Quarkonium Hybrid Decays

Quarkonium Working Group (QWG) 2022 Workshop

GSI Helmholtzzentrum, Darmstadt, Germany

Sep 28, 2022

Abhishek Mohapatra (TU Munich)

Nora Brambilla (TU Munich)
Wai Kin Lai (South China Normal University)
Antonio Vairo (TU Munich)

In preparation

Technical University of Munich

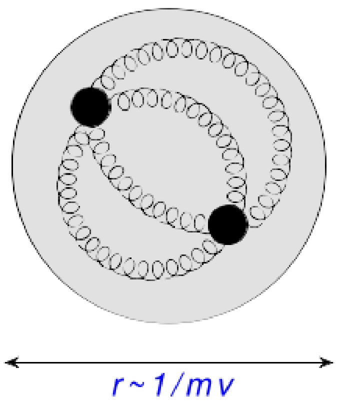
DFG Deutsche Forschungsgemeinschaft

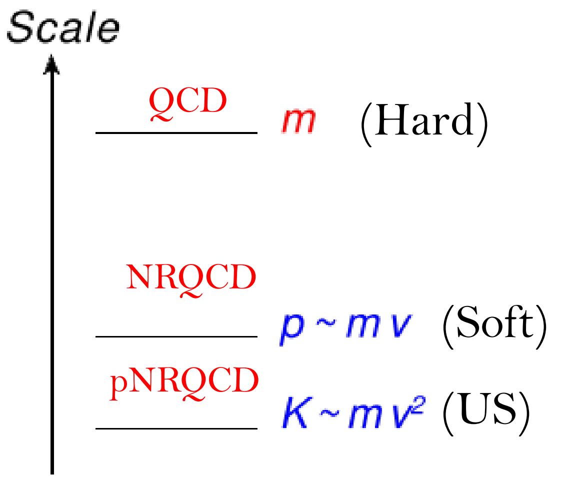
Outline

- Introduction to X Y Z mesons
- EFT for Quarkonium Hybrids
  - BO-EFT effective theory
  - Quarkonium Hybrid Spectrum
- Decay Rates for hybrid
- Summary and Outlook

Introduction

- Quarkonium: Color singlet bound state of $Q\bar{Q}$ (Q = c, b).



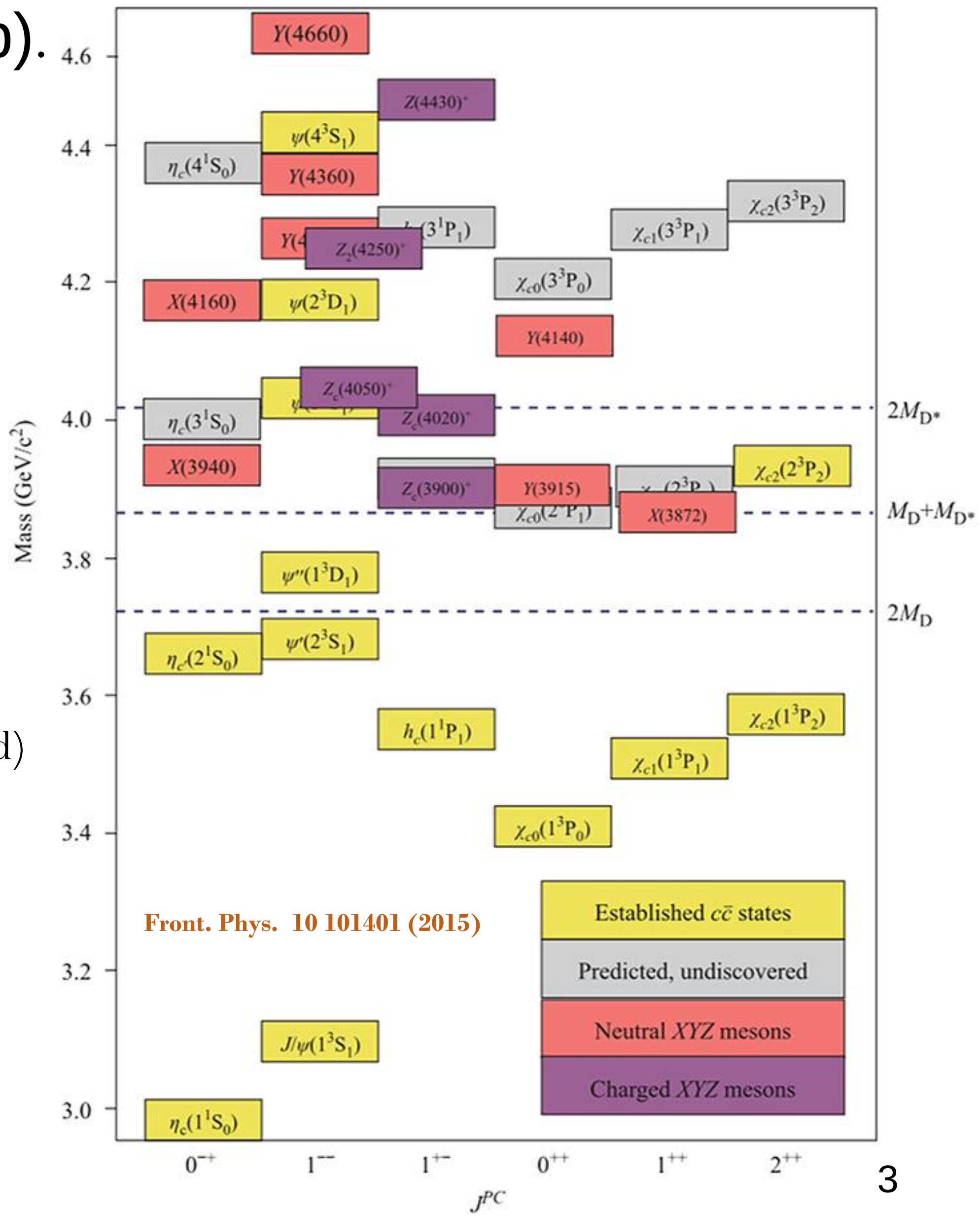


- Hierarchy of Energy Scales in $Q\bar{Q}$:

$$m \gg mv \gg mv^2, \Lambda_{\rm QCD}$$ (perturbative dynamics: Weakly Coupled)

$$m \gg mv, \Lambda_{\rm QCD} \gg mv^2$$ (nonperturbative dynamics: Strongly Coupled)

- pNRQCD: Relevant EFT for Quarkonium.



Front. Phys. 10 101401 (2015)

Bodwin, Braaten & Lepage (1994), Mehen and Fleming, Phys. Rev. D73, 034502 (2005)

Brambilla, Vairo, and Rosch, Phys. Rev. D72, 034021 (2005) Luke, Manohar & Rothstein (2000)

Introduction

- Quark Model:

  Mesons: quark-antiquark states   Baryons: 3-quark states

- QCD spectrum also allows for more complex structures called as Exotics.

- Exotic states: XYZ mesons

  ✓ Quarkonium-like states that don't fit traditional $Q\bar{Q}$ spectrum.

  ✓ In some cases exotic quantum numbers (charged Z_c and Z_b states)

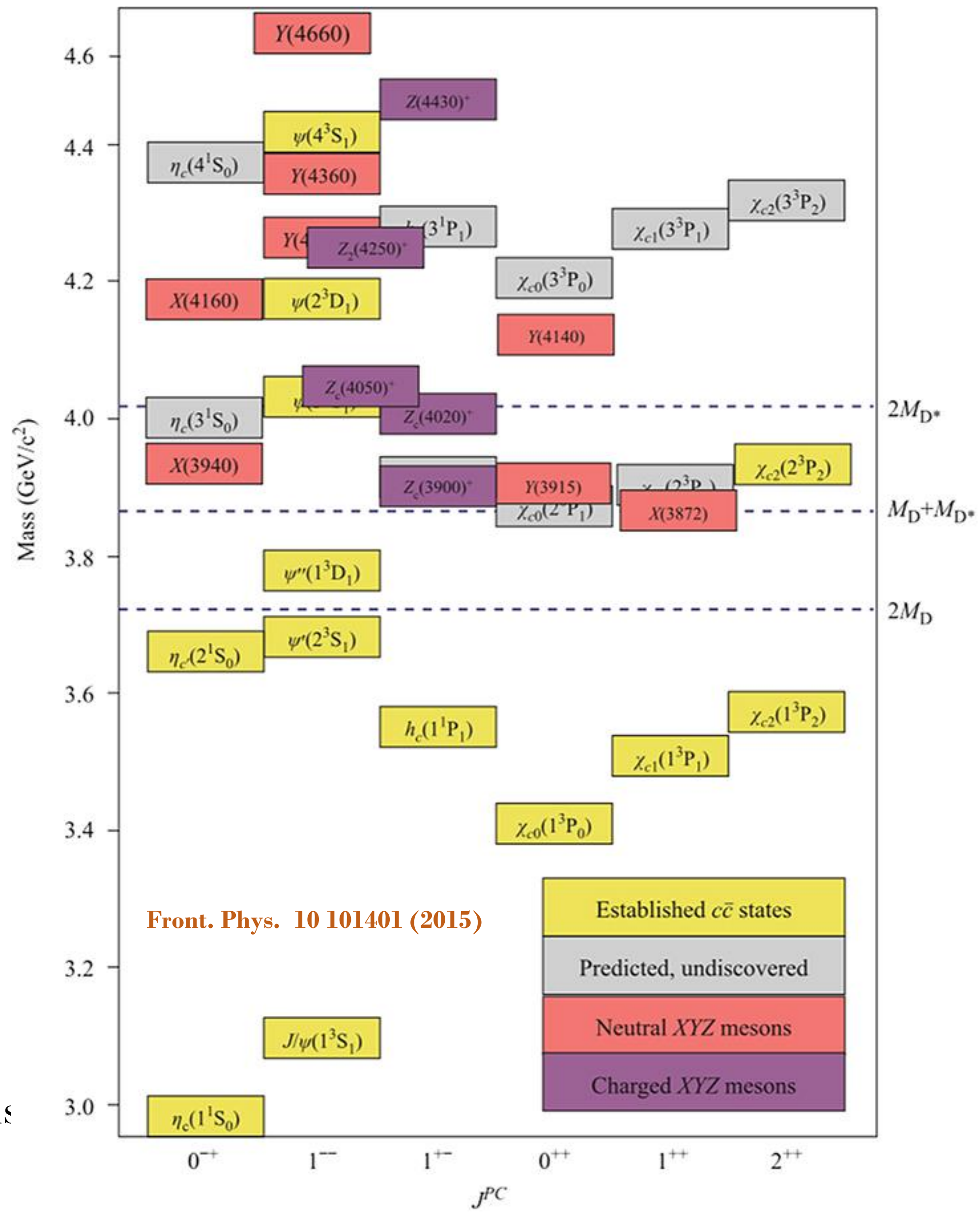
  For review see Brambilla et al. Phys. Reports. 873 (2020)

- X(3872): First exotic state discovered in 2003 by Belle.

  Phys. Rev. Lett. 91, 262001 (2003)

- Several **new** heavy quark exotic states have been discovered since 2003 (masses & decay rates measured in various channels

  PDG 2021



Front. Phys. 10 101401 (2015)

Introduction

- Exotics broadly classified as
  - ❖ Structures with active gluons ❖ Multiquark states
- Several interpretations of Exotics:

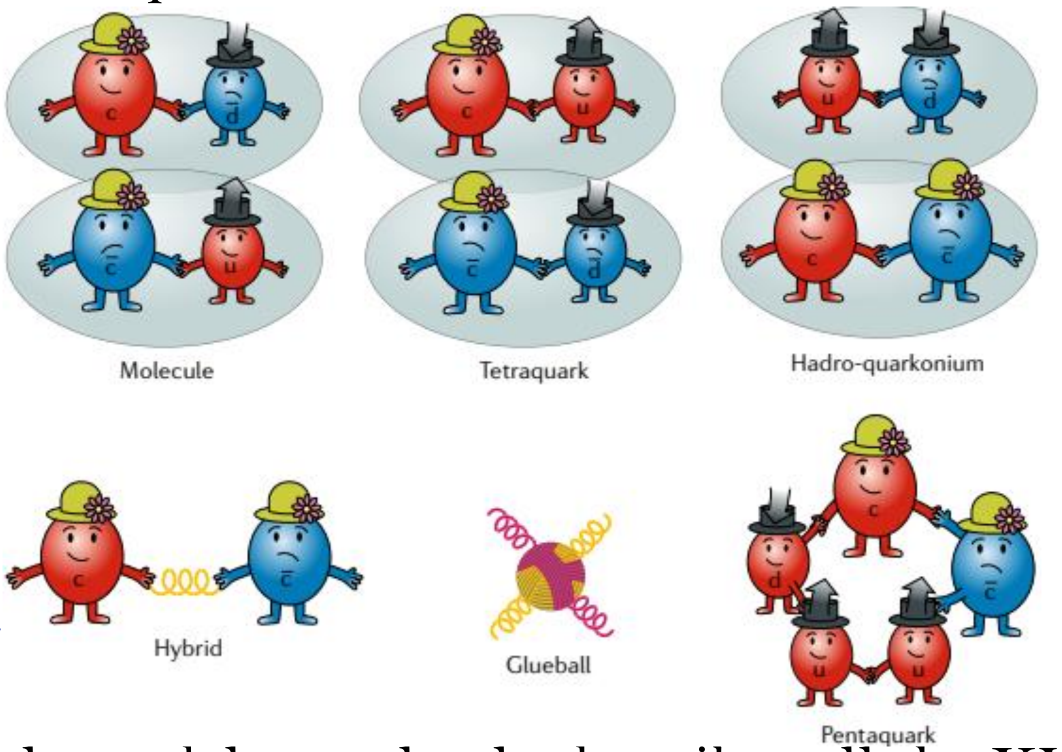
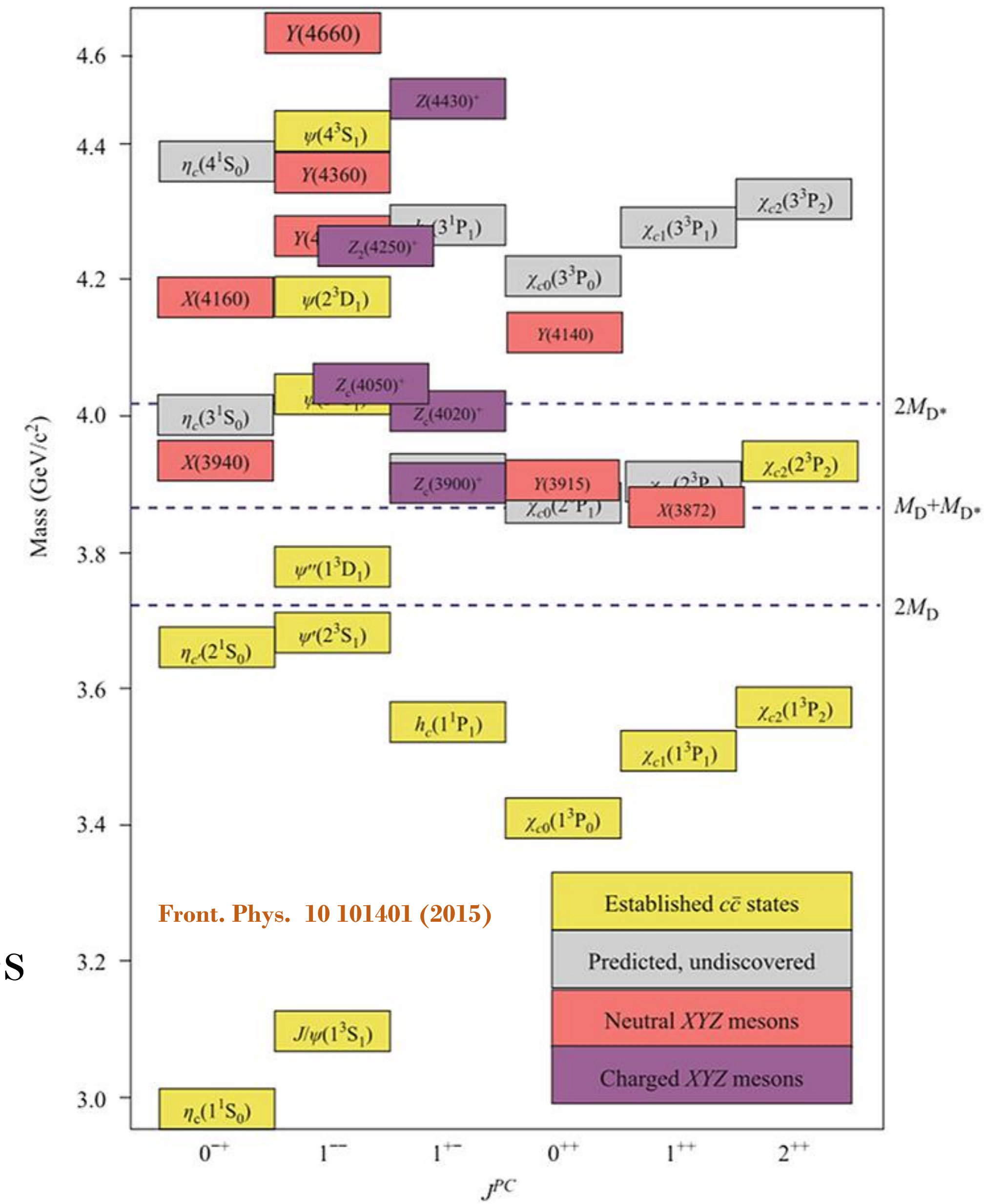


Figure from Nat Rev Phys 1, 480-494 (2019)



Front. Phys. 10 101401 (2015)

- No single model completely describes all the XYZ states
- Hybrids ($Q\bar{Q}g$): Focus of this talk. Use EFT + lattice to have model independent description.

Quarkonium hybrids: EFT

- Hybrids ($Q\bar{Q}g$): Color singlet combination of color octet $Q\bar{Q}$ + gluonic excitations.

- Separation of scales in hybrids:

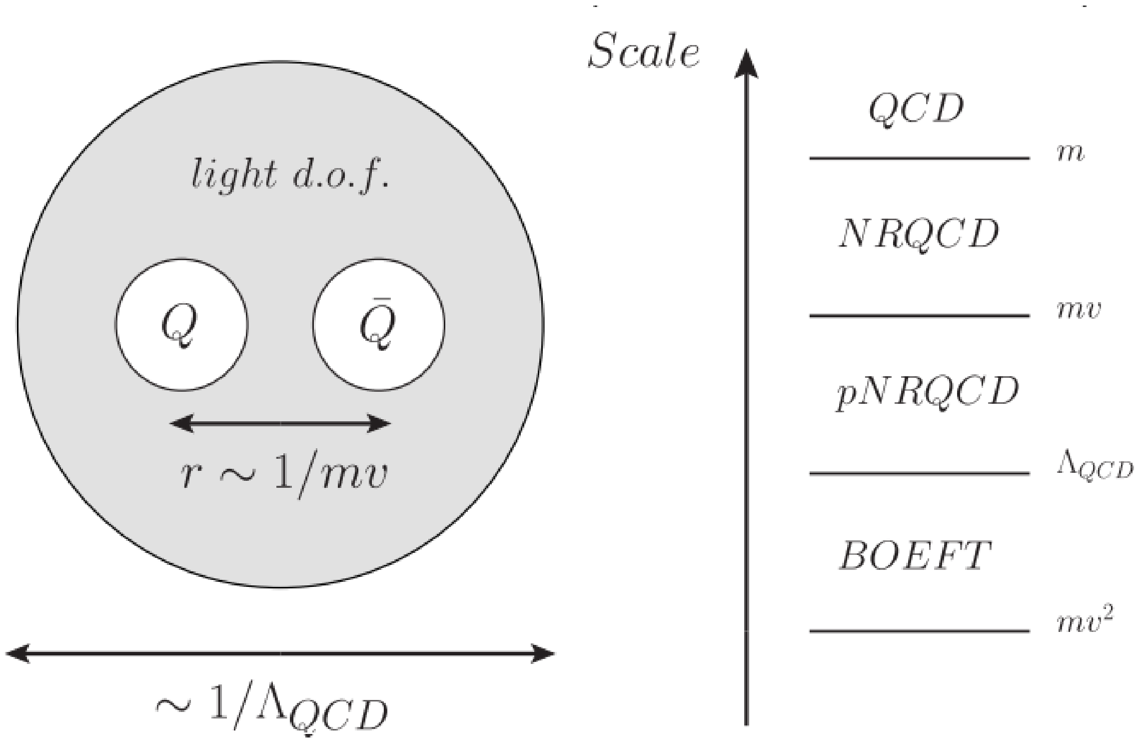
$$m \gg mv \gg \Lambda_{QCD} \gg mv^2$$

  - ❖ light d.o.f: Λ_{QCD}
  - ❖ Relative separation between heavy quarks: $r \sim 1/mv$
  - ❖ Heavy Quark K.E scale: mv^2

- Time-scale for dynamics of $Q\bar{Q}$: $\sim \frac{1}{mv^2} \gg \frac{1}{\Lambda_{QCD}}$

Born-Oppenheimer Approximation

Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)



- Appropriate EFT framework for Hybrids: **Born-Oppenheimer EFT (BOEFT)**

QCD → NRQCD → pNRQCD → BOEFT

Brambilla, Krein, Castellà , Vairo Phys. Rev. D. 97, (2018)

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

R. Oncala, J. Soto, Phys. Rev. D96 (2017)

Quarkonium hybrids: BOEFT

- Static limit ($m \rightarrow \infty$): Quantum #'s for hybrid

Irreducible representations of $D_{\infty h}$

- $\boldsymbol{K}$: angular momentum of light d.o.f.
  $\lambda = \hat{\boldsymbol{r}} \cdot \boldsymbol{K} = 0, \pm 1, \pm 2, \pm 3, \ldots$
  $\Lambda = |\lambda| = 0, 1, 2, 3, \ldots \; (\Sigma, \Pi, \Delta, \Phi, \ldots)$
- Eigenvalue of CP: $\eta = +1\,(g),\; -1\,(u)$
- σ: eigenvalue of relfection about a plane containing $\hat{\boldsymbol{r}}$ (only for Σ states)

- Static Energies (Σ, Π, Δ): Eigenvalue of NRQCD Hamiltonian in the static limit.

- For $r \rightarrow 0$: static energies are degenerate. Characterized by $O(3)\times C$ symmetry group.

Labelled by: $(K^{PC}, \Lambda_\eta^\sigma)$

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Gluonic static energies

M. Foster and C. Michael, Phys. Rev. D59 (1999)

K. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. 90 (2003)

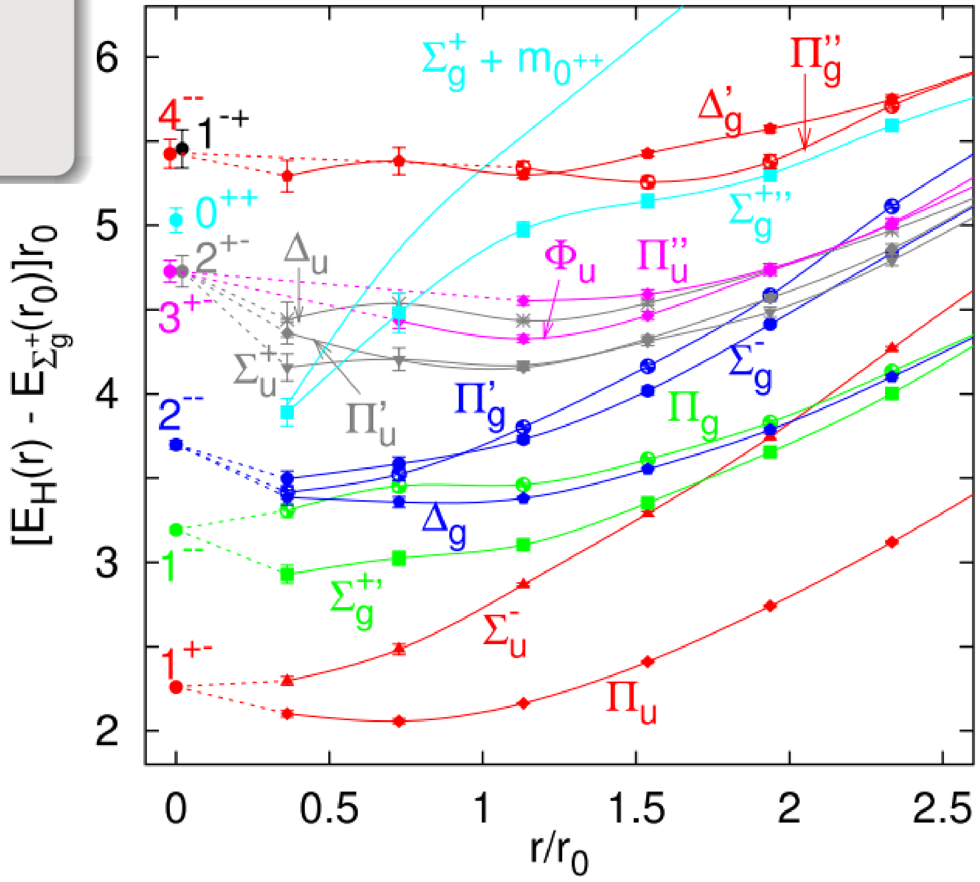


Fig from G. S. Bali and A. Pineda, Phys. Rev. D69 (2004)

Quarkonium hybrids: BOEFT

- Static limit (m→ ∞): Quantum #'s for hybrid

Irreducible representations of $D_{\infty h}$

- $\boldsymbol{K}$: angular momentum of light d.o.f.
  $\lambda = \hat{\boldsymbol{r}} \cdot \boldsymbol{K} = 0, \pm 1, \pm 2, \pm 3, \ldots$
  $\Lambda = |\lambda| = 0, 1, 2, 3, \ldots \ (\Sigma, \Pi, \Delta, \Phi, \ldots)$
- Eigenvalue of CP: $\eta = +1\,(g), \ -1\,(u)$
- σ: eigenvalue of relfection about a plane containing $\hat{\boldsymbol{r}}$ (only for Σ states)

Gluonic operators characterizing Hybrids in Wilson loop

- Static Energies (Σ, Π, Δ): Eigenvalue of NRQCD Hamiltonian in the static limit.

Λ_η^σ	K^{PC}	O_n
Σ_u^-	1^{+-}	$\hat{\boldsymbol{r}} \cdot \boldsymbol{B}, \ \hat{\boldsymbol{r}} \cdot (\boldsymbol{D} \times \boldsymbol{E})$
Π_u	1^{+-}	$\hat{\boldsymbol{r}} \times \boldsymbol{B}, \ \hat{\boldsymbol{r}} \times (\boldsymbol{D} \times \boldsymbol{E})$
$\Sigma_g^{+\prime}$	1^{--}	$\hat{\boldsymbol{r}} \cdot \boldsymbol{E}, \ \hat{\boldsymbol{r}} \cdot (\boldsymbol{D} \times \boldsymbol{B})$
Π_g	1^{--}	$\hat{\boldsymbol{r}} \times \boldsymbol{E}, \ \hat{\boldsymbol{r}} \times (\boldsymbol{D} \times \boldsymbol{B})$
Σ_g^-	2^{--}	$(\hat{\boldsymbol{r}} \cdot \boldsymbol{D})(\hat{\boldsymbol{r}} \cdot \boldsymbol{B})$
Π_g'	2^{--}	$\hat{\boldsymbol{r}} \times ((\hat{\boldsymbol{r}} \cdot \boldsymbol{D})\boldsymbol{B} + \boldsymbol{D}(\hat{\boldsymbol{r}} \cdot \boldsymbol{B}))$
Δ_g	2^{--}	$(\hat{\boldsymbol{r}} \times \boldsymbol{D})^i(\hat{\boldsymbol{r}} \times \boldsymbol{B})^j + (\hat{\boldsymbol{r}} \times \boldsymbol{D})^j(\hat{\boldsymbol{r}} \times \boldsymbol{B})^i$
Σ_u^+	2^{+-}	$(\hat{\boldsymbol{r}} \cdot \boldsymbol{D})(\hat{\boldsymbol{r}} \cdot \boldsymbol{E})$
Π_u'	2^{+-}	$\hat{\boldsymbol{r}} \times ((\hat{\boldsymbol{r}} \cdot \boldsymbol{D})\boldsymbol{E} + \boldsymbol{D}(\hat{\boldsymbol{r}} \cdot \boldsymbol{E}))$
Δ_u	2^{+-}	$(\hat{\boldsymbol{r}} \times \boldsymbol{D})^i(\hat{\boldsymbol{r}} \times \boldsymbol{E})^j + (\hat{\boldsymbol{r}} \times \boldsymbol{D})^j(\hat{\boldsymbol{r}} \times \boldsymbol{E})^i$

- For r→ 0: static energies are degenerate. Characterized by O(3)×C symmetry group.

Labelled by: $(K^{PC}, \Lambda_\eta^\sigma)$

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Focus on these two for low lying hybrids

Quarkonium hybrids: BOEFT

- BOEFT d.o.f involve color singlet fields $\hat{\Psi}_{\kappa\lambda}(\boldsymbol{r},\boldsymbol{R},t) \propto P^{i}_{\kappa\lambda} O^{a\dagger}(\boldsymbol{r},\boldsymbol{R},t) G^{ia}_{\kappa}(\boldsymbol{R},t)$
  - $O^{a\dagger}(\boldsymbol{r},\boldsymbol{R},t) G^{ia}_{\kappa}(\boldsymbol{R},t)$: Gluelump operator. Eigenvector of Hamiltonian in (m⟶ ∞):

$$H^{(0)} O^{a\dagger}(\boldsymbol{r},\boldsymbol{R},\boldsymbol{t}) G^{ia}_{\kappa}(\boldsymbol{R},t)|0\rangle = (V_0(r) + \Lambda_\kappa)\, O^{a\dagger}(\boldsymbol{r},\boldsymbol{R},t) G^{ia}_{\kappa}(\boldsymbol{R},t)|0\rangle$$

Λ_κ : Gluelump energy

  - $P^{i}_{\kappa\lambda}$: Projection operators of light d.o.f along heavy quark-antiquark axis.

- BOEFT Lagrangian:

$$L_{\mathrm{BOEFT}} = \int d^3R\, d^3r \sum_{\kappa} \sum_{\lambda\lambda'} \hat{\Psi}^{\dagger}_{\kappa\lambda}(\boldsymbol{r},\boldsymbol{R},t) \left\{ i\partial_t - V_{\kappa\lambda\lambda'}(r) + P^{i\dagger}_{\kappa\lambda} \frac{\boldsymbol{\nabla}^2_r}{m} P^{i}_{\kappa\lambda'} \right\} \hat{\Psi}_{\kappa\lambda'}(\boldsymbol{r},\boldsymbol{R},t) + \ldots$$

- Schrödinger Eq: Dynamics of $Q\bar{Q}$ at scale $mv^2 \ll \Lambda_{QCD}$

Schrödinger equation

$$\left[-P^{i\dagger}_{\kappa\lambda} \frac{\boldsymbol{\nabla}^2_r}{m} P^{i}_{\kappa\lambda'} + V_{\kappa\lambda\lambda'}(r) \right] \Psi^{n}_{\kappa\lambda'}(\boldsymbol{r}) = E_n \Psi^{n}_{\kappa\lambda}(\boldsymbol{r})$$

Hybrid wf

- Coupled Eq. due to projection operators. Mixes Σ_u and Π_u states.

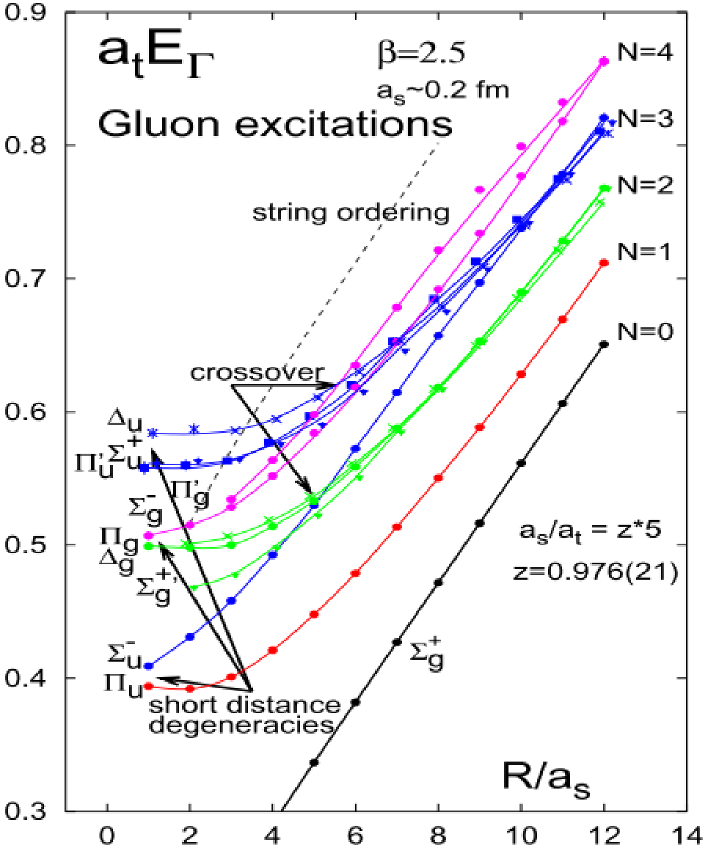
Brambilla, Krein, Castellà , Vairo Phys. Rev. D. 97, (2018) Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Quarkonium hybrids: Spectrum

- Lattice potentials for solving the Schrödinger Eq:

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Gluonic Static energies from lattice:



K. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. 90 (2003)

$$m_c^{RS} = 1.477\,(40)\ \text{GeV}$$
$$m_b^{RS} = 4.863\,(55)\ \text{GeV}$$

Quarkonium Potential:

$$V_{\Sigma_g^+}(r) = -\frac{\kappa_g}{r} + \sigma_g r + E_g^{Q\bar{Q}}$$

$$\kappa_g = 0.489\,, \quad \sigma_g = 0.187\,\text{GeV}^2 \qquad E_g^{c\bar{c}} = -0.254\,\text{GeV},\ E_g^{b\bar{b}} = -0.195\,\text{GeV},$$

RS-Scheme Hybrid Potential:

$$E_n^{(0)}(r) = \begin{cases} V_o^{RS}(\nu_f) + \Lambda_H^{RS}(\nu_f) + b_n r^2, & r < 0.25\,fm \\ \frac{a_1}{r} + \sqrt{a_2\, r^2 + a_3} + a_4, & r > 0.25\,fm \end{cases}$$

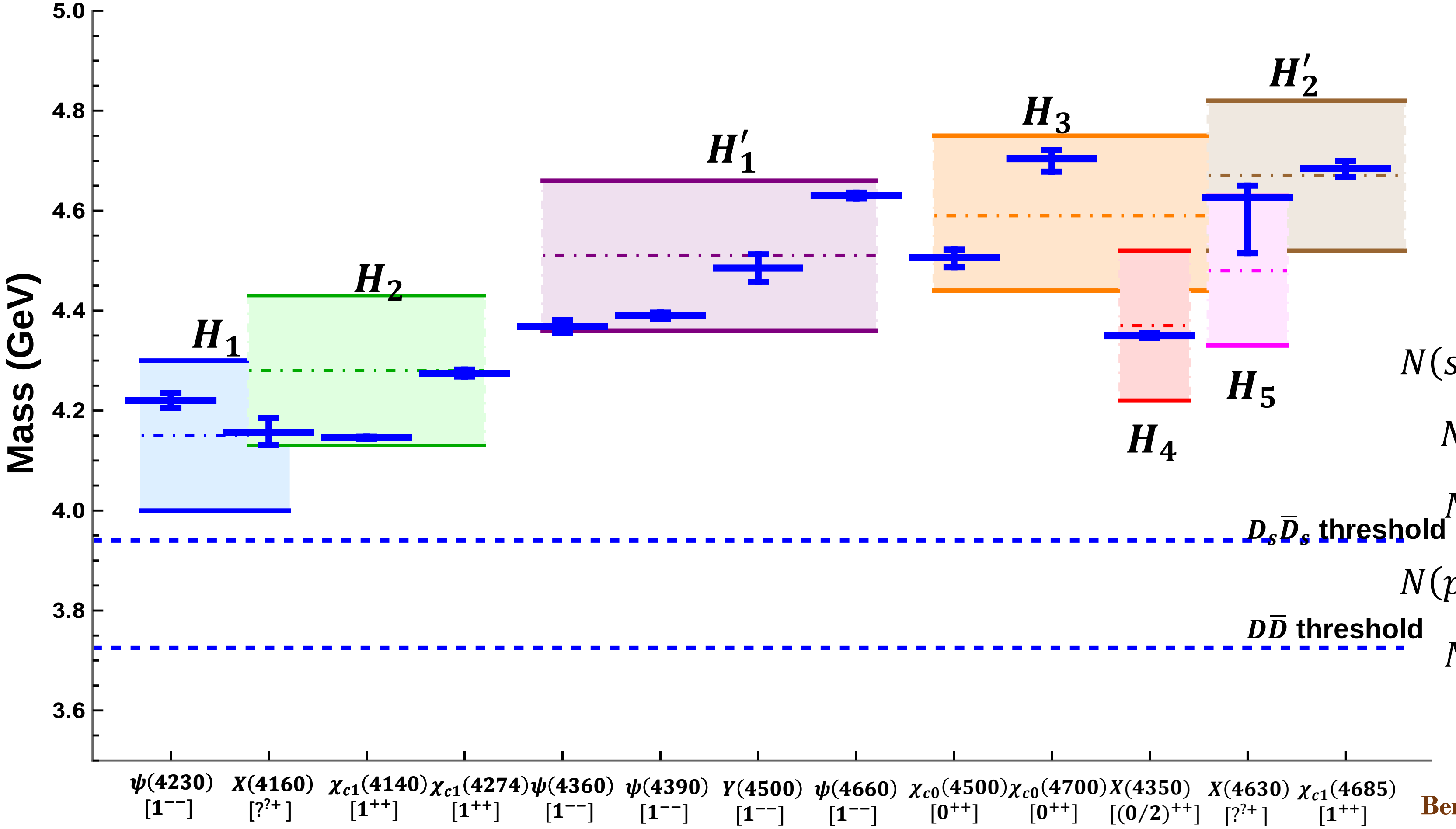
$$a_1^\Sigma = 0.000\,\text{GeVfm}\,, \quad a_2^\Sigma = 1.543\,\text{GeV}^2/\text{fm}^2, \quad a_3^\Sigma = 0.599\,\text{GeV}^2, \quad a_4^\Sigma = 0.154\,\text{GeV}\,,$$
$$a_1^\Pi = 0.023\,\text{GeVfm}\,, \quad a_2^\Pi = 2.716\,\text{GeV}^2/\text{fm}^2, \quad a_3^\Pi = 11.091\,\text{GeV}^2, \quad a_4^\Pi = -2.536\,\text{GeV}\,,$$
$$b_\Sigma = 1.246\,\text{GeV}/\text{fm}^2, \quad b_\Pi = 0.000\,\text{GeV}/\text{fm}^2$$

✓ Perturbative RS-scheme potentials V_o^{RS} upto order α_s^3.

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015) Bali and Pineda Phys. Rev. D. 69, (2004)
Kniehl, Penin, Schroder , Smirnov, Steinhauser Phys. Lett. B 607, (2005)

Quarkonium hybrids: Spectrum

- Charmonium hybrids: comparison with experimental results:



Other notation of hybrid states

Other notation		l	$J^{PC}\{s=0, s=1\}$	$E_n^{(0)}$
$N(s/d)_1$ →	H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$	Σ_u^-, Π_u
Np_1 →	H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$	Π_u
Np_0 →	H_3	0	$\{0^{++}, 1^{+-}\}$	Σ_u^-
$N(p/f)_2$ →	H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$	Σ_u^-, Π_u
Nd_2 →	H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$	Π_u

- Λ- doubling: opposite parity states non-degenerate.

PDG 2022

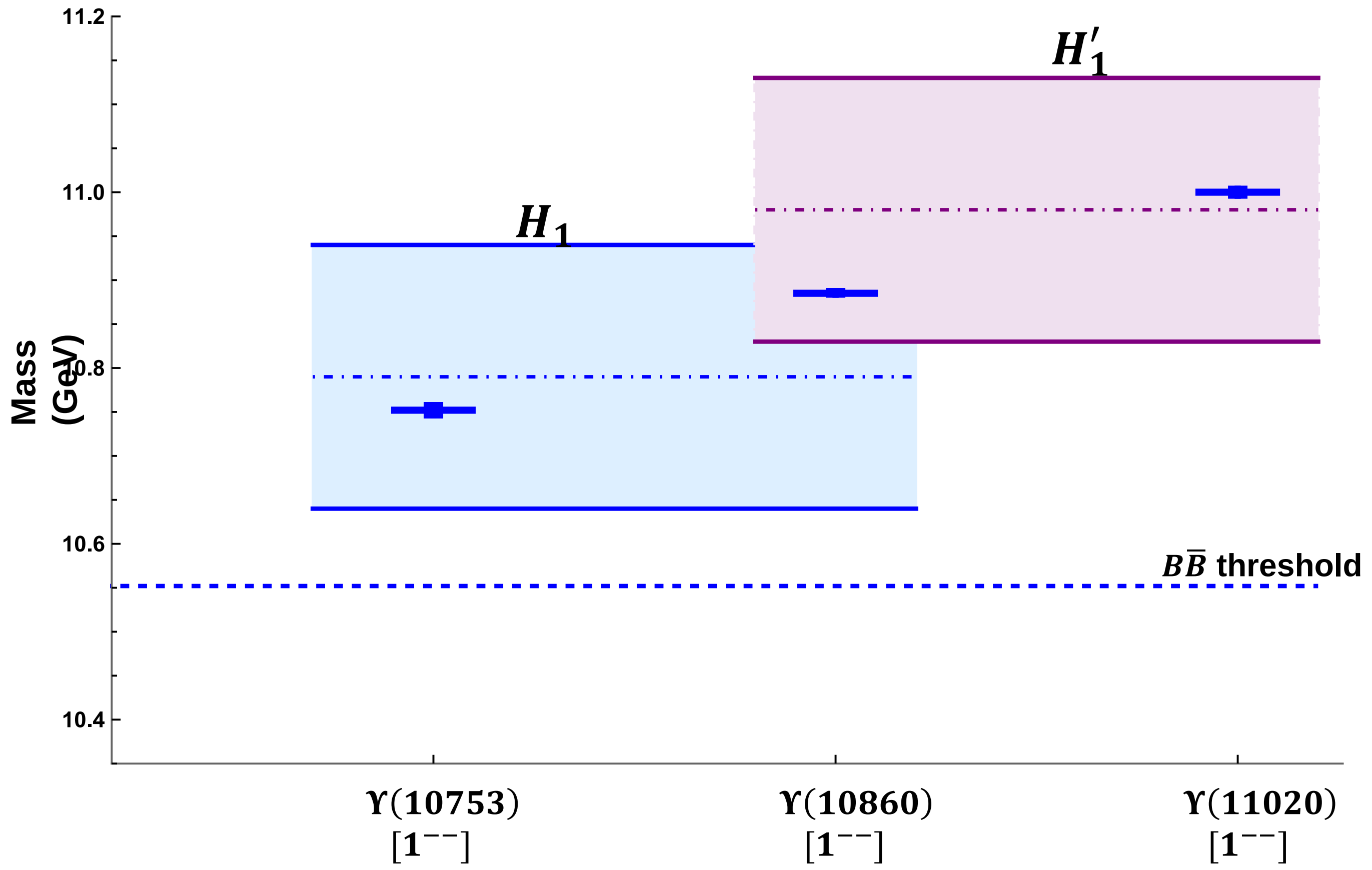
Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)

R. Oncala, J. Soto, Phys. Rev. D96 (2017)

Quarkonium hybrids: Spectrum

- Bottomonium hybrids: comparison with experimental results:



PDG 2022

Other notation of hybrid states

		l	$J^{PC}\{s=0, s=1\}$	$E_n^{(0)}$
$N(s/d)_1$ →	H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$	Σ_u^-, Π_u
Np_1 →	H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$	Π_u
Np_0 →	H_3	0	$\{0^{++}, 1^{+-}\}$	Σ_u^-
$N(p/f)_2$ →	H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$	Σ_u^-, Π_u
Nd_2 →	H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$	Π_u

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)

R. Oncala, J. Soto, Phys. Rev. D96 (2017)

Quakonium Hybrid Decays

- Dozens of XYZ states have been discovered (mass and decay rates measured) but physics still unknown.
- Several theoretical models for exotic states but no general consensus.
- Most of the exotic states discovered from decays to quarkonium. So, decays might provide information on the structure of XYZ.
- Consider the process: $\boldsymbol{H_m \to Q_n + X}$; H_m: low-lying hybrid, Q_n: low-lying quarkonium.
  - ✓ ΔE: Energy difference ⇒ $\Delta E \equiv \boldsymbol{E_{H_m} - E_{Q_n} \gtrsim 1}$ **GeV**.
  - ✓ Assume hierarchy of Scales: $\mathbf{mv \gg \Delta E \gg \Lambda_{QCD} \gg mv^2}$
- Start with **pNRQCD** effective theory and **obtain BOEFT** by **matching**: Integrate out modes of scale $\sim \Delta$ and $\sim \Lambda_{\mathrm{QCD}}$.

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Quakonium Hybrid Decays

- pNRQCD Lagrangian:

Weakly-coupled pNRQCD Lagrangian

$$L_{\text{pNRQCD}} = \int d^3R \Big\{ \int d^3r \Big(\text{Tr}\left[\text{S}^\dagger \left(i\partial_0 - h_s \right) \text{S} + \text{O}^\dagger \left(iD_0 - h_o \right) \text{O} \right]$$

$$+ g\text{Tr}\left[\text{S}^\dagger \boldsymbol{r} \cdot \boldsymbol{E}\,\text{O} + \text{O}^\dagger \boldsymbol{r} \cdot \boldsymbol{E}\,\text{S} + \frac{1}{2}\text{O}^\dagger \boldsymbol{r} \cdot \{\boldsymbol{E}, \text{O}\} \right] + \frac{g}{4m}\text{Tr}\left[\text{O}^\dagger \boldsymbol{L}_{Q\bar{Q}} \cdot [\boldsymbol{B}, \text{O}] \right]$$

$$+ \frac{gc_F}{m}\text{Tr}\left[\text{S}^\dagger (\boldsymbol{S}_1 - \boldsymbol{S}_2) \cdot \boldsymbol{B}\,\text{O} + \text{O}^\dagger (\boldsymbol{S}_1 - \boldsymbol{S}_2) \cdot \boldsymbol{B}\,\text{S} + \text{O}^\dagger \boldsymbol{S}_1 \cdot \boldsymbol{B}\,\text{O} - \text{O}^\dagger \boldsymbol{S}_2 \text{O} \cdot \boldsymbol{B} \right] - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu\, a} \Big\}$$

- BOEFT:

Potential term in BOEFT

BOEFT Hamiltonian

$$H_{\text{BOEFT}} = \int d^3x \int d^3R\, \text{Tr}\left[H^{i\dagger} \left(h_o \delta^{ij} + V^{ij}_{soft} \right) H^j \right]$$

- Decays are computed from local imaginary terms in the BOEFT Lagrangian.
- Imaginary term in V^{ij}_{soft} from 1-loop diagram in pNRQCD and then **<u>matching</u>** to BOEFT .

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Quakonium Hybrid Decays

- pNRQCD Lagrangian:

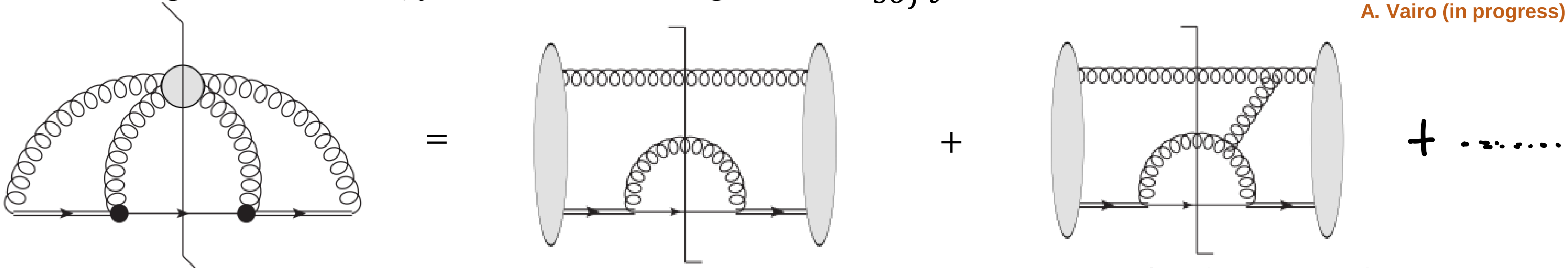
Weakly-coupled pNRQCD Lagrangian

$$L_{\mathrm{pNRQCD}} = \int d^3R \Big\{ \int d^3r \Big(\mathrm{Tr}\left[\mathrm{S}^\dagger \left(i\partial_0 - h_s \right) \mathrm{S} + \mathrm{O}^\dagger \left(iD_0 - h_o \right) \mathrm{O} \right]$$
$$+ g\mathrm{Tr}\left[\mathrm{S}^\dagger \boldsymbol{r} \cdot \boldsymbol{E}\, \mathrm{O} + \mathrm{O}^\dagger \boldsymbol{r} \cdot \boldsymbol{E}\, \mathrm{S} + \frac{1}{2} \mathrm{O}^\dagger \boldsymbol{r} \cdot \{ \boldsymbol{E}, \mathrm{O} \} \right] + \frac{g}{4m} \mathrm{Tr}\left[\mathrm{O}^\dagger \boldsymbol{L}_{Q\bar{Q}} \cdot [\boldsymbol{B}, \mathrm{O}] \right]$$
$$+ \frac{g c_F}{m} \mathrm{Tr}\left[\mathrm{S}^\dagger (\boldsymbol{S}_1 - \boldsymbol{S}_2) \cdot \boldsymbol{B}\, \mathrm{O} + \mathrm{O}^\dagger (\boldsymbol{S}_1 - \boldsymbol{S}_2) \cdot \boldsymbol{B}\, \mathrm{S} + \mathrm{O}^\dagger \boldsymbol{S}_1 \cdot \boldsymbol{B}\, \mathrm{O} - \mathrm{O}^\dagger \boldsymbol{S}_2 \mathrm{O} \cdot \boldsymbol{B} \right] - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu\, a} \Big\}$$

- Spin preserving decays
- Spin flipping decays

- 1-loop diagram in pNRQCD contributing to Im V^{ij}_{soft} in BOEFT:

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)



Black dot: vertex of pNRQCD

Spectator gluon approximation

example of α_s correction

Quakonium Hybrid Decays

- Spin-conserving decay due to $\boldsymbol{r} \cdot \boldsymbol{E}$ term :

$$\Gamma(H_m \to Q_n) = \frac{4\alpha_s\,(\Delta E)\,T_F}{3N_c}\,T^{ij}\,(T^{ij})^\dagger\,\Delta E^3$$

$|S_H = 1 > ---\to |S_Q = 1 >$
$|S_H = 0 > ---\to |S_Q = 0 >$

$$T^{ij} \equiv \langle H_m | r^j | Q_n \rangle = \int d^3\mathbf{r}\,\Psi^{i\dagger}_{(m)}(\mathbf{r})\,r^j\,\Phi^{Q\bar{Q}}_{(n)}(\mathbf{r})$$

$\Psi^i_{(m)}$: Hybrid wf
Φ^Q_n : Quarkonium wf

R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017). J. Castellà, E. Passemar, Phys. Rev. D104, 034019 (2021)

- Spin-flipping decay due to $\boldsymbol{S}.\boldsymbol{B}$ term:

$|\chi_H\rangle$: Hybrid spin wf
$|\chi_Q\rangle$: Quarkonium spin wf

$|S_H = 1 > ---\to |S_Q = 0 >$
$|S_H = 0 > ---\to |S_Q = 1 >$

$$T^{ij} \equiv \langle H_m | \left(S_1^j - S_2^j\right) | Q_n \rangle = \left[\int d^3\mathbf{r}\,\Psi^{i\dagger}_{(m)}(\mathbf{r})\,\Phi^{Q}_{(n)}(\mathbf{r})\right] \langle \chi_H | \left(S_1^j - S_2^j\right) | \chi_Q \rangle$$

- Depends on overlap of quarkonium and hybrid wavefunctions.
- Based on weakly-coupled pNRQCD hierarchy: $\mathbf{mv \gg \Delta E \gg \Lambda_{QCD} \gg mv^2}$

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Exotic States

State (PDG)	State (Former)	M (MeV)	Γ (MeV)	J^{PC}	Decay modes
$\chi_{c1}(4140)$	$Y(4140)$	4146.5 ± 3.0	19^{+7}_{-5}	1^{++}	$\phi J/\psi$
$X(4160)$		4156^{+29}_{-35}	139^{+113}_{-65}	$??^{+}$	$D^* \bar{D}^*$
$\psi(4230)$	$Y(4230)$	4222.7 ± 2.6	50 ± 9	1^{--}	$\pi^+\pi^- J/\psi$, $\omega\chi_{c0}(1P)$,
	$Y(4260)$				$\pi^+\pi^- h_c(1P)$
$\chi_{c1}(4274)$	$Y(4274)$	4286^{+8}_{-9}	51 ± 7	1^{++}	$\phi J/\psi$
$X(4350)$		$4350.6^{+4.7}_{-5.1}$	13^{+18}_{-10}	$(0/2)^{++}$	$\phi J/\psi$
$\psi(4360)$	$Y(4360)$	4372 ± 9	115 ± 13	1^{--}	$\pi^+\pi^- J/\psi$,
	$Y(4320)$				$\pi^+\pi^- \psi(2S)$
$\psi(4390)$	$Y(4390)$	4390 ± 6	139^{+16}_{-20}	1^{--}	$\eta J/\psi$, $\pi^+\pi^- h_c(1P)$
$\chi_{c0}(4500)$	$X(4500)$	4474 ± 4	77^{+12}_{-10}	0^{++}	$\phi J/\psi$
$Y(4500)$[a]		4484.7 ± 27.5	111 ± 34	1^{--}	
$X(4630)$[b]		4626^{+24}_{-111}	174^{+137}_{-78}	$??^{+}$	$\phi J/\psi$
$\psi(4660)$	$Y(4660)$	4630 ± 6	72^{+14}_{-12}	1^{--}	$\pi^+\pi^- \psi(2S)$, $\Lambda_c^+ \bar{\Lambda}_c^-$,
	$X(4660)$				$D_s^+ D_{s1}(2536)$
$\chi_{c1}(4685)$[c]		4684^{+15}_{-17}	126^{+40}_{-44}	1^{++}	$\phi J/\psi$
$\chi_{c0}(4700)$	$X(4700)$	4684^{+15}_{-17}	87^{+18}_{-10}	0^{++}	$\phi J/\psi$
$\Upsilon(10753)$		$10752.7^{+5.9}_{-6.0}$	36^{+18}_{-12}	1^{--}	$\pi\pi\Upsilon(1S, 2S, 3S)$
$\Upsilon(10860)$	$\Upsilon(5S)$	$10885.2^{+2.6}_{-1.6}$	37 ± 4	1^{--}	$\pi\pi\Upsilon(1S, 2S, 3S)$,
					$\pi^+\pi^- h_b(1P, 2P)$,
					$\eta\Upsilon(1S, 2S)$, $\pi^+\pi^-\Upsilon(1D)$
					(see PDG listings)
$\Upsilon(11020)$	$\Upsilon(6S)$	11000 ± 4	24^{+8}_{-6}	1^{--}	$\pi\pi\Upsilon(1S, 2S, 3S)$,
					$\pi^+\pi^- h_b(1P, 2P)$,
					(see PDG listings)

- ✓ Neutral meson states above the open-flavor thresholds which are potential candidates for hybrids
- ✓ Table adapted from PDG 2022
- ✓ Y(4500): New state recently seen by BESIII experiment.

M. Ablikim et al, arXiv: 2204.07800.

- ✓ X(4630): New state recently seen by LHCb experiment.
- ✓ $\chi_{c1}(4685)$: New state recently seen by LHCb experiment.

R. Aaji et al, Phys. Rev. Lett. 127, 082001 (2021)

N. Brambilla, W.K. Lai, AM A. Vairo (in progress)

Preliminary Results

- Spin-conserving rate:

H_m $[J^{PC}]$ (Mass) $\longrightarrow Q\bar{Q}$ $[J^{PC}]$	Γ (MeV)
Charmonium hybrid	
H_2 $[1^{++}]$ (4667) $\longrightarrow \eta_c(1S)$ $[0^{-+}]$	$65^{+27}_{-14}{}^{+20}_{-17}$
H_2 $[1^{++}]$ (5035) $\longrightarrow \eta_c(1S)$ $[0^{-+}]$	$31^{+11}_{-6}{}^{+8}_{-7}$
H_2 $[1^{++}]$ (5035) $\longrightarrow \eta_c(2S)$ $[0^{-+}]$	$45^{+20}_{-10}{}^{+16}_{-13}$
H_3 $[0^{++}]$ (5054) $\longrightarrow \eta_c(1S)$ $[0^{-+}]$	$45^{+16}_{-9}{}^{+11}_{-9}$
H_3 $[0^{++}]$ (5473) $\longrightarrow \eta_c(1S)$ $[0^{-+}]$	$18^{+6}_{-3}{}^{+4}_{-3}$
H_3 $[0^{++}]$ (5473) $\longrightarrow \eta_c(2S)$ $[0^{-+}]$	$26^{+10}_{-5}{}^{+7}_{-6}$
Bottomonium hybrid	
H_1 $[1^{--}]$ (10976) $\longrightarrow h_b(1P)$ $[1^{+-}]$	$15^{+8}_{-4}{}^{+7}_{-5}$
H_1 $[1^{--}]$ (11172) $\longrightarrow h_b(2P)$ $[1^{+-}]$	$22^{+14}_{-6}{}^{+13}_{-9}$
H_2 $[1^{++}]$ (10846) $\longrightarrow \eta_b(1S)$ $[0^{-+}]$	$29^{+13}_{-7}{}^{+10}_{-8}$
H_2 $[1^{++}]$ (11060) $\longrightarrow \eta_b(1S)$ $[0^{-+}]$	$28^{+11}_{-6}{}^{+9}_{-7}$
H_2 $[1^{++}]$ (11060) $\longrightarrow \eta_b(2S)$ $[0^{-+}]$	$0.22^{+0.12}_{-0.06}{}^{+0.11}_{-0.08}$
H_2 $[1^{++}]$ (11270) $\longrightarrow \eta_b(1S)$ $[0^{-+}]$	$22^{+8}_{-4}{}^{+6}_{-5}$
H_2 $[1^{++}]$ (11270) $\longrightarrow \eta_b(2S)$ $[0^{-+}]$	$6^{+3}_{-1}{}^{+2}_{-2}$
H_3 $[0^{++}]$ (11065) $\longrightarrow \eta_b(1S)$ $[0^{-+}]$	$69^{+28}_{-15}{}^{+21}_{-17}$
H_3 $[0^{++}]$ (11352) $\longrightarrow \eta_b(1S)$ $[0^{-+}]$	$34^{+12}_{-7}{}^{+9}_{-7}$
H_3 $[0^{++}]$ (11352) $\longrightarrow \eta_b(2S)$ $[0^{-+}]$	$42^{+19}_{-10}{}^{+16}_{-13}$
H_3 $[0^{++}]$ (11616) $\longrightarrow \eta_b(1S)$ $[0^{-+}]$	$19^{+6}_{-4}{}^{+4}_{-4}$
H_3 $[0^{++}]$ (11616) $\longrightarrow \eta_b(2S)$ $[0^{-+}]$	$20^{+8}_{-4}{}^{+6}_{-5}$

Decays not allowed in R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017)

$m_c^{RS} = 1.477(40)$ GeV

$m_b^{RS} = 4.863(55)$ GeV

Error bars from higher order corrections in α_s + Error bar from gluelump mass (± 0.15 GeV) .

Only those spin-conserving decays that satisfy weakly-coupled pNRQCD hierarchy: $\langle r \rangle_{mn}\Delta E \sim \Delta E/mv \ll 1$ and $\alpha_s(\Delta E) \lesssim 0.4$ for perturbative computation.

Spin-conserving decays for charm hybrid states: $H_1(4155)$, $H_1(4507)$, $H_2(4286)$, $H_3(4590)$, $H_4(4367)$ and bottom hybrid state $H_1(10786)$ cannot be computed **reliably** in the pNRQCD framework.

N. Brambilla, W.K. Lai, AM A. Vairo
(in progress)

Preliminary Results

- Spin-flipping rate:

$H_m\,[J^{PC}]$ (Mass) $\longrightarrow Q\bar{Q}\,[J^{PC}]$	Γ (MeV)
Charmonium hybrid decay	
$H_1\,[1^{--}]\,(4155) \longrightarrow J/\psi\,(1S)\,[1^{--}]$	$104^{+55}_{-26}\;^{+49}_{-37}$
$H_1\,[1^{--}]\,(4507) \longrightarrow J/\psi\,(1S)\,[1^{--}]$	$46^{+20}_{-10}\;^{+16}_{-13}$
$H_1\,[1^{--}]\,(4507) \longrightarrow J/\psi\,(2S)\,[1^{--}]$	$29^{+19}_{-8}\;^{+19}_{-13}$
$H_2\,[1^{++}]\,(4286) \longrightarrow \chi_c\,(1P)\;(0,1,2)^{++}$	$55^{+38}_{-16}\;^{+38}_{-26}$
$H_2\,[1^{++}]\,(4667) \longrightarrow \chi_c\,(1P)\;(0,1,2)^{++}$	$15^{+8}_{-4}\;^{+7}_{-5}$
$H_2\,[1^{++}]\,(5035) \longrightarrow \chi_c\,(1P)\;(0,1,2)^{++}$	$6^{+3}_{-1}\;^{+2}_{-2}$
$H_2\,[1^{++}]\,(5035) \longrightarrow \chi_c\,(1P)\;(0,1,2)^{++}$	$11^{+6}_{-3}\;^{+5}_{-4}$
$H_3\,[0^{++}]\,(4590) \longrightarrow \chi_c\,(1P)\;(0,1,2)^{++}$	$137^{+72}_{-34}\;^{+64}_{-49}$
$H_3\,[0^{++}]\,(5054) \longrightarrow \chi_c\,(1P)\;(0,1,2)^{++}$	$5^{+2}_{-1}\;^{+2}_{-1}$
$H_3\,[0^{++}]\,(5054) \longrightarrow \chi_c\,(2P)\;(0,1,2)^{++}$	$133^{+70}_{-33}\;^{+63}_{-48}$
$H_3\,[0^{++}]\,(5473) \longrightarrow \chi_c\,(3P)\;(0,1,2)^{++}$	$140^{+73}_{-35}\;^{+65}_{-50}$
$H_4\,[2^{++}]\,(4367) \longrightarrow \chi_c\,(1P)\;(0,1,2)^{++}$	$65^{+41}_{-18}\;^{+40}_{-28}$
Bottomonium hybrid decay	
$H_1\,[1^{--}\,[\,(10786) \longrightarrow \Upsilon\,(1S)\,[1^{--}]$	$9^{+4}_{-2}\;^{+3}_{-3}$
$H_1\,[1^{--}]\,(10976) \longrightarrow \Upsilon\,(1S)\,[1^{--}]$	$8^{+3}_{-2}\;^{+3}_{-2}$
$H_1\,[1^{--}]\,(10976) \longrightarrow \Upsilon\,(2S)\,[1^{--}]$	$0.3^{+0.2}_{-0.1}\;^{+0.2}_{-0.1}$
$H_1\,[1^{--}]\,(11172) \longrightarrow \Upsilon\,(1S)\,[1^{--}]$	$3^{+1}_{-1}\;^{+1}_{-1}$
$H_1\,[1^{--}]\,(11172) \longrightarrow \Upsilon\,(2S)\,[1^{--}]$	$0.3^{+0.1}_{-0.1}\;^{+0.1}_{-0.1}$
$H_1\,[1^{--}]\,(11172) \longrightarrow \Upsilon\,(3S)\,[1^{--}]$	$0.4^{+0.3}_{-0.1}\;^{+0.2}_{-0.2}$
$H_2\,[1^{++}]\,(10846) \longrightarrow \chi_b\,(1P)\;(0,1,2)^{++}$	$6^{+3}_{-1}\;^{+3}_{-2}$
$H_2\,[1^{++}]\,(11060) \longrightarrow \chi_b\,(1P)\;(0,1,2)^{++}$	$3^{+2}_{-1}\;^{+1}_{-1}$
$H_2\,[1^{++}]\,(11060) \longrightarrow \chi_b\,(2P)\;(0,1,2)^{++}$	$2^{+1}_{-0.5}\;^{+1}_{-1}$
$H_2\,[1^{++}]\,(11270) \longrightarrow \chi_b\,(1P)\;(0,1,2)^{++}$	$2^{+1}_{-0.4}\;^{+1}_{-1}$
$H_2\,[1^{++}]\,(11270) \longrightarrow \chi_b\,(2P)\;(0,1,2)^{++}$	$2^{+1}_{-1}\;^{+1}_{-1}$
$H_3\,[0^{++}]\,(11065) \longrightarrow \chi_b\,(1P)\;(0,1,2)^{++}$	$13^{+6}_{-3}\;^{+6}_{-4}$
$H_3\,[0^{++}]\,(11352) \longrightarrow \chi_b\,(1P)\;(0,1,2)^{++}$	$2^{+1}_{-1}\;^{+1}_{-1}$
$H_3\,[0^{++}]\,(11352) \longrightarrow \chi_b\,(2P)\;(0,1,2)^{++}$	$9^{+5}_{-2}\;^{+4}_{-3}$
$H_3\,[0^{++}]\,(11616) \longrightarrow \chi_b\,(2P)\;(0,1,2)^{++}$	$2^{+1}_{-0.4}\;^{+1}_{-1}$
$H_3\,[0^{++}]\,(11616) \longrightarrow \chi_b\,(3P)\;(0,1,2)^{++}$	$9^{+5}_{-2}\;^{+4}_{-3}$

$$m_c^{RS} = 1.477(40)\ \text{GeV}$$
$$m_b^{RS} = 4.863(55)\ \text{GeV}$$

Error bars from higher order corrections in α_s
+ Error bar from gluelump mass (± 0.15 GeV) .

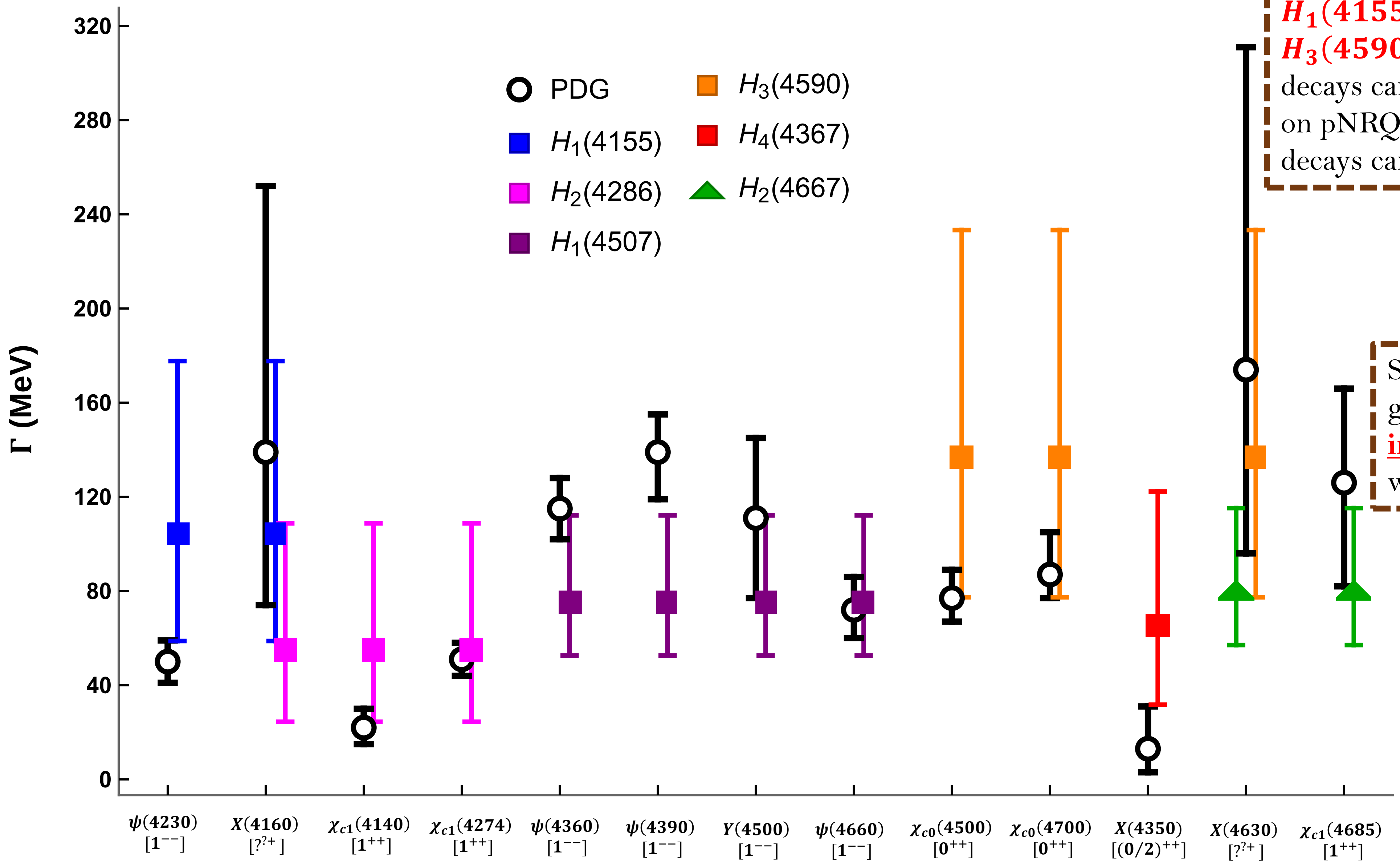
Only those spin-flipping decays that satisfy weakly-coupled pNRQCD hierarchy $\Delta E \gtrsim 0.8$ GeV and $\alpha_s(\Delta E) \lesssim 0.4$ for perturbative computation.

Total decay rate: Sum of all exclusive decays
= spin-conserving + spin-flipping decay rates.

N. Brambilla, W.K. Lai, AM A. Vairo
(in progress)

Preliminary Results

- Comparison with PDG states for charm exotic states:



$H_1(4155)$, $H_1(4507)$, $H_2(4286)$, $H_3(4590)$, $H_4(4367)$: Spin-conserving decays cannot be reliably estimated based on pNRQCD hierarchy. Only spin-flipping decays can be estimated.

Sum of spin-flipping decays gives **lower bound on the inclusive rate of hybrids** which is compared with PDG.

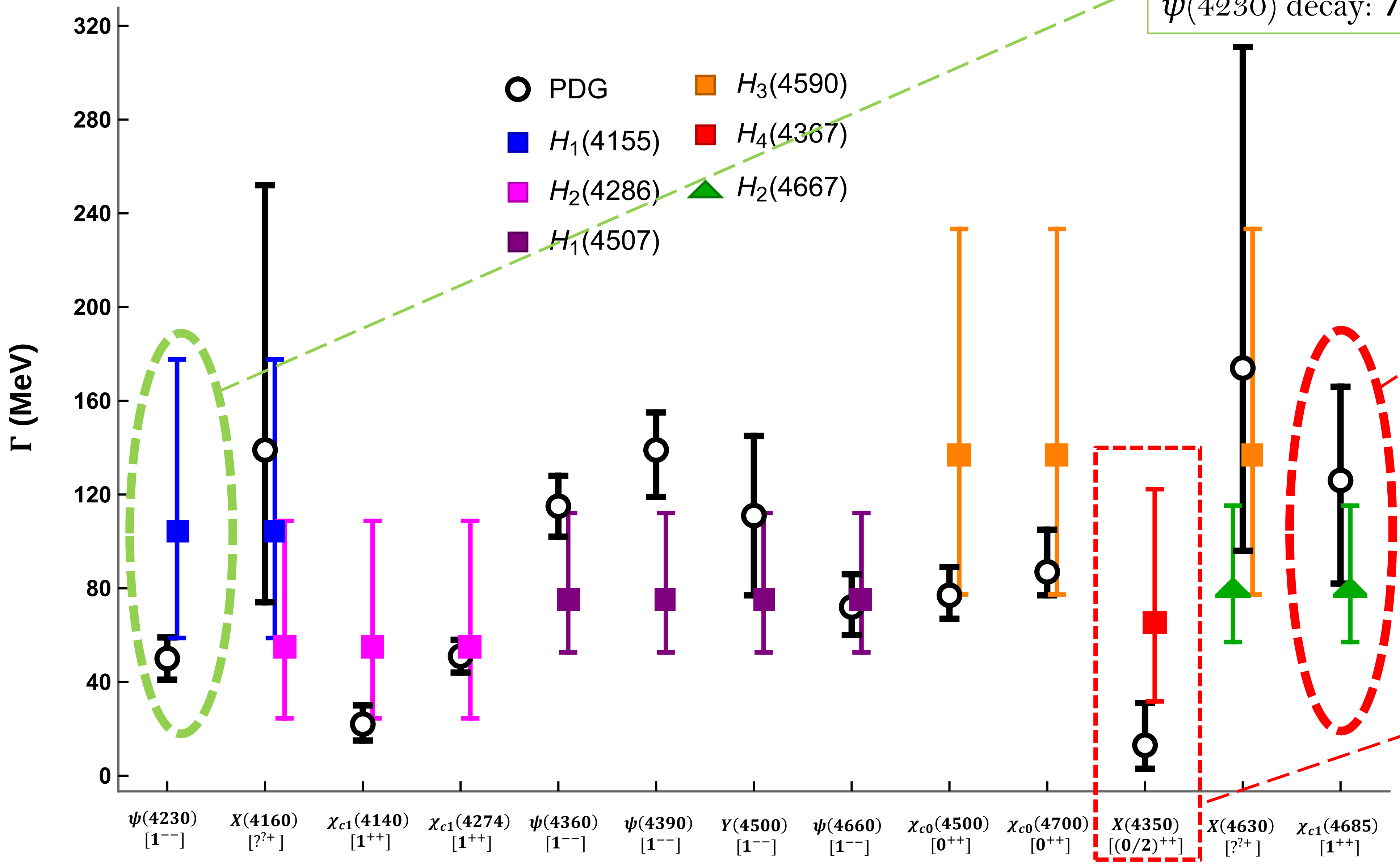
N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Preliminary Results

- Comparison with PDG states for charm exotic states:

Recent **BESIII** result for $\psi(4230)$ decay: **72 ± 33 MeV**

M. Ablikim et al, arXiv: 2204.07800.



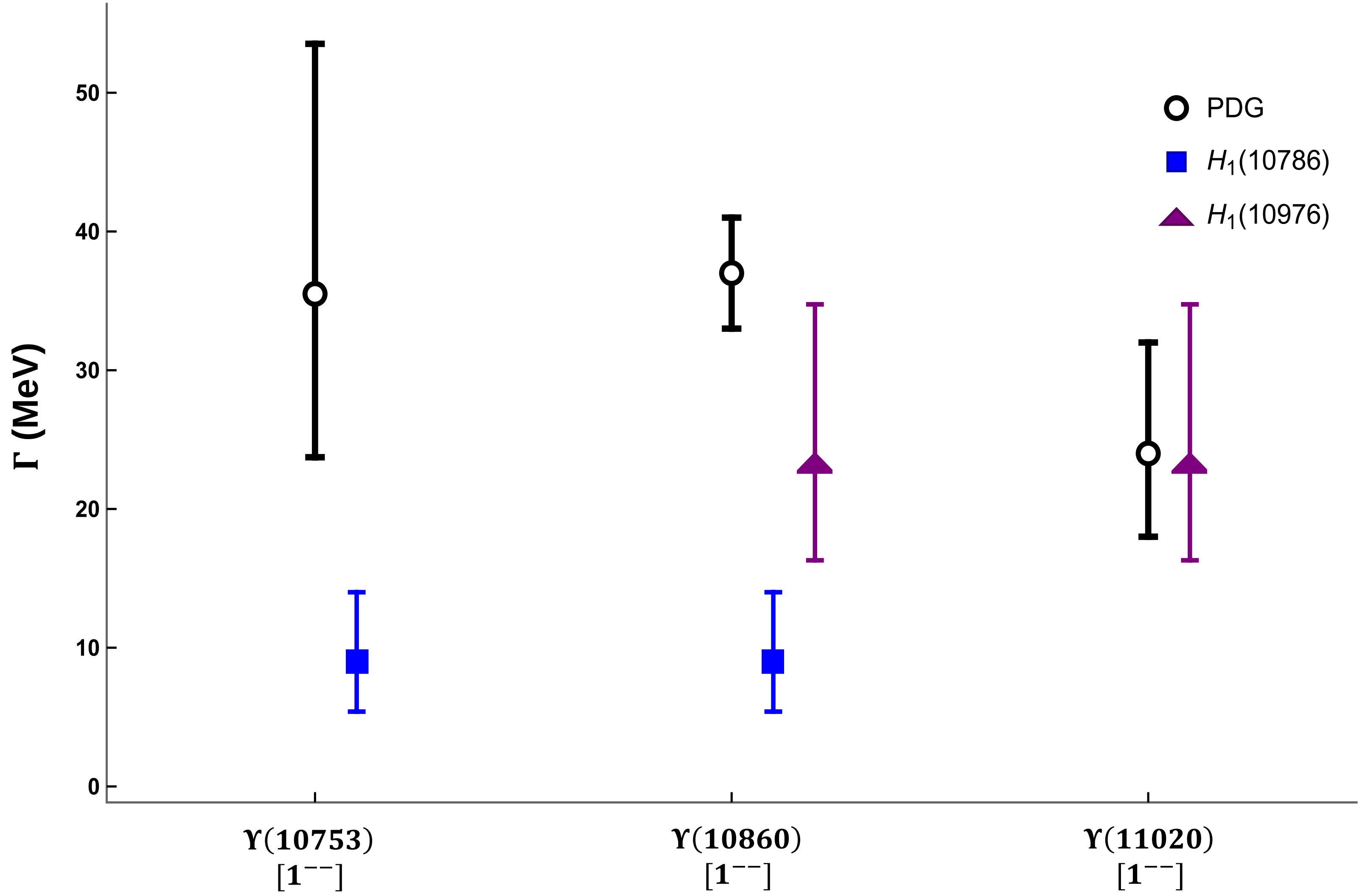
Possibly a hybrid state

Possibly not a hybrid state ?

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Preliminary Results

- Comparison with PDG states:



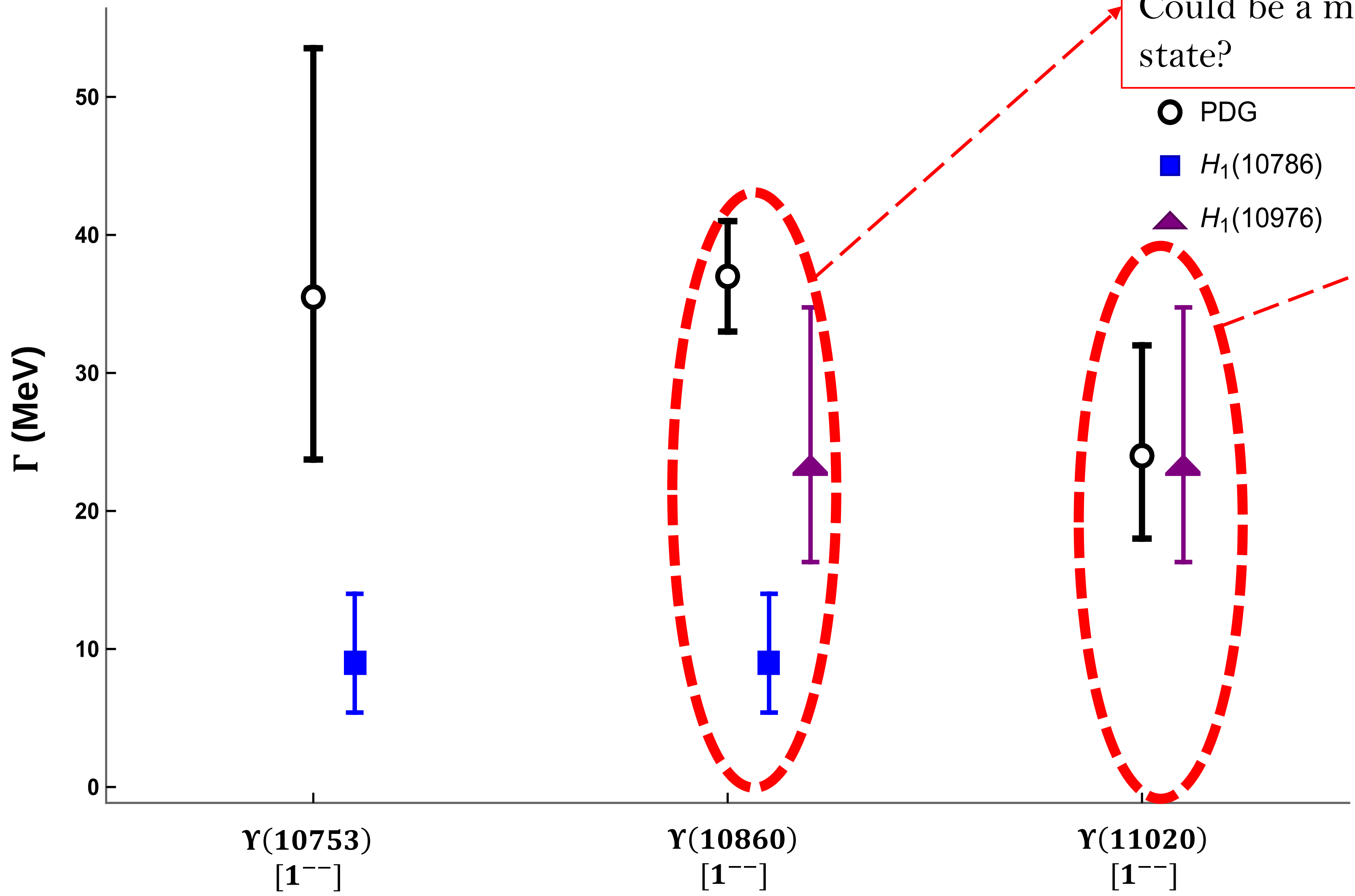
$H_1(10786)$: Spin-conserving decays cannot be reliably estimated based on pNRQCD hierarchy. Only spin-flipping decays can be estimated.

Sum of spin-flipping decays gives lower bound on the inclusive rate of hybrids which is compared with PDG.

N. Brambilla, W.K. Lai, AM A. Vairo
(in progress)

Preliminary Results

- Comparison with PDG states:



Υ(10860): close to B-meson pair threshold. Could be a molecular state ? Or it could be a Υ(5s) state?

Υ(11020): Possibly a hybrid state

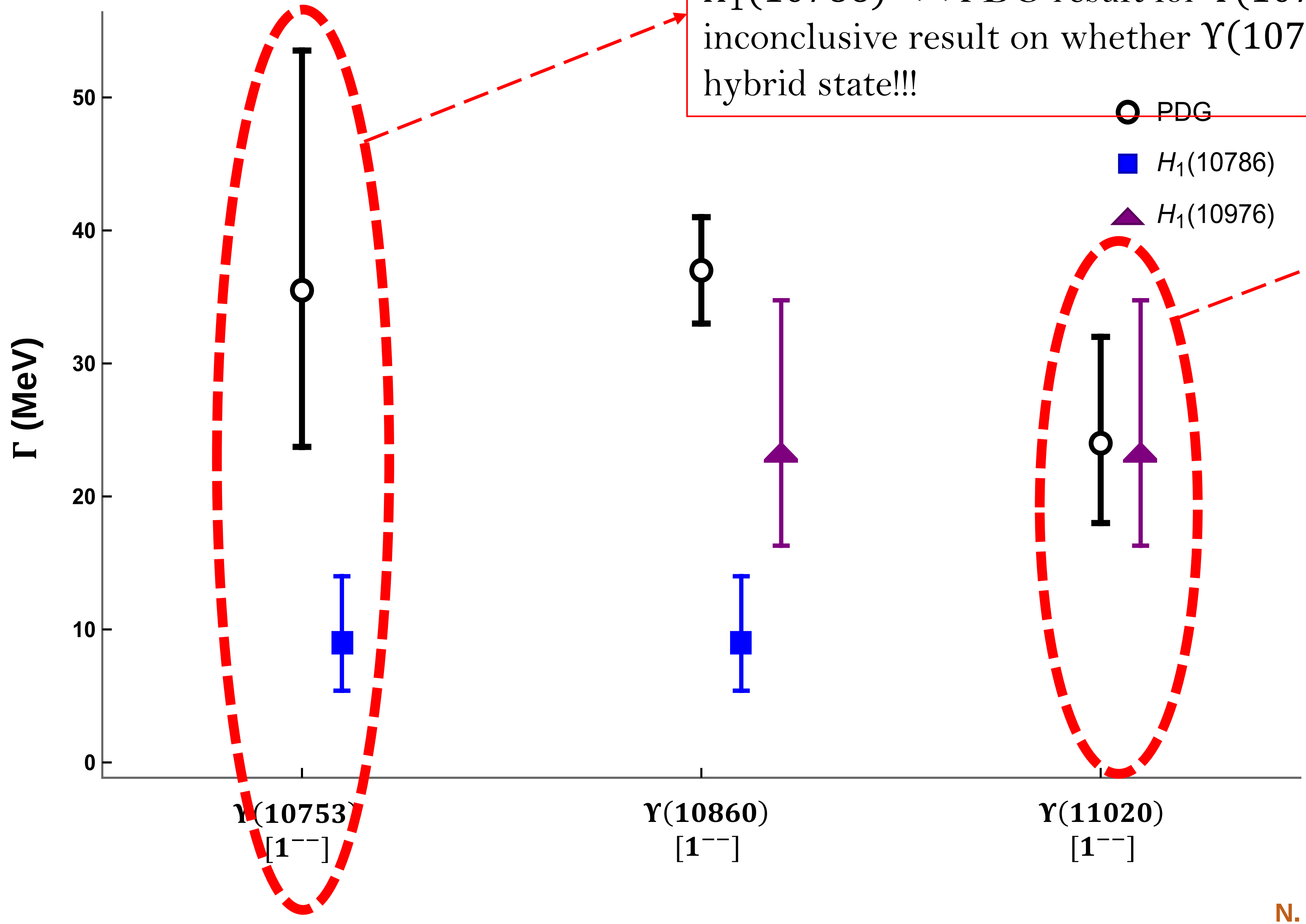
N. Brambilla, W.K. Lai, AM A. Vairo

(in progress)

Preliminary Results

- Comparison with PDG states:

Lower-bound estimate on the total decay rate of $H_1(10786)$ << PDG result for $\Upsilon(10753)$. So inconclusive result on whether $\Upsilon(10753)$ could be a hybrid state!!!



$\Upsilon(11020)$: Possibly a hybrid state

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Summary/Outlook

- BOEFT provides a model-independent & systematic way to study heavy quark hybrids (exotic) and decays.

- Spin-conserving decays for charm hybrid states: $\boldsymbol{H_1(4155), H_1(4507), H_2(4286), H_3(4590), H_4(4367)}$ and bottom hybrid state $\boldsymbol{H_1(10786)}$ cannot be computed **reliably** in the weakly-coupled pNRQCD framework. Sum of spin-flipping decays gives **lower-bound** on the inclusive rate of these hybrid states.

- **Our analysis suggests:**

  - $\boldsymbol{\chi_{c1}(4685)}$: could be the **charm hybrid state** $\boldsymbol{H_2[1^{++}](4667)}$**.**
  - $\boldsymbol{\Upsilon(11020)}$: could be the **bottom hybrid state** $\boldsymbol{H_1[1^{--}](10986)}$**.**
  - $\boldsymbol{X(4350)}$: possibly **not** a hybrid state.
  - $\boldsymbol{\psi(4230)}$: possibly **not** a hybrid state. However, decay rate from recent BESIII measurement consistent with our estimate of lower-bound on inclusive rate.
  - Nothing conclusive can be said about other exotic states.

Thank you!!

Backup Slides

Quarkonium hybrids: Spectrum

- Results for Hybrids from Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

$m_c^{RS} = 1.477(40)$ GeV

$m_b^{RS} = 4.863(55)$ GeV

multiplet	J^{PC}	$c\bar{c}$				$b\bar{c}$				$b\bar{b}$			
		m_H	$\langle 1/r \rangle$	E_{kin}	P_Π	m_H	$\langle 1/r \rangle$	E_{kin}	P_Π	m_H	$\langle 1/r \rangle$	E_{kin}	P_Π
H_1	$\{1^{--}, (0,1,2)^{-+}\}$	4.15	0.42	0.16	0.82	7.48	0.46	0.13	0.83	10.79	0.53	0.09	0.86
H_1'		4.51	0.34	0.34	0.87	7.76	0.38	0.27	0.87	10.98	0.47	0.19	0.87
H_2	$\{1^{++}, (0,1,2)^{+-}\}$	4.28	0.28	0.24	1.00	7.58	0.31	0.19	1.00	10.84	0.37	0.13	1.00
H_2'		4.67	0.25	0.42	1.00	7.89	0.28	0.34	1.00	11.06	0.34	0.23	1.00
H_3	$\{0^{++}, 1^{+-}\}$	4.59	0.32	0.32	0.00	7.85	0.37	0.27	0.00	11.06	0.46	0.19	0.00
H_4	$\{2^{++}, (1,2,3)^{+-}\}$	4.37	0.28	0.27	0.83	7.65	0.31	0.22	0.84	10.90	0.37	0.15	0.87
H_5	$\{2^{--}, (1,2,3)^{-+}\}$	4.48	0.23	0.33	1.00	7.73	0.25	0.27	1.00	10.95	0.30	0.18	1.00
H_6	$\{3^{--}, (2,3,4)^{-+}\}$	4.57	0.22	0.37	0.85	7.82	0.25	0.30	0.87	11.01	0.30	0.20	0.89
H_7	$\{3^{++}, (2,3,4)^{+-}\}$	4.67	0.19	0.43	1.00	7.89	0.22	0.35	1.00	11.05	0.26	0.24	1.00

Other notation of hybrid states

Other notation		l	$J^{PC}\{s=0, s=1\}$	$E_n^{(0)}$
$N(s/d)_1$ →	H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$	Σ_u^-, Π_u
Np_1 →	H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$	Π_u
Np_0 →	H_3	0	$\{0^{++}, 1^{+-}\}$	Σ_u^-
$N(p/f)_2$ →	H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$	Σ_u^-, Π_u
Nd_2 →	H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$	Π_u

Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)

R. Oncala, J. Soto, Phys. Rev. D96 (2017)

BOEFT Hamiltonian

$$H_{\rm BOEFT} = \int d^3x \int d^3R \,{\rm Tr}\left[H^{i\dagger}\left(h_o\delta^{ij} + V^{ij}_{soft}\right)H^j\right]$$

$V^{ij}_{soft} = \Lambda + b^{ij} r^2 + \cdots$
$\Lambda =$ gluelump mass ($= 0.87(15)$ GeV for lowest lying $\kappa = 1^{+-}$ gluelump)

For two insertions of the $\boldsymbol{r}\cdot\boldsymbol{E}$ vertex, the contribution to the two-point function is

$$I^{(2)}_{ij}(\boldsymbol{r},\boldsymbol{R},\boldsymbol{r}',\boldsymbol{R}')$$
$$= -\lim_{T\to\infty} g^2 \frac{T_F}{N_c}\int_{-T/2}^{T/2} dt \int_{-T/2}^{t} dt'\, e^{-ih_o(T/2-t)} r^k e^{-ih_s(t-t')} r^l e^{-ih_o(t'+T/2)}$$
$$\times \langle 0|G^{ib}(T/2)\phi^{ab}(T/2,t)E^{kb}(t)E^{lc}(t')\phi^{cd}(t',-T/2)G^{jd}(-T/2)|0\rangle \mathbb{I}\delta^3(\boldsymbol{r}-\boldsymbol{r}')\delta^3(\boldsymbol{R}-\boldsymbol{R}')$$

To seperate the scales Δ and $\Lambda_{\rm QCD}$, write $\boldsymbol{E} = \boldsymbol{E}_h + \boldsymbol{E}_s$, $\boldsymbol{E}_h \sim \Delta$, $\boldsymbol{E}_s \sim \Lambda_{\rm QCD}$.
Replace $\boldsymbol{E}$ by $\boldsymbol{E}_h$ to get the leading contribution.

$$\langle 0|G^{ib}(T/2)\phi^{ab}(T/2,t)E_h^{kb}(t)E_h^{lc}(t')\phi^{cd}(t',-T/2)G^{jd}(-T/2)|0\rangle$$
$$= \langle 0|G^{ib}(T/2)\phi^{ab}(T/2,t)\phi^{cd}(t',-T/2)G^{jd}(-T/2)|0\rangle\langle 0 E_h^{kb}(t)E_h^{lc}(t')|0\rangle$$
$$= \langle 0|G^{ib}(T/2)\phi^{ab}(T/2,t)\phi^{bd}(t',-T/2)G^{jd}(-T/2)|0\rangle \frac{\delta^{kl}}{3}\int\frac{d^3k}{(2\pi)^3}|\boldsymbol{k}|e^{-i|\boldsymbol{k}|(t-t')}$$
$$\approx \frac{\delta^{kl}}{3}e^{i\Lambda(t-t')}\langle 0|G^{ib}(T/2)\phi^{ab}(T/2,-T/2)G^{jb}(-T/2)|0\rangle\int\frac{d^3k}{(2\pi)^3}|\boldsymbol{k}|e^{-i|\boldsymbol{k}|(t-t')}$$
$$= \delta^{ij}\frac{\delta^{kl}}{3}e^{i\Lambda(t-t'-T)}\int\frac{d^3k}{(2\pi)^3}|\boldsymbol{k}|e^{-i|\boldsymbol{k}|(t-t')}\,.$$

Decay Rate: Calculation Details.

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

- Spin-conserving:

$$\Gamma_{\rm Incl} = {\rm Re}\frac{2g^2}{3}\frac{T_F}{N_c}\int d^3\mathbf{r}\int_0^\infty dt\,\Psi^{i\dagger}_{(m)}(\mathbf{r})\left[e^{i\Lambda t}e^{ih_o t/2}r^k e^{-ih_s t}r^k e^{ih_o t/2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}|\mathbf{k}|e^{-i|\mathbf{k}|t}\right]\Psi^i_{(m)}(\mathbf{r})\,.$$

Using complete set of octet and singlet states

$$\Gamma_{\rm Incl} = \sum_{n'}\Gamma_{m,n'} + \int\frac{d^3\mathbf{p}_s}{(2\pi)^3}\Gamma_{m,p_S}$$

$$\Gamma_{m,q} = \frac{4\alpha_s T_F}{3N_c}\int\frac{d^3\mathbf{l}}{(2\pi)^3}\int\frac{d^3\mathbf{l}'}{(2\pi)^3}\, f^i_{m\,\mathbf{l}}\, g^k_{\mathbf{l}\,q}\, g^{k\dagger}_{\mathbf{l}'\,q}\, f^{i\dagger}_{m\,\mathbf{l}'}\,(\Lambda + E^o_{\mathbf{l}}/2 + E^o_{\mathbf{l}'}/2 - E^s_q)^3$$

$$q = (n', p_s)$$

Inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo
(in progress)

Spin-preserving inclusive decay rate for $H_m \to Q_n + X$

$$\Gamma(H_m \to Q_n + X) = \frac{4\alpha_s T_F}{3N_c} \sum_{n'} |h_{nn'}|^2 \sum_{q,q'} \int dE \int dE' \, f^i_{mq}(E) g^j_{qn}(E) \times g^{j\dagger}_{q'n}(E') f^{i\dagger}_{mq'}(E') (\Lambda + E/2 + E'/2 - E^s_n)^3$$

Assumption:
$f^i_{mq}(E) \neq 0$ only for $E_m \approx E + \Lambda$

$h_{nn'} \approx 1$ and $E^Q_m \approx E^s_m$ (replace singlet with quarkonium)

Spin-preserving inclusive decay rate for $H_m \to Q_n + X$

$$\Gamma(H_m \to Q_n + X) = \frac{4\alpha_s T_F}{3N_c} (E_m - E^Q_n)^3 \, T^{ij} \, (T^{ij})^*$$

$$T^{ij} \equiv \int d^3r \, \Psi^{i\dagger}_m(\boldsymbol{r}) r^j \, \Phi^Q_n(\boldsymbol{r})$$

- Above result looks similar to the one in R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017). In general has **tensor structure T^{ij}** that agrees with J. Castellà, E. Passemar, arXiv:2104.03975.

Inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

- Spin-conserving decay due to $\boldsymbol{r}\cdot\boldsymbol{E}$ term :

$|S_H = 1 > \dashrightarrow |S_Q = 1 >$
$|S_H = 0 > \dashrightarrow |S_Q = 0 >$

$$\Gamma_{\rm Incl} = \sum_{n'} \Gamma_{m,n'} + \int \frac{d^3\mathbf{p}_s}{(2\pi)^3}\,\Gamma_{m,p_s}$$

bound singlet states — continuum singlet states

$$\Gamma_{m,q} = \frac{4\alpha_s T_F}{3N_c}\int \frac{d^3\mathbf{l}}{(2\pi)^3}\int \frac{d^3\mathbf{l}'}{(2\pi)^3}\, f^i_{m\,\mathbf{l}}\; g^k_{\mathbf{l}\,q}\; g^{k\dagger}_{\mathbf{l}'\,q}\; f^{i\dagger}_{m\,\mathbf{l}'}\,(\Lambda + E^o_{\mathbf{l}}/2 + E^o_{\mathbf{l}'}/2 - E^s_q)^3$$

$q = (n', p_s)$

Depends on several Overlap functions:

$$f^i_{(m)\,\mathbf{l}} \equiv \langle H_m|\Phi^o_{\mathbf{l}}\rangle = \int d^3\mathbf{r}\,\Psi^{i\dagger}_{(m)}(\mathbf{r})\,\Phi^o_{\mathbf{l}}(\mathbf{r})\,,$$

$$g^k_{\mathbf{l}\,q} \equiv \langle \Phi^o_{\mathbf{l}}|r^k|\Phi^s_q\rangle = \int d^3\mathbf{r}\,\Phi^{o\dagger}_{\mathbf{l}}(\mathbf{r})\,r^k\,\Phi^s_q(\mathbf{r}),$$

$\Psi^i_{(m)}$: Hybrid wf
$\Phi^o_{\mathbf{l}}$: Octet wf
Φ^s_q : Singlet wf

✓ Cubic factor $(\Lambda + E^o_{\mathbf{l}}/2 + E^o_{\mathbf{l}'}/2 - E^s_q)^3 \sim \Delta E^3$

✓ Including continuum states can account for decay to meson-meson thresholds.

- For singlet wf : $V_s = -\,{}^{4\,\alpha_s(mv)}/_{3}$, where $v \sim 1/\sqrt{3}$ for charm and $\sim 1/\sqrt{10}$ for bottom

Semi-inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

- Spin-conserving decay due to $\boldsymbol{r}\cdot\boldsymbol{E}$ term :

$$\Gamma(H_m \to Q_n) = \sum_{n'} |w_{nn'}|^2 \Gamma_{m,n'} + \int \frac{d^3\mathbf{p}_s}{(2\pi)^3} |w_{np_s}|^2 \Gamma_{m,ps}$$

bound singlet states

continuum singlet states

$$\Gamma_{m,q} = \frac{4\alpha_s T_F}{3N_c} \int \frac{d^3\mathbf{l}}{(2\pi)^3} \int \frac{d^3\mathbf{l}'}{(2\pi)^3} f^i_{m\,\mathbf{l}}\, g^k_{\mathbf{l}\,q}\, g^{k\dagger}_{\mathbf{l}'\,q}\, f^{i\dagger}_{m\,\mathbf{l}'} \left(\Lambda + E^o_{\mathbf{l}}/2 + E^o_{\mathbf{l}'}/2 - E^s_q\right)^3$$

$$q = (n', p_s)$$

Depends on several Overlap functions:

$$f^i_{(m)\,\mathbf{l}} \equiv \langle H_m|\Phi^o_{\mathbf{l}}\rangle = \int d^3\mathbf{r}\, \Psi^{i\dagger}_{(m)}(\mathbf{r})\, \Phi^o_{\mathbf{l}}(\mathbf{r})\,,$$

$$g^k_{\mathbf{l}\,q} \equiv \langle \Phi^o_{\mathbf{l}}|r^k|\Phi^s_q\rangle = \int d^3\mathbf{r}\, \Phi^{o\dagger}_{\mathbf{l}}(\mathbf{r})\, r^k\, \Phi^s_q(\mathbf{r}),$$

$$w_{qn} = \int d^3\mathbf{r}\, \Phi^{s\dagger}_q(\mathbf{r}) \Phi^Q_{(n)}(\mathbf{r})$$

Quarkonium wf

✓ Significant overlap between quarkonium and continuum singlet states except for 1s quarkonium.

Decay rate **different** from R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017) and J. Castellà, E. Passemar, Phys. Rev. D104, 034019 (2021).

Inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

- Spin-flipping decay due to $\boldsymbol{S}.\boldsymbol{B}$ term:

$|S_H = 1 > ---\to |S_Q = 0 >$
$|S_H = 0 > ---\to |S_Q = 1 >$

$$\Gamma_{\rm Incl} = \sum_{n'} \Gamma_{m,n'} + \int \frac{d^3\mathbf{p}_s}{(2\pi)^3}\,\Gamma_{m,p_S}$$

bound singlet states — continuum singlet states

$$\Gamma_{m,q} = \frac{4\alpha_s T_F}{3N_c m_Q^2} \int \frac{d^3\mathbf{l}}{(2\pi)^3} \int \frac{d^3\mathbf{l}'}{(2\pi)^3}\, f^i_{m\,\mathbf{l}}\, g^k_{\mathbf{l}\,q}\, g^{k\dagger}_{\mathbf{l}'\,q}\, f^{i\dagger}_{m\,\mathbf{l}'}\, (\Lambda + E^o_{\mathbf{l}}/2 + E^o_{\mathbf{l}'}/2 - E^s_q)^3$$

$q = (n', p_s)$

Depends on several Overlap functions:

$$f^i_{(m)\,\mathbf{l}} \equiv \langle H_m | \Phi^o_{\mathbf{l}} \rangle = \int d^3\mathbf{r}\, \Psi^{i\dagger}_{(m)}(\mathbf{r})\, \Phi^o_{\mathbf{l}}(\mathbf{r})\,,$$

$$g^k_{\mathbf{l}\,q} \equiv \langle \Phi^o_{\mathbf{l}} | \left(S^k_1 - S^k_2\right) | \Phi^s_{n'} \rangle = \left[\int d^3\mathbf{r}\, \Phi^{o\dagger}_{\mathbf{l}}(\mathbf{r})\, \Phi^s_{n'}(\mathbf{r})\right] \langle \chi_o | \left(S^k_1 - S^k_2\right) | \chi_s \rangle$$

$|\chi_{o,s}\rangle$: Spin wf of octet and singlet

- $Q_m \to Q_n + X$ spin-flipping decays: Decay rate suppressed by additional $(\boldsymbol{r}\cdot\boldsymbol{E})^2 \sim v^2$ vertex factor.

Difficulties with continuum singlet states

W. Gordon, Ann. Phys. (Leipzig) 2, 1031 (1929)

A. Maquet, Phys. Rev. A 15, 1088 (1977)

- Dipole matrix element:

$$g^k_{\mathbf{k}_o\,\mathbf{p}_s} \equiv \langle \Phi^o_{\mathbf{k}_o}|\mathbf{r}|\Phi^s_{\mathbf{p}_s}\rangle = \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\left[\int r^2 A_l^*(k_o,r)\, r\, B_{l'}(p_s,r)\, dr\right]\left[d\Omega\, P_l\left(\hat{\mathbf{k}}_o\cdot\hat{\mathbf{r}}\right)\hat{r}\, P_{l'}\left(\hat{\mathbf{p}}_s\cdot\hat{\mathbf{r}}\right)\right]$$

Continuum radial wf for Coulomb octet and singlet

After integrating over $d\Omega$: $l' = l+1$ or $l' = l-1$

✓ Radial matrix element:

$$R_{l,l+1}(k_o,p_s) = C_{l,l+1}(k_o,p_s)\, \mathcal{J}^{-2ip_s,-2ik_o,1}_{l+2+i\eta_s,l+1+i\eta_o,2l+4,2},$$

$$R_{l,l-1}(k_o,p_s) = C_{l,l-1}(k_o,p_s)\, \mathcal{J}^{-2ik_o,-2ip_s,1}_{l+1+i\eta_o,l+i\eta_s,2l+2,2}$$

$\eta_o = m_Q\alpha_s/12k_o$

$\eta_s = -4m_Q\alpha_s/6p_s$

Smooth function of octet and singlet momentum k_o and p_s

Singular function:
"Diagonal Singularity" for $k_o \to p_s$

Madajczyk, Trippenbach, J . Phys. A: Math. Gen. 22 2369 (1989)

Veniard, Piraux, Phys. Rev. A 41, 4019 (1989)

Difficulties with continuum singlet states

$$\mathcal{J}^{-2ip_s,-2ik_o,1}_{l+2+i\eta_s,l+1+i\eta_o,2l+4,2} = -\frac{(2l+3)!e^{-\frac{\pi m_Q}{4}\left(\frac{\alpha_s}{6k_o}+\frac{4\alpha_s}{3p_s}\right)}e^{-\frac{\pi m_Q}{4}\left(-\frac{\alpha_s}{6k_o}+\frac{4\alpha_s}{3p_s}\right)\mathrm{sgn}(p_s-k_o)}}{4\left(p_s-k_o\right)^{2l+4}p_s^2}\left|\frac{p_s-k_o}{k_o+p_s}\right|^{i\frac{m_Q}{2}\left(\frac{-\alpha_s}{6k_o}+\frac{4\alpha_s}{3p_s}\right)}\Bigg({}_2F_1\left[l+2+i\eta_s,l+1+i\eta_o,2l+2,-\frac{4p_sk_o}{(p_s-k_o)^2}\right]\left(2ip_s+\frac{3m_Q\alpha_s}{2}\right)$$
$$+3m_Q\alpha_s\left(\frac{p_s-k_o}{k_o+p_s}\right){}_2F_1\left[l+1+i\eta_s,l+1+i\eta_o,2l+2,-\frac{4p_sk_o}{(p_s-k_o)^2}\right]+{}_2F_1\left[l+i\eta_s,l+1+i\eta_o,2l+2,-\frac{4p_sk_o}{(p_s-k_o)^2}\right]\left(-2ip_s+\frac{3m_Q\alpha_s}{2}\right)\left(\frac{p_s-k_o}{k_o+p_s}\right)^2\Bigg)$$

$$\mathcal{J}^{-2ik_o,-2ip_s,1}_{l+1+i\eta_o,l+i\eta_s,2l+2,2} = \frac{(2l+1)!e^{\frac{-\pi m_Q}{4}\left(\frac{\alpha_s}{6k_o}+\frac{4\alpha_s}{3p_s}\right)}e^{-\frac{\pi m_Q}{4}\left(-\frac{\alpha_s}{6k_o}+\frac{4\alpha_s}{3p_s}\right)\mathrm{sgn}(p_s-k_o)}}{4\left(p_s-k_o\right)^{2l+2}k_o^2}\left|\frac{p_s-k_o}{k_o+p_s}\right|^{i\frac{m_Q}{2}\left(\frac{-\alpha_s}{6k_o}+\frac{4\alpha_s}{3p_s}\right)}\Bigg({}_2F_1\left[l+1+i\eta_o,l+i\eta_s,2l,-\frac{4p_sk_o}{(p_s-k_o)^2}\right]\left(-2ik_o+\frac{3m_Q\alpha_s}{2}\right)$$
$$-3m_Q\alpha_s\left(\frac{p_s-k_o}{k_o+p_s}\right){}_2F_1\left[l+i\eta_o,l+i\eta_s,2l,-\frac{4p_sk_o}{(p_s-k_o)^2}\right]+{}_2F_1\left[l-1+i\eta_o,l+i\eta_s,2l,-\frac{4p_sk_o}{(p_s-k_o)^2}\right]\left(2ik_0+\frac{3m_Q\alpha_s}{2}\right)\left(\frac{p_s-k_o}{k_o+p_s}\right)^2\Bigg)$$

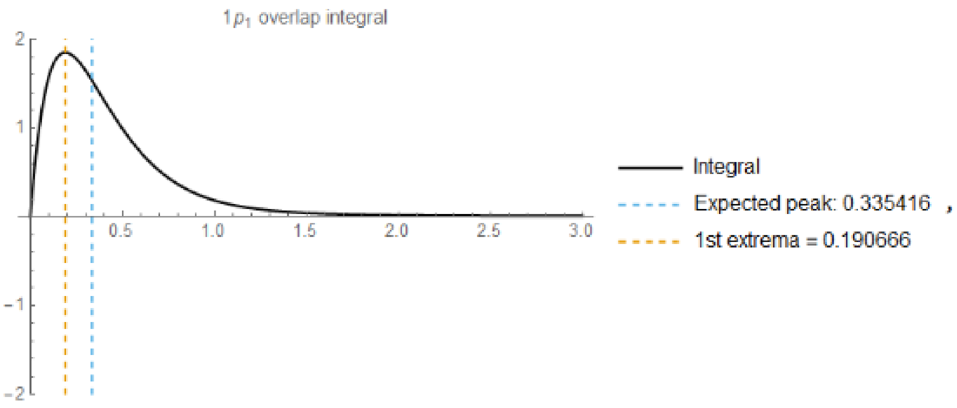
"Diagonal Singularity" for $k_o \to p_s$: Singular Gauss hypergeometric $2F_1$ function

Inclusive Decays

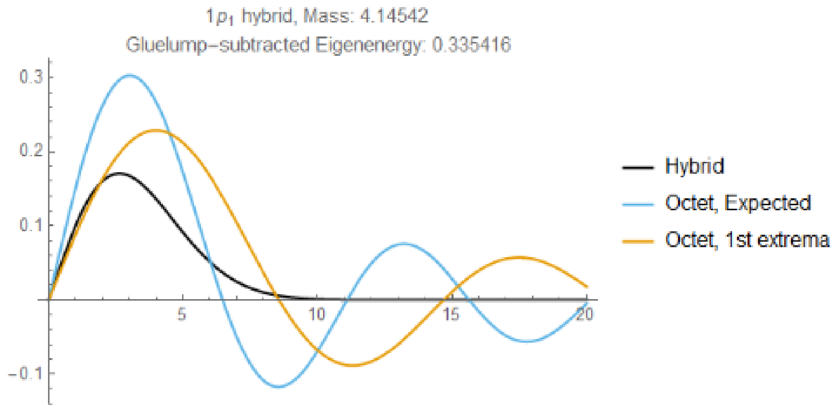
It is interesting to see how $f^i_{mq}(E) = \left[\int d^3r\, \Psi^{i\dagger}_m(\boldsymbol{r})\Phi^o_{E,q}(\boldsymbol{r})\right]$ looks like as a function of E:

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

$$H_2\text{-multiplet, } l=1, J^{PC} = [1^{++}, (0,1,2)^{+-}]$$
$$H_2(4145):$$



Radial integral of $f^i_{mq}(E)$ vs E (GeV)



Radial hybrid wave function vs r (GeV^{-1})

- The actual peak is slightly off (at a lower E) from the expected peak at $E = E_m - \Lambda$.
- The peak is broad, with width ~ 1 GeV. The assumption that $f^i_{mq}(E)$ is nonzero only when $E_m \approx E + \Lambda$ is not true.

Singlet-Quarkonium overlap

2) Overlap of Coulomb singlet bound states & Quarkonium

$$w_{nn'} = \langle \Phi_n^{Q\bar{Q}} \mid \Phi_{n'}^{s} \rangle$$

$\Phi_n^{Q\bar{Q}}$ → Quarkonium wf

$\Phi_{n'}^{s}$ → Coulomb Singlet bound state wf.

3) Overlap of continuum Singlet & Quarkonium

$$w_n = \int \frac{d^3k}{(2\pi)^3} \left| \langle \Phi_n^{Q\bar{Q}} \mid \Phi_k^{s} \rangle \right|^2$$

Φ_k^{s} → continuum singlet states

Charm

n	$\sum_{n'} \lvert w_{nn'} \rvert^2$	w_n
1s	0.73	0.26
2s	0.26	0.72
1p	0.18	0.76
2p	0.20	0.73

Bottom.

n	$\sum_{n'} \lvert w_{nn'} \rvert^2$	w_n
1s	0.90	0.10
2s	0.42	0.55
1p	0.45	0.51
2p	0.37	0.59