Static Energy in $(2+1+1)$-Flavor Lattice QCD: Scale Setting and Charm Effects

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Based on [N. Brambilla, et al., arXiv/2206.03156[hep-lat]]

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Motivation

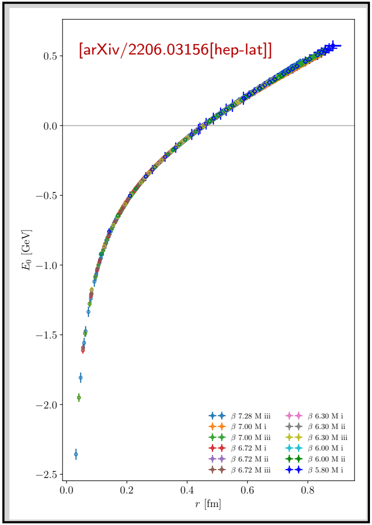
- Static energy $E_0(r)$ studied on the lattice since earliest times (**confinement**) [1], defined as ground state of Wilson loop.
- High accuracy, numerically cheap observable on the lattice.
- Of major importance for **scale setting** in the past – still now?
- Wilson loops and related correlators contribute in most **effective field theories** for heavy quarks and quarkonia.
- **No scheme change** required between lattice and continuum.
- To date, extraordinarily **detailed lattice studies** in pure gauge theory ($N_\mathrm{f} = 0$) or $(2+1)$-flavor QCD [2] $\Rightarrow$ $\Lambda_{\overline{MS}}^{N_\mathrm{f}=0,3}$.
- So far, dearth of results in $N_\mathrm{f} = (2+1+1)$-flavor lattice QCD.

[1] [K. Wilson, PRD 10 (1974) 2445]

[2] [A. Bazavov, et al., PRD 100 (2019) 11, 114511]

Summary

- We calculate the static energy in $(2+1+1)$-**flavor QCD** covering $r \approx 0.03 - 0.9\,\mathrm{fm}$.
- We **self-consistently convert** the distance r/a and the static energy aE_0 **to physical units** using scales obtained from E_0.
- We resolve dependence of the scales and the effective string tension on **sea quark masses**.
- We study in detail **effects of the charm sea** for $r \gtrsim 1/m_c$.



Lattice setup

- $(2+1+1)$-flavor QCD **ensembles with HISQ** [3] and 1-loop Symanzik gauge action generated by MILC collaboration [4].
- **Three light quark masses** ($m_l/m_s \approx 1/27,\ 1/10,\ 1/5$), physical strange and charm sea.
- **Six lattice spacings** via f_{p4s} scale ($a_{f_{p4s}} \approx 0.03 - 0.15\,\mathrm{fm}$).
- We compute the **Wilson line correlator in Coulomb gauge**, with or without one step of HYP smearing [5].

$$W(\boldsymbol{r},\tau,a) = \prod_{u=0}^{\tau/a-1} U_4(\boldsymbol{r},ua,a),$$

$$C(\boldsymbol{r},\tau,a) = \left\langle \frac{1}{N_\sigma^3} \sum_{\boldsymbol{x}} \sum_{\boldsymbol{y}=R(\boldsymbol{r})} \frac{1}{N_c\, N_{\boldsymbol{r}}} \mathrm{tr}\left[W^\dagger(\boldsymbol{x}+\boldsymbol{y},\tau,a)\, W(\boldsymbol{x},\tau,a)\right] \right\rangle,$$

[3] [E. Follana, et al., PRD 75 (2007) 054502]

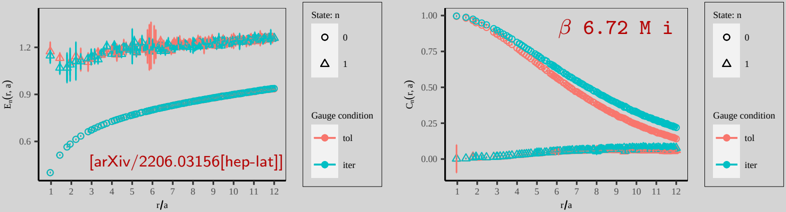
[4] [A. Bazavov, et al., PRD 98 (2018) 7, 074512]

[5] [A. Hasenfratz, F. Knechtli, PRD 64 (2001) 034504]

Spectrum of Wilson line correlators

- **Improved action** $\Rightarrow$ spectral representation holds at $\tau/a \geq 2$.
- **Physical $T_{\mathbf{max}}$ and $r_{\mathbf{max}}$** $\simeq 0.6 - 0.9$ **fm** implies much fewer time slices on coarse lattices (factor five variation in a).
- On coarse lattices **limitation in number of states**, N_{st}.

$$C(\boldsymbol{r}, \tau, a) = \mathrm{e}^{-\tau E_0(\boldsymbol{r},a)} \left(C_0(\boldsymbol{r}, a) + \sum_{n=1}^{N_{\mathrm{st}}-1} C_n(\boldsymbol{r}, a) \prod_{m=1}^{n} \mathrm{e}^{-\tau \Delta_m(\boldsymbol{r},a)} \right) + \ldots,$$



Energies $aE_{0,1}$ and overlap factors $C_{0,1}$ in different gauges.

Fits

- Up to $N_{\boldsymbol{r}} \sim 500$ $|\boldsymbol{r}|/a$ values after averaging, cover factor 24.
- Up to $T_{\max}/a \times N_{\boldsymbol{r}} \sim 10^4$ correlated data $\Rightarrow$ automation!
- **Fit range** [i.e. $\tau_{\min,N_{st}}(\boldsymbol{r}/a)$] **varies with N_{st} and $|\boldsymbol{r}|$**

$$\begin{aligned}
|\boldsymbol{r}| + 0.2\text{ fm} \le \tau_{\min,1} &\le 0.3\text{ fm} && \text{for } N_{st}=1 \Rightarrow \text{ prior values,}\\
\tfrac{2}{3}|\boldsymbol{r}| + 0.1\text{ fm} \le \tau_{\min,2} &\le \tau_{\min,1} - 2a && \text{for } N_{st}=2 \Rightarrow \text{ final choice,}\\
\tfrac{1}{3}|\boldsymbol{r}| \le \tau_{\min,3} &\le \tau_{\min,2} - 2a && \text{for } N_{st}=3 \Rightarrow \text{ cross-check}
\end{aligned}$$

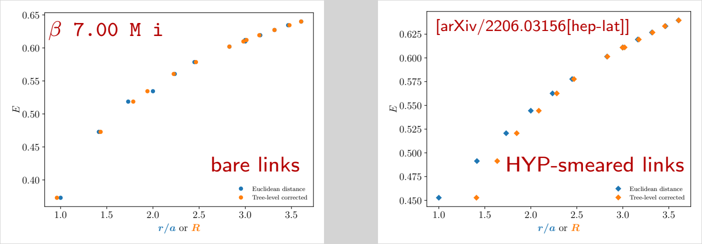
- We use Bayesian fits with **loose, linear priors** ($\geq 10\%$ width).
- Priors for excited state overlap factors with $\geq 100\%$ width: no unacceptable examination of data, robust for prior variation.
- Priors for $\Delta_1 = E_1 - E_0$ are from SU(3) pure gauge theory [6].

[6] [K. Juge, et al., PRL90, 161601 (2003)]

Discretization artifacts

- $E_0(\boldsymbol{r}, a)$ is available only at discrete distances, and depends on the direction of $\boldsymbol{r}$ $\Rightarrow$ not smooth in $|\boldsymbol{r}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.
- Distance r_I defined via tree-level [7] gluon propagator $D_{\mu\nu}(k)$

$$E_0^{\mathrm{tree}}(\boldsymbol{r}, a) = -C_F g_0^2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} D_{44}(k) \equiv -\frac{C_F g_0^2}{4\pi} \frac{1}{r_I}.$$



Tree-level-correction (aka: -improvement) in use for $|\boldsymbol{r}|/a \leq 6$.

[7] Extension to one loop ongoing: [G. v. Hippel, et al., in preparation: TUM-EFT 171/22]

Determination of scales

- The scales are defined in terms of **the force** $F(r) \equiv \frac{\mathbf{d}E_0(r)}{\mathbf{d}r}$

$$r_i^2 F(r_i) = c_i, \quad i = 0, 1, 2$$

with $c_0 = 1.65$ [8], $c_1 = 1$ [9], and $c_2 = 1/2$ [10].

- The r_i correspond to **different regimes**. $(2+1)$-flavor QCD:

$$r_0 \approx 0.475 \text{ fm}, \quad r_1 \approx 0.3106 \text{ fm}, \quad r_2 \approx 0.145 \text{ fm},$$

- A **Cornell Ansatz** encodes the main features of $E_0(r)$ locally

$$E(R, a) = -\frac{A}{R} + B + \Sigma R,$$

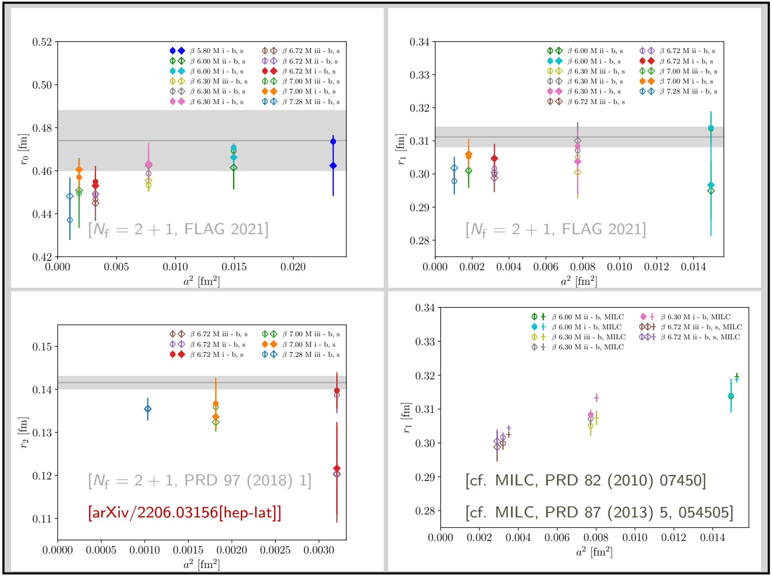
- $r_i/a = \sqrt{(c_i - A)/\Sigma}$; identify Σ at large r with string tension
- **Consistent definition** of $F(r, a)$ across various a is difficult

[8] [R. Sommer, NPB 411 (1994) 839]

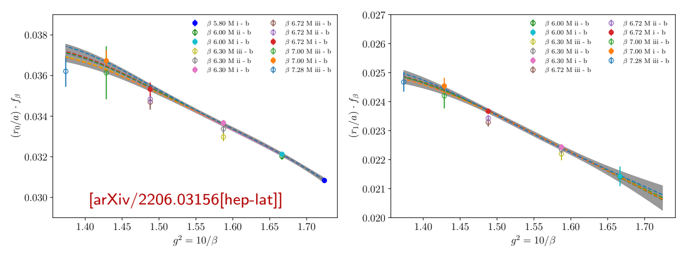
[9] [C. Bernard, et al., PRD 62 (2000) 034503]

[10] [A. Bazavov, et al., PRD 97 (2018) 1, 014510]

Lattice spacing, N_f, and quark mass dependence



Smoothing (via Allton-Ansatz)



$$\frac{a}{r_i} = \frac{C_0 f_\beta + C_2 g_0^2 f_\beta^3 + C_4 g_0^4 f_\beta^3}{1 + D_2 g_0^2 f_\beta^2}, \quad f_\beta = \text{two-loop } \beta \text{ fct. } (N_\mathrm{f} = 4)$$

$$C_0 = C_{00} + C_{01l}\frac{am_\mathrm{l}}{f_\beta} + C_{01s}\frac{am_\mathrm{s}}{f_\beta} + C_{01}\frac{am_\mathrm{tot}}{f_\beta} + C_{02}\frac{(am_\mathrm{tot})^2}{f_\beta},$$

$$C_2 = C_{20} + C_{21}\frac{am_\mathrm{tot}}{f_\beta}, \quad am_\mathrm{tot} = 2am_\mathrm{l} + am_\mathrm{s} + am_\mathrm{c},$$

Continuum extrapolation

- Using smoothed results we extrapolate to the continuum limit: $r_0/r_1,\ r_1/r_2;\ a_{f_{p4s}} r_0/a,\ a_{f_{p4s}} r_1/a;\ \sqrt{\sigma a^2}\, r_0/a$
- Lattice spacing dependence via $x = (a/r_i)^2,\ \ i = 0, 1$
- m_l dependence via $y = m_l/m_s$ (sea or tuned [4] m_s), where we use PDG values [11] in the continuum GMOR

$$m_l/m_s = 1/(2M_K^2/M_\pi^2 - 1) \quad \Rightarrow \quad \begin{cases} m_l/m_s|_{M_\pi^2 = M_{\pi^\pm}^2} & = 0.04128, \\ m_l/m_s|_{M_\pi^2 = M_{\pi^0}^2} & = 0.03851. \end{cases}$$

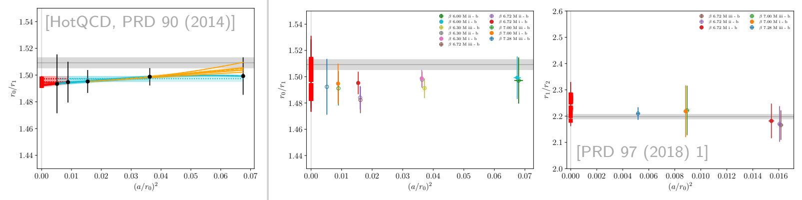
- Logs in the Symanzik expansion via $\alpha = 1$ or $\alpha_b = g_0^2/(4\pi u_0^2)$

$$\xi = \xi_0 + \alpha^2[\xi_1 x + \xi_2 x y^{1,2}] + \xi_3 x^2 + \xi_4 y \qquad (\mathrm{l,q;l,qm;mc}),$$

[4] [A. Bazavov, et al., PRD 98 (2018) 7, 074512]

[11] We use: $2M_K^2 = M_{K^\pm}^2 + M_{K^0}^2$ and $M_\pi^2 = M_{\pi^\pm}^2$ or $M_{\pi^0}^2$

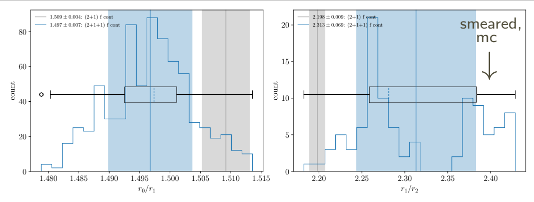
The ratios r_0/r_1 and r_1/r_2



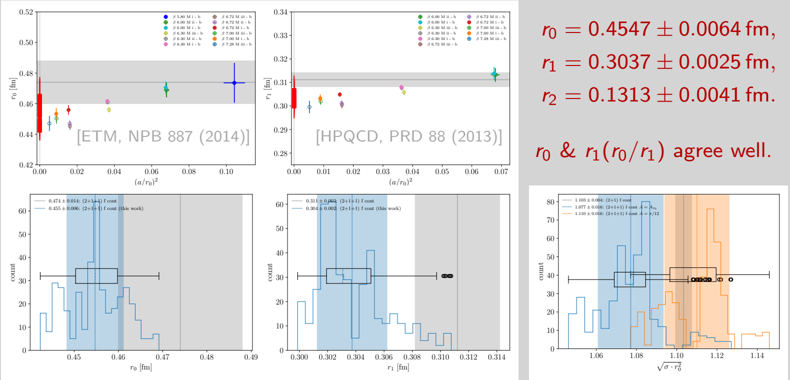
$$r_0/r_1 = 1.4968 \pm 0.0069,$$
$$r_1/r_2 = 2.313 \pm 0.069.$$

AIC favors weighted average:

central r_0/r_1, lowest r_1/r_2.



The scales r_0 and r_1 and the string tension



$$r_0 = 0.4547 \pm 0.0064\,\text{fm},$$
$$r_1 = 0.3037 \pm 0.0025\,\text{fm},$$
$$r_2 = 0.1313 \pm 0.0041\,\text{fm}.$$

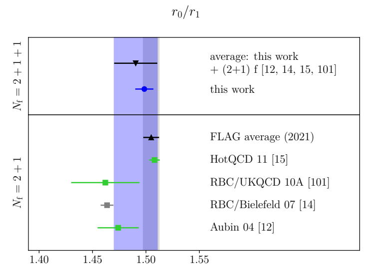
r_0 & $r_1(r_0/r_1)$ agree well.

The string tension is consistent with $(2+1)$-flavor QCD [12]:

$$\sqrt{\sigma r_0^2} = 1.077 \pm 0.016 \quad (A = A_{r_0}), \qquad \sqrt{\sigma r_0^2} = 1.110 \pm 0.016 \quad (A = \pi/12).$$

[12] [RBC/Bielefeld, PRD 77 (2008) 014511]

Comparison to published results



r_0/r_1 averaged between $N_f = 2+1$ & $N_f = 2+1+1$:

$r_0/r_1 = 1.490(20)$, $\chi^2/\text{dof} \approx 14.800/3$

cf. $r_0/r_1 = 1.5049(74)$, $\chi^2/\text{dof} \approx 4.741/2$

r_0 or r_1 averaged within $(2+1+1)$-flavor QCD:

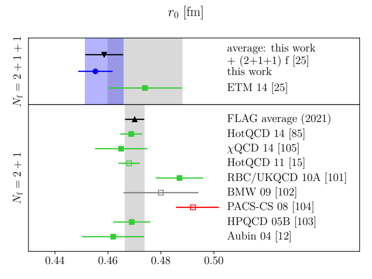
$r_0 = 0.4586(71)$ fm, $\chi^2/\text{dof} \approx 1.472/1$,

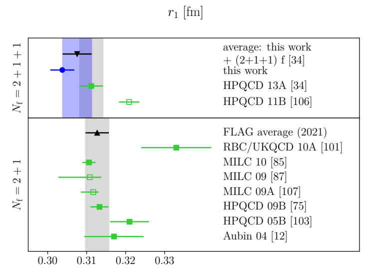
cf. $r_0 = 0.4701(36)$ fm, $\chi^2/\text{dof} \approx 3.790/4$

$r_1 = 0.3076(37)$ fm, $\chi^2/\text{dof} \approx 3.021/1$,

cf. $r_1 = 0.3127(30)$ fm, $\chi^2/\text{dof} \approx 7.281/4$

[(2+1)-flavor QCD excluding RBC/Bielefeld result.]





Static energy in perturbation theory with massless sea

- The static energy is defined as

$$E_0(r) = \lim_{t\to\infty} \frac{\mathrm{i}}{t} \ln \left\langle \operatorname{tr} \mathcal{P} \exp \left[\mathrm{i}g \oint_{r\times t} \mathrm{d}z^\mu A_\mu(z) \right] \right\rangle ,$$

and–for massless sea quarks–perturbatively known at $\mathrm{N}^3\mathrm{LL}$

$$E_0(r) = \Lambda - \frac{C_\mathrm{F}\alpha_\mathrm{s}}{r} \left(1 + \#\alpha_\mathrm{s}^1 + \#\alpha_\mathrm{s}^2 + \#\alpha_\mathrm{s}^2 + \#\alpha_\mathrm{s}^2 \ln\alpha_\mathrm{s} + \#\alpha_\mathrm{s}^3 \ln\alpha_\mathrm{s} + \#\alpha_\mathrm{s}^3 \ln^2\alpha_\mathrm{s} + \dots \right),$$

- Integrating the force eliminates the leading renormalon

$$E_0(r) = \int_{r^*}^{r} \mathrm{d}r' \; F(r') + \mathrm{const.}$$

- $E_0(r)$ depends on N_f via its coefficients $\#$ and via α_s
- α_s depends on a scale $\nu \sim 1/r$ (we use $\nu = 1/r$)

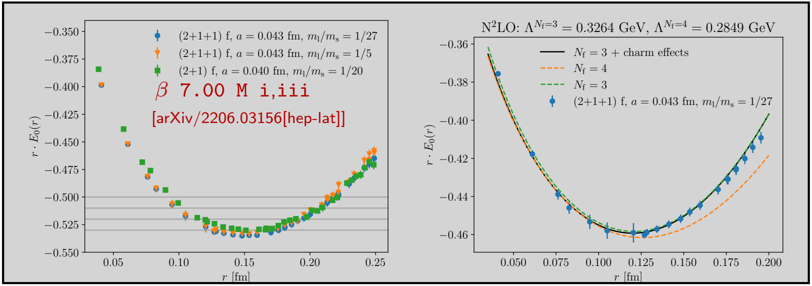
Massive charm sea

- Massive charm quark decouples for large r:

$$
\begin{array}{llll}
rm_{\mathrm{c}} \gg 1 & \Rightarrow \text{ charm is effectively infinitely heavy} & \Rightarrow & N_{\mathrm{f}} = 3 \\
rm_{\mathrm{c}} \ll 1 & \Rightarrow \text{ charm is effectively massless} & \Rightarrow & N_{\mathrm{f}} = 4
\end{array}
$$

- For $rm_{\mathrm{c}} \sim 1 \;\Rightarrow$ massive correction $\delta V_m^{(N_{\mathrm{f}})}(r)$ known at N^{2}LO

$$
E_{0,m}^{(N_{\mathrm{f}})}(r) = \int_{r^*}^{r} \mathrm{d}r' \; F^{(N_{\mathrm{f}})}(r') + \delta V_m^{(N_{\mathrm{f}})}(r) + \mathrm{const},
$$



Conclusions

- We have computed the static energy $E_0(r)$ in $(2+1+1)$-flavor QCD over a wide range of lattice spacings and quark masses.
- First simultaneous determination of both the scales r_0, r_1 and their ratio r_0/r_1, and the ratio r_1/r_2 and the string tension σ.
- r_1/a on coarse lattices is slightly lower than earlier results [13]; procedural differences in defining r_1/a lead to larger errors.
- r_0/r_1 and $r_0\sqrt{\sigma}$ are consistent with $(2+1)$-flavor QCD results.
- r_1/r_2 differs significantly from $(2+1)$-flavor QCD results.
- We study the charm quark contribution across its decoupling.
- $E_0(r)$ at $r \lesssim 0.2\,\text{fm}$ is inconsistent with $(2+1)$-flavor QCD, but well reproduced by perturbation theory at two-loop order.

[13] [MILC, PRD 82 (2010) 07450]; [MILC, PRD 87 (2013) 5, 054505]

Data sets

MILC $(2+1+1)$-flavor gauge ensembles [1] used in this study.

our naming	$N_\sigma^3 \times N_\tau$	β	$a_{f_{p4s}}$ (fm)	u_0	am_l	am_s	am_c	m_l/m_s	$(am_s)_{\text{tuned}}$	M_π (MeV)	#conf.
β 5.80 M i	$32^3 \times 48$	5.80	0.15294	0.85535	0.00235	0.0647	0.831	phys	0.06852	131	1041
β 6.00 M ii	$32^3 \times 64$	6.00	0.12224	0.86372	0.00507	0.0507	0.628	1/10	0.05296	217	1000
β 6.00 M i	$48^3 \times 64$				0.00184			phys		132	709
β 6.30 M iii	$32^3 \times 96$	6.30	0.08786	0.874164	0.0074	0.037	0.44	1/5	0.03627	316	1008
β 6.30 M ii	$48^3 \times 96$				0.00363	0.0363	0.43	1/10		221	1031
β 6.30 M i	$64^3 \times 96$				0.0012		0.432	phys		129	1074
β 6.72 M iii	$48^3 \times 144$	6.72	0.05662	0.885773	0.0048	0.024	0.286	1/5	0.02176	329	1017
β 6.72 M ii	$64^3 \times 144$				0.0024			1/10		234	1103
β 6.72 M i	$96^3 \times 192$				0.0008	0.022	0.26	phys		135	1268
β 7.00 M iii	$64^3 \times 192$	7.00	0.0426	0.892186	0.00316	0.0158	0.188	1/5	0.01564	315	1165
β 7.00 M i	$144^3 \times 288$				0.000569	0.01555	0.1827	phys		134	478
β 7.28 M iii	$96^3 \times 288$	7.28	0.03216	0.89779	0.00223	0.01115	0.1316	1/5	0.01129	309	821

Time and distance intervals in the full data set.

$a_{f_{p4s}}$ (fm)	β	T_{min}/a	T_{max}/a	T_{max} (fm)	r_{max}/a	r_{max} (fm)
0.15294	5.8	1	9	1.35	6	0.92
0.12224	6.0	1	6	0.73	6	0.73
0.08786	6.3	1	8	0.70	8	0.70
0.05662	6.72	1	10	0.57	12	0.68
0.0426	7.0	1	20	0.85	20	0.85
0.03216	7.28	1	28	0.91	24	0.78

[1] [A. Bazavov, et al., PRD 98 (2018)]

Fits

- For each N_{st}, $\tau_{\mathrm{min},N_{\mathrm{st}}}(\boldsymbol{r}/a)$ is varied by $\pm a \Rightarrow$ fits are robust.
- 100 Jackknife pseudoensembles to propagate statistical errors.
- Randomized thinning of τ range to invert correlation matrix [14].
- Or full τ range, but restricted to the diagonal $\Rightarrow$ consistent.
- We use Bayesian fits with loose, linear priors ($\geq 10\%$ width).

[14] Modified eigenvalue smoothing à la [C. Michael, A. McKerrell, PRD 51 (1995) 3745].

A consistent definition of the force on the lattice?

Wanted: **independent determinations** of the three scales!

- Small R: discretization artifacts $\sim 0.1 - 1\% \gg$ stat. errors
- The slope determining the $1/R$ coefficient A decreases with R
- Stat. errors of $E(R,a)$ data increases with R or a
- Density of $E(R,a)$ data increases with R or a
- Correlations between many data at similar $R \Rightarrow$ many small eigenvalues $\Rightarrow$ cannot invert correlation matrix

Can we possibly **trust the statistical errors** we obtain for r_i/a?

A consistent definition of the force on the lattice?

Asymmetric, randomized thinning in intervals (expected $r_i/a \pm x\%$)

- $x = 30$ (or 35 for the finest lattice)
- Out of $10 = \sqrt{N_J}$ pick 3(7) below (above) expected R
- Extend interval if necessary by demanding at least six(fourteen) below(above) to permit randomized picking
- If not possible relax to at least five(eleven) below(above)
- Pick $N_P = 100$ (or $N_P = 200$ for the finest lattice) R values

Defining the force through asymmetric, random picking