QCD Static Force in Gradient Flow

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Outlines

QCD Static Potential and Force

QCD Static Force in Gradient Flow

One-loop Calculations and Results

Summary

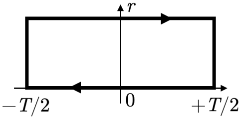
Static Potential

- Potential between a static quark and a static anti-quark.
- Definition in Euclidean QCD:

$$V(r) = -\lim_{T\to\infty} \frac{1}{T} \log\langle W_{r\times T}\rangle, \tag{1}$$

- Wilson loop:

$$W_{r\times T} = \mathrm{tr}_{\mathrm{color}} P \exp\left[i g_s \oint_C dz^\mu A_\mu(z) \right]. \tag{2}$$



Static Potential

- Encodes important information about QCD interactions for a wide range of distances.
- Useful in the extraction of α_s from Lattice QCD calculations.
- Applied in Quarkonium Physics, the static potential determines the quarkonium wavefunctions.
- pQCD calculations in dimensional regularization contain an $\mathcal{O}(\Lambda_{\mathrm{QCD}})$ renormalon ambiguity, which is associated with an arbitrary overall constant shift of $V(r)$.
- In lattice regularization, there is a linear divergence that is proportional to the inverse of the lattice spacing $\sim a^{-1}$.

Static Force

- Define the QCD static force by the spatial derivative of $V(r)$ as:

$$F(r) \equiv \frac{\partial}{\partial r} V(r) = -i \lim_{T \to \infty} \frac{1}{T} \frac{\int_{-T/2}^{+T/2} dx_0 \, \langle W_{r \times T} \hat{\mathbf{r}} \cdot g_s \mathbf{E}(x_0, \mathbf{r}) \rangle}{\langle W_{r \times T} \rangle}. \quad (3)$$

- Free of the constant shift (cancel between numerator and denominator), which makes it convenient for comparing with lattice studies.
- Can be computed from the finite differences of the lattice data of the static potential.
- However, data at short distances are still sparse, and the computation of the force from the finite differences leads to large uncertainties.

Static Force from Lattice

- It has been suggested to compute the force directly from the definition of $F(r)$, and a direct Lattice QCD calculation was carried out. Brambilla, Leino, Philipsen, Reisinger, Vairo and Wagner, PRD 105 (2022) 5, 054514
- Sizable discretization errors and the convergence to the continuum limit is rather slow.
- This may be understood from the convergence of the Fourier transform of the perturbative QCD calculation in momentum space.

Convergence of Fourier Transform

- For the potential, we have

$$\int \frac{d^3q}{(2\pi)^3}\frac{e^{iq\cdot r}}{q^2} = \int_0^{+\infty} dq \frac{\sin(qr)}{2\pi^2 qr}. \tag{4}$$

Convergence is not great, but we can still complete it through $\int_0^{+\infty} dq = \lim_{\Lambda\to+\infty}\int_0^{+\infty} dq$

- It doesn't work for the force

$$\partial_r \int \frac{d^3q}{(2\pi)^3}\frac{e^{iq\cdot r}}{q^2} = \int_0^{+\infty} dq \frac{qr\cos(qr) - \sin(qr)}{2\pi^2 q} \tag{5}$$

In this case, the Fourier transform cannot be defined as a limit of a cutoff.

- Therefore, it is likely that a cutoff regularization (e.g. lattice) will run into problems.

Gradient Flow

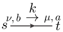
- It has been expected that gradient flow may improve the situation.
- Gradient flow: In operator definition, replace gluon fields A with flowed field B, which depends on both space time x and flow time t. Narayanan and Neuberger, JHEP 03 (2006) 064; Lüscher, Commun. Math. Phys. 293 (2010) 899, JHEP 08 (2010) 071; Lüscher and Weisz, JHEP 02 (2011) 051.
- B is a solution of the flow equation (usually set $\lambda = 1$)

$$\frac{\partial}{\partial t} B_\mu(x;t) = D_\nu G_{\nu\mu} + \lambda D_\mu \partial_\nu B_\nu,$$
$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu], \quad D_\mu = \partial_\mu + [B_\mu, \cdot], \quad (6)$$

  with the initial condition: $B_\mu(x; t=0) = g_s A_\mu(x)$.
- Advantages: No need for additional renormalization besides fields and QCD parameters renormalization in gradient flow formalism. No operator renormalization mixing, great advantage for comparing pQCD and lattice QCD results.

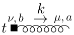
Gradient Flow Feynman Rules

- Flow propagator:

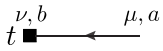


$= \theta(t-s)e^{-(t-s)k^2}\delta_{\mu\nu}\delta^{ab}$

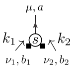
- Flow vertices:



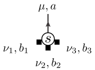
$= e^{-tk^2}\delta_{\mu\nu}\delta^{ab}$



$= \delta_{\mu\nu}\delta^{ab}$



$= g\int_0^\infty ds\, X^{(2,0)}(k_1+k_2, k_1, k_2)^{ab_1b_2}_{\mu\nu_1\nu_2}$



$= g^2\int_0^\infty ds\, X^{(3,0)}{}^{ab_1b_2b_3}_{\mu\nu_1\nu_2\nu_3}$

$$X^{(2,0)}(p,q_1,q_2)^{ab_1b_2}_{\mu\nu_1\nu_2} = if^{ab_1b_2}[(q_2-q_1)_\mu\delta_{\nu_1\nu_2} + 2q_{1\,\nu_2}\delta_{\mu\nu_1} - 2q_{2\,\nu_1}\delta_{\mu\nu_2}],$$

$$X^{(3,0)}(p,q_1,q_2,q_3)^{ab_1b_2b_3}_{\mu\nu_1\nu_2\nu_3} = f^{ab_1c}f^{b_2b_3c}(\delta_{\mu\nu_3}\delta_{\nu_1\nu_2} - \delta_{\mu\nu_2}\delta_{\nu_1\nu_3}) + f^{ab_3c}f^{b_1b_2c}(\delta_{\mu\nu_2}\delta_{\nu_1\nu_3} - \delta_{\mu\nu_1}\delta_{\nu_2\nu_3}) + f^{ab_2c}f^{b_3b_1c}(\delta_{\mu\nu_1}\delta_{\nu_2\nu_3} - \delta_{\mu\nu_3}\delta_{\nu_1\nu_2}).$$

Lüscher and Weisz, JHEP 02 (2011) 051;

Harlander and Neumann, JHEP06 (2016) 161;

Artz, Harlander, Lange, Neumann, Prausa, JHEP06 (2019) 121

Static Force in Gradient Flow

- In lattice calculation, usually start with the definition

$$F(r) = -i \lim_{T\to\infty} \frac{1}{T} \frac{\int_{-T/2}^{+T/2} dx_0 \, \langle W_{r\times T} \hat{\mathbf{r}} \cdot g_s \mathbf{E}(x_0, \mathbf{r}) \rangle}{\langle W_{r\times T} \rangle}, \quad (7)$$

  but replace $A \to B$ in Wilson line, and $F^{i0} \to G^{i0}$.

- In perturbative calculation, it is not necessary to start with above definition for $F(r,t)$. Instead, we just compute $V(r,t)$ with gradient flow and apply ∂_r later.
- We perform the calculation with Feynman gauge, in which we can neglect fields at large times.

Leading Order Results

- At leading order, $\tilde{B}^a_\mu(p;t) = e^{-p^2 t} g \tilde{A}^a_\mu(p) +$ higher orders, then

$$V(r,t) = -g_s^2 C_F \int \frac{d^3q}{(2\pi)^3} \frac{e^{iq\cdot r}}{q^2} e^{-2q^2 t} + \mathcal{O}(g_s^4). \tag{8}$$

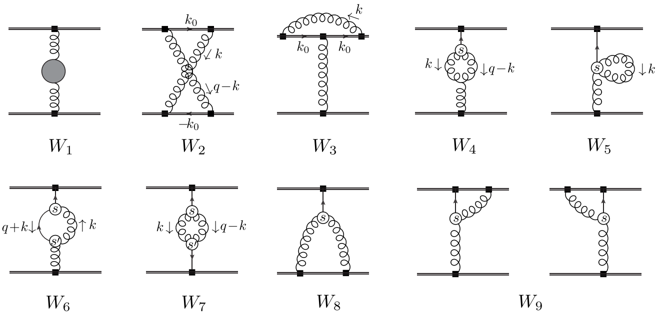
  Basically, smoothed over a Gaussian distribution.

- $e^{-2q^2 t}$ is a strong convergence factor for nonzero t, so that the static force is well defined. Explicit calculation gives

$$\begin{aligned} F(r,t) &= \frac{\alpha_s(\mu) C_F}{r^2} \left[\text{erf}\left(\frac{r}{\sqrt{8t}}\right) - \frac{r}{\sqrt{2\pi t}} \text{exp}\left(-\frac{r^2}{8t}\right) + \mathcal{O}(\alpha_s) \right] \\ &\overset{t\to 0}{=} \frac{\alpha_s(\mu) C_F}{r^2} + \mathcal{O}(\alpha_s^2). \end{aligned} \tag{9}$$

- B is a smoothed version of A over a length scale of $(8t)^{1/2}$.

One-loop Diagrams



$W_1 - W_6$ are divergent, $W_7 - W_9$ are finite.

Results in Momentum Space

- After renormalization of g_s in $\overline{\rm MS}$ scheme, we have ($\bar{t} = q^2 t$)

$$\begin{aligned}\tilde{V}(q;t) \quad = \quad & -\frac{4\pi\alpha_s(\mu)C_F e^{-2q^2 t}}{q^2}\left\{1 + \frac{\alpha_s(\mu)}{4\pi}\left[\beta_0 \log(\mu^2/q^2) + a_1\right.\right. \\ & \left.\left. + C_A\, W^F_{\rm NLO}(\bar{t})\right]\right\} + O(\alpha_s^3),\end{aligned} \tag{10}$$

where

$$a_1 = \frac{31}{9}C_A - \frac{10}{9}n_f, \quad \beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f, \quad W^F_{\rm NLO}(\bar{t}) = \sum_{n=2}^{9} W^F_n(\bar{t}). \tag{11}$$

- In the small $\bar{t}$ region, we have

$$W^F_{\rm NLO}(\bar{t}) = \bar{t}\left(-\frac{22\gamma_{\rm E}}{3} + \frac{277}{18} - \frac{31\log 2}{3} + c_L \log \bar{t}\right) + O(\bar{t}^2), \tag{12}$$

where $c_L = -22/3$, which will be crucial in obtaining the asymptotic behaviour of the static force in position space.

Results in Position Space

- In position space, we have

$$F(r;t) = \frac{\partial}{\partial r}\int \frac{d^3q}{(2\pi)^3}\tilde{V}(q;t)e^{iq\cdot r}$$
$$= \frac{1}{r^2}\int_0^\infty d|q|q^2\,\frac{|q|r\cos(|q|r)-\sin(|q|r)}{2\pi^2|q|}\tilde{V}(q;t). \quad (13)$$

- At α_s^2 order, we decompose $F(r;t)$ as

$$F(r;t) = \frac{\alpha_s(\mu)C_F}{r^2}\left[\left(1+\frac{\alpha_s}{4\pi}a_1\right)\mathcal{F}_0(r;t)+\frac{\alpha_s}{4\pi}\beta_0\mathcal{F}^L_{\rm NLO}(r;t;\mu)+\frac{\alpha_s C_A}{4\pi}\mathcal{F}^F_{\rm NLO}(r;t)\right], \quad (14)$$

where

Tree level: $$\mathcal{F}_0(r;t) = -\int_0^\infty d|q|q^2\,\frac{|q|r\cos(|q|r)-\sin(|q|r)}{2\pi^2|q|}\,\frac{4\pi e^{-2q^2t}}{q^2}, \quad (15)$$

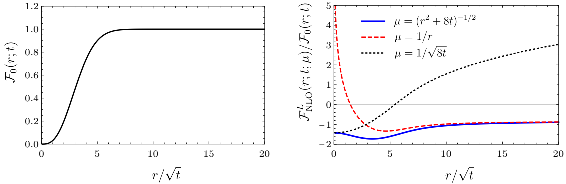
One-loop logs: $$\mathcal{F}^L_{\rm NLO}(r;t;\mu) = -\int_0^\infty d|q|q^2\,\frac{|q|r\cos(|q|r)-\sin(|q|r)}{2\pi^2|q|}\,\frac{4\pi e^{-2q^2t}}{q^2}\log(\mu^2/q^2), \quad (16)$$

One-loop extra finite: $$\mathcal{F}^F_{\rm NLO}(r;t) = -\int_0^\infty d|q|q^2\,\frac{|q|r\cos(|q|r)-\sin(|q|r)}{2\pi^2|q|}\,\frac{4\pi e^{-2q^2t}}{q^2}W^F_{\rm NLO}(\bar{t}). \quad (17)$$

$\mathcal{F}_0(r;t)$ & $\mathcal{F}^L_{\rm NLO}(r;t;\mu)$

$$\mathcal{F}_0(r;t) = \text{erf}\left(\frac{r}{\sqrt{8t}}\right) - \frac{r}{\sqrt{2\pi t}}\exp\left(-\frac{r^2}{8t}\right) \overset{t\to 0}{=} 1, \tag{18}$$

$$\begin{aligned}
\mathcal{F}^L_{\rm NLO}(r;t;\mu) &= \log(\mu^2 r^2)\mathcal{F}_0(r;t) + \log\left(\frac{8t}{r^2}e^{\gamma_{\rm E}}\right)\mathcal{F}_0(r;t) \\
&\quad - \frac{r}{\sqrt{2\pi t}}\left[e^{-\frac{r^2}{8t}}M^{(1,0,0)}\left(0,\frac{1}{2},\frac{r^2}{8t}\right) + M^{(1,0,0)}\left(\frac{1}{2},\frac{3}{2},-\frac{r^2}{8t}\right)\right] \\
&\overset{t\to 0}{\approx} \log(\mu^2 r^2) + 2(\gamma_{\rm E}-1) - 12\frac{t}{r^2}.
\end{aligned} \tag{19}$$

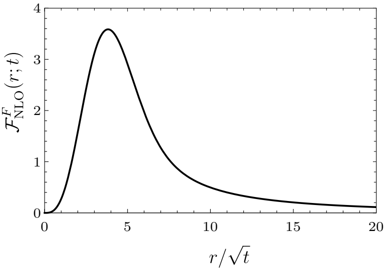


- With $\mu = (r^2+8t)^{-1/2}$, $\mathcal{F}^L_{\rm NLO}(r;t;\mu)/\mathcal{F}_0(r;t;\mu)$ is of order 1 for all values of $r/\sqrt{t}$.

$\mathcal{F}^F_{\mathrm{NLO}}(r;t)$

- In the limit $\xi \to +\infty$ ($t \to 0$, fixed r), we have

$$\mathcal{F}^F_{\mathrm{NLO}}(r;t) \approx -\frac{6c_L}{\xi^2} = -6c_L\frac{t}{r^2}, \quad c_L = -\frac{22}{3}. \tag{20}$$



Asymptotic Results in Position Space

- The multi-scale nature of the static force in gradient flow complicates the calculation due to complicated functions of $r/\sqrt{t}$ which are not analytical at $t=0$. We are not able to obtain fully analytical results.
- The extracting of the asymptotic behavior is nontrivial because of the non-analyticity at $t=0$. Explicitly, we have (in the limit $t \to 0$)

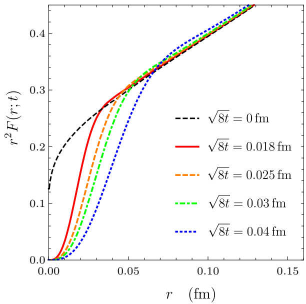
$$r^2 F(r;t) \approx r^2 F(r;t=0) + \frac{\alpha_s^2 C_F}{4\pi}\left[-12\beta_0 - 6C_A c_L\right]\frac{t}{r^2}, \tag{21}$$

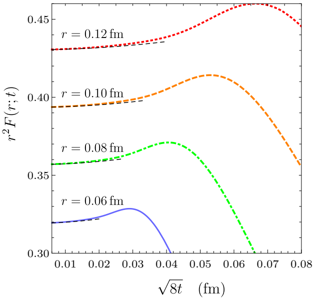
where $[-12\beta_0 - 6C_A c_L] = 8n_f$ and $F(r;t=0)$ is the usual QCD result for the static force

$$F(r;t=0) = \frac{\alpha_s(\mu)C_F}{r^2}\left\{1 + \frac{\alpha_s}{4\pi}\left[a_1 + 2\beta_0 \log(\mu r e^{\gamma_{\rm E}-1})\right]\right\} + O(\alpha_s^3). \tag{22}$$

- Surprisingly, the coefficient of the t/r^2 term vanishes at NLO in the pure SU(3) gauge theory ($n_f = 0$, quenched case).

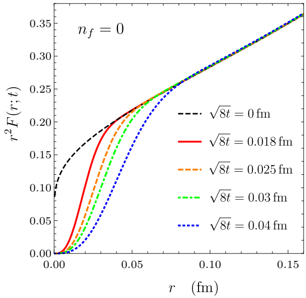
Nemrical Results in Position Space $n_f = 4$

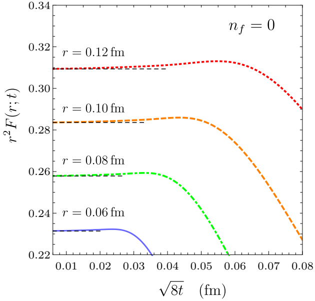




- Black dashed lines represent the QCD results.
- We choose $\mu = (r^2 + 8t)^{-1/2}$.
- The gradient flow results approach to the QCD results as $t \to 0$.

Nemrical Results in Position Space $n_f = 0$





- More plat than $n_f = 4$ case because the coefficient of the t/r^2 term vanish at NLO in the limit $t \to 0$.

Summary

- We have computed the QCD static force using gradient flow at NLO in α_s.
- The gradient flow makes the Fourier transform of the static force in momentum space better converging because of the factor e^{-2tq^2} for positive t.
- We have derived the asympototic expression of static force in gradient flow near $t = 0$ at NLO in α_s.
- Our results in position space may be useful for extrapolating to zero flow time in lattice QCD computations of static force.
- Similar analysis could be extended to similar quantities involving Wilson lines and gluon field strengths, such as higher order potentials in potential NRQCD, hybrid potentials, and gluonic correlators that appear in quarkonium decay rates.