Potential in $N_f = 2$ QCD

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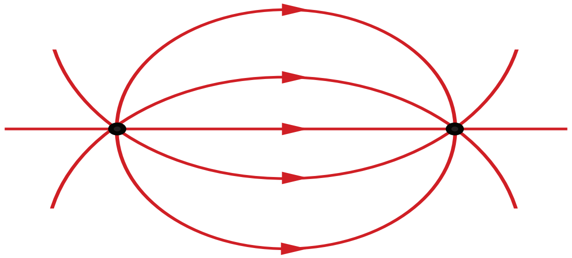
QwG

Introduction

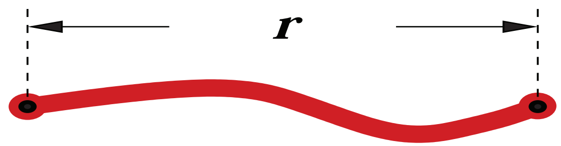
Energy levels of a static (infinitely heavy) quark and anti-quark pair at distance r

Pure gauge theory:



$r < 0.1$ fm

asymptotic freedom
perturbation theory



$r >> 1$ fm

confinement
effective bosonic string theory

With sea quarks:

Around $\mathbf{r}_\mathrm{b} \approx \mathbf{1.5\,fm}$ formation of two static-light mesons (string breaking)

Introduction

Static quark ψ_h

$$P_+ \, \psi_h = \psi_h \qquad ; \quad \overline{\psi}_h \, P_+ = \overline{\psi}_h$$

For static anti-quark $\psi_{\bar{h}}$: $P_+ = (1+\gamma_0)/2 \longrightarrow P_- = (1-\gamma_0)/2$

Static lattice Lagrangians [Eichten and Hill, 1990]

$$\mathcal{L}_h \quad = \quad \frac{1}{1+a\delta m}\overline{\psi}_h(x)[D_0 + \delta m]\psi_h(x)$$

$D_0\psi_h(x) = \frac{1}{a}[\psi_h(x) - U(x-a\hat{0},0)^\dagger\psi_h(x-a\hat{0})]$
δm is a mass counter term and yields an energy shift $\widehat{\delta m} = \frac{1}{a}\ln(1+a\delta m)$ for any state

Static propagators

$$\left\langle \psi_h(x)\,\overline{\psi}_h(y) \right\rangle \quad = \quad \theta(x_0 - y_0)\delta(\vec{x}-\vec{y})\mathrm{e}^{-\widehat{\delta m}(x_0-y_0)}P(y,x)^\dagger P_+$$

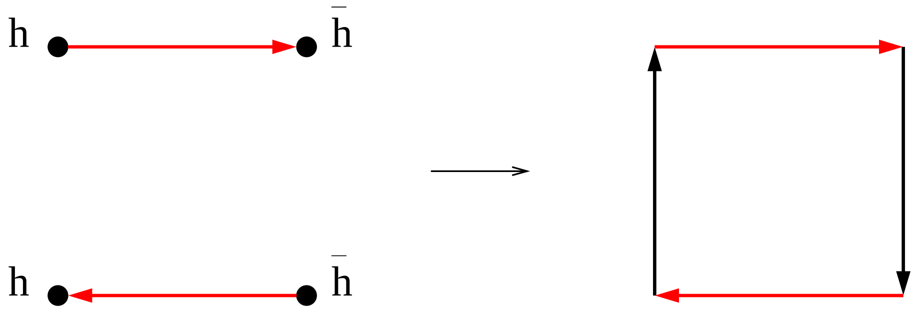
cf review on heavy quark effective theory [Sommer, Les Houches, 2009]

The static energies for $\mathbf{r} < \mathbf{r}_\mathrm{b}$ can be extracted from **Wilson loops** $\mathbf{W}(r,T)$

Introduction

Annihilation/creation of a quark anti-quark pair at $t = 0/t = T$

$$
\begin{aligned}
O(0, r\hat{k}) &= \overline{\psi}_h(0) P(0; r\hat{k}) \gamma_5 \psi_{\bar{h}}(r\hat{k}) \\
\bar{O}(T\hat{0} + r\hat{k}, T\hat{0}) &= -\overline{\psi}_{\bar{h}}(T\hat{0} + r\hat{k}) P^\dagger(T\hat{0}; T\hat{0} + r\hat{k}) \gamma_5 \psi_h(T\hat{0})
\end{aligned}
$$



$$
\frac{1}{2} \langle O(0, r\hat{k}) \bar{O}(T\hat{0} + r\hat{k}, T\hat{0}) \rangle_{U,\psi,\overline{\psi},\psi_h,\overline{\psi}_h,\psi_{\bar{h}},\overline{\psi}_{\bar{h}}} = \mathrm{e}^{-2T\widehat{\delta m}} \langle \mathbf{W}(r, T) \rangle
$$

$$
\langle W(r, T) \rangle = \left\langle \mathrm{tr} \left\{ P(0; r\hat{k})\, P(r\hat{k}; r\hat{k} + T\hat{0})\, P^\dagger(T\hat{0}; T\hat{0} + r\hat{k})\, P^\dagger(0, T\hat{0}) \right\} \right\rangle_{U,\psi,\overline{\psi}}
$$

Static quark anti-quark potential

$$
V_{\mathrm{reno.}}(r) = -\lim_{T \to \infty} \partial_T \ln(\langle W(r, T) \rangle) + 2\widehat{\delta m}
$$

Introduction

Due to confinement

$$\langle W(r,T)\rangle \text{ “} = \langle \pm \rangle \text{ “} \quad \approx \quad \exp(-\sigma r T)$$

but

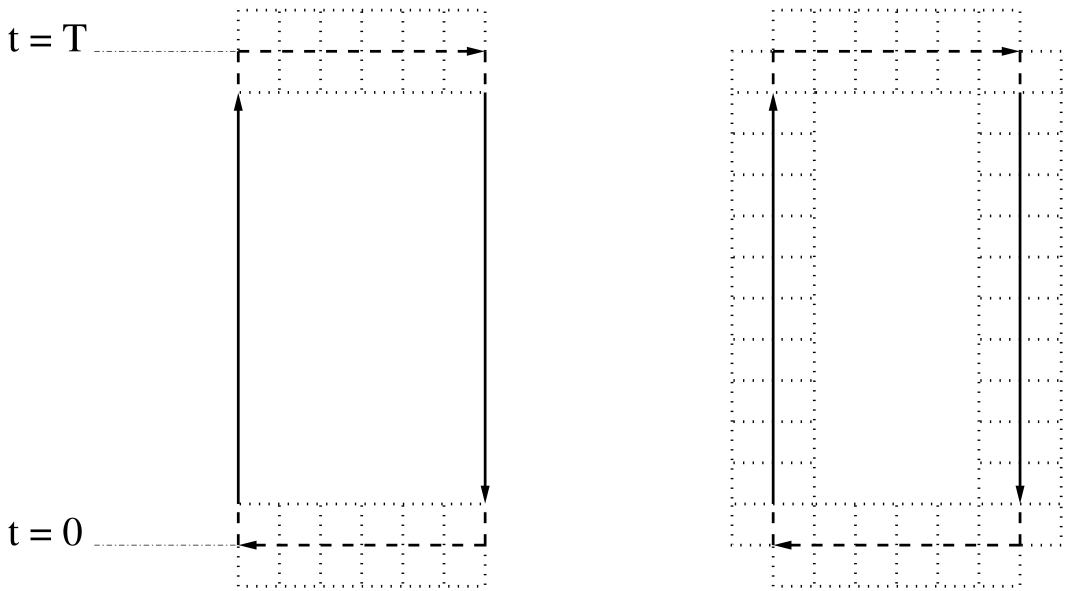
$$\left\langle W(r,T)^2 \right\rangle \text{ “} = \langle + \rangle \text{ “} \quad \approx \quad \text{const}$$

$\Rightarrow$ noise-to-signal ratio grows exponentially with the area of the loop

- in pure gauge theory there is a cure: exponential suppression of the statistical noise through multi-hit method [Parisi, Petronzio and Rapuano, 1983] and more efficient multilevel algorithm [Lüscher and Weisz, 2001]. But these methods are not applicable ...
- with dynamical fermions. Here we use the method of HYP smearing the links in the Wilson loops [A. Hasenfratz and Knechtli, 2001; A. Hasenfratz, R. Hoffmann and Knechtli, 2002; ALPHA, Della Morte et al., 2004]

Techniques

HYP smeared Wilson loops [Donnellan, Knechtli, BL and Sommer, 2011]



- (left figure) HYP smearing of space-like links corresponds (in the Hamiltonian formalism) to an operator $\hat{O}^\dagger$ that creates a $|Q\overline{Q}(r)\rangle$ state
- (right figure) HYP smearing of the time-like links corresponds to the choice of a static quark action (and a modification of the operator $\hat{O}$)

Techniques

- HYP2 static quark action: $\alpha_1 = 1.0\,, \quad \alpha_2 = 1.0\,, \quad \alpha_3 = 0.5$.
  Binding energy of a meson made of a static and a dynamical quark

$$E_{\rm stat}|_{\delta m=0} \quad \sim \quad \frac{1}{a} e^{(1)} g_0^2 + \ldots$$

  HYP2 smearing minimizes $e^{(1)}$ [Della Morte, Shindler and Sommer, 2005]
- Static energies $V_n(r)$

$$\langle W(r,T) \rangle \quad \sim \quad \sum_n c_n c_n^* \mathrm{e}^{-V_n(r)(T-2a)} \quad \text{(with } N_t \to \infty \text{ time-slices)}$$

  c_n depends on $\hat{O}$; $V_n(r)$ on the static quark action; relies on existence of transfer matrix for Wilson fermions without clover term and Wilson plaquette action [Lüscher, 1977]
- Lattice artifacts with $\mathcal{O}(a)$ improved dynamical fermions

$$V_n^{\rm HYP2}(r) - 2E_{\rm stat}^{\rm HYP2} = V_n^{\rm continuum}(r) - 2E_{\rm stat}^{\rm continuum} + \mathcal{O}(a^2)$$

  relies on automatic $\mathcal{O}(a)$ improvement of heavy quark effective theory [Kurth and Sommer, 2001; Necco and Sommer, 2002]

Techniques

- variational basis: space-like links ($\leftrightarrow$ operator $\hat{O}$) are smeared using n_l iterations of *spatial* HYP smearing

$$\langle W(r,t)\rangle \quad \longrightarrow \quad C_{lm}(r,T)$$

we take a basis with $M = 3$ levels ($n_{2,3,5} = 8, 12, 20$ at $\beta = 5.3$)

- generalized eigenvalue method to extract V_α [Lüscher and Wolff, 1990]

$$\begin{aligned} C(t)\,\psi_\alpha &= \lambda_\alpha(t,t_0)\,C(t_0)\,\psi_\alpha\,, \qquad \alpha = 0,1,\ldots,M-1 \\ E_\alpha(t+\frac{a}{2},t_0) &= \ln\left(\lambda_\alpha(t,t_0)/\lambda_\alpha(t+a,t_0)\right) \end{aligned}$$

if $t_0 + a \leq t \leq 2t_0$ [Blossier et al., 2009]

$$E_\alpha(t+\frac{a}{2},t_0) \quad = \quad E_\alpha + \beta_\alpha \mathrm{e}^{-(E_M-E_\alpha)(t+\frac{a}{2})}$$

we fit this expression to determine the systematic error σ_{sys}

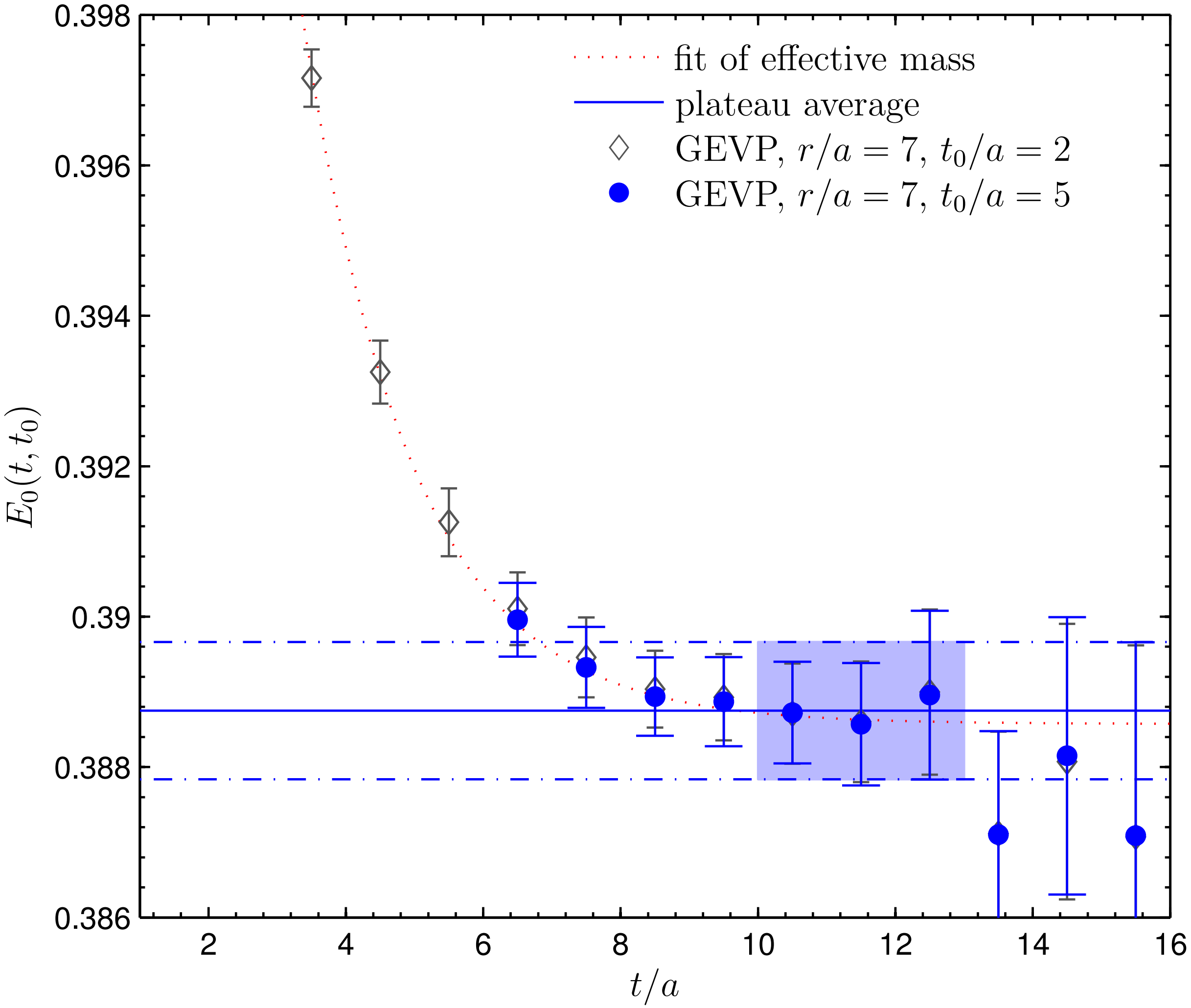
- ground state static potential V = plateau average of $E_0(t,t_0)$
starts at $t = 2t_0$; the value of t_0 is determined by the requirement that σ_{sys} is smaller than $1/4$ of the statistical error (at $t = 2t_0$)

Potential

CLS ensemble E5g (https://twiki.cern.ch/twiki/bin/view/CLS/WebHome)

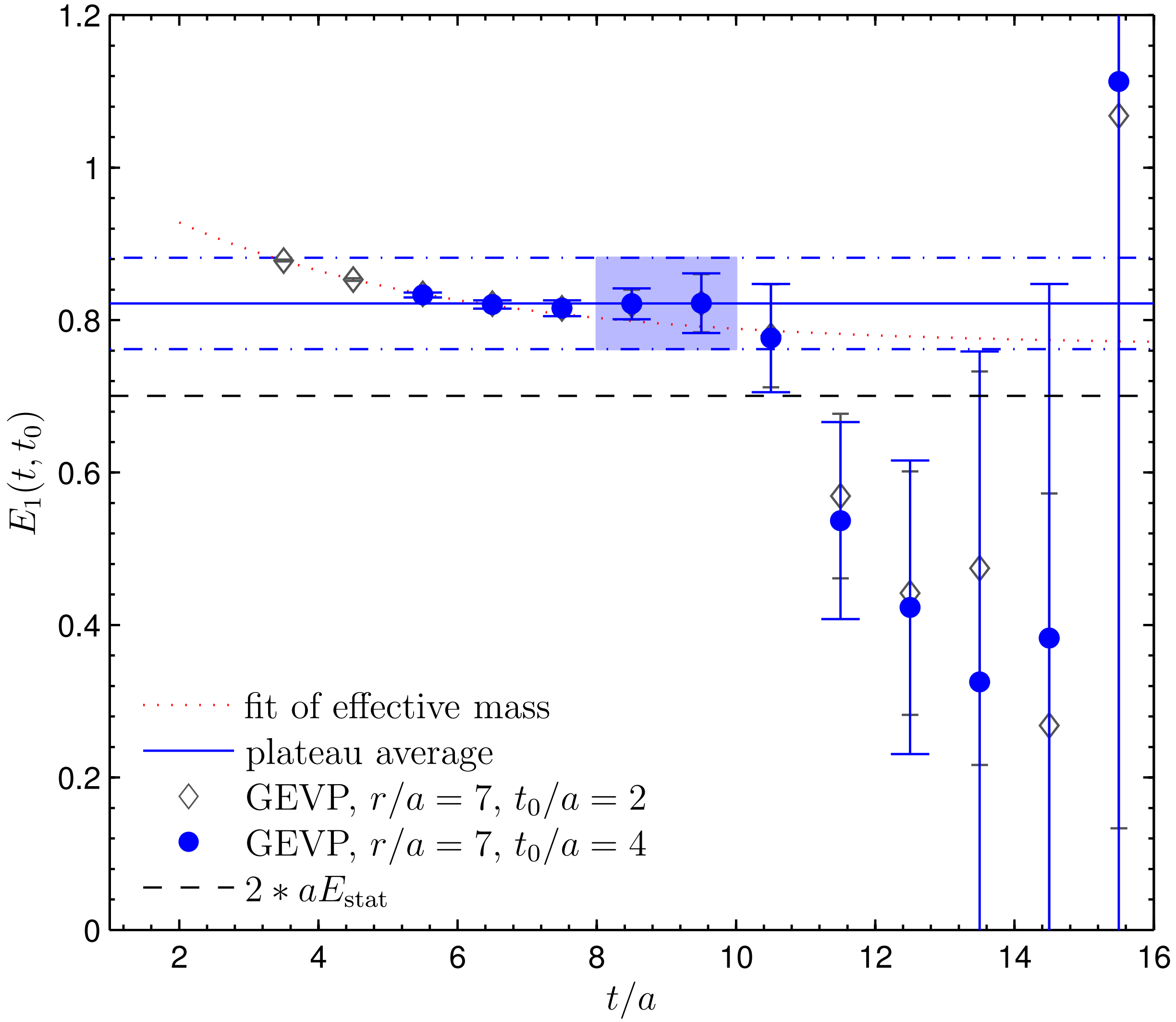
- Wilson gauge action and $N_\mathrm{f} = 2$ flavors of $\mathcal{O}(a)$ improved Wilson quarks with periodic boundary conditions for all fields apart from anti-periodic boundary conditions for the fermions in time
- $\beta = 5.3$, $\kappa = 0.13625$, 64×32^3
- deflation accelerated DD-HMC algorithm [Lüscher 2005; 2007] with trajectory length $\tau = 4$
- 1000 Wilson loops measurements separated by approximately 6 molecular dynamics units

Potential



fit of effective mass
plateau average
GEVP, $r/a = 7$, $t_0/a = 2$
GEVP, $r/a = 7$, $t_0/a = 5$
$E_0(t, t_0)$
t/a
0.398
0.396
0.394
0.392
0.39
0.388
0.386
2
4
6
8
10
12
14
16

Potential

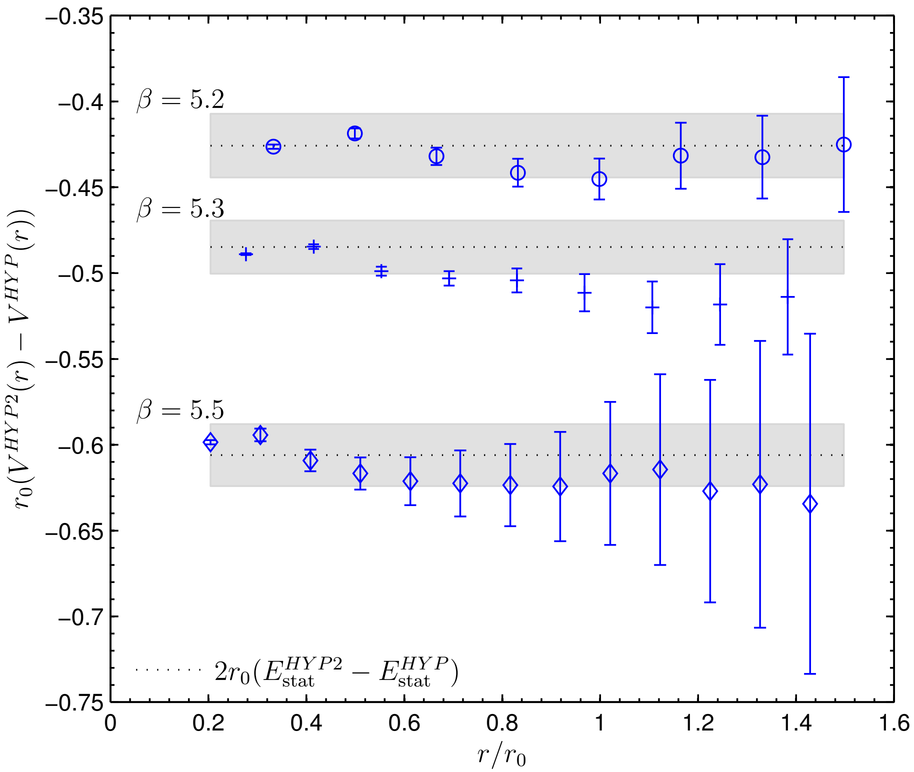

1.2
1
0.8
0.6
0.4
0.2
0
2
4
6
8
10
12
14
16
$E_1(t, t_0)$
t/a
fit of effective mass
plateau average
GEVP, $r/a = 7$, $t_0/a = 2$
GEVP, $r/a = 7$, $t_0/a = 4$
$2 * aE_{\text{stat}}$

Potential

Comparison with HYP static action: $\alpha_1 = 0.75\,,\quad \alpha_2 = 0.6\,,\quad \alpha_3 = 0.3$

$$r_0\,[V^{\mathrm{HYP2}}(r) - V^{\mathrm{HYP}}(r)] \quad = \quad 2\,r_0\,(E^{\mathrm{HYP2}}_{\mathrm{stat}} - E^{\mathrm{HYP}}_{\mathrm{stat}}) + \frac{a^2\,r_0}{r^3}\,G(\Lambda r, m_q r)$$

At three lattice spacings ($\beta = 5.2$, 5.3 and 5.5) for $r_0 m_{\mathrm{PS}}(m_q) \approx 1$

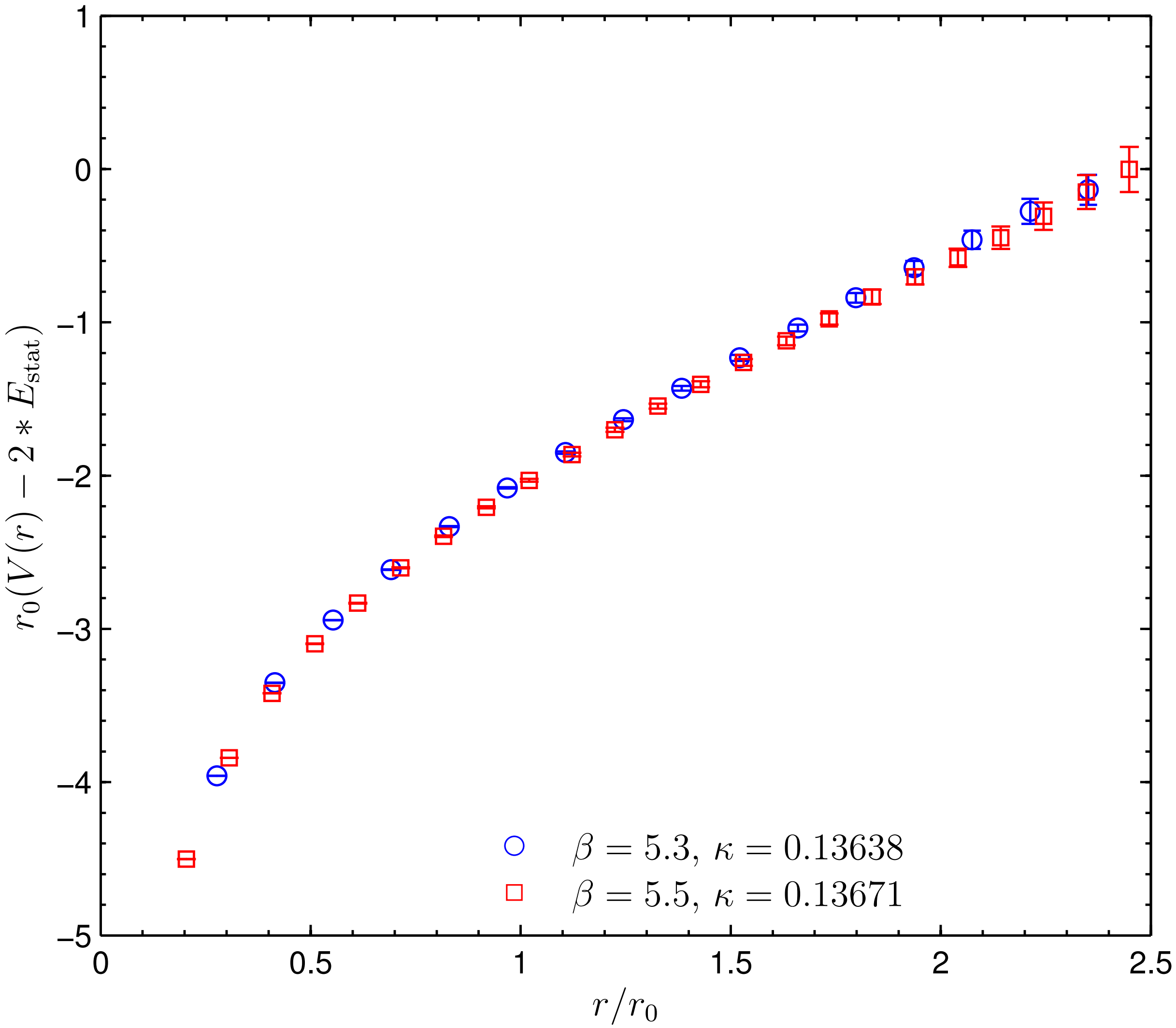


Potential

Comparison of two ensembles at quark mass corresponding to $(r_0 m_{\mathrm{PS}})(m_q) \approx 0.62 \ldots 0.64$:

- F7 at $\beta = 5.3$, $\kappa = 0.13638$, 96×48^3:
  $r_0/a(\mathrm{F7}) = 7.05(4)$
- O7 at $\beta = 5.5$, $\kappa = 0.13671$, 128×64^3:
  $r_0/a(\mathrm{O7}) = 9.63(12)$

Potential


1
0
−1
−2
−3
−4
−5
0
0.5
1
1.5
2
2.5
$r_0(V(r) - 2 * E_{\mathrm{stat}})$
r/r_0
$\beta = 5.3,\ \kappa = 0.13638$
$\beta = 5.5,\ \kappa = 0.13671$

Shape of the potential

- First derivative of the potential: static force $F(r)$
  Defines a running coupling (qq-scheme)

$$\bar{g}^2_{\mathrm{qq}}(\mu) \quad = \quad \frac{4\pi}{C_\mathrm{F}} r^2 F(r) \, , \;\; \mu = 1/r$$

  where $C_\mathrm{F} = 4/3$ for SU(3)

- Second derivative of the potential: slope $c(r) = \frac{1}{2} r^3 F'(r)$
  Defines a running coupling (c-scheme)

$$\bar{g}^2_{\mathrm{c}}(\mu) \quad = \quad -\frac{4\pi}{C_\mathrm{F}} c(r) \, , \;\; \mu = 1/r$$

  Like for the force, improved lattice definition [Lüscher and Weisz, 2002]

$$c(\tilde{r}) \quad = \quad \frac{1}{2}\tilde{r}^3 [V(r+a) + V(r-a) - 2V(r)]/a^2$$

  $\tilde{r} = r + \mathcal{O}(a^2)$ is chosen such that at tree level $c_\mathrm{tree}(\tilde{r}) = -C_\mathrm{F}\, g_0^2/(4\pi)$

Shape of the potential

Renormalization group (RG) equation in the scheme S (S=qq,c)

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \bar{g}_{\mathrm{S}}(\mu) = \beta_{\mathrm{S}}(\bar{g}_{\mathrm{S}}(\mu))$$

It is solved by

$$\frac{\Lambda_{\mathrm{S}}}{\mu} = \left(b_0 \bar{g}_{\mathrm{S}}^2\right)^{-b_1/(2b_0^2)} \mathrm{e}^{-1/(2b_0 \bar{g}_{\mathrm{S}}^2)} \exp\left\{-\int_0^{\bar{g}_{\mathrm{S}}} \mathrm{d}x \left[\frac{1}{\beta_{\mathrm{S}}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

with the universal coefficients ($C_{\mathrm{A}} = 3$, N_{f} dynamical fermions)

$$b_0^{(\mathrm{S})} = b_0 = \frac{1}{(4\pi)^2}(11C_{\mathrm{A}}/3 - 2N_{\mathrm{f}}/3)$$

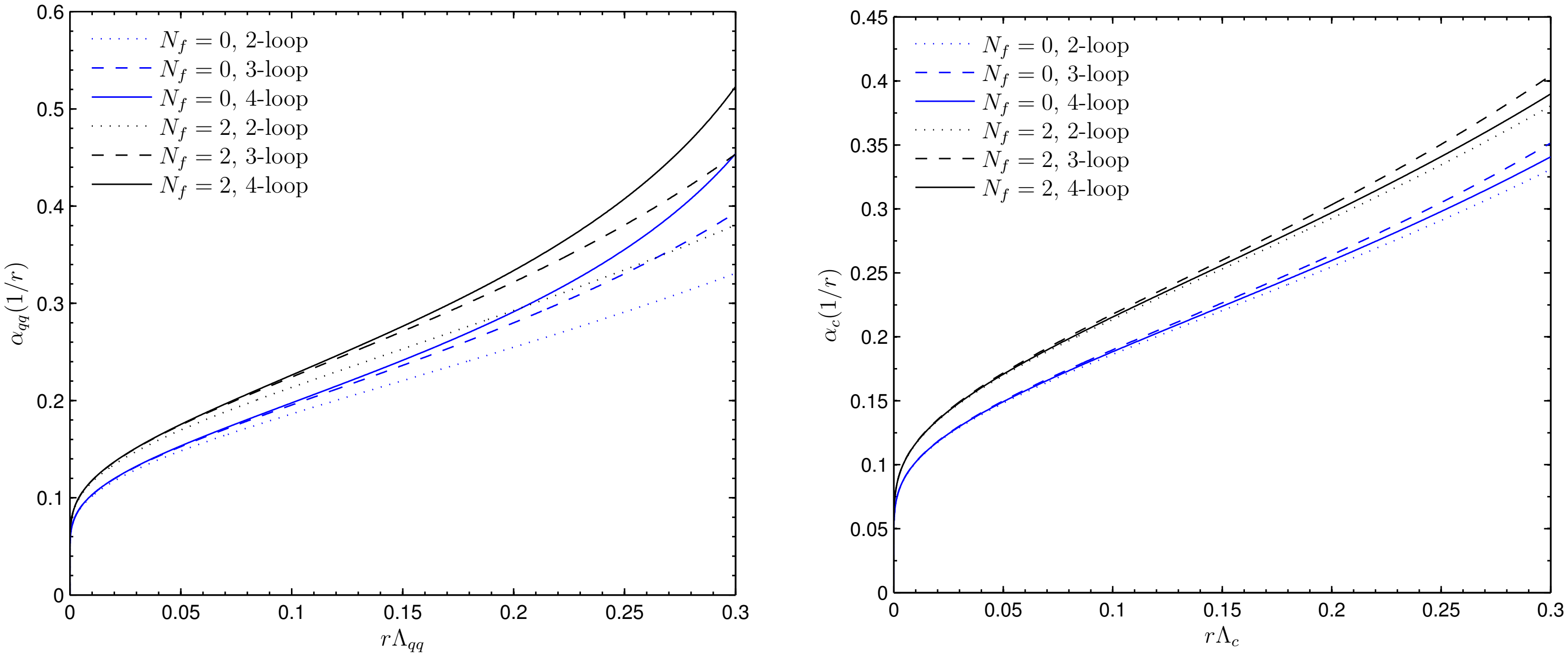
$$b_1^{(\mathrm{S})} = b_1 = \frac{1}{(4\pi)^4}(34C_{\mathrm{A}}^2/3 - 10C_{\mathrm{A}}N_{\mathrm{f}}/3 - 2C_{\mathrm{F}}N_{\mathrm{f}})$$

Perturbative solution: use for β_{S} the truncated perturbative expansion

Shape of the potential

In perturbation theory, β_{S} is known up to 4 loop [Brambilla, Pineda, Soto and Vairo, 1999 and 2000; Smirnov, Smirnov and Steinhauser, 2010; Anzai, Kiyo and Sumino, 2010]

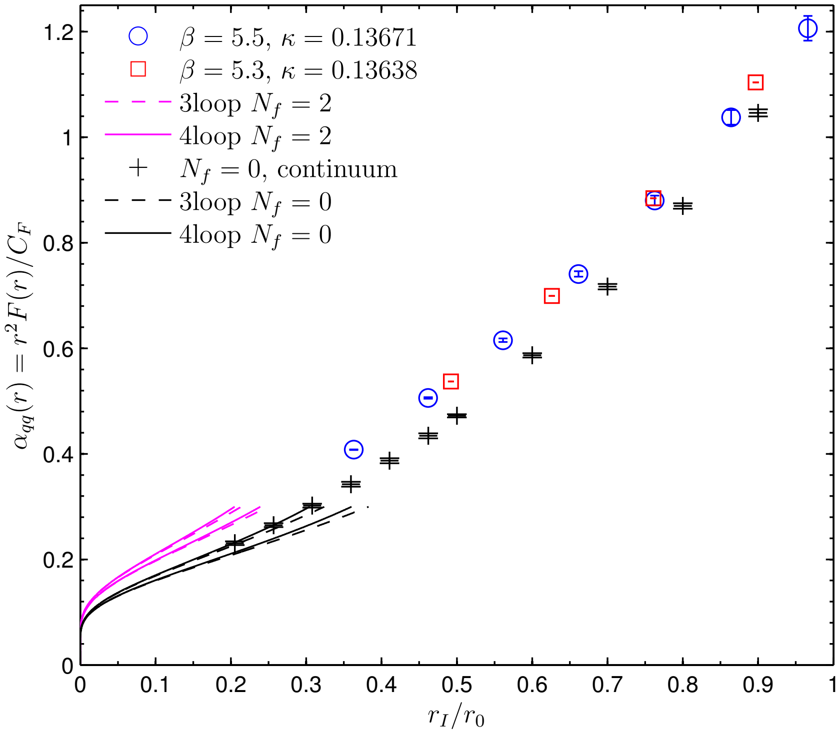
$$\beta_{\mathrm{S}}(\bar{g}_{\mathrm{S}}) \quad = \quad -\bar{g}_{\mathrm{S}}^3\Big[\sum_{n=0}^{3} b_n^{(\mathrm{S})}\bar{g}_{\mathrm{S}}^{2n} + b_{3,l}^{(\mathrm{S})}\bar{g}_{\mathrm{S}}^6 \log(C_{\mathrm{A}}\bar{g}_{\mathrm{S}}^2/(8\pi)) + \mathrm{O}(\bar{g}_{\mathrm{S}}^8)\Big]$$

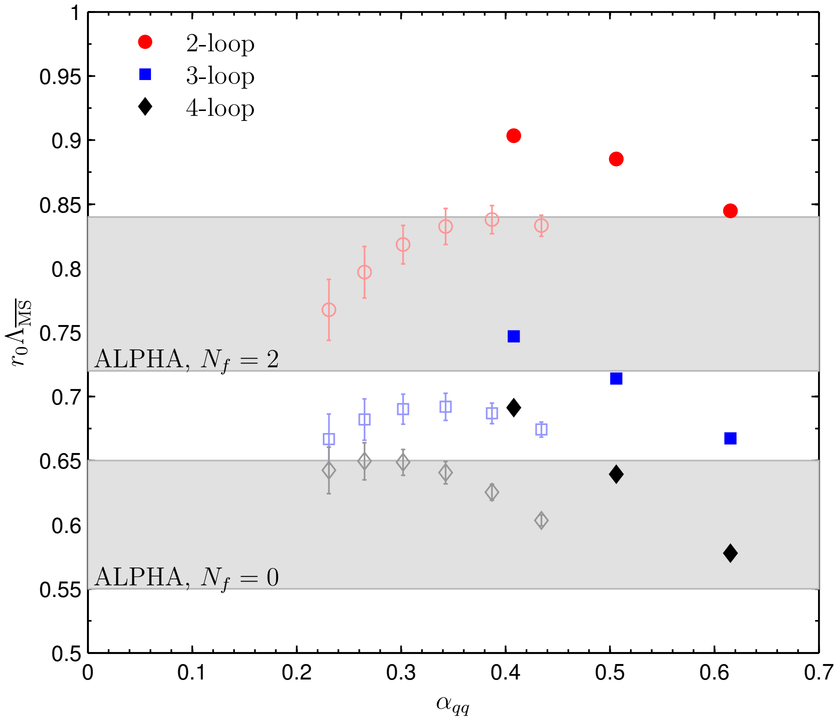


$\alpha_{\mathrm{S}} = \bar{g}_{\mathrm{S}}^2/(4\pi)$: perturbation theory behaves much better in the c-scheme but can be trusted only at small distances r where $\alpha_{\mathrm{qq}}(1/r) \approx 1/4$

Shape of the potential

Results for $\alpha_{\mathrm{qq}}(1/r) = r^2\, F(r)/C_{\mathrm{F}}$





Our lattice spacing is not small enough to make contact with perturbation theory

No chance to extract the Λ parameter solving the RG equation for $\alpha_{\mathrm{qq}}(1/r)$

ALPHA values from Schrödinger Functional coupling:
$r_0\Lambda_{\overline{\mathrm{MS}}}^{N_{\mathrm{f}}=0} = 0.60(5)$ [hep-lat/9810063], $r_0\Lambda_{\overline{\mathrm{MS}}}^{N_{\mathrm{f}}=2} = 0.78(3)(5)$ [Lattice 2011]

Shape of the potential

Results for $c(r) = r^3\, F'(r)/2$

- in the $N_\mathrm{f} = 0$ theory $c(r)$ approaches the asymptotic value $c(\infty) = -\pi/12$ with corrections $\mathcal{O}(1/r^2)$ predicted from the bosonic effective theory [Lüscher, Symanzik and Weisz, 1980; Lüscher 1980; Lüscher and Weisz, 2002 and 2004]
- comparison with phenomenological potential models: Cornell [Eichten et al., 1980]

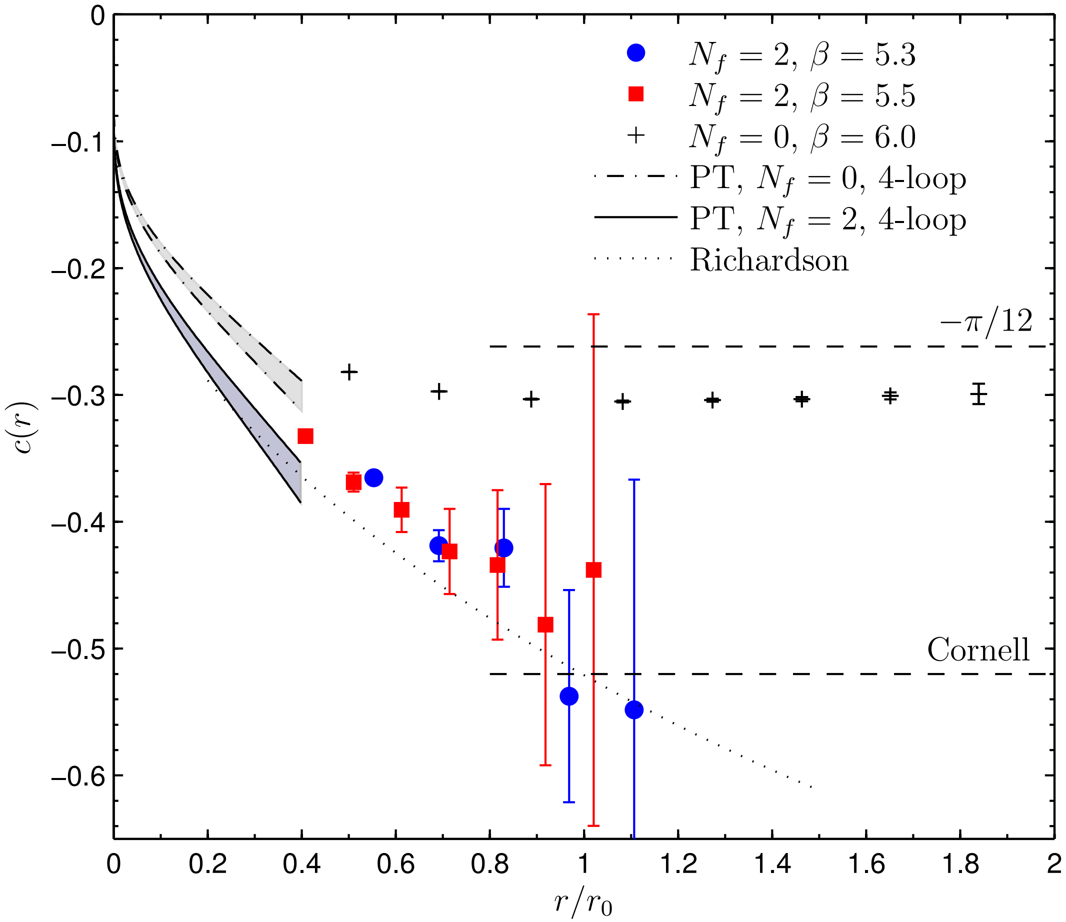
$$V_\mathrm{Cornell} = -\frac{\kappa}{r} + \sigma\, r\,, \quad \kappa = 0.52 \quad \Rightarrow\; c = -\kappa$$

  Richardson [Richardson, 1979]

$$V_\mathrm{Richardson}(r) \;=\; \frac{1}{6\pi b_0}\Lambda\left(\Lambda\, r - \frac{f(\Lambda\, r)}{\Lambda\, r}\right)$$

  at small distances $r\,\Lambda \ll 1$: $V_\mathrm{Richardson}(r) \;\sim\; -1/[6\pi b_0\, r\, \ln(1/\Lambda\, r)]$
  at large distances $r\,\Lambda \gg 1$: $V_\mathrm{Richardson}(r) \;\sim\; \mathrm{const} \times r$
- c is an interesting but difficult quantity for holographic QCD models [Giataganas and Irges, 2011]

Shape of the potential



$r_0\Lambda_{\overline{\rm MS}}^{N_{\rm f}=0} = 0.60(5)$ [hep-lat/9810063], $\qquad$ $r_0\Lambda_{\overline{\rm MS}}^{N_{\rm f}=2} = 0.78(3)(5)$ [Lattice 2011]

Conclusions and Outlook

- We determine the static potential from Wilson loops using the HYP2 static action
- Cut-off effects appear to be small and the scale r_0/a can be accurately determined
- We cannot resolve the first excitation of the potential, to this end fermionic correlators are needed
- The running coupling obtained from the static force is compared to perturbation theory: we cannot reach small enough distances in order to extract the Λ parameter
- We observe large effects from dynamical fermions in the slope $c(r)$ The statistical precision is worse than in the pure gauge case. Improvement due to the inclusion of fermionic correlators?
- We plan to study quark mass effects and string breaking

Scale r_0

Scale r_0 from the static force $F(r) = V'(r)$ [Sommer, 1993]

$$r^2 F(r)\Big|_{r=r_0} = 1.65$$

$r_0 \approx (0.45 \ldots 0.5)\,\mathrm{fm}$ through phenomenological potential models

- improved lattice definition of the force [Sommer, 1993]

$$F(r_\mathrm{I}) = [V(r) - V(r-a)]/a$$

  $r_\mathrm{I} = r - a/2 + \mathcal{O}(a^2)$ is chosen such that at tree level $F_\mathrm{tree}(r_\mathrm{I}) = C_\mathrm{F}\, g_0^2/(4\pi r_\mathrm{I}^2)$
- we get

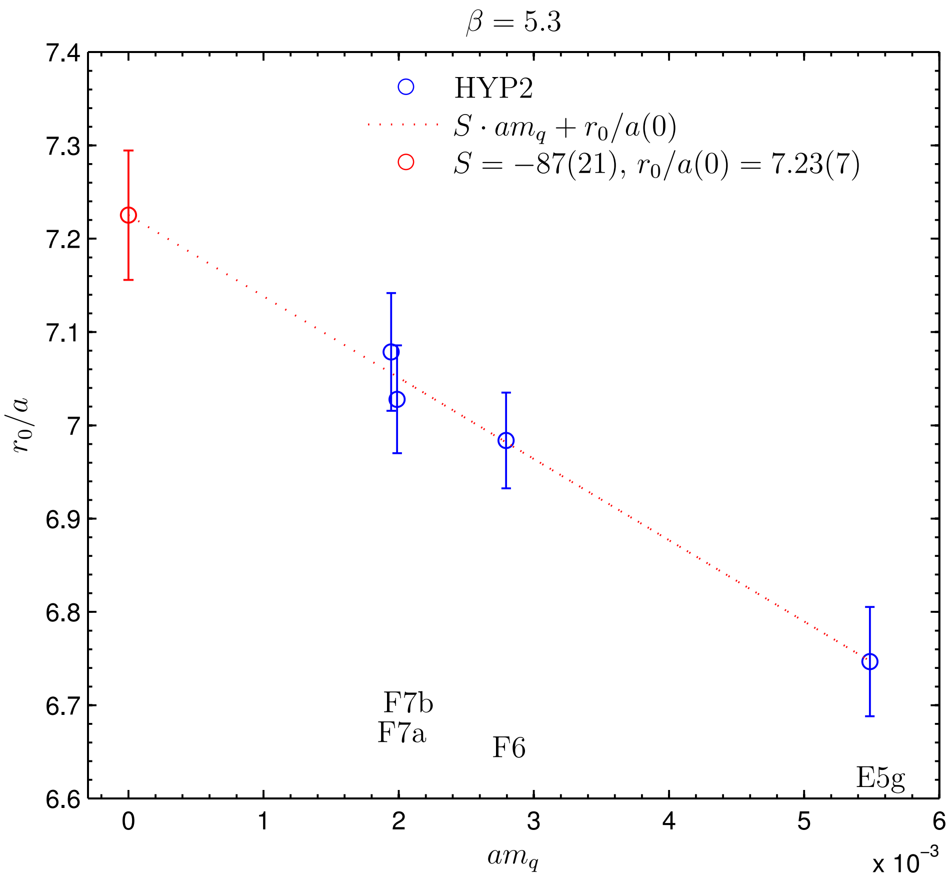
$$\frac{r_0}{a}(\mathrm{E5g}) = 6.75(6)$$

- error analysis: we use the method of [Wolff, 2004] with a correction for the slow modes [Schäfer, Sommer and Virotta, 2010] giving the upper bound

$$\tau_\mathrm{int}(r_0) = 6 \text{ (molecular dynamics units)}$$

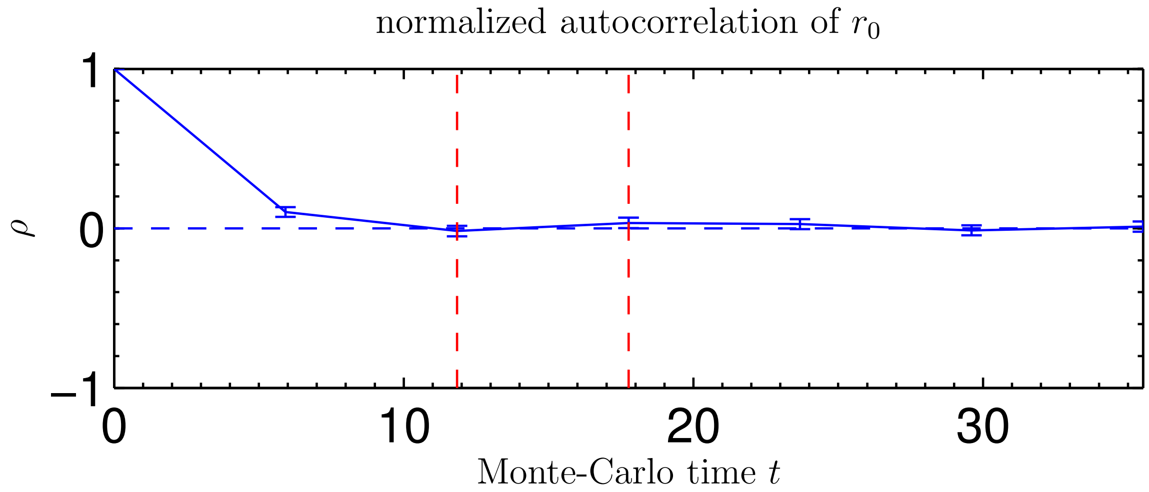
Scale r_0

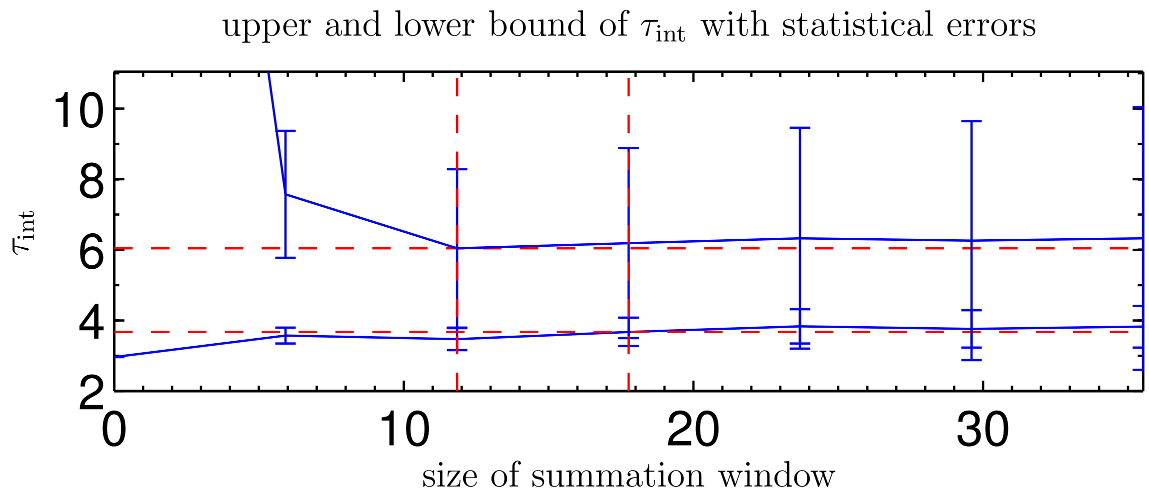
Chiral extrapolation of r_0: update of [BL and Knechtli, 2010]



$$\frac{r_0}{a}(m_q = 0)\Big|_{\beta=5.3} = 7.23(7)$$

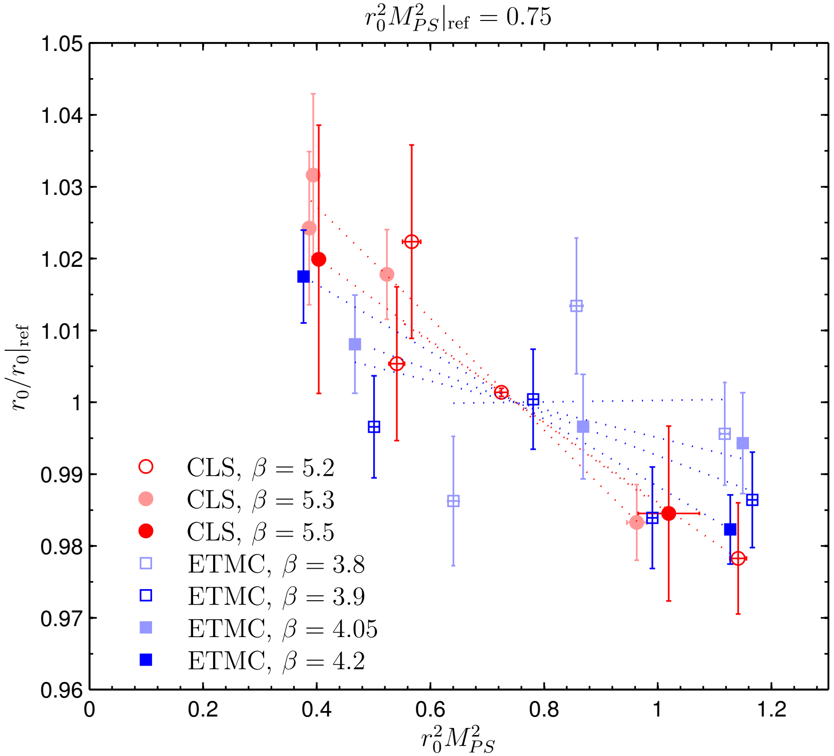
Autocorrelation of r_0





$\tau_{\text{exp}} = 39$ from [Schäfer, Sommer and Virotta, 2010]

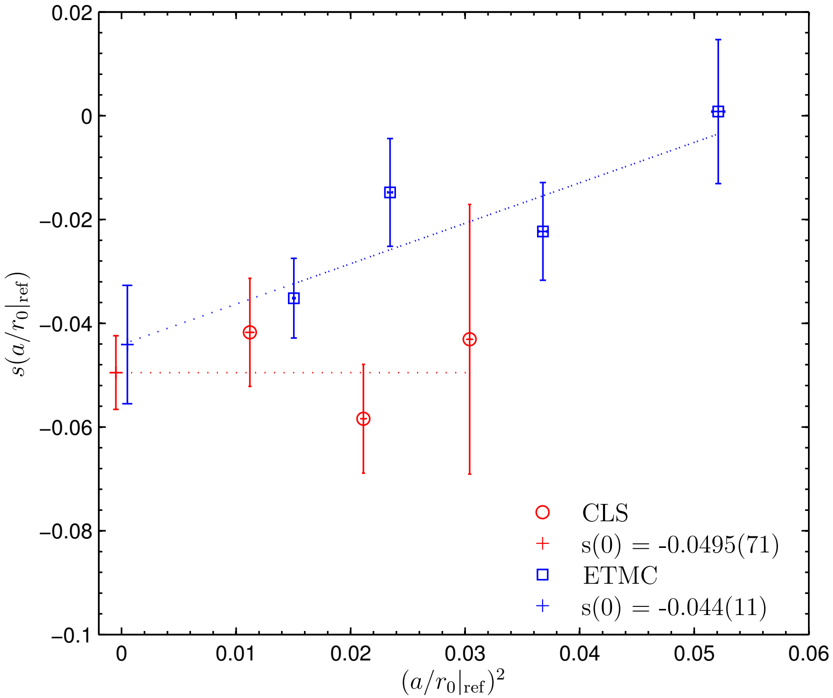
Mass dependence and cutoff effects in r_0/a



Comparison of improved Wilson and twisted mass [`arXiv:0911.5061`]:

- use combinations w/o Z-factors
- plot $r_0/r_0|_{\text{ref}}$ versus $(r_0 M_{\text{PS}})^2$
- choose $(r_0 M_{\text{PS}})^2|_{\text{ref}} = 0.75$ and cut $(r_0 M_{\text{PS}})^2 \leq 1.2$

Mass dependence and cutoff effects in r_0/a



Fit Taylor-expansion around $(r_0 M_{\mathrm{PS}})^2|_{\mathrm{ref}}$:

$$\frac{r_0}{r_0|_{\mathrm{ref}}}(x) = 1 + s(a/r_0|_{\mathrm{ref}}) \cdot (x-1)$$

- slopes $s(a/r_0|_{\mathrm{ref}})$ well determined
- tmQCD slopes increase as $a \to 0$
- line up with Clover for smallest a
- Clover error larger due to accounting for long tails