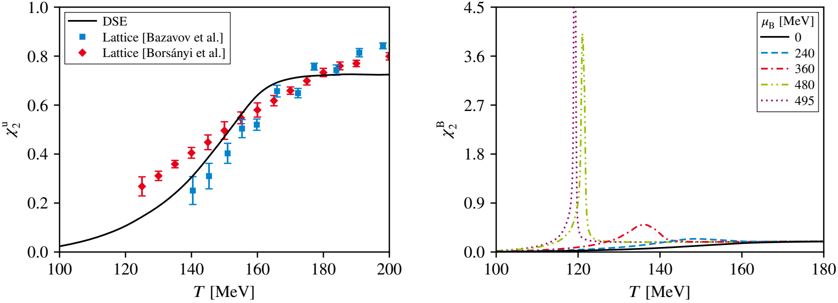
Result: dressed (i.e., nonperturbative) quark and unquenched gluon propagators

Thermodynamics from DSEs: status quo

- Our current status: quark and baryon number fluctuations

  P.I., Buballa, Fischer, Gunkel, PRD 100 (2019) 074011

$$\chi_{ijk}^{\mathrm{uds}} = -T^{(i+j+k)-4} \frac{\partial^{i+j+k}\Omega}{\partial\mu_{\mathrm{u}}^{i}\,\partial\mu_{\mathrm{d}}^{j}\,\partial\mu_{\mathrm{s}}^{k}}$$



- Starting point for fluctuations: $n_f = \partial\Omega/\partial\mu_f$
- Full thermodynamics extremely difficult within DSEs

  $\Rightarrow$ Approach needed to access Ω

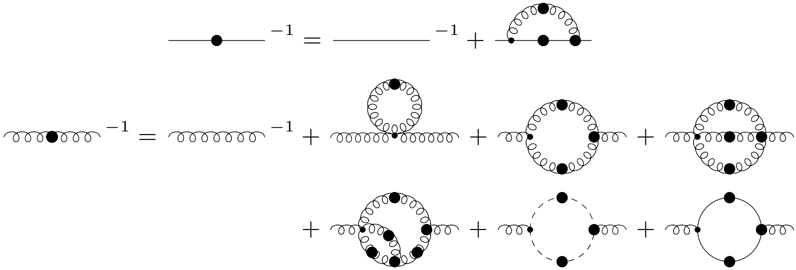
Thermodynamics from DSEs: the ‘problem’ with Ω

Master equation

$$0 = \int \mathcal{D}[\phi] \, \frac{\delta}{\delta \phi} \exp\left(-\mathcal{S}[\phi] + \int \mathrm{d}^4 x \, J\phi \right)$$

$$\Downarrow$$

$$\frac{\delta \Gamma_{1\mathrm{PI}}}{\delta \langle \phi \rangle} = \frac{\delta \mathcal{S}}{\delta \phi} \left[\phi \to \pm \left(\frac{\delta}{\delta J} + \langle \phi \rangle \right) \right] 1$$



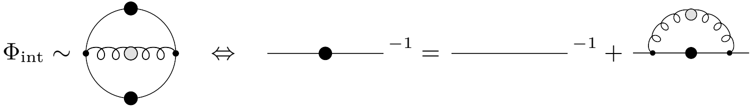
$\Gamma_{1\mathrm{PI}} \sim -\Omega$ at physical point $\Rightarrow$ ‘integration’ necessary

Thermodynamics from DSEs: the ‘problem’ with Ω

- 2PI formalism (quarks only):
  Cornwall, Jackiw, Tomboulis, PRD 10 (1974) 2428

$$-\frac{V}{T}\,\Omega[S] = \mathrm{Tr}\log\frac{S^{-1}}{T} - \mathrm{Tr}\left[\mathbb{1} - S_0^{-1}S\right] + \Phi_{\mathrm{int}}[S]$$

- Physical propagator from stationary condition: $\delta\Omega/\delta S = 0$
  $\Rightarrow\ S^{-1} = S_0^{-1} + \Sigma$ with $\Sigma \sim \delta\Phi_{\mathrm{int}}/\delta S$
- Closed form for Φ_{int}: quark-gluon vertex must not depend on quark



- So far: thermodynamics from DSEs only in rainbow-ladder truncation
  Blaschke, Roberts, Schmidt, PLB 425 (1998) 232
  Xu, Yan, Cui, Zong, IJMPA 30 (2015) 1550217
  Gao et al., PRD 93 (2016) 094019
- Needed: Ω from a **truncation-independent** method

Thermodynamics from DSEs: Maxwell method

P.I., Fischer, Steinert, PRD 103 (2021) 054012

- Consider: $\Omega = \Omega(T, \mu; m)$
- Current-quark mass: external source for bilinear $\bar{\psi}\psi$
  $\Rightarrow \langle\bar{\psi}\psi\rangle(T, \mu; m) = \partial\Omega(T, \mu; m)/\partial m$
- Integrate:

$$\Omega(T, \mu; m_2) - \Omega(T, \mu; m_1) = \int_{m_1}^{m_2} \mathrm{d}m' \, \langle\bar{\psi}\psi\rangle(T, \mu; m')$$

- Ω and $\langle\bar{\psi}\psi\rangle$ are divergent but **suitable derivatives are finite!**
- Derivative w.r.t. T:

$$s(T, \mu; m_2) - s(T, \mu; m_1) = -\int_{m_1}^{m_2} \mathrm{d}m' \, \frac{\partial\langle\bar{\psi}\psi\rangle}{\partial T}(T, \mu; m')$$

- Integration limits: let $m_1 = m$ and $m_2 \to \infty$

General relation between s and $\langle\bar{\psi}\psi\rangle$

$$s(T, \mu; m) = s_{\mathrm{YM}}(T) + \int_{m}^{\infty} \mathrm{d}m' \, \frac{\partial\langle\bar{\psi}\psi\rangle}{\partial T}(T, \mu; m')$$

Thermodynamics from DSEs: Maxwell method

Maxwell-like relation

$$\frac{\partial^2 \Omega}{\partial m \partial T} = \frac{\partial^2 \Omega}{\partial T \partial m} \quad \Rightarrow \quad -\frac{\partial s}{\partial m} = \frac{\partial \langle \bar{\psi}\psi \rangle}{\partial T}$$

- Pressure at vanishing chemical potential:

$$p(T,0) = p(T_0,0) + \int_{T_0}^{T} \mathrm{d}T'\, s(T',0)$$

- Nonzero chemical potential:

$$p(T,\mu) = p(T_0,0) + \int_{T_0}^{T} \mathrm{d}T'\, s(T',0) + \int_{0}^{\mu} \mathrm{d}\mu'\, n(T,\mu')$$

- **Applicable as soon as the quark condensate is available!**

Quark condensate from the propagator

$$\langle \bar{\psi}\psi \rangle \sim \mathrm{Tr}\, S$$

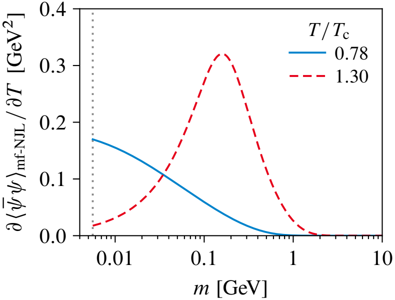
Detour: NJL model

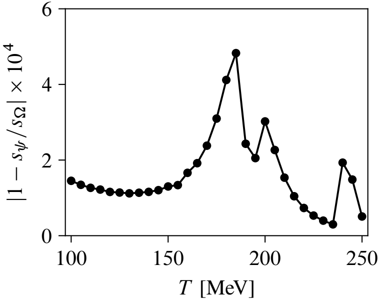
P.I., Fischer, Steinert, PRD 103 (2021) 054012

- Mean-field NJL Lagrangian:

$$\mathcal{L}_{\text{mf-NJL}} = \bar{\psi}(\mathrm{i}\partial\!\!\!/ - M)\psi - \frac{(M-m)^2}{4G}$$

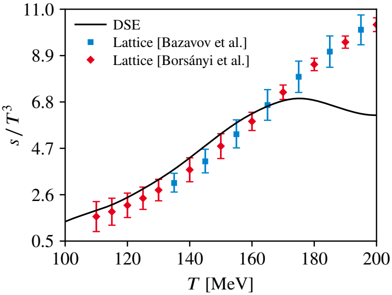
- Quark condensate: $\langle\bar{\psi}\psi\rangle_{\text{mf-NJL}} = (m - M)/2G$

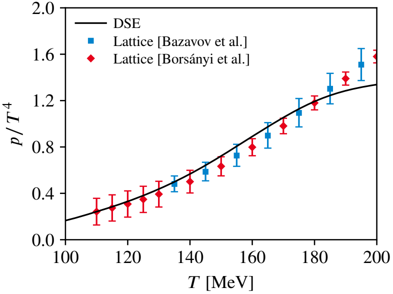


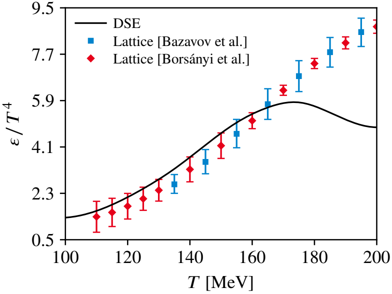


Thermodynamics from DSEs: results at $\mu = 0$

P.I., Fischer, Steinert, PRD 103 (2021) 054012

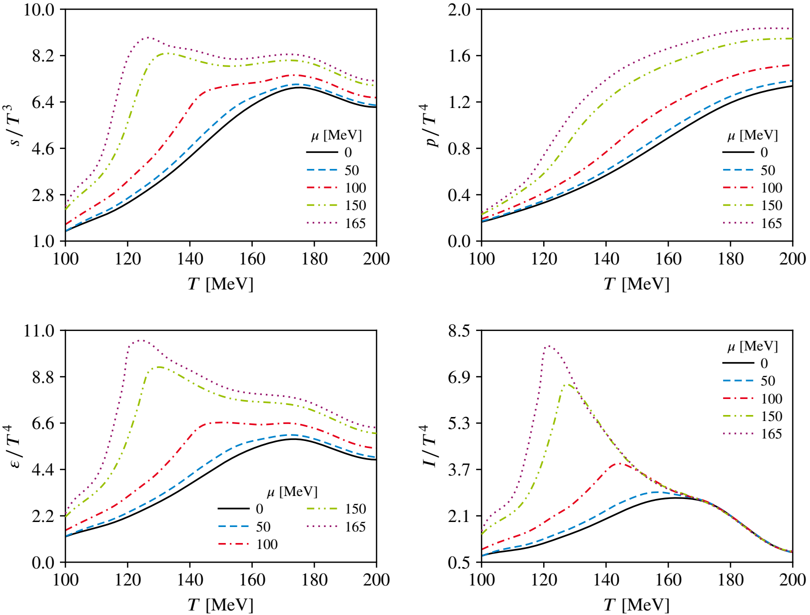






Thermodynamics from DSEs: results at $\mu \neq 0$

P.I., Fischer, Steinert, PRD 103 (2021) 054012



Summary

EoS from DSEs:

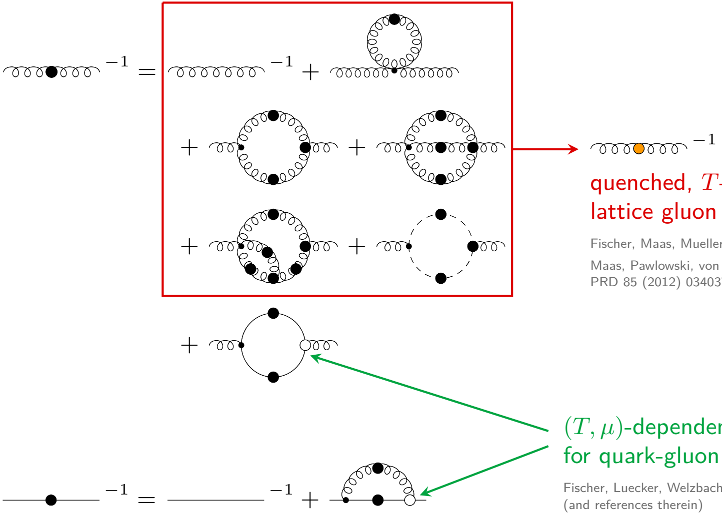
- Proposed a truncation-independent method to compute thermodynamic quantities within the DSE approach
- Based on a general relation between $\langle\bar{\psi}\psi\rangle$ and s
- EoS in $(2+1)$-flavor QCD at $\mu = 0$ and $\mu \neq 0$:
  - Satisfying agreement with lattice results below and around T_{c}
  - Deviations at high temperatures related to the truncation of the quark-gluon vertex

Outlook:

- Systematic control over error budget
- Off-diagonal fluctuations
- Strangeness neutrality
- Need high-quality quark-gluon vertex for thermodynamics at large temperatures and/or densities

Backup slides

Backup: how to truncate?



quenched, T-dependent lattice gluon propagator

Fischer, Maas, Mueller, EPJC 68 (2010) 165

Maas, Pawlowski, von Smekal, Spielmann, PRD 85 (2012) 034037

(T, μ)-dependent ansatz for quark-gluon vertex

Fischer, Luecker, Welzbacher, PRD 90 (2014) 034022 (and references therein)

Backup: ansatz for quark-gluon vertex

$$S_f^{-1}(p) = \mathrm{i}(\omega_p + \mathrm{i}\mu_f)\gamma_4 C_f(p) + \mathrm{i}\vec{p\!\!/} A_f(p) + B_f(p)$$

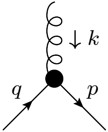
Vertex ansatz:

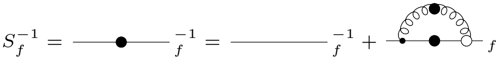
$$\Gamma_\nu^f(q,p,k) = \widetilde{Z}_3\,\Gamma(k^2)\,\gamma_\nu \left(\delta_{4\nu}\,\frac{C_f(q)+C_f(p)}{2} + (1-\delta_{4\nu})\,\frac{A_f(q)+A_f(p)}{2} \right)$$

Phenomenological dressing function:

$$\Gamma(k^2) = \frac{d_1}{d_2 + k^2} + \frac{1}{1+\Lambda^2/k^2}\left(\frac{\alpha_\mathrm{s}\beta_0}{4\pi}\log\left(1+k^2/\Lambda^2\right)\right)^{2\delta}$$

- Abelian STI (leading term of Ball-Chiu vertex)
  Ball, Chiu, PRD 22 (1980) 2542
- Perturbative running in the ultraviolet (quantitative)
- Ansatz for IR (qualitative)
  - d_1 fixed via T_c
  - d_2 fixed to match scale of quenched lattice gluon





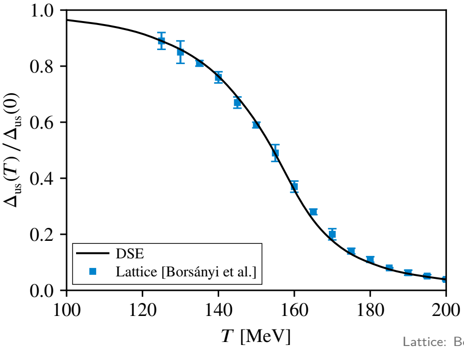
Chiral order parameter:

Quark condensate

$$\langle\bar{\psi}\psi\rangle_f \sim \mathrm{Tr}\, S_f$$

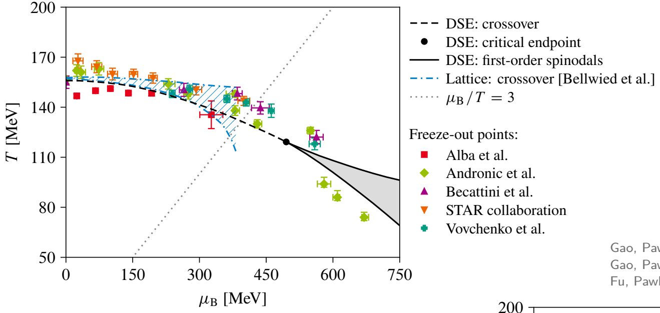
Subtracted condensate

$$\Delta_{ff'} = \langle\bar{\psi}\psi\rangle_f - \frac{m_f}{m_{f'}}\langle\bar{\psi}\psi\rangle_{f'}$$



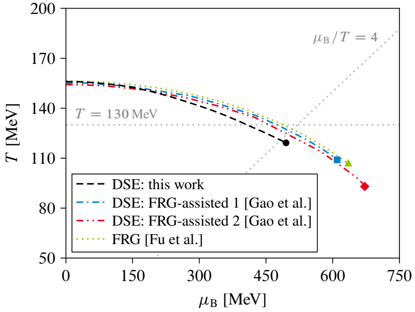
Lattice: Borsányi et al., JHEP09(2010)073

Backup: QCD phase diagram



P.I., Buballa, Fischer, Gunkel, PRD 100 (2019) 074011

Gao, Pawlowski, PLB 820 (2021) 136584
Gao, Pawlowski, PRD 102 (2020) 034027
Fu, Pawlowski, Rennecke, PRD 101 (2020) 054032



- CEPs cluster around

$490\,\mathrm{MeV} \lesssim \mu_\mathrm{B} \lesssim 680\,\mathrm{MeV}$

$90\,\mathrm{MeV} \lesssim T \lesssim 120\,\mathrm{MeV}$

- No CEP for $\mu_\mathrm{B}/T \lesssim 4$

Backup: fluctuations of conserved charges

Fluctuations from QCD's grand-canonical potential

$$\chi_{ijk}^{\mathrm{BQS}} = -T^{(i+j+k)-4} \frac{\partial^{i+j+k}\,\Omega}{\partial\mu_{\mathrm{B}}^{i}\,\partial\mu_{\mathrm{Q}}^{j}\,\partial\mu_{\mathrm{S}}^{k}}$$

- Relation to cumulants of probability distribution:

$$C_n^X = V\,T^3\,\chi_n^X$$

- Statistical quantities:

$$\sigma_X^2 = C_2^X\,, \qquad S_X = C_3^X\,(C_2^X)^{-3/2}\,, \qquad \kappa_X = C_4^X\,(C_2^X)^{-2}$$

- Prominent quantities are ratios:

$$\frac{\chi_3^{\mathrm{B}}}{\chi_2^{\mathrm{B}}} = S_{\mathrm{B}}\,\sigma_{\mathrm{B}}\,, \qquad \frac{\chi_4^{\mathrm{B}}}{\chi_2^{\mathrm{B}}} = \kappa_{\mathrm{B}}\,\sigma_{\mathrm{B}}^2$$

Reviews: Luo, Xu, Nucl. Sci. Tech. 28 (2017) 112
Bzdak, Esumi, Koch, Liao, Stephanov, Xu, Phys. Rep. 853 (2020) 1