pNREFTs for dark matter models

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in collaboration with N. Brambilla, G. Qerimi, A. Vairo (in preparation)
V. Shtabovenko 2106.06472, 2112.10145

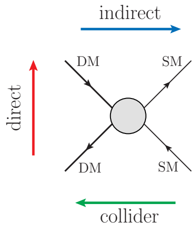
Particle interpretation of DM and freeze-out

- DM from many compelling (gravitational) observations

- DM as a particle: many candidates (Bertone and Hooper [1605.04909])
- Any model has to comply with

$$\Omega_{\rm DM} h^2(M_{\rm DM}, M_{\rm DM'}, \alpha_{\rm DM}, \alpha_{\rm SM}) = 0.1200 \pm 0.0012$$

◇ from CMB anysotropies with ΛCDM *Planck Collab. Results 2018*



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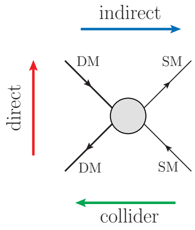

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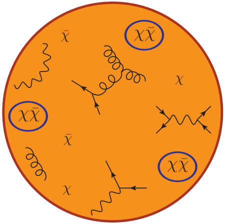

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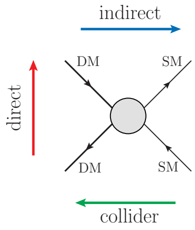
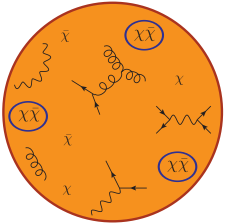




Thermal freeze-out Gondolo and Gelmini (1991)

- Boltzmann equation for DM (χ)

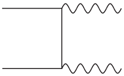
$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle \sigma v \rangle (n_\chi^2 - n_{\chi,{\rm eq}}^2)$$

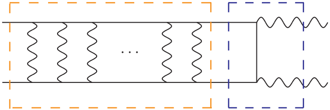
- relevant processes $\chi\chi \leftrightarrow \mathrm{SM\,SM}$, $\chi\chi \leftrightarrow \chi'\chi'$
- decoupling from $H \sim n_{\rm eq} \langle \sigma_{\rm ann} v_{\rm rel} \rangle$

$$H \simeq \frac{T^2}{M_{\rm Pl}}, \quad \langle \sigma_{\rm ann} v_{\rm rel} \rangle \simeq \frac{\alpha^2}{M^2}, \quad \frac{T}{M} \approx \frac{1}{25}$$

Going towards a realistic picture

- DM and/or coannihilating partners interact with gauge bosons and scalars





$E \sim Mv,\ Mv^2$ $\quad$ $E \sim M$

- repeated soft interactions: Sommerfeld enhancement and bound states Hisano, Matsumoto, Nojiri [hep-ph/0212022], [hep-ph/0307216]; B. von Harling and K. Petraki [1407.7874]; Beneke, Hellmann, Ruiz-Femenia [1411.6924] ...

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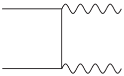

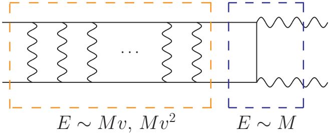

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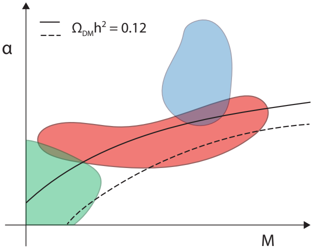
$$\frac{dn_X}{dt} + 3Hn_X = -\langle \sigma_{\rm eff}\, v_{\rm rel}\rangle (n_X^2 - n_{X,\rm eq}^2)\ ,$$

$$\langle \sigma_{\rm eff}\, v_{\rm rel}\rangle = \langle \sigma_{\rm ann}\, v_{\rm rel}\rangle + \sum_n \langle \sigma^n_{\rm bsf}\, v_{\rm rel}\rangle \frac{\Gamma^n_{\rm ann}}{\Gamma^n_{\rm ann} + \Gamma^n_{\rm bsd}}$$

J. Ellis, F. Luo, and K. A. Olive [1503.07142], M.Garny and J.Heisig [2112.01499]



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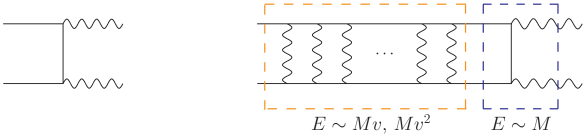

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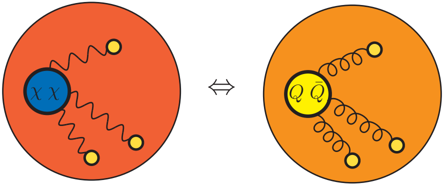


Abelian DM model

- $M \gg M\alpha \gg M\alpha^2 \gtrsim T$

  possible problematic region $P \sim \sqrt{MT}$: multiple expansion endangered for $T \gtrsim M\alpha^2$

- Relativistic model Lagrangian

$$\mathcal{L} = \bar{X}(i\not{D} - M)X - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{\rm portal}$$

- Non-relativistic theory: integrate out the scale M

$$\begin{aligned}
\mathcal{L}_{\rm NRQED_{DM}} &= \psi^\dagger \left(iD_0 - M + \frac{\mathbf{D}^2}{2M} + c_{\rm F}\frac{\boldsymbol{\sigma}\cdot g\mathbf{B}}{2M} + c_{\rm D}\frac{\boldsymbol{\nabla}\cdot g\mathbf{E}}{8M^2} + ic_{\rm S}\frac{\boldsymbol{\sigma}\cdot(\mathbf{D}\times g\mathbf{E} - g\mathbf{E}\times\mathbf{D})}{8M^2} \right)\psi \\
&+ \chi^\dagger \left(iD_0 + M - \frac{\mathbf{D}^2}{2M} - c_{\rm F}\frac{\boldsymbol{\sigma}\cdot g\mathbf{B}}{2M} + c_{\rm D}\frac{\boldsymbol{\nabla}\cdot g\mathbf{E}}{8M^2} + ic_{\rm S}\frac{\boldsymbol{\sigma}\cdot(\mathbf{D}\times g\mathbf{E} - g\mathbf{E}\times\mathbf{D})}{8M^2} \right)\chi \\
&- \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{d_2}{M^2}F^{\mu\nu}\mathbf{D}^2F_{\mu\nu} + \frac{d_s}{M^2}\psi^\dagger\chi\,\chi^\dagger\psi + \frac{d_v}{M^2}\psi^\dagger\,\boldsymbol{\sigma}\,\chi\cdot\chi^\dagger\,\boldsymbol{\sigma}\,\psi \\
&+ \mathcal{L}_{\rm portal}\,,
\end{aligned}$$

pNRQED$_{\mathrm{DM}}$

- integrating out the scale $M\alpha$: EFT for DM pairs and ultrasoft dark photons

$$\mathcal{L}_{\mathrm{pNRQED_{DM}}} = \int d^3\boldsymbol{r}\, \phi^\dagger(t,\boldsymbol{r},\boldsymbol{R})\, [i\partial_0 - H(\boldsymbol{r},\boldsymbol{p},\boldsymbol{P},\boldsymbol{S}_1,\boldsymbol{S}_2) + g\, \boldsymbol{r}\cdot\boldsymbol{E}(t,\boldsymbol{R})]\, \phi(t,\boldsymbol{r},\boldsymbol{R}) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

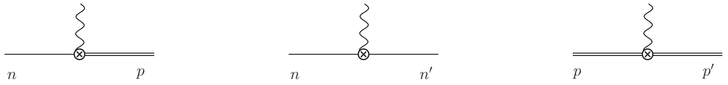
- Hamitonian and potential

$$H(\boldsymbol{r},\boldsymbol{p},\boldsymbol{P},\boldsymbol{S}_1,\boldsymbol{S}_2) = 2M + \frac{\boldsymbol{p}^2}{M} + \frac{\boldsymbol{P}^2}{4M} - \frac{\boldsymbol{p}^4}{4M^3} + V(\boldsymbol{r},\boldsymbol{p},\boldsymbol{P},\boldsymbol{S}_1,\boldsymbol{S}_2) + \dots$$

$$V(\boldsymbol{r},\boldsymbol{p},\boldsymbol{P},\boldsymbol{S}_1,\boldsymbol{S}_2) = V^{(0)} + \frac{V^{(1)}}{M} + \frac{V^{(2)}}{M^2} + \dots$$

- account for above-threshold and below-threshold states (for heavy quakonium X. Yao and T. Mehen [1811.07027])

$$\phi_{ij}(t,\boldsymbol{r},\boldsymbol{R}) = \int \frac{d^3\boldsymbol{P}}{(2\pi)^3}\left[\sum_n e^{-iE_nt+i\boldsymbol{P}\cdot\boldsymbol{R}}\, \Psi_n(\boldsymbol{r})\, S_{ij}\, \phi_n(\boldsymbol{P}) + \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} e^{-iE_pt+i\boldsymbol{P}\cdot\boldsymbol{R}}\, \Psi_p(\boldsymbol{r})\, S_{ij}\, \phi_p(\boldsymbol{P})\right]$$



pNRQED$_{\rm DM}$ at work: annihilation and decay

- unbound-pair annihilations and bound state decay originate from the same set of operators

$$
\begin{aligned}
(\sigma_{\rm ann} v_{\rm rel})(\boldsymbol{p}) &= \frac{1}{2M^2} \langle\, \boldsymbol{p}, 0 | \int d^3\boldsymbol{r}\, \phi^\dagger(\boldsymbol{r}, \boldsymbol{R}, t)\ \left[-{\rm Im}\delta {\rm V}^{\rm ann}(\boldsymbol{r})\right]\ \phi(\boldsymbol{r}, \boldsymbol{R}, t)\, |\boldsymbol{p}, 0\rangle \\
&= \frac{{\rm Im}(d_s) + 3{\rm Im}(d_v)}{M^2} |\Psi_{\boldsymbol{p}0}(0)|^2 = (\sigma^{\rm NR}_{\rm ann} v_{\rm rel})\, S_{\rm ann}(\zeta)
\end{aligned}
$$

- $S_{\rm ann}(\zeta) = \frac{2\pi\zeta}{1-e^{-2\pi\zeta}}$, $\zeta = \frac{1}{a_0 p} = \frac{\alpha}{v_{\rm rel}}$

$$
\Gamma_{\rm para} = \frac{4{\rm Im}(d_s)}{M^2} \frac{|R_{n0}(0)|^2}{4\pi} = \frac{M\alpha^5}{2}\left[1 + \frac{\alpha}{\pi}\left(\frac{\pi^2}{4} - 5 + \frac{4}{3}\log 2\right)\right]
$$

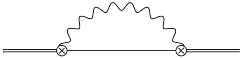
$$
\Gamma_{\rm ortho} = \frac{4{\rm Im}(d_v)}{M^2} \frac{|R_{n0}(0)|^2}{4\pi} = \frac{2(\pi^2-9)M\alpha^6}{9\pi}
$$

- orhto-darkonium is an $\mathcal{O}(\alpha^3)$ effect $\Rightarrow$ $(\sigma^{\rm NR}_{\rm ann} v_{\rm rel})$ at the same order

$$
(\sigma^{\rm NR}_{\rm ann} v_{\rm rel})_{\rm NLO} = \frac{\pi\alpha^2}{M^2}\left[1 + \frac{\alpha}{\pi}\left(\frac{19}{12}\pi^2 - 17 + \frac{4}{3}\log 2\right)\right]
$$

pNRQED$_{\rm DM}$ at work: bound-state formation

- $(X\bar{X})_p \to \gamma + (X\bar{X})_n$



$$\Sigma_p^{11} = -ig^2 \frac{d-2}{d-1} \mu^{4-d} r^i \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \frac{i}{p^0 - k^0 - H + i\epsilon} \times k_0^2 \left[\frac{i}{k_0^2 - |\boldsymbol{k}|^2 + i\epsilon} + 2\pi\delta(k_0^2 - |\boldsymbol{k}|^2) n_B(|k_0|) \right] r^i$$

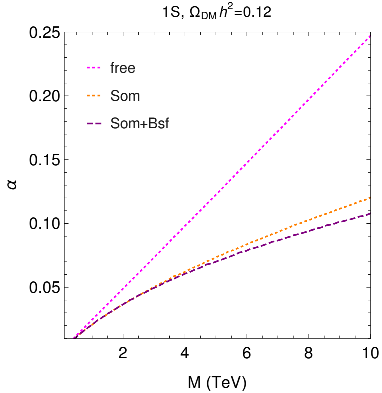
$$(\sigma_{\rm bsf}\, v_{\rm rel})(\boldsymbol{p}) \equiv \sum_n (\sigma_{(X\bar{X})_p \to \gamma + (X\bar{X})_n} v_{\rm rel})_n = \frac{g^2}{3\pi} \sum_n \left[1 + n_{\rm B}(\Delta E_n^p)\right] |\langle n | \boldsymbol{r} | \boldsymbol{p} \rangle|^2 (\Delta E_n^p)^3$$

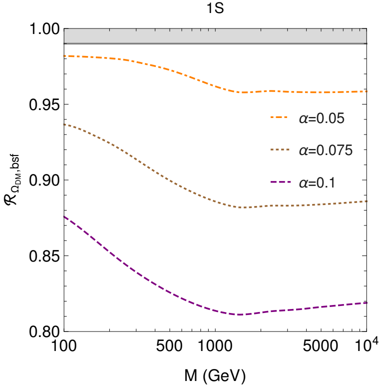
- Our work: matrix elements in a closed form for any bound-state see also M.Garny and J.Heisig [2112.01499]

$$(\sigma_{1\rm S\, bsf} v_{\rm rel})(\boldsymbol{p}) = \frac{\alpha^2 \pi^2 \, 2^{10} \, \zeta^5}{3\, M^2 \, (1+\zeta^2)^2} \frac{e^{-4\zeta \,\mathrm{arccot}\, \zeta}}{1 - e^{-2\pi\zeta}} \left[1 + n_{\rm B}(\Delta E_{1\rm S}^{p1})\right] \, , \quad \Delta E_{1\rm S}^{p1} = \frac{M v_{\rm rel}^2}{4} \left(1 + \frac{\alpha^2}{v_{\rm rel}^2}\right)$$

agreement with B. von Harling and K. Petraki [1407.7874], T. Binder, B. Blobel, J. Harz and K. Mukaida [2002.07145]

BSD and dark matter energy density





- Right plot: energy density with $\mathcal{O}(\alpha^3)$ or $\mathcal{O}(\alpha^2)$ for

$$\sigma_{bsf} v_{rel}\,, \quad \sigma_{ann} v_{rel}\,, \quad \Gamma_{ann}\,, \quad \Gamma_{bsd}$$

pNREFT for (pseudo)scalar mediators

Simplified model

$$\mathcal{L} = \bar{X}(i\partial\!\!\!/ - M)X + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \bar{X}(g + ig_5\gamma_5)X\phi + \mathcal{L}_{\text{portal}}\,,$$

M. B. Wise and Y. Zhang [1407.4121]; K. Kainulainen, K. Tuominen and V. Vaskonen [1507.04931]

- $M \gg M\alpha \gg \pi T \gtrsim M\alpha^2 \gg m$

$$L_{\text{pNRY}_{\gamma_5}} = \int d^3\boldsymbol{r}\, d^3\boldsymbol{R}\, \varphi^\dagger(\boldsymbol{r},\boldsymbol{R},t)\left\{ i\partial_0 + \frac{\boldsymbol{\nabla}_{\boldsymbol{r}}^2}{M} + \frac{\boldsymbol{\nabla}_{\boldsymbol{R}}^2}{4M} + \frac{\boldsymbol{\nabla}_{\boldsymbol{r}}^4}{4M^3} - V(\boldsymbol{p},\boldsymbol{r},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) \right.$$
$$\left. -2g\phi(\boldsymbol{R},t) - g\frac{r^i r^j}{4}\left[\nabla_R^i\nabla_R^j\,\phi(\boldsymbol{R},t)\right] - g\phi(\boldsymbol{R},t)\frac{\boldsymbol{\nabla}_{\boldsymbol{r}}^2}{M^2}\right\}\varphi(\boldsymbol{r},\boldsymbol{R},t) + L_{\text{scalar}}$$

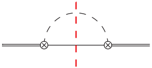
- Monopole, quadrupole and derivative interactions between the heavy pair and the mediator
- monopole contribution is zero $\langle\varphi_s|\varphi_p\rangle$, $\langle\varphi_s|\varphi_{s'}\rangle$, $\langle\varphi_b|\varphi_{b'}\rangle$

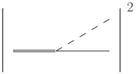
  exceptions for models with charged scalar and vector mediator, see R. Onacala and K. Petraki 1911.02605

Formation, Dissociation and annihilations

- $\varphi_s \to \varphi_b + \phi$

$$\sigma_{\rm bsf} v_{\rm rel}\Big|_T = \sigma_{\rm bsf} v_{\rm rel}\Big|_{T=0} \left[1 + n_{\rm B}(\Delta E_n^p)\right]$$





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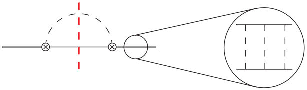

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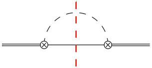
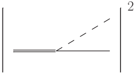
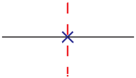
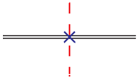




- Pair annihilations

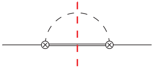
$$\Gamma^{nS}_{\rm ann} = \langle nS|\, 2{\rm Im}(-V^{\rm ann})\,|nS\rangle$$
$$\sigma_{\rm ann} v_{\rm rel} = \langle \boldsymbol{p}|\, 2{\rm Im}(-V^{\rm ann})\,|\boldsymbol{p}\rangle$$

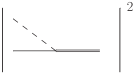




- $\varphi_b + \phi \to \varphi_b$

$$\Gamma^n_{\rm bsd} = \int_{|\boldsymbol{k}|\geq|E_n|} \frac{d^3\boldsymbol{k}}{(2\pi)^3}\; n_B(|\boldsymbol{k}|)\, \sigma^n_{\rm bsd}(|\boldsymbol{k}|)$$





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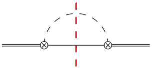

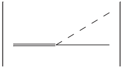

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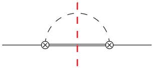

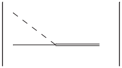

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- $\varphi_b \to \phi\phi$,

$$\Gamma^{nS}_{\rm ann} = \frac{|R_{nS}(0)|^2}{\pi M^2}\left\{{\rm Im}[f(^1S_0)] + \frac{E_n}{M}{\rm Im}[g(^1S_0)]\right\}$$
$$= \frac{M\,\alpha^4\alpha_5}{n^3}\left(1 + \frac{\alpha^2}{3n^2}\right)$$

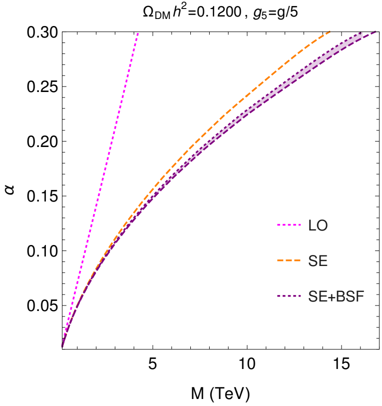
- $\varphi_s \to \phi\phi$

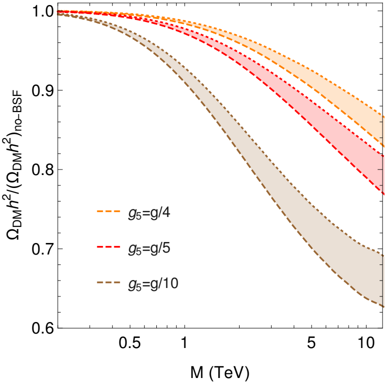
$$\sigma_{\rm ann} v_{\rm rel} = \frac{2\pi\alpha\alpha_5}{M^2}\left(1 - \frac{v^2_{\rm rel}}{3}\right) S(\zeta)$$
$$+ \frac{(9\alpha^2 - 2\alpha\alpha_5 + \alpha_5^2)v^2_{\rm rel}}{24M^2} S_p(\zeta)$$

Energy density

$$\frac{dn_X}{dt} + 3Hn_X = -\langle\sigma_{\rm eff}\, v_{\rm rel}\rangle(n_X^2 - n_{X,{\rm eq}}^2)\,, \quad \langle\sigma_{\rm eff}\, v_{\rm rel}\rangle = \langle\sigma_{\rm ann}\, v_{\rm rel}\rangle + \sum_n \langle\sigma^n_{\rm bsf}\, v_{\rm rel}\rangle \frac{\Gamma^n_{\rm ann}}{\Gamma^n_{\rm ann} + \Gamma^n_{\rm bsd}}$$

one-single Boltzmann equation, see J. Ellis, F. Luo, and K. A. Olive [1503.07142]

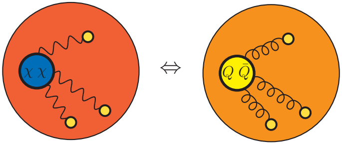




Conclusions

- DM particles and thermal freeze-out in the early universe:
  non-relativistic particles in a thermal environment
- Many models feature vector or scalar force carriers between DM particles

  ⇒ Similarities with heavy quarkonia/NR pairs at finite temperature



- Adapted NREFTs and pNREFT techniques for determining dark matter energy density
- Borrowed from NRQED, pNRQED and developed a NR and pNR for scalar mediators

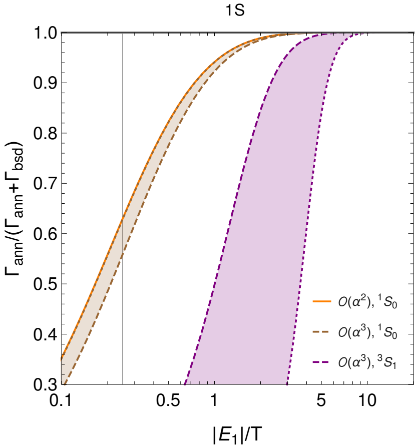
  ⇒ Large effects on the model parameter space

Outlook

- Inspect the thermal scale $\sqrt{MT}$ and Debye mass
- Consider non-abelian models, mixed mediator models
- Derive evolution equations from pNREFTs: connection with Open-Quantum Systems

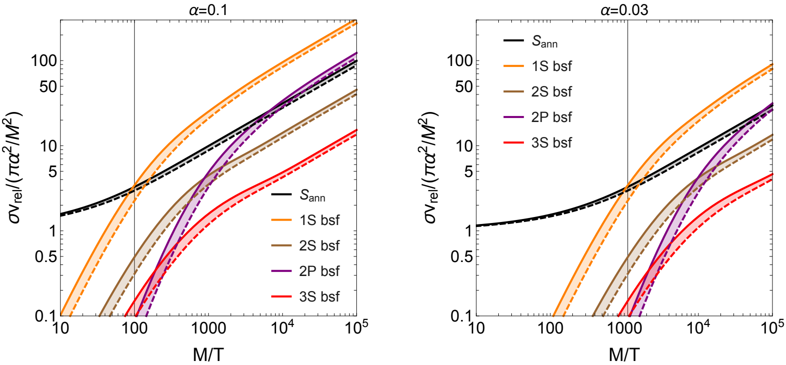
Back up

$$G^{\phi}(p_0) \approx \begin{pmatrix} \dfrac{i}{p_0 - H + i\epsilon} & 0 \\ 2\pi\delta(p_0 - H) & \dfrac{-i}{p_0 - H - i\epsilon} \end{pmatrix} .$$



Thermal average

$$\langle \sigma_{\rm ann} v_{\rm rel} \rangle = \sqrt{\frac{2}{\pi}} \left(\frac{M}{2T} \right)^{\frac{3}{2}} \int_0^\infty dv_{\rm rel}\, v_{\rm rel}^2\, e^{-\frac{M}{4T} v_{\rm rel}^2} \sigma_{\rm ann} v_{\rm rel} \,,$$



S

$$
\begin{aligned}
&\mathcal{L}_{\rm NRY} = \\
&\psi^\dagger \left(i\partial_0 + c_1\, g\phi + c_2 \frac{\boldsymbol{\nabla}^2}{2M} + c_3 \frac{\phi^2}{M} + c_4 \frac{\phi^3}{M^2} + c_{\rm D} \frac{g\,\{\boldsymbol{\nabla}, \{\boldsymbol{\nabla}, \phi\}\}}{8M^2} + i c_{\rm S} \frac{g \sigma^i \epsilon^{ijk} \boldsymbol{\nabla}^j \phi \boldsymbol{\nabla}^k}{4M^2} \right) \psi \\
&+ \chi^\dagger \left(i\partial_0 + c_1'\, g\phi + c_2' \frac{\boldsymbol{\nabla}^2}{2M} + c_3' \frac{\phi^2}{M} + c_4' \frac{\phi^3}{M^2} + c_{\rm D}' \frac{g\,\{\boldsymbol{\nabla}, \{\boldsymbol{\nabla}, \phi\}\}}{8M^2} + i c_{\rm S}' \frac{g \sigma^i \epsilon^{ijk} \boldsymbol{\nabla}^j \phi \boldsymbol{\nabla}^k}{4M^2} \right) \chi \\
&+ \mathcal{L}_{\hbox{4-fermions}} \\
&+ \frac{d_1}{2} \partial_\mu \phi\, \partial^\mu \phi - d_2 \frac{m^2}{2} \phi^2 + \frac{d_3}{4!} \phi^4 + \frac{d_4}{M^2} (\partial^\mu \phi) \partial^2 (\partial_\mu \phi) + \frac{d_5}{M^2} (\phi \partial^\mu \phi)(\phi \partial_\mu \phi)\,,
\end{aligned}
$$

NRY and pNRY

- four-fermion operators

$$(\mathcal{L}_{\text{4-fermions}})_{d=6} = \frac{f(^1S_0)}{M^2}\psi^\dagger\chi\,\chi^\dagger\psi + \frac{f(^3S_1)}{M^2}\psi^\dagger\,\boldsymbol{\sigma}\,\chi\cdot\chi^\dagger\,\boldsymbol{\sigma}\,\psi$$

$$\begin{aligned}(\mathcal{L}_{\text{4-fermions}})_{d=8} &= \frac{f(^1P_1)}{M^4}\mathcal{O}(^1P_1) + \frac{f(^3P_0)}{M^4}\mathcal{O}(^3P_0) + \frac{f(^3P_1)}{M^4}\mathcal{O}(^3P_1) \\ &+ \frac{f(^3P_2)}{M^4}\mathcal{O}(^3P_2) + \frac{g(^1S_0)}{M^4}\mathcal{P}(^1S_0) + \frac{g(^3S_1)}{M^4}\mathcal{P}(^3S_1) \\ &+ \frac{g(^3S_1,^3D_1)}{M^4}\mathcal{P}(^3S_1,^3D_1) + \cdots\end{aligned}$$

- matching coefficients

$$\mathrm{Im}[f(^1S_0)] = 2\pi\alpha\alpha_5\,, \quad \mathrm{Im}[f(^3S_1)] = 0$$

$$\begin{aligned}&\mathrm{Im}[f(^1P_1)] = \mathrm{Im}[f(^3P_1)] = 0\,, \\ &\mathrm{Im}[f(^3P_0)] = \frac{\pi}{6}(5\alpha - \alpha_5)^2\,, \quad \mathrm{Im}[f(^3P_2)] = \frac{\pi}{15}(\alpha+\alpha_5)^2\,, \\ &\mathrm{Im}[g(^1S_0)] = -\frac{8\pi}{3}\alpha\alpha_5\,, \quad \mathrm{Im}[g(^3S_1)] = \mathrm{Im}[g(^3S_1,^3D_1)] = 0\,.\end{aligned}$$

BSF and BSD

- in-vaccum bsf cross section

$$\sigma_{\rm bsf} v_{\rm rel}\Big|_{T=0} = \frac{\alpha}{120}\sum_n (\Delta E_n^p)^5 \left[|\langle \boldsymbol{p}|\boldsymbol{r}^2|n\rangle|^2 + 2|\langle \boldsymbol{p}|r^i r^j|n\rangle|^2 \right]$$
$$+2\alpha \sum_n \Delta E_n^p \left| \left\langle \boldsymbol{p} \right| \frac{\nabla_{\boldsymbol{r}}^2}{M^2} \left| n \right\rangle \right|^2$$
$$-\frac{\alpha}{3}\sum_n (\Delta E_n^p)^3 \,{\rm Re}\left[\left\langle \boldsymbol{p} \right| \frac{\nabla_{\boldsymbol{r}}^2}{M^2} \left| n \right\rangle \langle n|\boldsymbol{r}^2|p\rangle \right] .$$

- bound-state dissociation and dissociation cross section

$$\Gamma_n = \int_{|\boldsymbol{k}|\geq |E_n|} \frac{d^3\boldsymbol{k}}{(2\pi)^3}\; n_B(|\boldsymbol{k}|)\,\sigma^n_{\rm bsd}(|\boldsymbol{k}|) .$$

$$\sigma^n_{\rm bsd}(|\boldsymbol{k}|) = \alpha M^{\frac{3}{2}}\sqrt{|\boldsymbol{k}|+E_n}\left\{ \frac{|\boldsymbol{k}|^3}{240}\left[|\langle \boldsymbol{p}|\boldsymbol{r}^2|n\rangle|^2 + 2|\langle \boldsymbol{p}|r^i r^j|n\rangle|^2 \right] \right.$$
$$\left. + \frac{1}{|\boldsymbol{k}|}\left| \left\langle \boldsymbol{p} \right| \frac{\nabla_{\boldsymbol{r}}^2}{M^2} \left| n \right\rangle \right|^2 - \frac{|\boldsymbol{k}|}{6}{\rm Re}\left[\left\langle \boldsymbol{p} \right| \frac{\nabla_{\boldsymbol{r}}^2}{M^2} \left| n \right\rangle \langle n|\boldsymbol{r}^2|p\rangle \right] \right\}\Big|_{|\boldsymbol{p}|=\sqrt{M(|\boldsymbol{k}|+E_n)}} .$$