Renormalons of static QCD potential

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Static QCD potential

- Necessary to describe quarkonium
- Good quantity for precise α_s determination

2012 Bazavov et al.
2018 Takaura et al.
2020 Ayala et al.

In order to give precise theoretical calculations,
renormalon uncertainties should be understood.

Renormalons

Perturbative calculation: $V_S(r) = -\frac{C_F}{r}\sum_{n\geq 0} a_n \alpha_s^{n+1}$

$a_n \sim n!(b_0/u)^n$ induces ambiguity to the perturbative series and "renormalon uncertainties" appear.

Known facts:

- The renormalon uncertainties in the large-β_0 approximation are $\delta V_S(r)|_{\text{large-}\beta_0} \sim \Lambda_{\overline{\rm MS}},\ r^2\Lambda^3_{\overline{\rm MS}}, \ldots$
  u=1/2 u=3/2
- The exact form of the u=1/2 renormalon is known to be $\delta V_S(r) \propto \Lambda_{\overline{\rm MS}}$.
- The momentum-space potential exhibits good convergence.

Renormalons

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  u=1/2 u=3/2
- The exact form of the u=1/2 renormalon is known to be $\delta V_S(r) \propto \Lambda_{\overline{\rm MS}}$.

  → How about the second (u=3/2) renormalon?
- The momentum-space potential exhibits good convergence.

  → How can we explain this?

Contents

Today, I am going to talk about three issues.

- The u=3/2 renormalon in $V_s(r)$ beyond large-β_0 approx.
- Renormalons in the q-space potential
- Estimate of the size of the u=3/2 renormalon

How to determine u=3/2 renormalon structure

Question: What is the r dependence of the u=3/2 renormalon?

In general, renormalon uncertainties take the form

$$\delta C_1(Q^2) = N \left(\frac{\Lambda_{\overline{\rm MS}}}{Q}\right)^d \alpha_s^{\gamma_0/b_0}(Q^2)[1 + s_1\alpha_s(Q^2) + s_2\alpha_s^2(Q^2) + \ldots]$$

Basic logic to determine the exact form

・Cancellation of renormalon uncertainty of $C_1(Q^2)$ against the uncertainty of the second term in the OPE

$$S(Q^2) = C_1(Q^2) + C_2(Q^2;\mu)\frac{\boxed{\langle 0|\mathcal{O}(\mu)|0\rangle}}{Q^4} + \cdots$$

Nonperturbative

・The above form is determined by understanding the Q-dependence of the second term.

pNRQCD

Brambilla, Pineda, Soto, Vairo

The static QCD pot. can be studied by multipole expansion:

$$V_{\rm QCD}(r) = V_S(r) + \delta E_{\rm US}(r) + \cdots$$

Here

$V_S(r)$: 1/*r* part and genuine perturbative part $V_S(r) = -\frac{C_F}{r}\sum_{n\geq 0} a_n \alpha_s^{n+1}$

$\delta E_{\rm US}(r)$: *r*[2] correction to the potential

$$\delta E_{\rm US}(r) = -i\frac{V_A^2(r)}{6}\int_0^\infty dt\, e^{-it\Delta V(r)}\langle g\vec{r}\cdot\vec{E}^a(t,\vec{0})\varphi_{\rm adj}(t,0)^{ab} g\vec{r}\cdot\vec{E}^b(0,\vec{0})\rangle$$

The u=3/2 renormalon in V_s(r) should be cancelled against a UV originated ambiguity of $\delta E_{\rm US}$.

(confirmed in the large-β_0 approx.)

u=3/2 renormalon

2020 Sumino, HT
2020 Ayala, Lobregat, Pineda

$$\delta E_{\rm US}(r) = -i\frac{V_A^2(r)}{6}\int_0^\infty dt\, e^{-it\Delta V(r)}\langle g\vec{r}\cdot\vec{E}^a(t,\vec{0})\varphi_{\rm adj}(t,0)^{ab}g\vec{r}\cdot\vec{E}^b(0,\vec{0})\rangle$$

The UV contribution $t\sim 0$ cancels the u=3/2 renormalon in $V_s(r)$.

$$\delta E_{\rm US}(r)|_{\rm UV} \simeq -i\frac{V_A^2(r)}{6}\int_{t\sim 0} dt\,\langle g\vec{r}\cdot\vec{E}^a(t,\vec{0})\varphi_{\rm adj}(t,0)^{ab}g\vec{r}\cdot\vec{E}^b(0,\vec{0})\rangle$$

$$\propto r^2\Lambda_{\overline{\rm MS}}^3 V_A^2(r)$$

w/ $V_A^2(r) = 1+\mathcal{O}(\alpha_s^2(1/r))$ (Anomalous dim. $\gamma_0=\gamma_1=0$)

Therefore, the exact form of the u=3/2 renormalon is

$$\delta V_S(r)|_{u=3/2} = N_{3/2}r^2\Lambda_{\overline{\rm MS}}^3(1+\mathcal{O}(\alpha_s^2(1/r)))$$

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Suppression of q-space renormalons

2020 Sumino, HT

$$-4\pi C_F \frac{\alpha_V(q^2)}{q^2} = \int d^3r\, e^{-iq\cdot r} V_S(r)$$

A renormalon uncertainty of $\delta v_s(r) = \delta(r V_S(r)) = (r^2 \Lambda_{\overline{\rm MS}}^2)^u$
gives the q-space potential renormalon as

$$\delta\alpha_V(q^2) = -\frac{q^2}{4\pi C_F}\int d^3r\, e^{-iq\cdot r}\frac{1}{r}(r^2\Lambda_{\overline{\rm MS}}^2)^u = \frac{1}{C_F}\left(\frac{\Lambda_{\overline{\rm MS}}^2}{q^2}\right)^u \Gamma(2u+1)\cos(\pi u)$$

u=1/2 renormalon

$$\delta v_s(r)|_{u=1/2} \propto (r^2\Lambda_{\overline{\rm MS}}^2)^{1/2} \longrightarrow \delta\alpha_V(q^2)|_{u=1/2} = 0$$

because of cos(π/2)=0

u=1/2 renormalon is absent in q-space.

1998 Beneke Diagrammatic analysis

Suppression of q-space renormalons

2020 Sumino, HT

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u=3/2 renormalon

$$\delta v_s(r)|_{u=3/2} \propto (r^2\Lambda_{\overline{\rm MS}}^2)^{3/2}(1 + s_2\alpha_s^2(1/r) + \cdots)$$

$$\alpha_s(1/r) = \alpha_s(\mu) + b_0\alpha_s^2(\mu)\log(r^2\mu^2) + \cdots$$

$$= (r^2\Lambda_{\overline{\rm MS}}^2)^{3/2} \times \text{[Polynomial of log(r}\mu\text{)]}$$

↓

$$\delta\alpha_V(q^2)|_{u=3/2} \propto \left(\frac{\Lambda_{\overline{\rm MS}}^2}{q^2}\right)^{3/2} \alpha_s^3$$

Very much suppressed

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Normalization of renormalon

u=1/2 renormalon is clearly visible in the current perturbative series.

What is the size of the u=3/2 renormalon $N_{3/2}$?

For $rV_S(r) = \sum_{n\geq 0} d_n^v(\mu r)\alpha_s^{n+1}(\mu)$

Borel transform

$$B_v(t) = \sum_{n\geq 0} \frac{d_n^v(\mu r)}{n!} t^n \simeq (\mu^2 r^2)^{3/2} \frac{N_{3/2}}{(1-2b_0 t/3)^{1+\nu}} \sum_{k\geq 0} c_k(\mu r)\left(1-\frac{2b_0 t}{3}\right)^k + \cdots$$

Method A Lee

$$N_{3/2} = T_{t=0}[(1-2b_0 t/3)^{1+\nu} B_v(t)/(\mu^2 r^2)^{3/2}]|_{t=3/(2b_0)}$$

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For $rV_S(r) = \sum_{n\geq 0} d_n^v(\mu r)\alpha_s^{n+1}(\mu)$

Asymptotic form

$$d_n^{v\,(\mathrm{asym})} = N_{3/2}(\mu^2 r^2)^{3/2}\frac{\Gamma(n+1+\nu)}{\Gamma(1+\nu)}\left(\frac{2b_0}{3}\right)^n \sum_{k\geq 0} c_k(\mu r)\frac{\nu(\nu-1)\cdots(\nu-k+1)}{(n+\nu)(n+\nu-1)\cdots(n+\nu-k+1)}$$

Method B   Ayala, Cvetic, Pineda

$$N_{3/2} = \lim_{n\to\infty}\frac{d_n}{d_n^{v\,(\mathrm{asym})}/N_{3/2}}$$

Test of the methods

I use model series

$$V_S(r)\Big|_{\mathrm{N^kLL}} = -4\pi C_F \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r} \frac{\alpha_V(q^2)\big|_{\mathrm{N^kLL}}}{q^2}$$

e.g. $\alpha_V(q^2)|_{\mathrm{LL}} = \alpha_s(q^2) = \alpha_s(\mu^2) \sum_{n\geq 0} [b_0 \alpha_s(\mu^2) \log(\mu^2/q^2)]^n$

We can calculate $N_{3/2}$ exactly for these model series w/o using the above methods.

In the following we consider QCD force $dV_S(r)/dr$ to eliminate the u=1/2 renormalon and to make the u=3/2 renormalon the leading renormalon.

When does the u=3/2 renormalon dominate?

pert. coeff. for force

If the u=3/2 renormalon dominates perturbative coefficients d_n^f ,

$$d \log (d_n^f)/dL \simeq 3/2 \qquad (L \equiv \log(\mu^2 r^2))$$

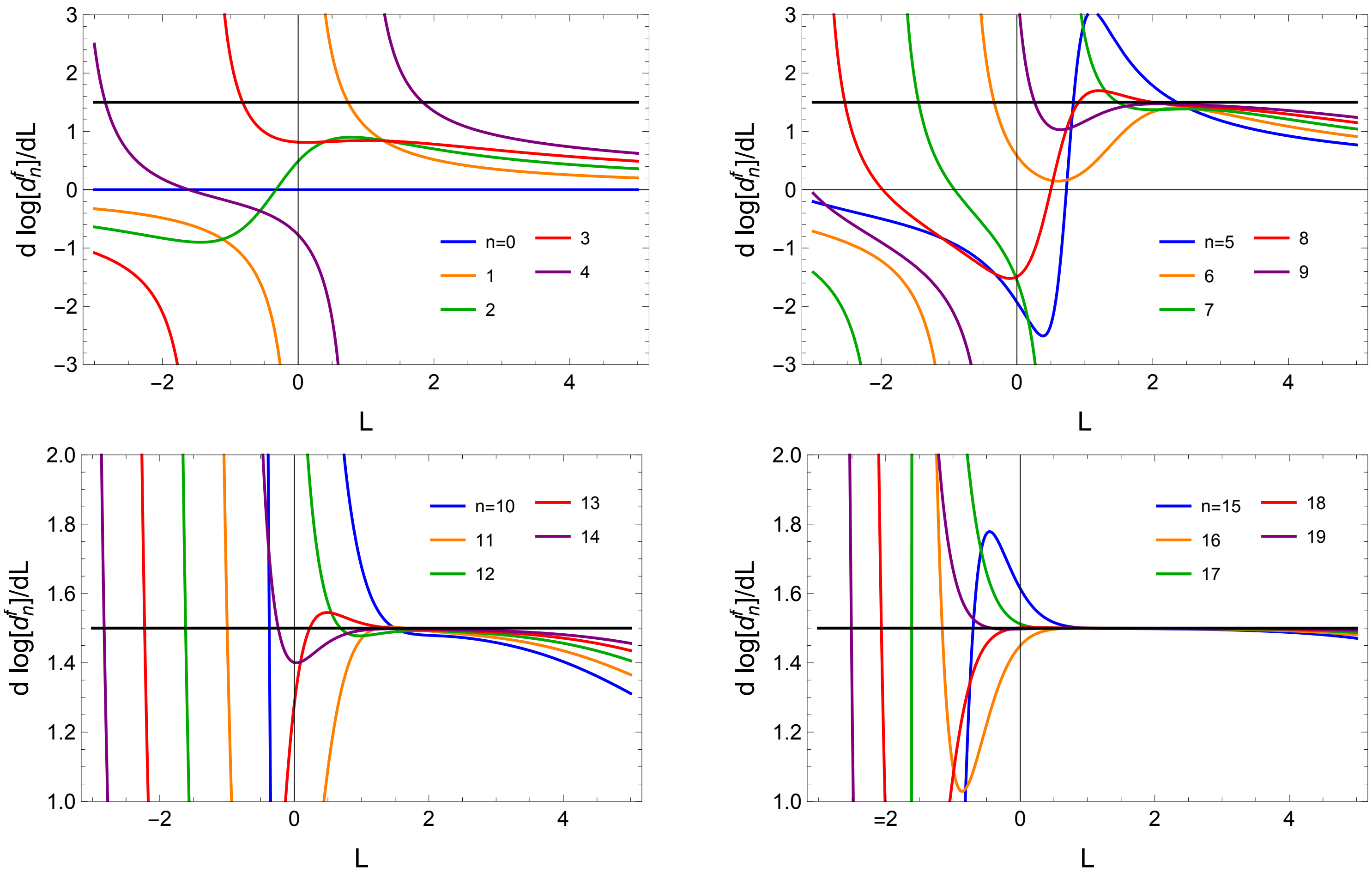
because of

$$d_n^{f\,(\mathrm{asym})} = 2N_{3/2}(\mu^2 r^2)^{3/2} \frac{\Gamma(n+1+\nu)}{\Gamma(1+\nu)} \left(\frac{2b_0}{3}\right)^n \sum_{k\geq 0} c_k(\mu r) \frac{\nu(\nu-1)\cdots(\nu-k+1)}{(n+\nu)(n+\nu-1)\cdots(n+\nu-k+1)}$$

When the u=3/2 renormalon dominates?

2021 HT

Result for N^{3}LL model series



Efficiency test of Method A and B

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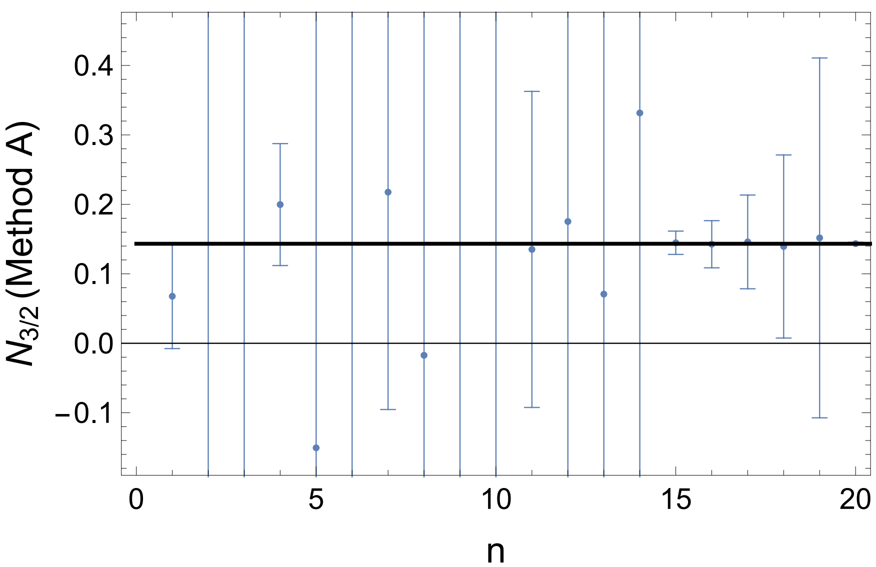
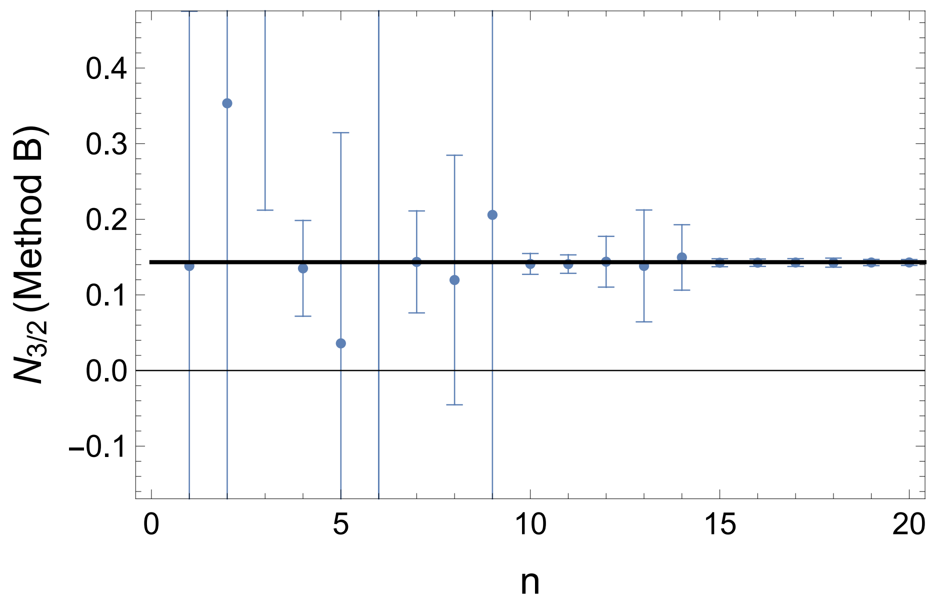
Minimal sensitivity scale used



Minimal sensitivity scale used



Method B is superior

This agrees with a conclusion of 2020 Ayala, Lobregat, Pineda

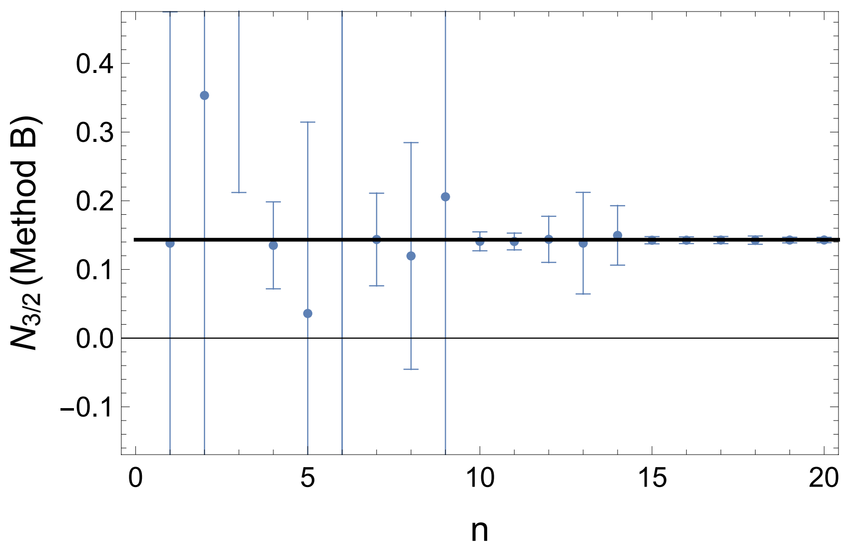
But the error doesn't show simple convergent behavior at $n \lesssim 13$

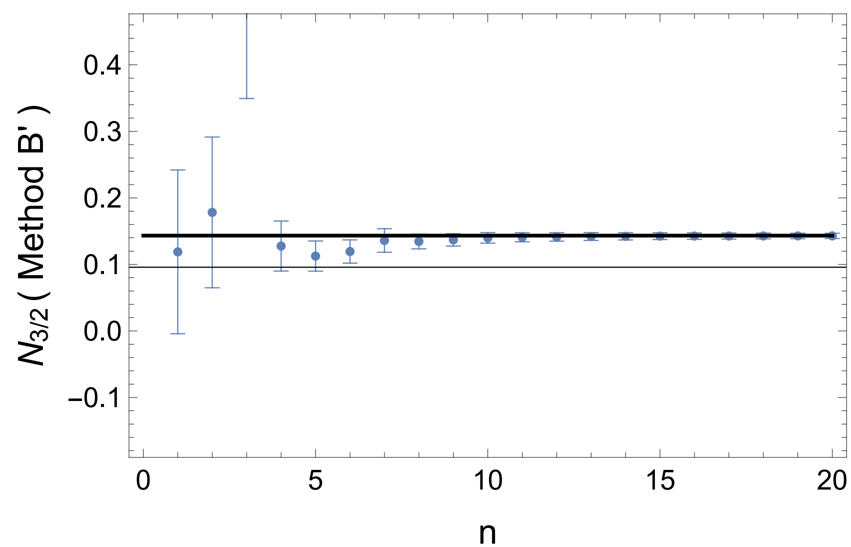
Improvement for Method B

2021 HT

Instead of minimal sensitivity scale, let's use the scale $d\log(d_n^f)/dL \simeq 3/2$

Minimal sensitivity scale used





Becomes much better

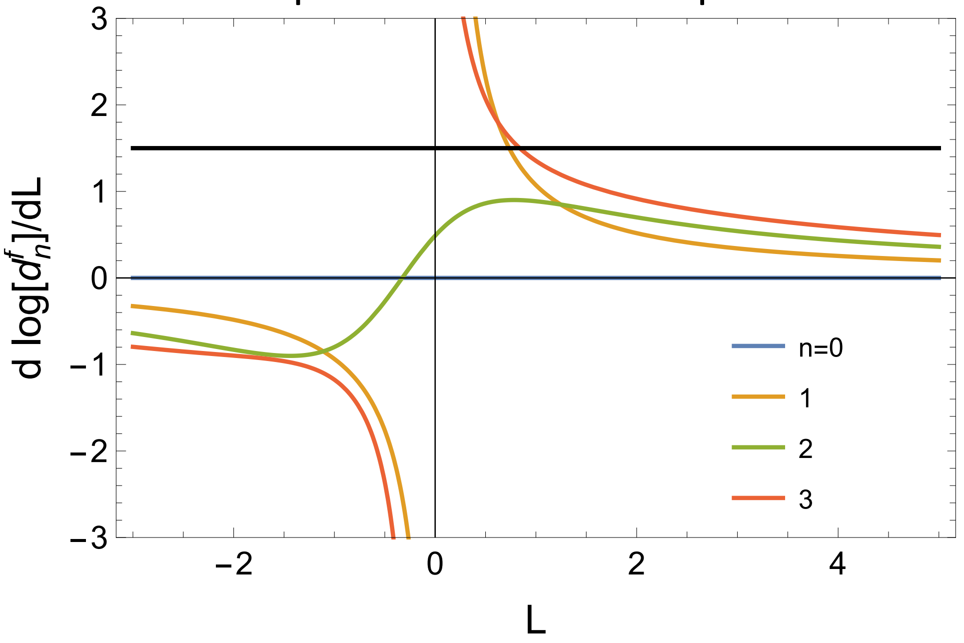
In Method B' larger renormalization scale is chosen, where renormalon behavior seems to strongly appear.

$$d_n^{f\,(\mathrm{asym})} = 2N_{3/2}(\mu^2 r^2)^{3/2}\frac{\Gamma(n+1+\nu)}{\Gamma(1+\nu)}\left(\frac{2b_0}{3}\right)^n \sum_{k\geq 0} c_k(\mu r)\frac{\nu(\nu-1)\cdots(\nu-k+1)}{(n+\nu)(n+\nu-1)\cdots(n+\nu-k+1)}$$

Estimate of $N_{3/2}$ from current perturbative series

2021 HT

Current perturbative series up to NNNLO



Using Method B', we obtain

$$N^f_{3/2} = 0.35(11)$$

2020 Ayala, Lobregat, Pineda $N^f_{3/2} = 0.37(17)$ using Method B

Summary

- The u=3/2 renormalon uncertainty is

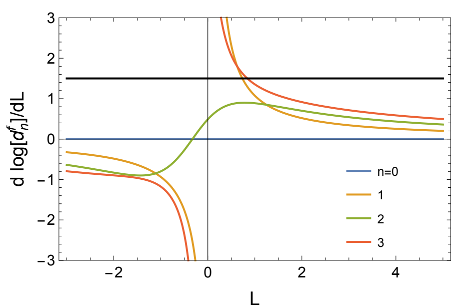
$$\delta V_S(r)|_{u=3/2} = N_{3/2} r^2 \Lambda_{\overline{\rm MS}}^3 (1 + \mathcal{O}(\alpha_s^2(1/r)))$$

  and it turned out to be close to $\sim r^2 \Lambda_{\overline{\rm MS}}^3$

- A simple formula concludes that, in momentum space the u=1/2 renormalon is absent and the u=3/2 renormalon is suppressed by α_s^3.

- I suggested an improved method to estimate renormalon normalization and gave an estimate $N^f_{3/2} = 0.35(11)$.

Can the u=3/2 renormalon be seen more clearly in the next order?



Back up

Renormalons originally encoded in $\alpha_V(q^2)$

Suppose that momentum-space potential $\alpha_V(q^2) = \sum_{n\geq 0} a_n \alpha_s^{n+1}$ has renormalon divergence

$$a_n \sim \left(\frac{\mu^2}{q^2}\right)^{u_0} \left(\frac{b_0}{u_0}\right)^n \Gamma(n+1+\nu)$$

To the coordinate-space potential $V_S(r) = \sum_{n\geq 0} d_n \alpha_s^{n+1}$, this behavior gives

$$d_n = -4\pi C_F \int \frac{d^3q}{(2\pi)^3} a_n \frac{1}{q^2} e^{-iq\cdot r} \sim \frac{1}{r}(\mu^2 r^2)^{u_0} \frac{\Gamma(\frac{1}{2}-u_0)}{\Gamma(1+u_0)} \left(\frac{b_0}{u_0}\right)^n \Gamma(n+1+\nu)$$

non-zero for $u_0>0$

This argument suggests

> If the $u=u_0$ renormalon exists in momentum space,
> the $u=u_0$ renormalon exists in coordinate space.

namely

> If the $u=u_0$ renormalon does not exist in coordinate space,
> the $u=u_0$ renormalon does not exist in momentum space.

Error estimate of normalization constant

Systematic errors

(i) Scale variation around minimal sensitivity scale by factor $\mathbf{1/\sqrt{2}}$ and $\mathbf{\sqrt{2}}$

(ii) Difference from previous order result

(iii) Impact of 1/n correction