

Static Energy in $(2 + 1 + 1)$ -Flavor Lattice QCD: Scale Setting and Charm Effects

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Based on [N. Brambilla, et al., [arXiv/2206.03156\[hep-lat\]](https://arxiv.org/abs/2206.03156)]

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Motivation

- Static energy $E_0(r)$ studied on the lattice since earliest times (**confinement**) ¹, defined as ground state of Wilson loop.
- High accuracy, numerically cheap observable on the lattice.
- Of major importance for **scale setting** in the **past** – **still now?**
- Wilson loops and related correlators contribute in most **effective field theories** for heavy quarks and quarkonia.
- **No scheme change** required between lattice and continuum.
- To date, extraordinarily **detailed lattice studies** in pure gauge theory ($N_f = 0$) or (2 + 1)-flavor QCD ²) $\Lambda_{\overline{MS}}^{N_f=0,3}$.
- So far, dearth of results in $N_f = (2 + 1 + 1)$ -flavor lattice QCD.

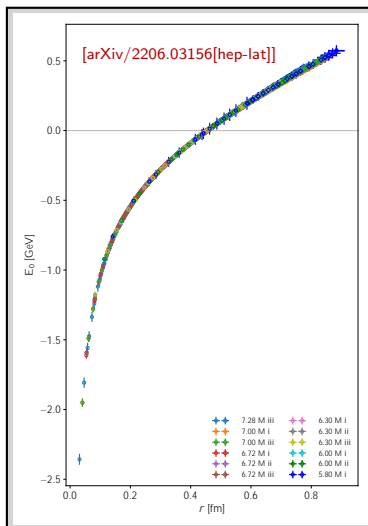
¹[K. Wilson, PRD 10 (1974) 2445]

²[A. Bazavov, et al., PRD 100 (2019) 11, 114511]



Summary

- We calculate the static energy in $(2 + 1 + 1)$ -**flavor QCD** covering $r \in [0.03, 0.9]$ fm.
- We **self-consistently convert** the distance $r=a$ and the static energy aE_0 to **physical units** using scales obtained from E_0 .
- We resolve dependence of the scales and the effective string tension on **sea quark masses**.
- We study in detail **effects of the charm sea** for $r \lesssim 1/m_c$.



Lattice setup

- (2 + 1 + 1)-flavor QCD **ensembles with HISQ**³ and 1-loop Symanzik gauge action generated by MILC collaboration⁴.
- **Three light quark masses** ($m_l = m_s$ 1=27; 1=10; 1=5), physical strange and charm sea.
- **Six lattice spacings** via f_{p4s} scale ($a_{f_{p4s}}$ 0:03 0:15 fm).
- We compute the **Wilson line correlator in Coulomb gauge**, with or without one step of HYP smearing⁵.

$$W(r; ; a) = \prod_{u=0}^{=a-1} U_4(r; ua; a);$$

$$C(r; ; a) = \left\langle \frac{1}{N^3} \sum_x \sum_{y=R(r)} \frac{1}{N_c N_r} \text{tr} \left[W^y(x+y; ; a) W(x; ; a) \right] \right\rangle;$$

³[E. Follana, et al., PRD 75 (2007) 054502]

⁴[A. Bazavov, et al., PRD 98 (2018) 7, 074512]

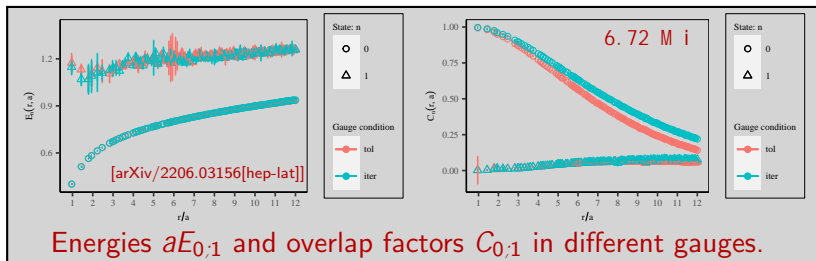
⁵[A. Hasenfratz, F. Knechtli, PRD 64 (2001) 034504]



Spectrum of Wilson line correlators

- **Improved action**) spectral representation holds at $a = a_2$
- **Physical T_{\max} and r_{\max}** ' 0:6 0:9 fm implies much fewer time slices on coarse lattices (factor five variation in a).
- On coarse lattices **limitation in number of states, N_{st}** .

$$C(r; ; a) = e^{-E_0(r;a)} \left(C_0(r;a) + \sum_{n=1}^{N_{\text{st}}} C_n(r;a) \prod_{m=1}^n e^{-\Delta_m(r;a)} \right) + \dots;$$



Fits

- Up to $N_r = 500$ $|jrj=a$ values after averaging, cover factor 24.
- Up to $T_{\max}=a$ $N_r = 10^4$ correlated data) **automation!**
- **Fit range** [i.e. $\min; N_{\text{st}}(r=a)$] **varies with N_{st} and jrj**

$jrj + 0:2$ fm	$\min;1$	0:3 fm	for $N_{\text{st}} = 1$)	prior values;
$\frac{2}{3}jrj + 0:1$ fm	$\min;2$	$\min;1$	$2a$ for $N_{\text{st}} = 2$)	final choice;
$\frac{1}{3}jrj$	$\min;3$	$\min;2$	$2a$ for $N_{\text{st}} = 3$)	cross-check

- We use Bayesian fits with **loose, linear priors** (10% width).
- Priors for excited state overlap factors with 100% width: no unacceptable examination of data, robust for prior variation.
- Priors for $\Delta_1 = E_1 - E_0$ are from SU(3) pure gauge theory ⁶.

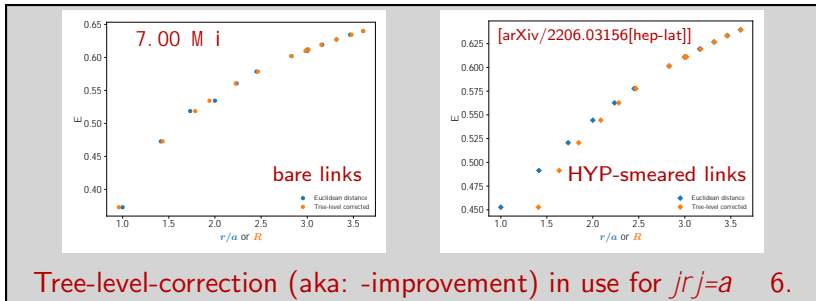
⁶[K. Juge, et al., PRL90, 161601 (2003)]



Discretization artifacts

- $E_0(r; a)$ is available only at discrete distances, and depends on the direction of r) not smooth in $|r| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.
- Distance r_l defined via tree-level ⁷ gluon propagator $D(k)$

$$E_0^{\text{tree}}(r; a) = C_F g_0^2 \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot r} D_{44}(k) = \frac{C_F g_0^2}{4} \frac{1}{r_l}$$



⁷Extension to one loop ongoing: [G. v. Hippel, et al., in preparation: TUM-EFT 171/22]



Determination of scales

- The scales are defined in terms of **the force** $F(r)$ $\frac{dE_0(r)}{dr}$

$$r_i^2 F(r_i) = c_i; \quad i = 0; 1; 2$$

with $c_0 = 1.65$ ⁸, $c_1 = 1$ ⁹, and $c_2 = 1=2$ ¹⁰.

- The r_i correspond to **different regimes**. (2 + 1)-flavor QCD:

$$r_0 = 0.475 \text{ fm}; \quad r_1 = 0.3106 \text{ fm}; \quad r_2 = 0.145 \text{ fm};$$

- A **Cornell Ansatz** encodes the main features of $E_0(r)$ locally

$$E(R; a) = \frac{A}{R} + B + \Sigma R;$$

- $r_i = a = \sqrt{(c_i - A) / \Sigma}$; identify Σ at large r with string tension
- **Consistent definition** of $F(r; a)$ across various a is difficult

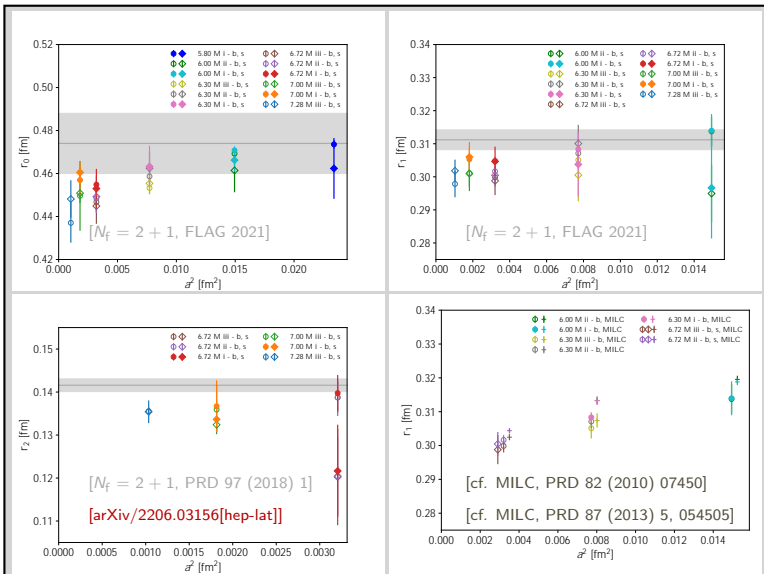
⁸[R. Sommer, NPB 411 (1994) 839]

⁹[C. Bernard, et al., PRD 62 (2000) 034503]

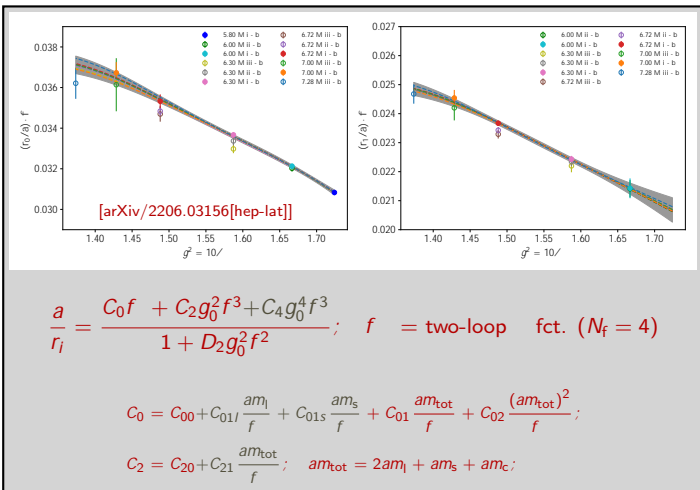
¹⁰[A. Bazavov, et al., PRD 97 (2018) 1, 014510]



Lattice spacing, N_f , and quark mass dependence



Smoothing (via Allton-Ansatz)



Continuum extrapolation

- Using smoothed results we extrapolate to the continuum limit:
 $r_0=r_1, r_1=r_2; a_{f_{p4s}} r_0=a, a_{f_{p4s}} r_1=a; \overline{a^2} r_0=a$
- Lattice spacing dependence via $x = (a=r_i)^2; i = 0;1$
- m_l dependence via $y = m_l=m_s$ (sea or tuned ⁴ m_s),
 where we use PDG values ¹¹ in the continuum GMOR

$$m_l=m_s = 1-(2M_K^2=M^2 \quad 1) \quad \left\{ \begin{array}{l} m_l=m_s/M^2=M^2 = 0.04128; \\ m_l=m_s/M^2=M^2_0 = 0.03851; \end{array} \right.$$

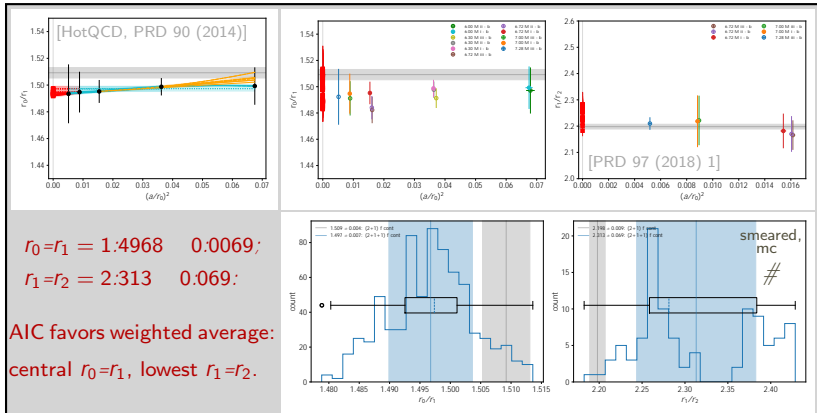
- Logs in the Symanzik expansion via $b = 1$ or $b = g_0^2=(4 \quad u_0^2)$
 $= 0 + \quad [\quad x + \quad 2xy^{1;2}] + \quad 3x^2 + \quad 4y \quad (l,q;l,qm;mc);$

⁴[A. Bazavov, et al., PRD 98 (2018) 7, 074512]

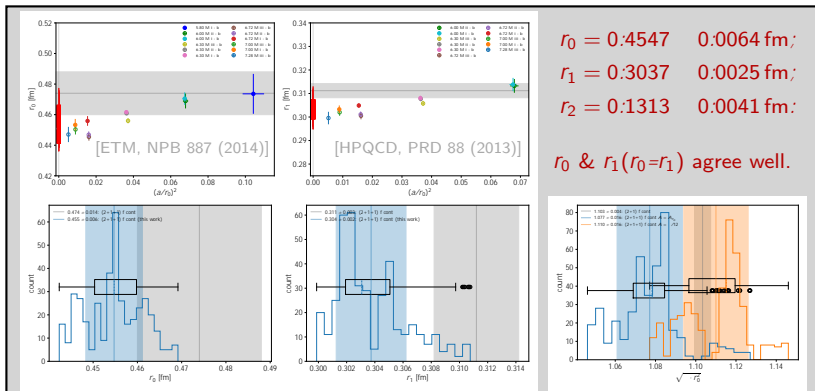
¹¹We use: $2M_K^2 = M_K^2 + M_{K^0}^2$ and $M^2 = M^2$ or M^2_0



The ratios $r_0=r_1$ and $r_1=r_2$



The scales r_0 and r_1 and the string tension

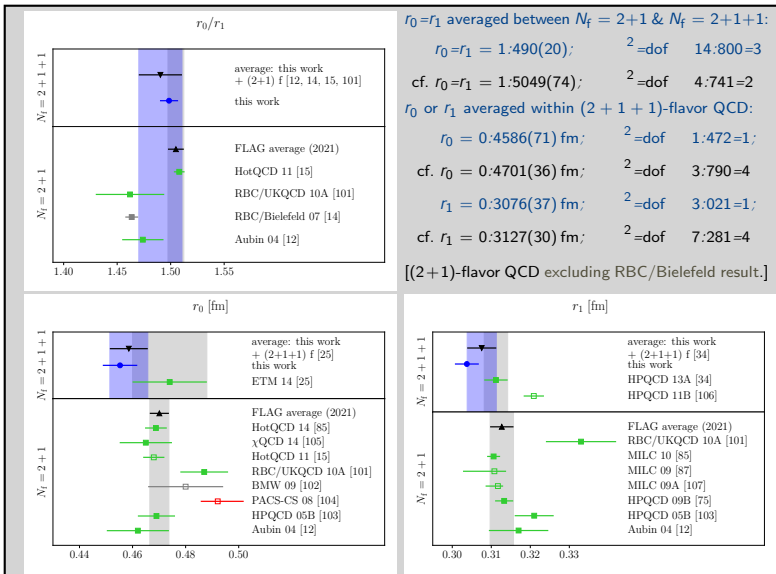


The string tension is consistent with $(2 + 1)$ -flavor QCD ¹²:

$$\sqrt{r_0^2} = 1.077 \quad 0.016 \quad (A = A_{r_0}); \quad \sqrt{r_0^2} = 1.110 \quad 0.016 \quad (A = =12):$$

¹²[RBC/Bielefeld, PRD 77 (2008) 014511]

Comparison to published results



Static energy in perturbation theory with massless sea

- The static energy is defined as

$$E_0(r) = \lim_{t \rightarrow \infty} \frac{i}{t} \ln \left\langle \text{tr} P \exp \left[ig \oint_{r,t} dz A(z) \right] \right\rangle;$$

and—for massless sea quarks—perturbatively known at N³LL

$$E_0(r) = \Lambda \frac{C_F}{r} \left(1 + \# \frac{1}{s} + \# \frac{2}{s} + \# \frac{2}{s} \ln s + \# \frac{3}{s} \ln s + \# \frac{3}{s} \ln^2 s + \dots \right);$$

- Integrating the force eliminates the leading renormalon

$$E_0(r) = \int_r^r dr^0 F(r^0) + \text{const.}$$

- $E_0(r)$ depends on N_f via its coefficients $\#$ and via s
- s depends on a scale $\mu = 1/r$ (we use $\mu = 1/r$)



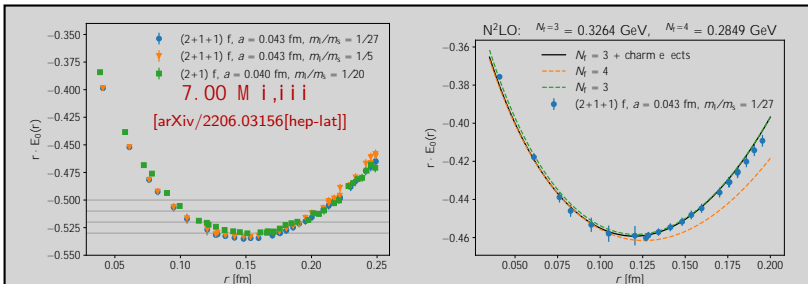
Massive charm sea

- Massive charm quark decouples for large r :

$$\begin{array}{ll}
 rm_c \gg 1 & \text{charm is effectively infinitely heavy} & N_f = 3 \\
 rm_c \ll 1 & \text{charm is effectively massless} & N_f = 4
 \end{array}$$

- For $rm_c \ll 1$ massive correction $V_m^{(N_f)}(r)$ known at N²LO

$$E_{0;m}^{(N_f)}(r) = \int_r^r dr^0 F^{(N_f)}(r^0) + V_m^{(N_f)}(r) + \text{const.}$$



Conclusions

- We have computed the static energy $E_0(r)$ in $(2+1+1)$ -flavor QCD over a wide range of lattice spacings and quark masses.
- First simultaneous determination of both the scales r_0 , r_1 and their ratio $r_0=r_1$, and the ratio $r_1=r_2$ and the string tension .
- $r_1=a$ on coarse lattices is slightly lower than earlier results ¹³; procedural differences in defining $r_1=a$ lead to larger errors.
- $r_0=r_1$ and $r_0^{\rho_-}$ are consistent with $(2+1)$ -flavor QCD results.
- $r_1=r_2$ differs significantly from $(2+1)$ -flavor QCD results.
- We study the charm quark contribution across its decoupling.
- $E_0(r)$ at $r \in [0, 2]$ fm is inconsistent with $(2+1)$ -flavor QCD, but well reproduced by perturbation theory at two-loop order.

¹³[MILC, PRD 82 (2010) 07450]; [MILC, PRD 87 (2013) 5, 054505]



Data sets

MILC (2 + 1 + 1)-flavor gauge ensembles¹ used in this study.

our naming	N^3	N		a_{pts} (fm)	u_0	am_l	am_s	am_c	$m_l=m_s$	$(am_s)_{\text{tuned}}$	M (MeV)	#conf.
5.80 M i	32^3	48	5.80	0.15294	0.85535	0.00235	0.0647	0.831	phys	0.06852	131	1041
6.00 M ii	32^3	64				0.00507			1=10		217	1000
6.00 M i	48^3	64	6.00	0.12224	0.86372	0.00184	0.0507	0.628	phys	0.05296	132	709
6.30 M iii	32^3	96				0.0074	0.037	0.44	1=5		316	1008
6.30 M ii	48^3	96	6.30	0.08786	0.874164	0.00363		0.43	1=10	0.03627	221	1031
6.30 M i	64^3	96				0.0012	0.0363	0.432	phys		129	1074
6.72 M iii	48^3	144				0.0048		0.286	1=5		329	1017
6.72 M ii	64^3	144	6.72	0.05662	0.885773	0.0024	0.024	0.26	1=10	0.02176	234	1103
6.72 M i	96^3	192				0.0008	0.022	0.26	phys		135	1268
7.00 M iii	64^3	192				0.00316	0.0158	0.188	1=5		315	1165
7.00 M i	144^3	288	7.00	0.0426	0.892186	0.000569	0.01555	0.1827	phys	0.01564	134	478
7.28 M iii	96^3	288	7.28	0.03216	0.89779	0.00223	0.01115	0.1316	1=5	0.01129	309	821

Time and distance intervals in the full data set.

$a_{f_{p4s}}$ (fm)		$T_{\min}=a$	$T_{\max}=a$	T_{\max} (fm)	$r_{\max}=a$	r_{\max} (fm)
0.15294	5.8	1	9	1.35	6	0.92
0.12224	6.0	1	6	0.73	6	0.73
0.08786	6.3	1	8	0.70	8	0.70
0.05662	6.72	1	10	0.57	12	0.68
0.0426	7.0	1	20	0.85	20	0.85
0.03216	7.28	1	28	0.91	24	0.78

¹[A. Bazavov, et al., PRD 98 (2018)]



Fits

- For each N_{st} , $\min_{r=a} N_{\text{st}}$ is varied by a) fits are robust.
- 100 Jackknife pseudoensembles to propagate statistical errors.
- Randomized thinning of r range to invert correlation matrix ¹⁴.
- Or full r range, but restricted to the diagonal) consistent.
- We use Bayesian fits with loose, linear priors ($\approx 10\%$ width).

¹⁴Modified eigenvalue smoothing a la [C. Michael, A. McKerrell, PRD 51 (1995) 3745].



A consistent definition of the force on the lattice?

Wanted: **independent determinations** of the three scales!

- Small R : discretization artifacts 0:1 1% stat. errors
- The slope determining the $1=R$ coefficient A decreases with R
- Stat. errors of $E(R; a)$ data increases with R or a
- Density of $E(R; a)$ data increases with R or a
- Correlations between many data at similar R) many small eigenvalues) cannot invert correlation matrix

Can we possibly **trust the statistical errors** we obtain for $r_j=a$?



A consistent definition of the force on the lattice?

Asymmetric, randomized thinning in intervals (expected $r_i = a \quad x\%$)

- $x = 30$ (or 35 for the finest lattice)
- Out of $10 = \rho \overline{N}_J$ pick 3(7) below (above) expected R
- Extend interval if necessary by demanding at least six(fourteen) below(above) to permit randomized picking
- If not possible relax to at least five(eleven) below(above)
- Pick $N_P = 100$ (or $N_P = 200$ for the finest lattice) R values



Defining the force through asymmetric, random picking

