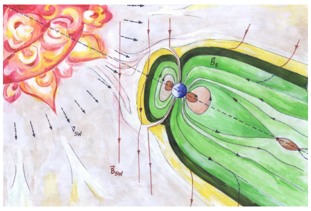
Energy principle for the determination of instabilities in MHD systems

C.-V. Meister, Technische Universität Darmstadt, Germany



Artistic picture of the Sun-Earth system.
Blue - Earth, green - magnetosphere, dark green - magnetopause, yellow - magnetosheath, white - bow shock.

Motivation

Magnetohydrodynamic (MHD) instabilities play a key role in astrophysics and high energy density physics. Thus, at the Technische Universität Darmstadt, new projects are being under preparation that consider the dynamics of MHD instabilities.
Usually, wave growth rates are studied using kinetic theory, as also wave excitation by inverse Landau damping is possible. Here, based on former works by K.-H. Spatschek (Theoretische Plasmaphysik, Teubner Studienbücher, 1991) and P.A. Sturrock (Plasma Physics, Cambridge University Press, 1994), a simple MHD method is presented which should help to roughly evaluate parameter regions of MHD instabilities.

Energy principle

The momentum balance of the linearized MHD system of equations is expressed by the Lagrange velocity $\vec{\xi}$ using the relation between the Euler velocity $\vec{u}(\vec{r},t)$ and the Lagrange velocity of a fluid element which was situated at $\vec{r}_o$ at time t_o

$$\frac{\partial\vec{\xi}(\vec{r}_o,t)}{\partial t} = \vec{u}(\vec{r},t) \approx \vec{u}(\vec{r}_o,t) + \vec{\xi}\nabla\vec{u} + ... \approx \vec{u}(\vec{r}_o,t) \qquad (1)$$

Then, the linearized equation may be represented by

$$\rho_o \frac{\partial^2\vec{\xi}}{\partial t^2} = \vec{F}(\vec{\xi}). \qquad (2)$$

Further it is assumed that the separation of variables by

$$\vec{\xi}(\vec{r}_o,t) = \vec{\xi}_k(\vec{r}_o)\tau_k(t) \qquad (3)$$

is possible, so that the function $\tau_k(t)$ may be represented by a plain wave

$$\tau_k \sim \mathrm{exp}[i(\omega_k t + \varphi_k)]. \qquad (4)$$

ω_k are the solutions of the equation

$$-\rho_o\omega_k^2\vec{\xi}_k = \vec{F}(\vec{\xi}_k). \qquad (5)$$

Because of the linearity of the problem the solution of eq. (2) may be represented by

$$\vec{\xi}(\vec{r}_o,t) = \sum_k a_k\vec{\xi}_k(\vec{r}_o)\mathrm{exp}[i(\omega_k t + \varphi_k)]. \qquad (6)$$

Thus, the MHD system is (exponentially) stable if - and only if - all eigenvalues of $\vec{F}/\rho_o$ are negative.
For the potential energy density W_F, $\vec{F} = -\nabla W_F$, it follows

$$W_F = -\int_0^{\vec{\xi}} \vec{F}(\vec{\eta})\, d\vec{\eta} = -\frac{\vec{\xi}}{2}\vec{F} \qquad (7)$$

In case of negative potential energy, the system is unstable, or, with other words, for the stability of the hydrodynamic system it is necessary and sufficient that the potential energy has no negative values.
Taking the hydrodynamic stability condition for ideal plasmas into account

$$\nabla\, p_o = \frac{1}{4\pi}(\nabla\times\vec{B}_o)\times\vec{B}_o, \qquad (8)$$

Spatschek and Sturrock found, that eq. (7) may be used for the determination of instabilities in plasmas, surrounded by a vacuum and placed in a fixed conducting surface, provided that additional vacuum and surface contributions of the potential energy are taken into account. For adiabatic systems follows:
volume contribution to potential energy:

$$W_F = \frac{1}{2}\int_{V_p}\left[\frac{Q^2}{4\pi} + \frac{1}{4\pi}(\nabla\times\vec{B}_o)(\vec{\xi}\times\vec{Q}) + (\nabla\vec{\xi})\vec{\xi}\cdot\nabla p_o + \gamma p_o(\nabla\vec{\xi})^2\right]\, d\vec{r}, \qquad (9)$$

surface and vacuum contributions to potential energy:

$$W_S + W_V = \frac{1}{2}\int_S (\vec{\xi}\cdot\vec{n})^2\vec{n}\cdot\nabla\left(p_o + \frac{B_o^2}{8\pi} - \frac{B_V^2}{8\pi}\right)\, d\vec{S} + \frac{1}{8\pi}\int_S \vec{B}_V\cdot\delta\vec{B}_V\,\xi\cdot\, d\vec{S}, \qquad (10)$$

contributions to potential energy by variation of vacuum magnetic field:

$$W_V = \frac{1}{8\pi}\int_{V_V}(\delta\vec{B}_V)^2\, d\vec{r}. \qquad (11)$$

$$\vec{Q} = \nabla\times(\vec{\xi}\times\vec{B}_o). \qquad (12)$$

Application to an internally homogeneous linear pinch

- In a homogeneous plasma, an axial magnetic field $B_z\vec{n}_z$ exists.
- On a metallic cylinder surrounding the plasma, axial electric currents flow.
- These currents generate an external azimuthal magnetic field $B_\varphi\vec{n}_\varphi$.
- Between plasma and metallic cylinder, a vacuum exists.
- In the metallic cylinder, $\nabla\times\vec{B}_z = 0$ and $\nabla p_o = 0$.

Thus, one has to determine, amongst others, the minimum of the potential energy

$$W_F = \frac{1}{8\pi}\int\left[\vec{Q}^2 + 4\pi\gamma p_o(\nabla\times\xi)^2\right]\, d\vec{r}. \qquad (13)$$

It is found by the variational method using the Euler equation (Spatschek)

$$\frac{1}{4\pi}(\nabla\times\vec{Q})\times\vec{B}_z + \gamma p_o\nabla(\nabla\vec{\xi}), \qquad (14)$$

which gives for the minimum potential energy of the plasma in the cylinder

$$\text{Min } W_F = \frac{1}{2}\int\left(\gamma p_o\nabla\xi - \frac{\vec{B}_z\vec{Q}}{4\pi}\right)(\vec{\xi}\cdot d\vec{F}). \qquad (15)$$

To calculate the integral of eq. (14), ξ has to be determined from eq. (2). Assuming that ξ is periodic in z and φ, one finds from (2) ($\vec{B}_z$ = const)

$$-\omega\rho\xi_z = \gamma p_o\frac{\partial}{\partial z}\nabla\xi, \qquad (16)$$

$$-\omega\rho\xi_r = \gamma p_o\frac{\partial}{\partial r}\nabla\xi - \frac{mk}{4\pi}\frac{B_z^2\xi_\varphi}{r} - \frac{k^2}{4\pi}B_z^2\xi_r + B_z^2\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r\xi_r), \qquad (17)$$

$$-\omega\rho\xi_\varphi = \frac{\gamma p_o}{r}\frac{\partial}{\partial\varphi}\nabla\xi + \frac{im}{4\pi}\frac{B_z^2}{r^2}\frac{\partial}{\partial r}(r\xi_r) - \frac{ik}{4\pi}B_z^2\frac{\partial}{\partial r}\xi_\varphi. \qquad (18)$$

In case of $\partial/\partial\varphi = 0$ (i.e. $m = 0$), from eqs. (16, 17) the expression

$$\frac{\partial^2\xi_z}{\partial r^2} + \frac{1}{r}\frac{\partial\xi_z}{\partial r} - \alpha^2\xi_z = 0, \quad \alpha^2 = \frac{(v_s^2k^2 - \omega^2)(v_A^2k^2 - \omega^2)}{v_s^2v_A^2k^2 - (v_s^2 + v_A^2)\omega^2} \qquad (19)$$

follows, which was already derived by Sturrock studying the sausage instability. Thus

$$\xi_z(r) = \frac{\text{const}}{\alpha^2}I_o(\alpha r), \qquad (20)$$

where I_o is the modified Bessel function of first kind of order zero.

Tasks for future work

- Numerical solution of the system of equations (15-18) for $m \neq 0$.
- Further development of the energy principle for anisotropic systems.
- Application of the energy principle to planetary transition layers.
- Determination of growth rates of Kelvin-Helmholtz and Rayleigh-Taylor instabilities in astrophysical plasmas and high energy density systems.

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