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Insights on the $Z_b(10610)$ and $Z_b(10650)$ from the dipion decays

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EMMI Workshop

Experimental and theoretical status of and perspectives for XYZ states

12-15 April 2021

based on

VB, E.Epelbaum, A.A.Filin, C.Hanhart, R.V. Mizuk, A.Nefediev, and S. Ropertz

Phys. Rev. D 103, 034016 (2021)

$Z_b(10610)$ and $Z_b(10650)$ from $\Upsilon(10860)$ decays at Belle

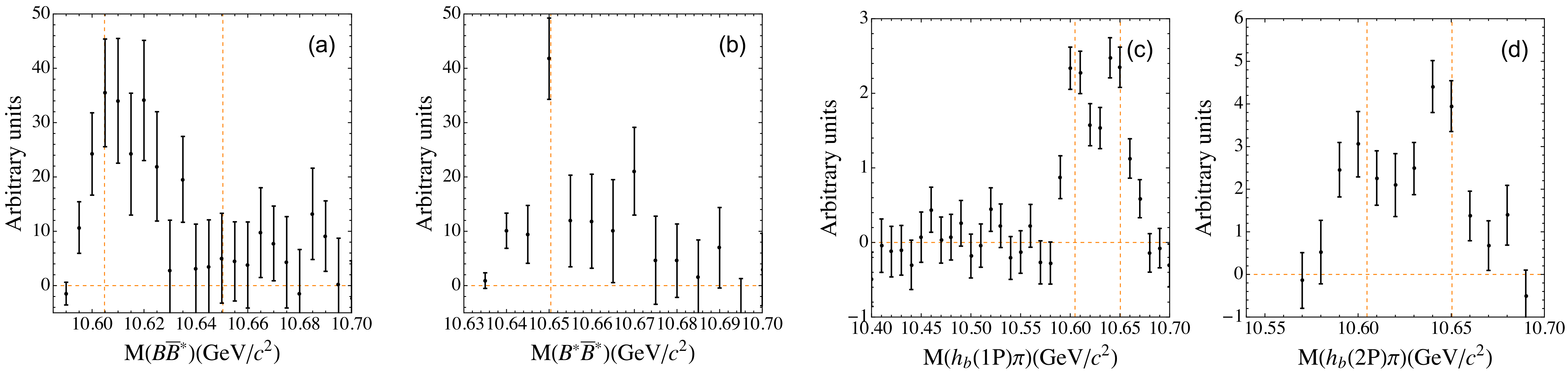
$$\Upsilon(10860) \to \pi\pi\Upsilon(nS), \quad n = 1, 2, 3$$

$$\Upsilon(10860) \to \pi\pi h_b(mP), \quad m = 1, 2$$

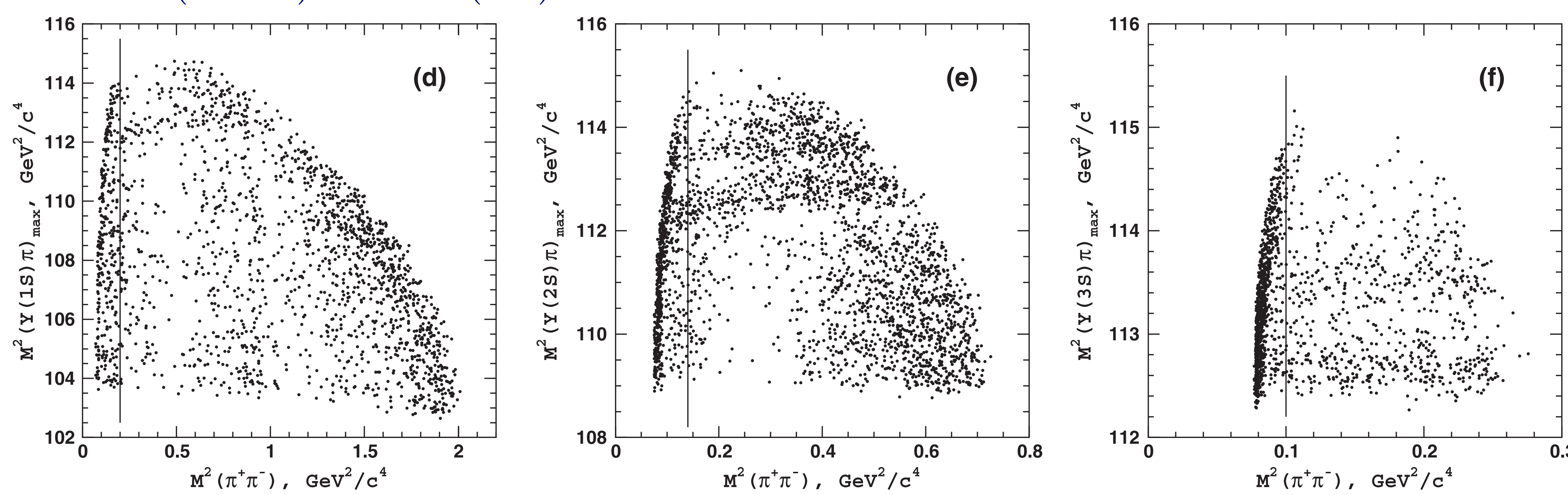
$$\Upsilon(10860) \to \pi B^{(*)}\bar{B}^*$$

Bondar et al. PRL108, 122001(2012)
Garmash et al.PRL116, 212001(2016)
PRD91, 072003 (2015)

— Invariant mass distributions in *BB*, B*B** and $h_b(mP)\pi$ channels



— Dalitz plots for $\Upsilon(10860) \to \pi\pi\Upsilon(nS)$ and its projections to Υ(nS)π and $\pi\pi$



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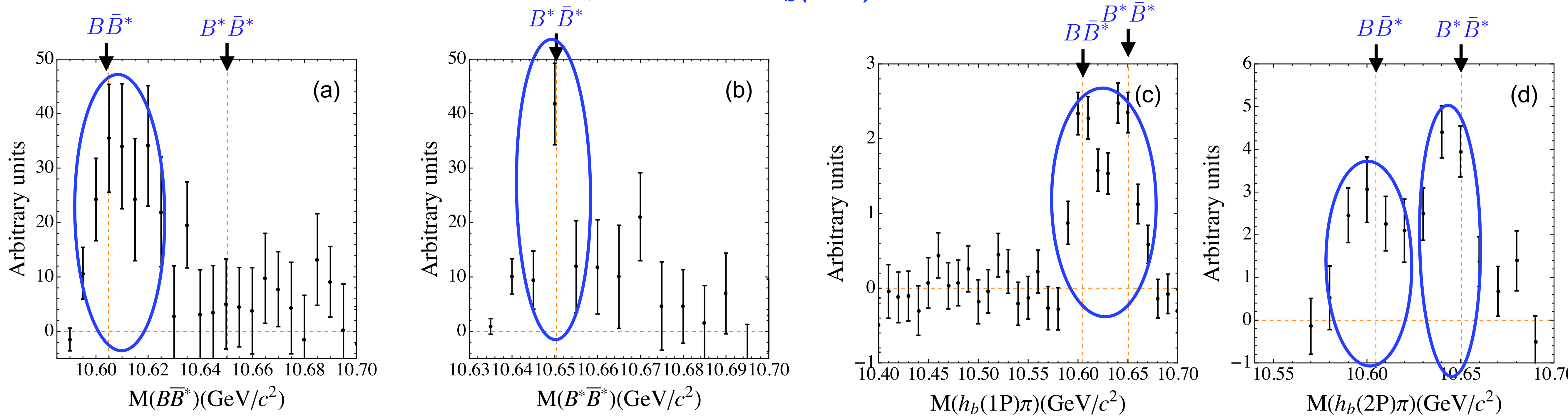

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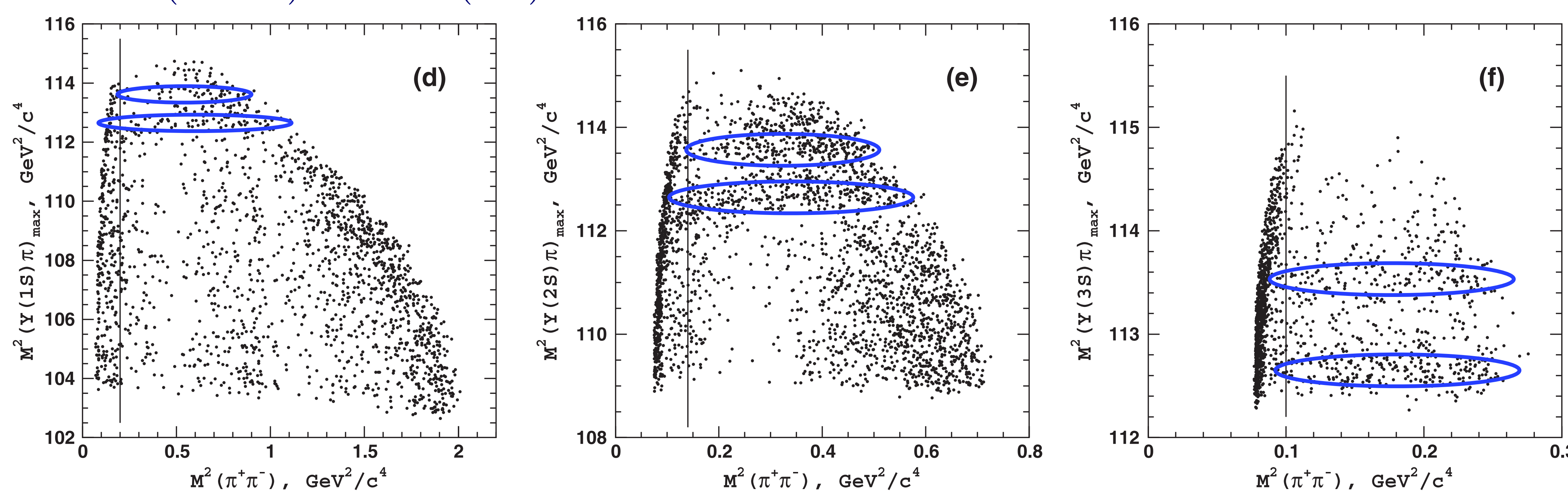

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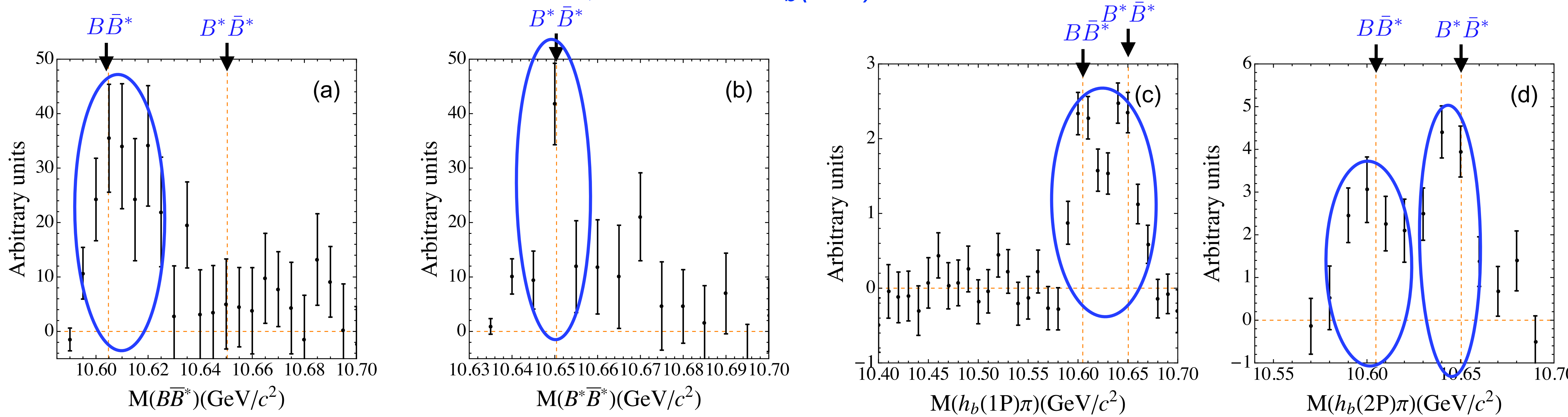


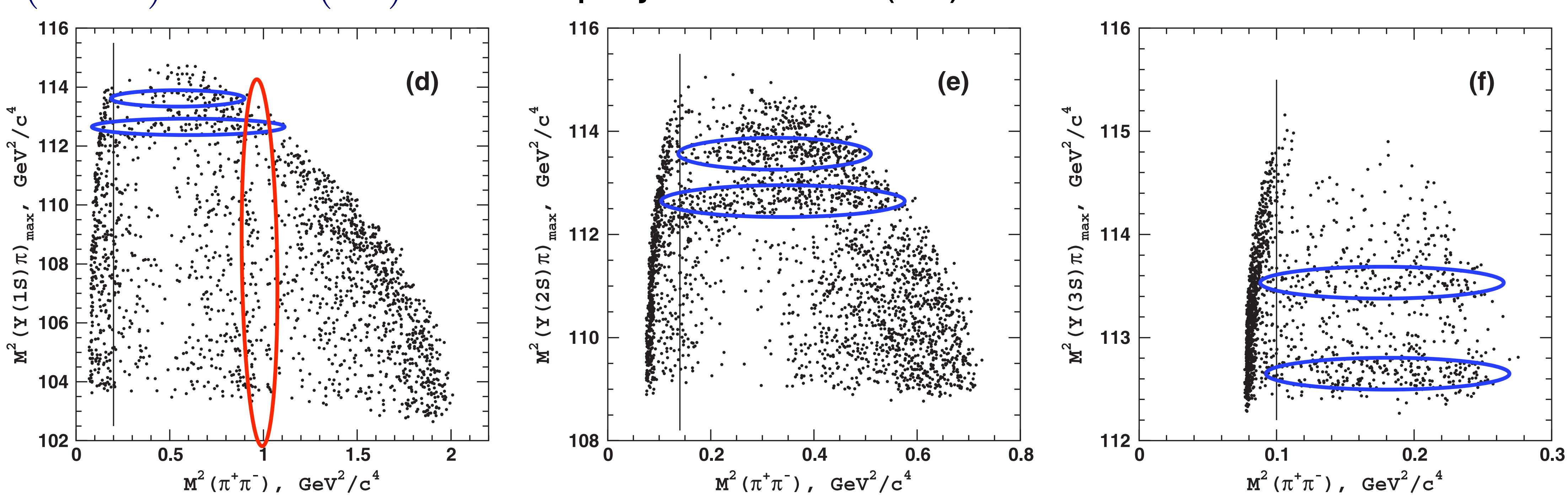
— Dalitz plot for $\Upsilon(10860) \to \pi\pi\Upsilon(nS)$ and its projections to Υ(nS)π and $\pi\pi$

contains

Z_b's in Υ(nS)pi

but also $\pi\pi$ FSI,

e.g. from $f_0(980)$



$Z_b(10610)$ and $Z_b(10650)$. Known Facts

- Location: very near S-wave thresholds of two hadrons $B\bar{B}^*$, $B^*\bar{B}^*$

- Dominant decays to these open-flavour channels

- $Z_b^{(\prime)}$ are clearly exotics: seen in charged modes $\pi^\pm h_b(mP)$, $\pi^\pm \Upsilon(nS)$

  $\Longrightarrow$ must be made of ≥4 quarks

- $\mathrm{Br}[\underbrace{\Upsilon(10860) \to \pi\pi h_b}_{\text{Heavy quark spin flip}}(mP)] \simeq \mathrm{Br}[\underbrace{\Upsilon(10860) \to \pi\pi\Upsilon}_{\text{No spin flip}}(nS)]$

Talk by Alexander Bondar on Monday

Natural explanation :

(a) the decays go through Zb's and

(b) Zb's are molecules

$$|Z_b\rangle = -\frac{1}{\sqrt{2}}\Big[(1^-_{b\bar{b}} \otimes 0^-_{q\bar{q}})_{S=1} + (0^-_{b\bar{b}} \otimes 1^-_{q\bar{q}})_{S=1}\Big]$$

$$|Z_b'\rangle = +\frac{1}{\sqrt{2}}\Big[(1^-_{b\bar{b}} \otimes 0^-_{q\bar{q}})_{S=1} - (0^-_{b\bar{b}} \otimes 1^-_{q\bar{q}})_{S=1}\Big]$$

Bondar, Garmash, Milstein, Mizuk, and **Voloshin**
PRD 84, 054010 (2011)

$\Longrightarrow$ $Z_b(10610)/Z_b(10650)$ are strong candidates for hadronic molecules

This Talk is about

- Z_b(10610) and Z_b(10650) from decays: $\Upsilon(10860) \to \pi Z_b^{(\prime)} \to \pi B^{(*)}\bar{B}^*$

  $\Upsilon(10860) \to \pi Z_b^{(\prime)} \to \pi\pi h_b(mP)$

  Wang, VB, Filin, Hanhart, Nefediev, and Wynen PRD 98, 074023 (2018)

  — Simultaneous analysis of these line shapes in chiral EFT approach with coupled-channels

  — Fixing LECs and extracting Z_b poles and residues

  — Predictions for HQSS partners is not discussed here, see

  VB, Epelbaum, Filin, Hanhart, Nefediev, and Wang PRD 99, 094013 (2019)

- Insights into the nature of the Z_b(10610) and Z_b(10650) from $\Upsilon(10860) \to \Upsilon(nS)\, \pi^+\pi^-$ (n=1,2,3)

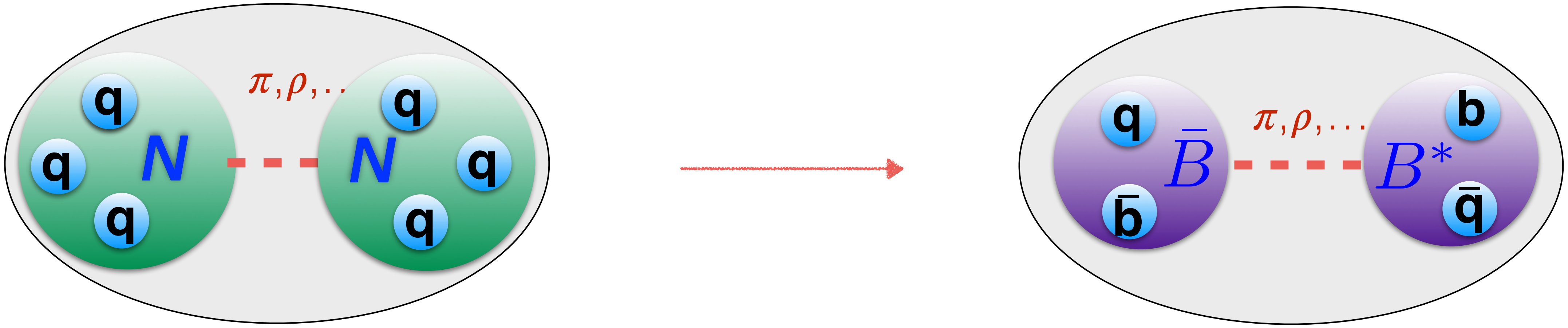
  VB, Epelbaum, Filin, Hanhart, Mizuk, Nefediev and Ropertz PRD103, 034016 (2021)

  — Multichannel Dalitz plot analysis using the same EFT approach from above as input

  — Focus is put on pipi/KK final-state interactions (FSI) and the consistency check

EFT for heavy-hadron molecules

our work PRD 98, 074023 (2018)



Underlying idea: Forces are similar to NN potential Voloshin, Okun (1976)

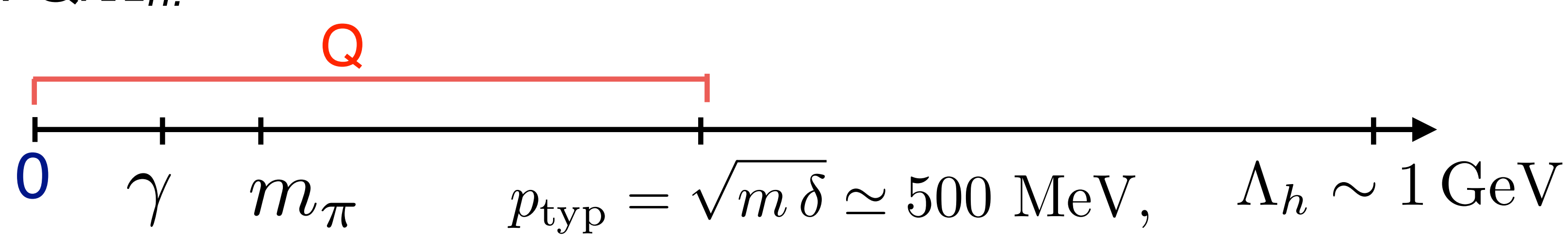
New insights: coupled-channel effects and consequences from HQSS

Goals: Analyse exp. line shapes including the energy range between $B\bar{B}^*$ and $B^*\bar{B}^*$

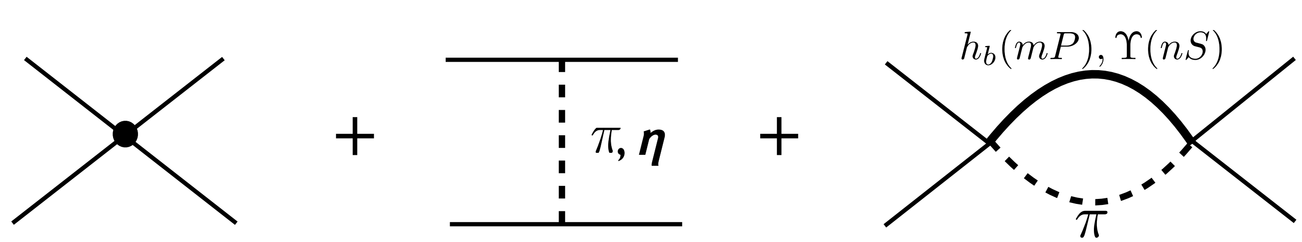
Tools: Coupled-channel potentials respecting HQSS and chiral symmetry

Elastic potential to a given order in Q/Λ_h:

- typical soft scale Q



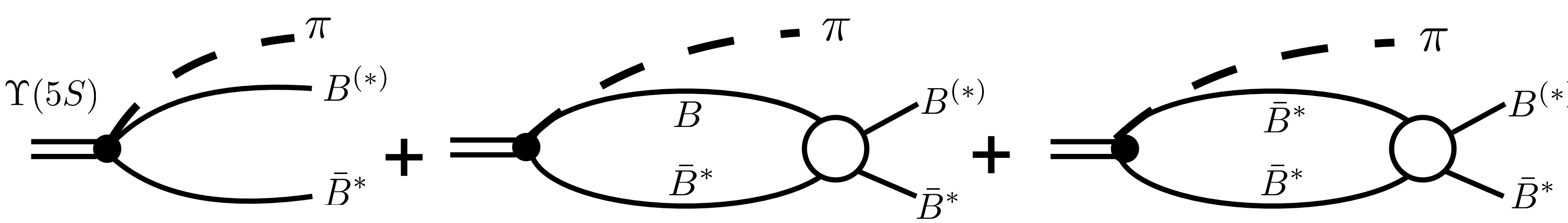
- $V_{\rm LO}^{\rm eff}$ =



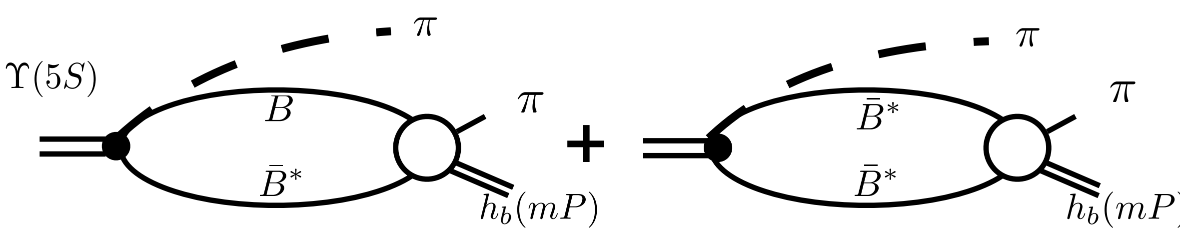
Formalism for line shapes $\Upsilon(10860) \to \pi Z_b^{(\prime)} \to \pi\alpha$

- Production amplitudes for the events dominated by the Zb's poles:

$U_{\rm el} =$



$U_{\rm inel} =$



$$T_{\alpha\beta} = V^{\rm eff}_{\alpha\beta} - \sum_\gamma \int \frac{d^3q}{(2\pi)^3} V^{\rm eff}_{\alpha\gamma} G_\gamma T_{\gamma\beta}$$

- Input: experimental distributions for

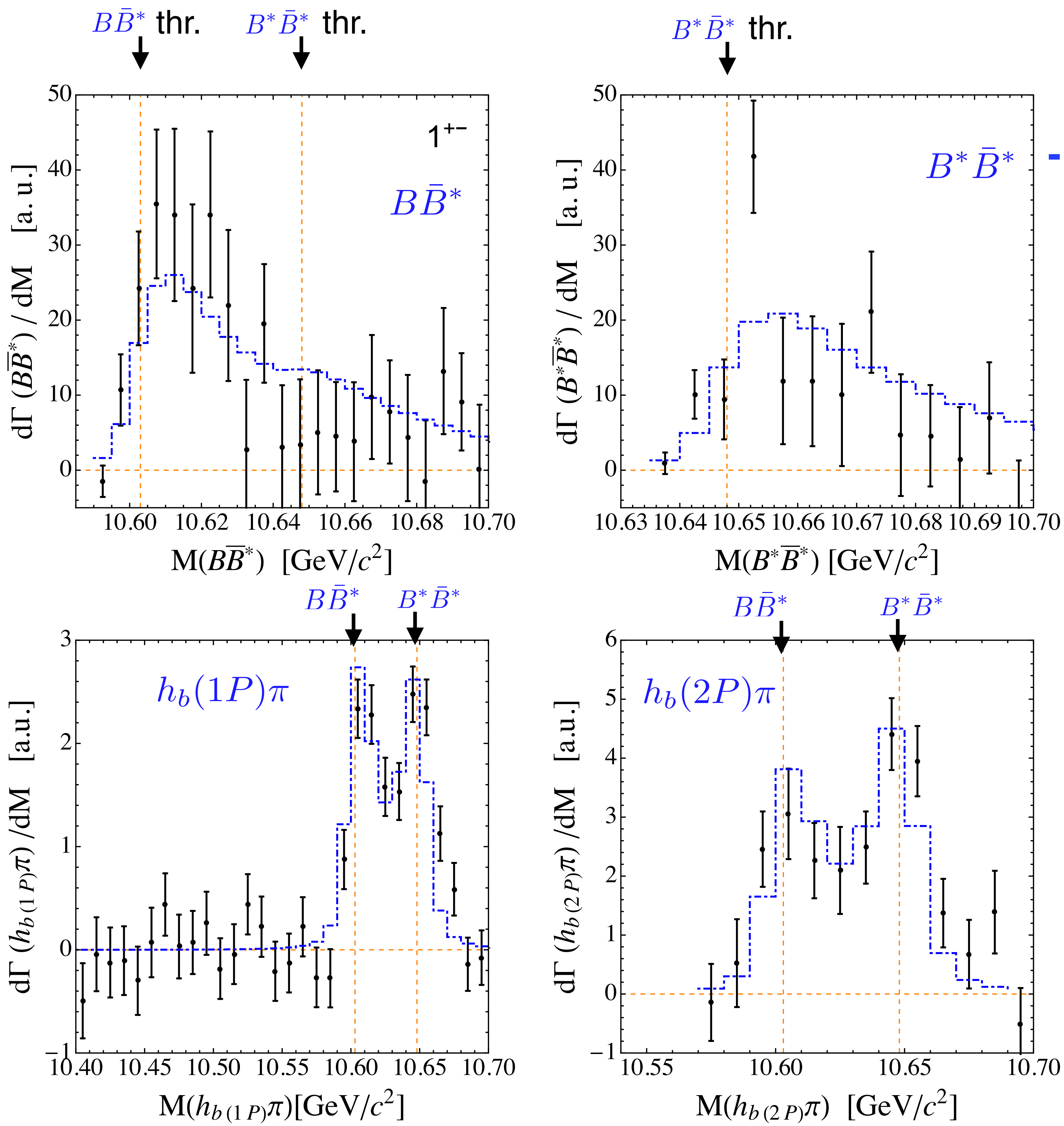
$\Upsilon(10860) \to \pi Z_b^{(\prime)} \to \pi\alpha$ $\quad \alpha = BB^*, \quad B^*B^*, \quad h_b(1P)\pi, \quad h_b(2P)\pi$

and branching fractions for $\quad \alpha = B\bar{B}^*,\ B^*\bar{B}^*,\ \ h_b(1P)\pi,\ h_b(2P)\pi,\ \Upsilon(1S)\pi,\ \Upsilon(2S)\pi,\ \Upsilon(3S)\pi$

Belle: Bondar et al. (2012), Garmash et al. (2016)

our work:
PRD 98, 074023 (2018)

Results: pionless theory at LO



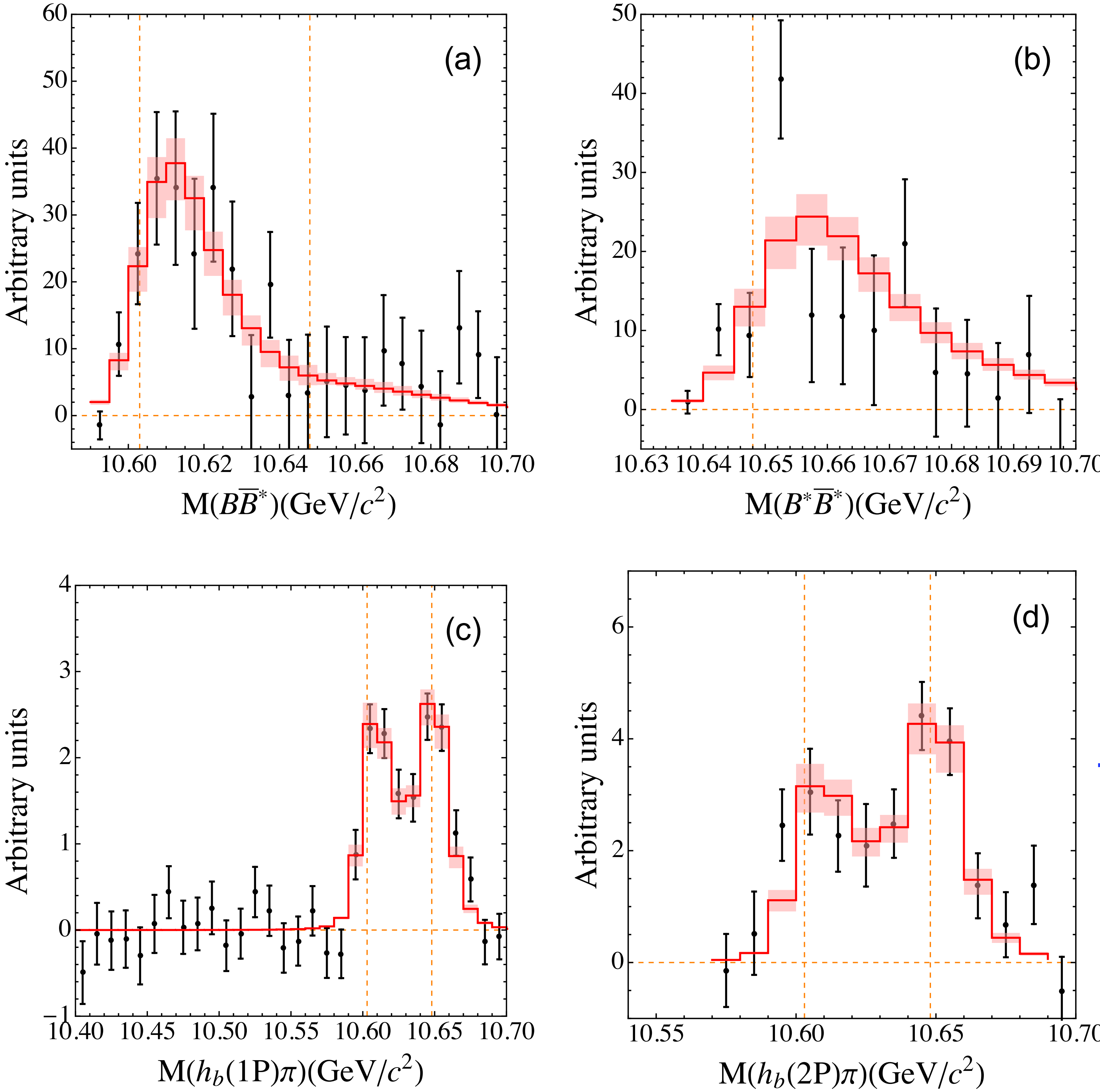
$$\chi^2 \equiv \frac{\chi^2}{\text{dof.}}$$

πless: LO CT's χ^2=1.29

— HQSS is preserved in the potentials

— Z_b's are virtual states

Final remarks from $\Upsilon(10860) \to \pi Z_b^{(\prime)} \to \pi B^{(*)} B^* \\ \to \pi\pi h_b(mP)$



— Visible effect from tensor part of OPE

χ^2=1.29 → χ^2=0.95

Zb's: virtual states → resonances

— All LECs are extracted from the best fit including 1σ errors

— Natural suppression of higher-order terms

— Data are consistent with HQSS respecting interactions

J^{PC}	State	Threshold	E_B w.r.t. threshold, [MeV]	Residue at pole
1^{+-}	Z_b	BB^*	$(-2.3 \pm 0.5) - i(1.1 \pm 0.1)$	$(-1.2 \pm 0.2) + i(0.3 \pm 0.2)$
1^{+-}	Z_b'	$B^*\bar{B}^*$	$(1.8 \pm 2.0) - i(13.6 \pm 3.1)$	$(1.5 \pm 0.2) - i(0.6 \pm 0.3)$

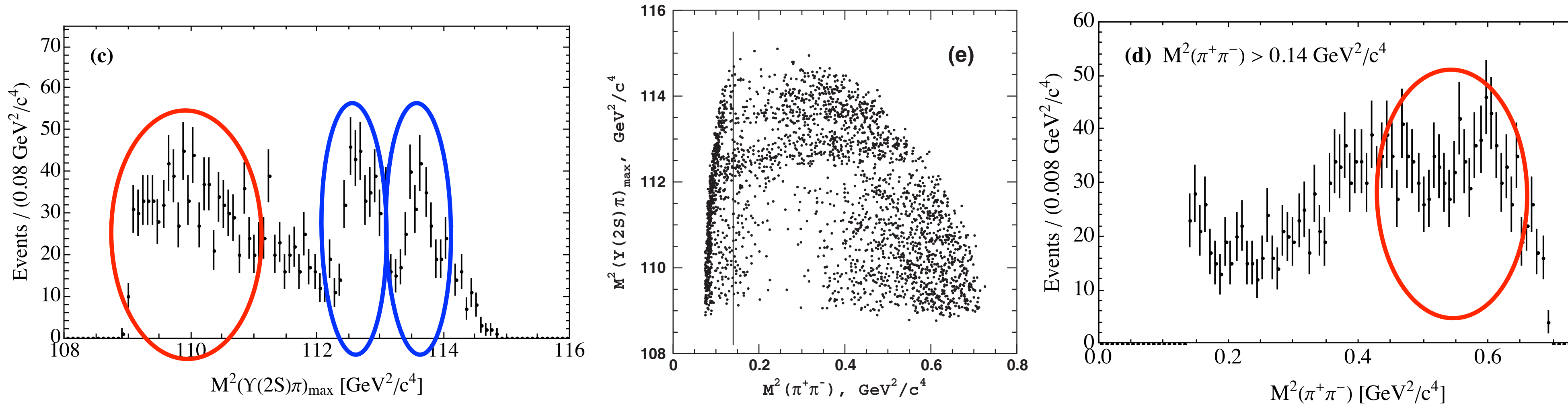
$\Upsilon(10860) \to \Upsilon(nS)\ \pi^+\pi^-$: Goals and Tools

our work:
PRD103, 034016 (2021)

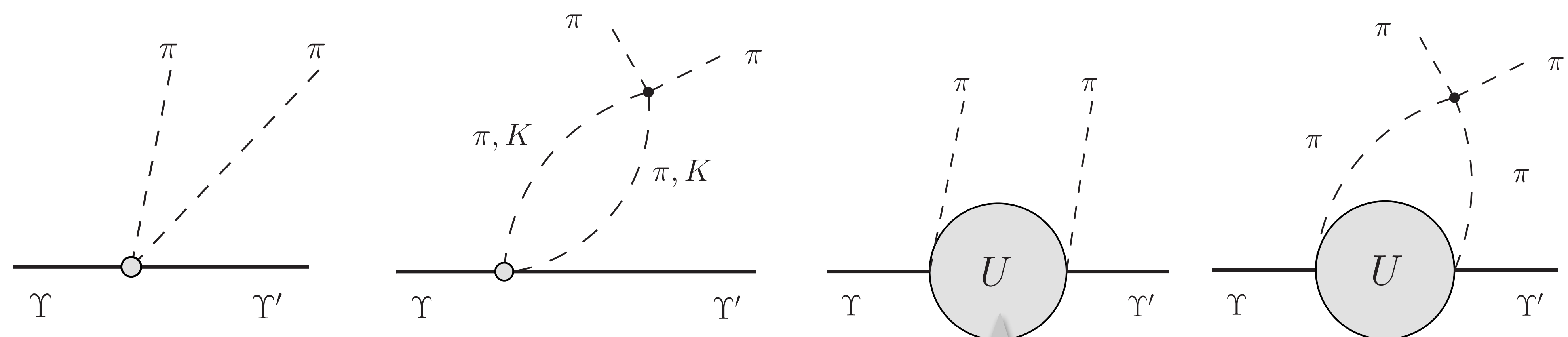
-High-statistic data by Belle

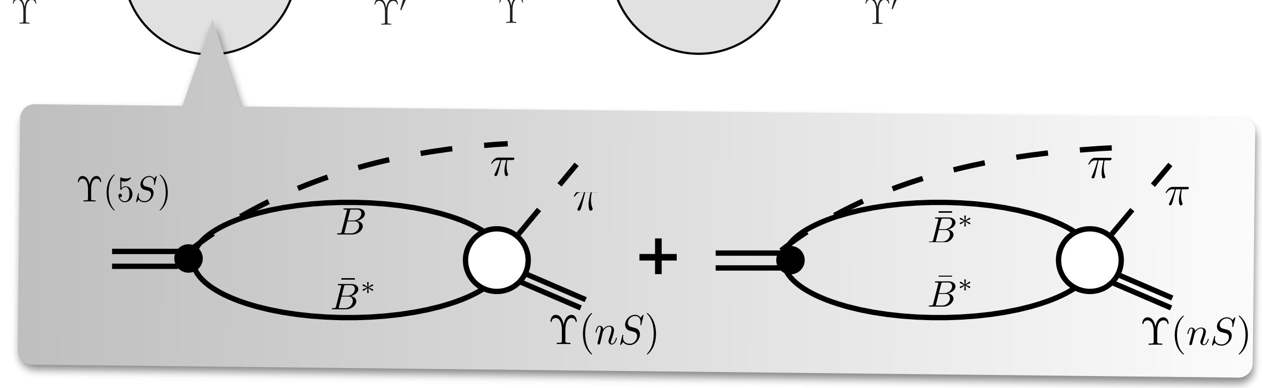
-Both $\pi\pi$ and $\pi\Upsilon$ play a role simultaneously!

→ Analyse Dalitz plot



- Production: contact and coupled-channel via B-meson loops, as formulated above





U is a *parameter-free input* from a simple but realistic pionless scheme

- Dispersive approach to account for the $\pi\pi$-KK FSI

- Important consistency check with previous results!

Dispersion relations for $\pi\pi$-K$\bar{\text{K}}$ FSI

S-wave projection: $$M_0(s) = \frac{1}{2}\int_{-1}^{+1} dz\, M(s,t,u) \equiv M_0^L + M_0^R$$

Left-hand cut piece

Right-hand cut piece with FSI

Dispersive reconstruction of $\hat{M}_0^R(s)$ via $\hat{M}_0^L(s)$

$$\hat{M}_0(s) = \hat{M}_0^L(s) + \overbrace{\frac{\hat{\Omega}_0(s)}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\hat{\Omega}_0^{-1}(s')\hat{T}(s')\hat{\sigma}(s')\hat{M}_0^L(s')}{s'-s-i0}}$$

$\pi\pi$-K$\bar{\text{K}}$ scattering amplitude: $$\hat{T}(s) = \begin{pmatrix} T_{\pi\pi\to\pi\pi} & T_{\pi\pi\to K\bar{K}} \\ T_{K\bar{K}\to\pi\pi} & T_{K\bar{K}\to K\bar{K}} \end{pmatrix} = \begin{pmatrix} \frac{\eta e^{2i\delta}-1}{2i\sigma_\pi} & g e^{i\psi} \\ g e^{i\psi} & \frac{\eta e^{2i(\psi-\delta)}-1}{2i\sigma_K} \end{pmatrix}$$

R. Garcia-Martin et al.,PRD83, 074004 (2011), I. Caprini et al., EPJC72, 1860 (2012), P. Buettiker et al., EPJC33, 409 (2004), L.Y.Dai et al., PRD90, 036004 (2014).

multichannel Omnès function: $$\hat{\Omega}_0(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\hat{T}^*(s')\hat{\sigma}(s)\hat{\Omega}_0(s')}{s'-s-i0}$$

$$\eta = \sqrt{1-4g^2\,\sigma_\pi\,\sigma_K\,\theta(s-4m_K^2)}$$

$$\hat{\sigma}(s) = \mathrm{diag}\{\sigma_\pi, \sigma_K\}$$

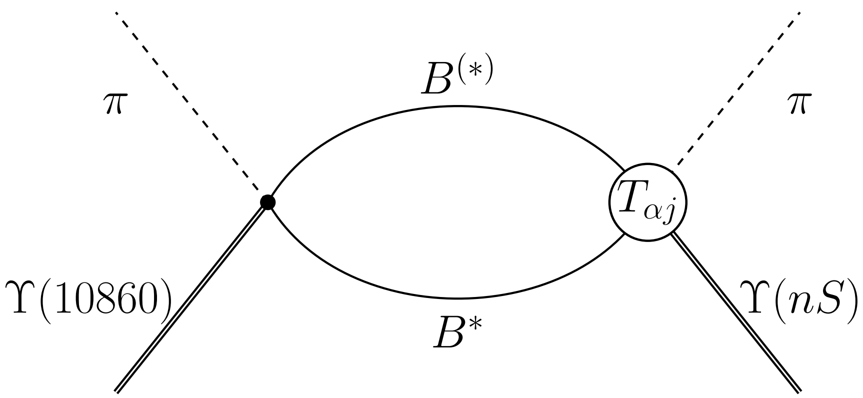
$$\sigma_P(s) = \sqrt{1-s_P^{\mathrm{th}}/s}$$

$$P = \pi, K$$

Production via $\pi\pi$ mode: $$\hat{M}_0^L = \left([M_0^L]_{\pi\pi}, 0\right)^T$$

Left-hand cut production amplitude

$M^L(t,u) =$



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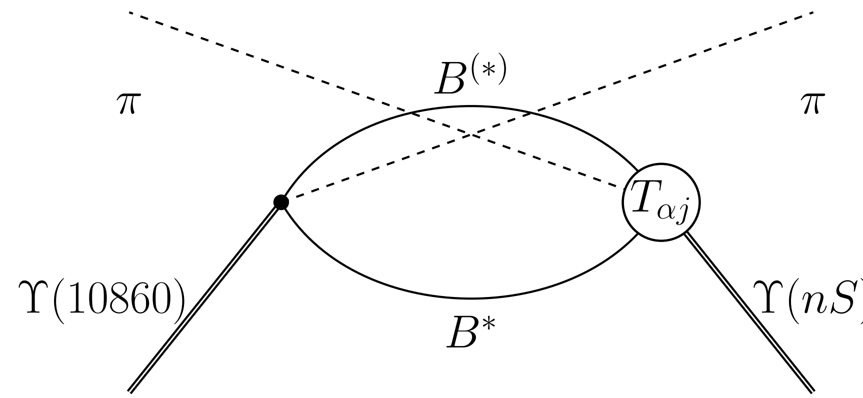

$$M^L(t,u) = U(t) + U(u) = -\frac{1}{\pi}\int_{(m_\pi + m_{\Upsilon(1S)})^2}^{\infty} d\mu^2 \, \mathrm{Im}\, U(\mu^2) \left(\frac{1}{t-\mu^2} + \frac{1}{u-\mu^2} \right) \xrightarrow[\text{projection}]{\text{S-wave}} M_0^L(s)$$

– Z_b(10610)/Z_b(10650) are poles in the coupled-channel amplitudes $M^L(t,u)$

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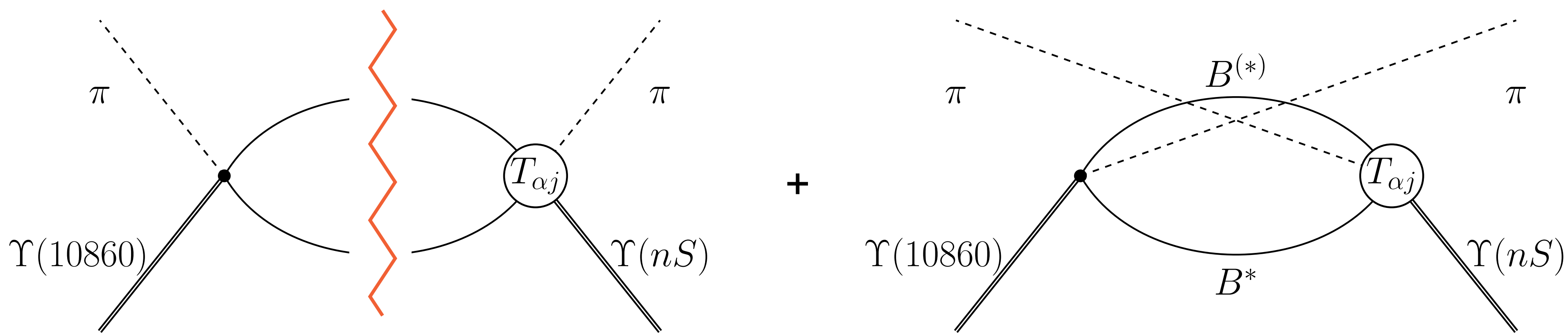

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– Z_b(10610)/Z_b(10650) are poles in the coupled-channel amplitudes $M^L(t,u)$

– $\mathrm{Im}\, M_0^L(s)$: Leading contribution is from the $B^{(*)}\bar{B}^*$ cuts, these states can be on shell
subleading one— from inelastic channels

Subtractions and matching to chiral contact amplitudes

$$\hat{M}_0(s) = \hat{M}_0^L(s) + \frac{\hat{\Omega}_0(s)}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\hat{\Omega}_0^{-1}(s')\hat{T}(s')\hat{\sigma}(s')\hat{M}_0^L(s')}{s' - s - i0}$$

— Dispersive Integral is convergent but details of $\pi\pi$ at large s are known badly

$\Longrightarrow$ 2 subtractions with real coefficients

- $\mathrm{Im}\, M_0^L(s)$ is under control, since it is driven by the $B^{(*)}\bar{B}^*$ cuts from the finite region of s
- In contrast, 2 complex coefficients were used in a related study of $e^+e^- \to \pi^+\pi^-\psi(2S)$

Molnar et al., PLB797, 134851 (2019)

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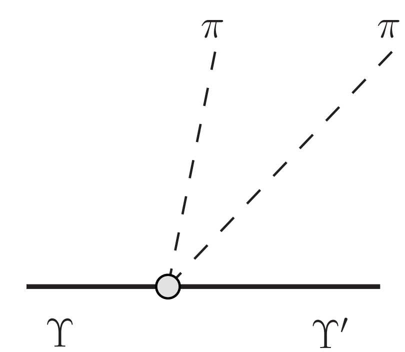
Molnar et al., PLB797, 134851 (2019)

— Matching to chiral expansion

$$\mathcal{L}_{\Upsilon\Upsilon'\Phi\Phi} = \frac{c_1}{2}\langle J^\dagger J'\rangle\langle u_\mu u^\mu\rangle + \frac{c_2}{2}\langle J^\dagger J'\rangle\langle u_\mu u_\nu\rangle v^\mu v^\nu + \mathrm{h.c.}$$

$$J = \boldsymbol{\Upsilon}\cdot\boldsymbol{\sigma} + \eta_b$$



$$u_\mu = i\left(u^\dagger\partial_\mu u - u\partial_\mu u^\dagger\right) \qquad u = \exp\left(\frac{i\Phi}{\sqrt{2}f}\right)$$

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}$$

$$\Longrightarrow \quad \boxed{\hat{M}_0^\chi(s) = \left(M_0^{\chi,\pi\pi}(s), \frac{2}{\sqrt{3}}M_0^{\chi,KK}(s)\right)^T}$$

$$\boxed{M_0^{\chi,PP}(s) = -\frac{2}{f_P^2}\sqrt{m_\Upsilon m_{\Upsilon'}}\left\{c_1\left(s-2m_P^2\right) + \frac{c_2}{2}\left[s + q^2\left(1 - \frac{\sigma_P^2(s)}{3}\right)\right]\right\}}$$

Chen et al. PRD93, 034030 (2016), PRD95, 034022 (2017)

Final results for $M(s,t,u)$

$$M(s,t,u) = \underbrace{M^L(t,u)}_{\substack{\text{coupled-channel}\\ \text{production amplitude,}\\ \text{contains all partial waves}}} + \hat{\Omega}_0(s)\left(\underbrace{\hat{M}_0^{\chi,\pi\pi}(s)}_{\substack{\text{chiral contact}\\ \text{term}}} + \underbrace{\hat{I}_0^{(2)}(s)}_{\substack{\text{dispersive}\\ \text{integral}}}\right) + \underbrace{\Omega_2(s)M_2^{\chi,\pi\pi}(s)P_2(z)}_{\text{D-wave contribution}}$$

$$\hat{I}_0^{(2)}(s) = \underbrace{\frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\hat{\Omega}_0^{-1}(s')\hat{T}(s')\hat{\sigma}(s')\,\mathrm{Re}M_0^L(s')}{s'-s-i0}}_{\text{twice-subtracted}} + \underbrace{\frac{i}{\pi}\int_{4m_\pi^2}^{\infty}ds'\frac{\hat{\Omega}_0^{-1}(s')\hat{T}(s')\hat{\sigma}(s')\,\mathrm{Im}M_0^L(s')}{s'-s-i0}}_{\text{unsubtracted}} + \underbrace{\hat{I}_0^{\mathrm{anom}}(s)}_{\substack{\text{anomalous}\\ \text{contribution}}}$$

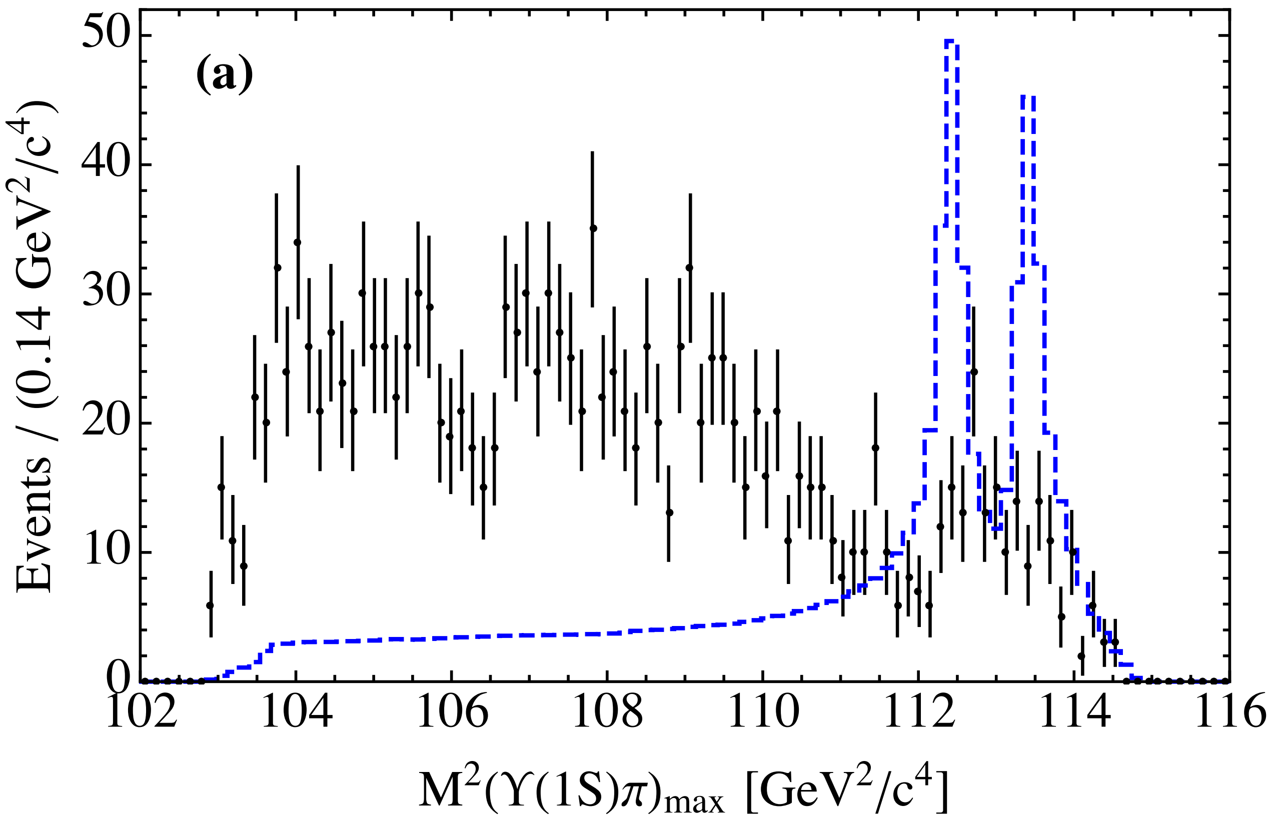
- All the parameters from a coupled-channel approach in $M^L(t,u)$ fixed from data to the decays

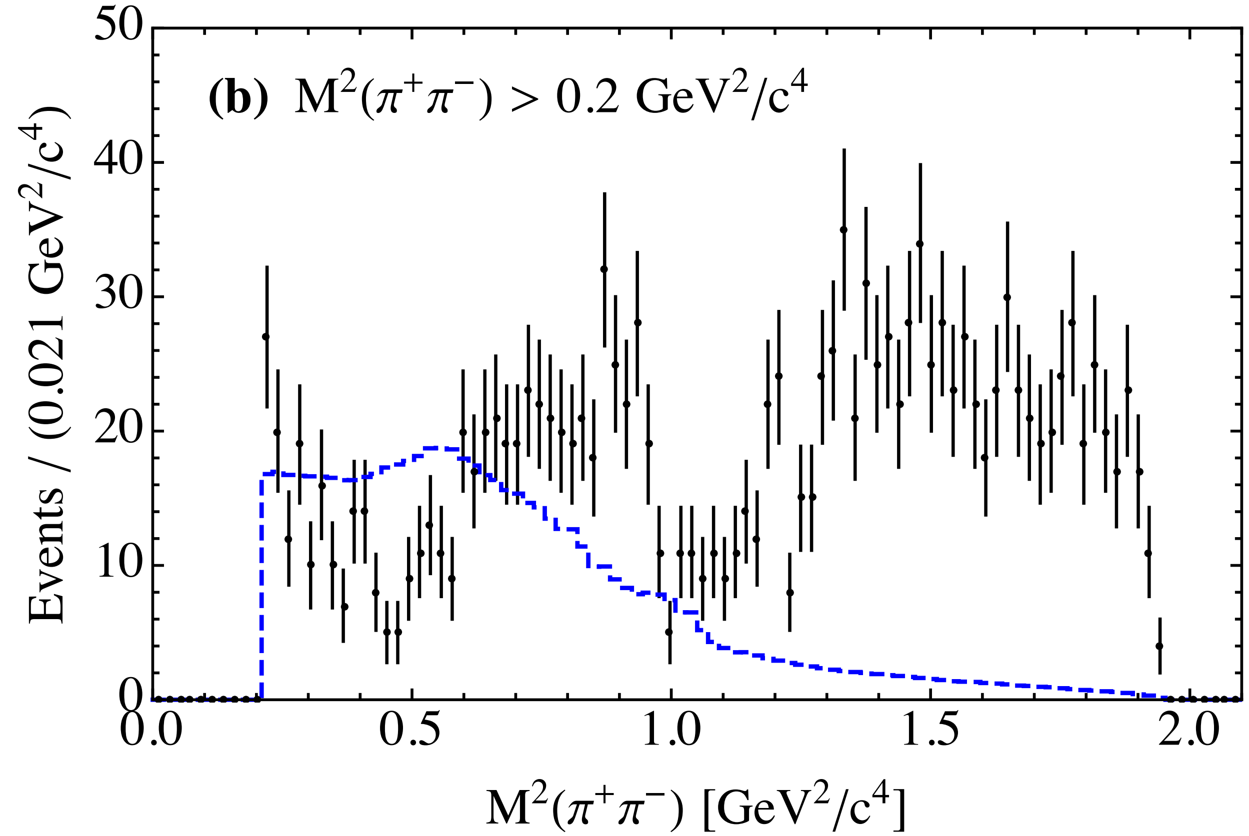
$$\Upsilon(10860) \to \pi Z_b^{(\prime)} \to \pi B^{(*)}\bar{B}^* \quad \text{and} \quad \Upsilon(10860) \to \pi Z_b^{(\prime)} \to \pi\pi h_b(mP)$$

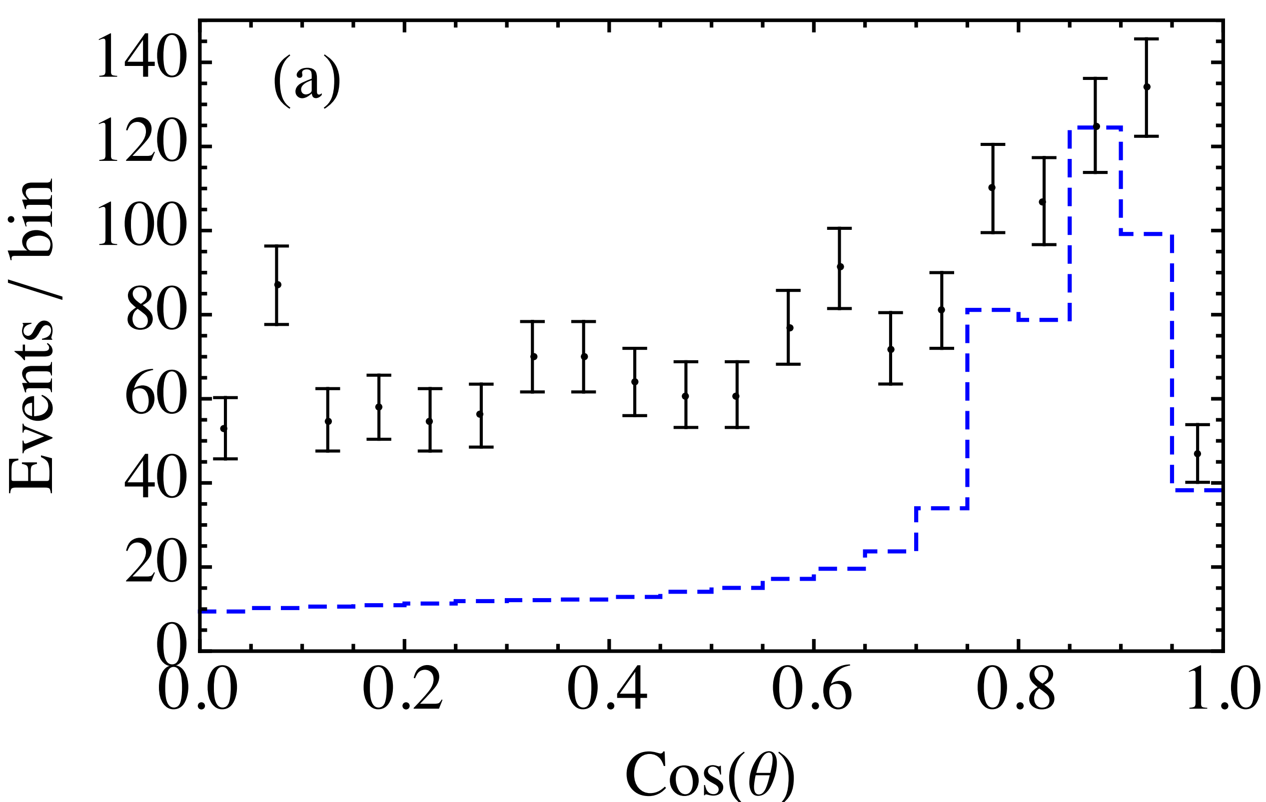
- Parameters in the fits: overall normalization $\mathcal{N}$ and chiral LECs c1 and c2

Dalitz plot projections: Individual Contrib's.

$\Upsilon(10860) \rightarrow \Upsilon(1S)\,\pi^+\pi^-$







$$M^L(t,u) = U(t) + U(u)$$ "Z_b"

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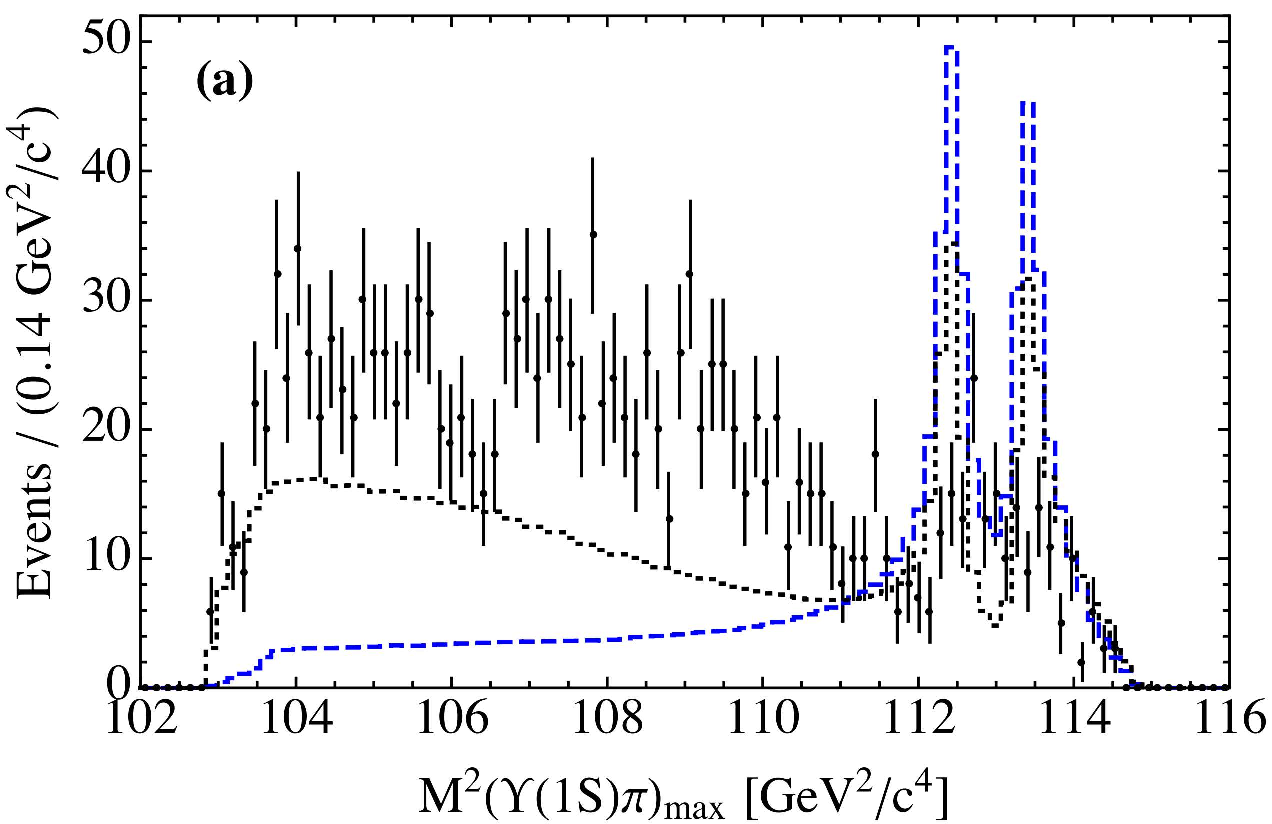

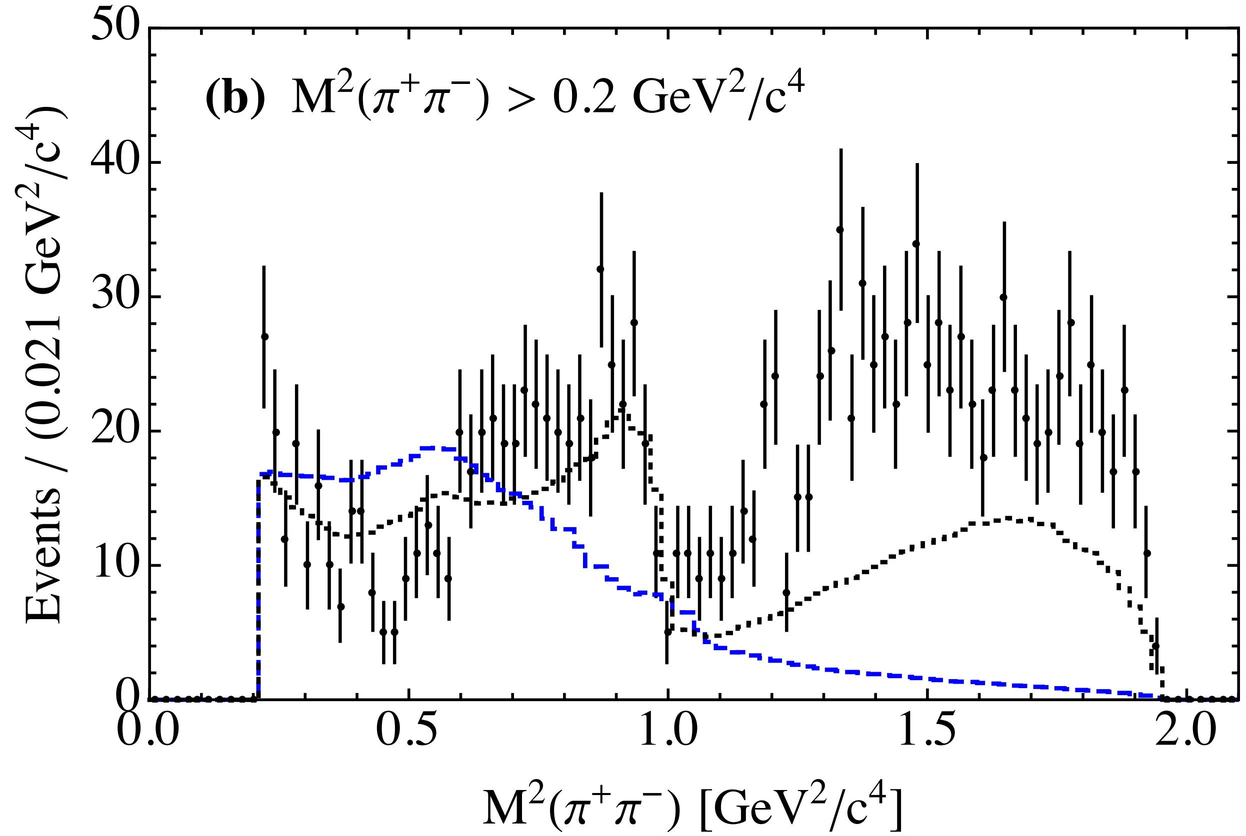

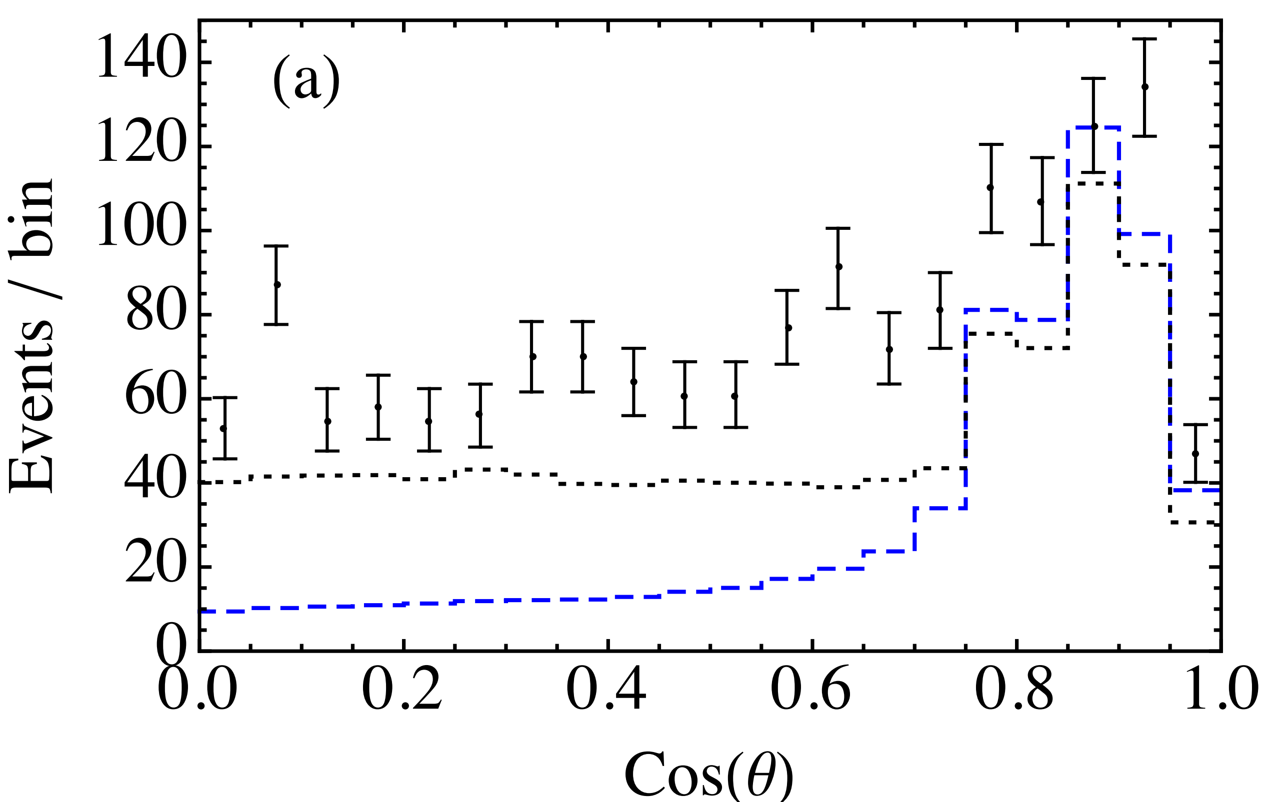

$M^L(t,u) = U(t) + U(u)$ "Z_b"

+ dispersive integral

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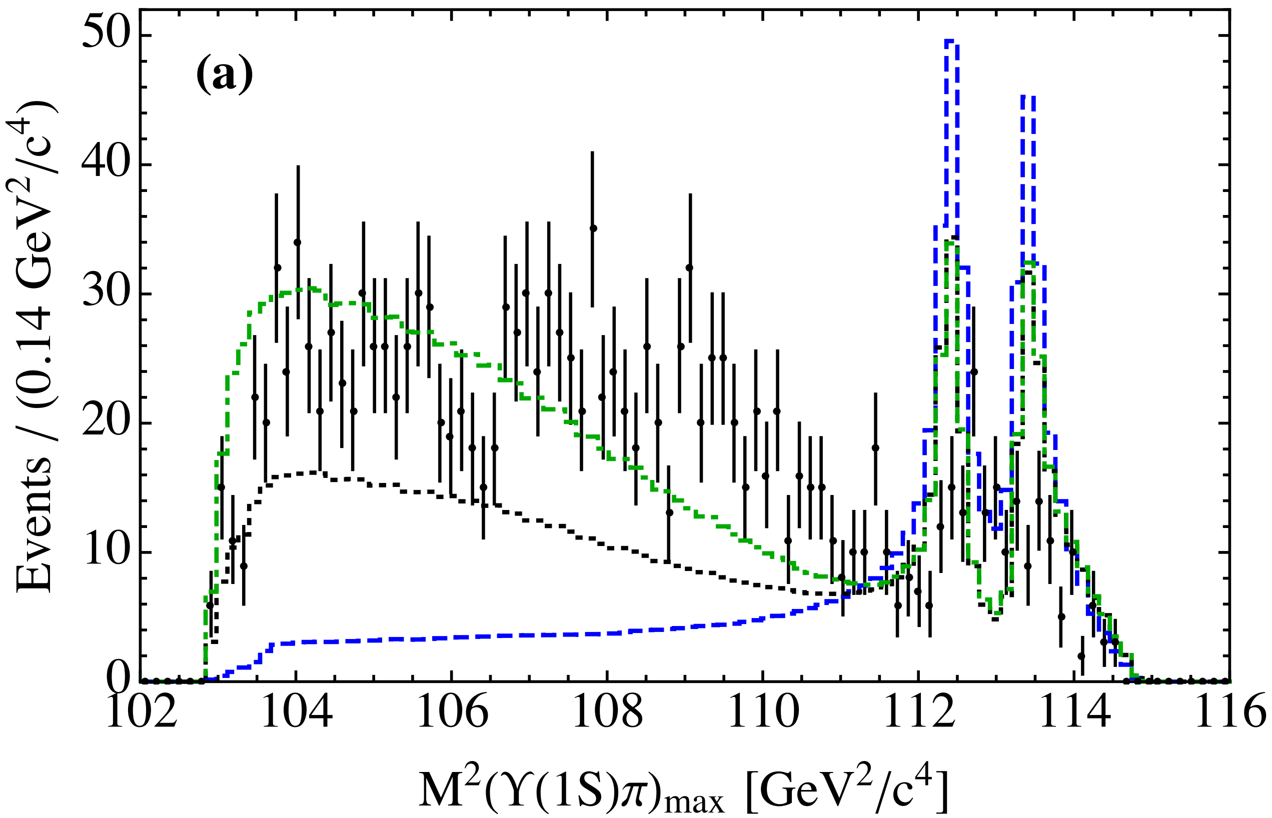
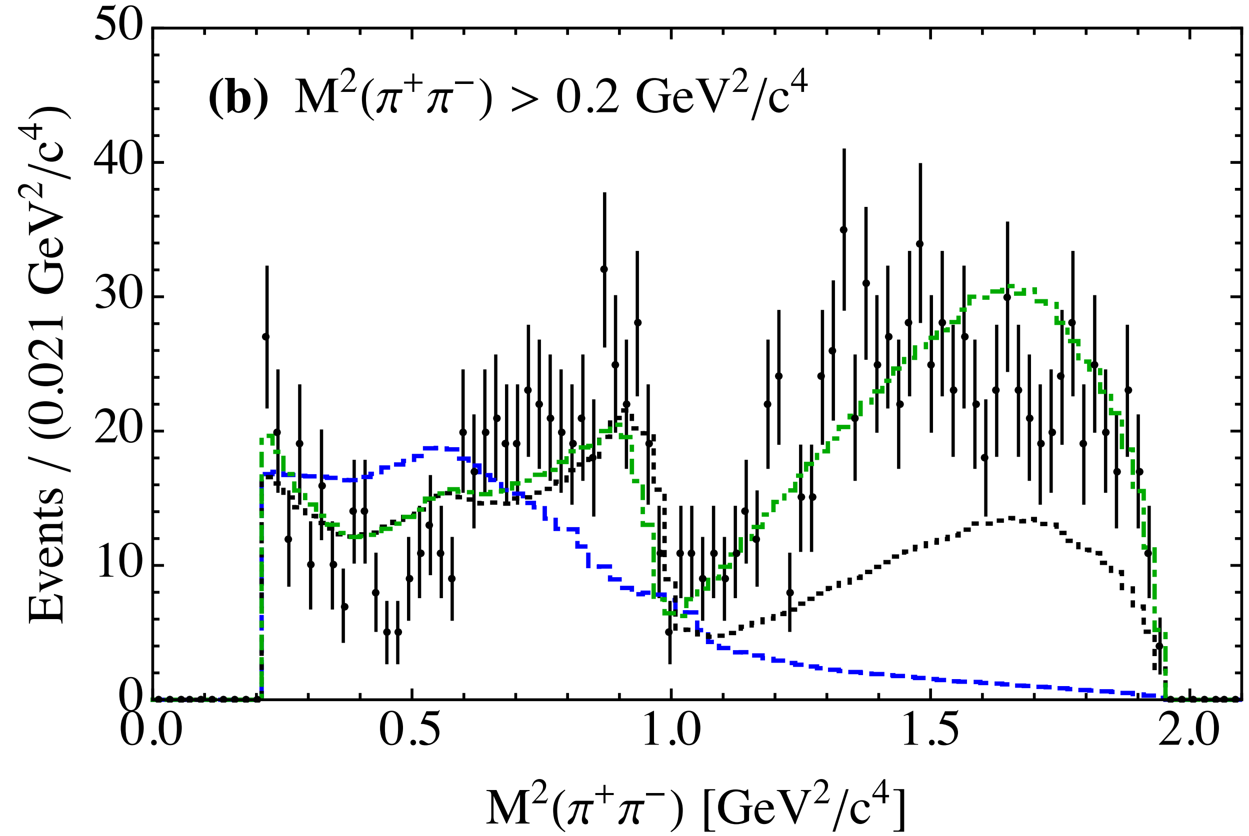
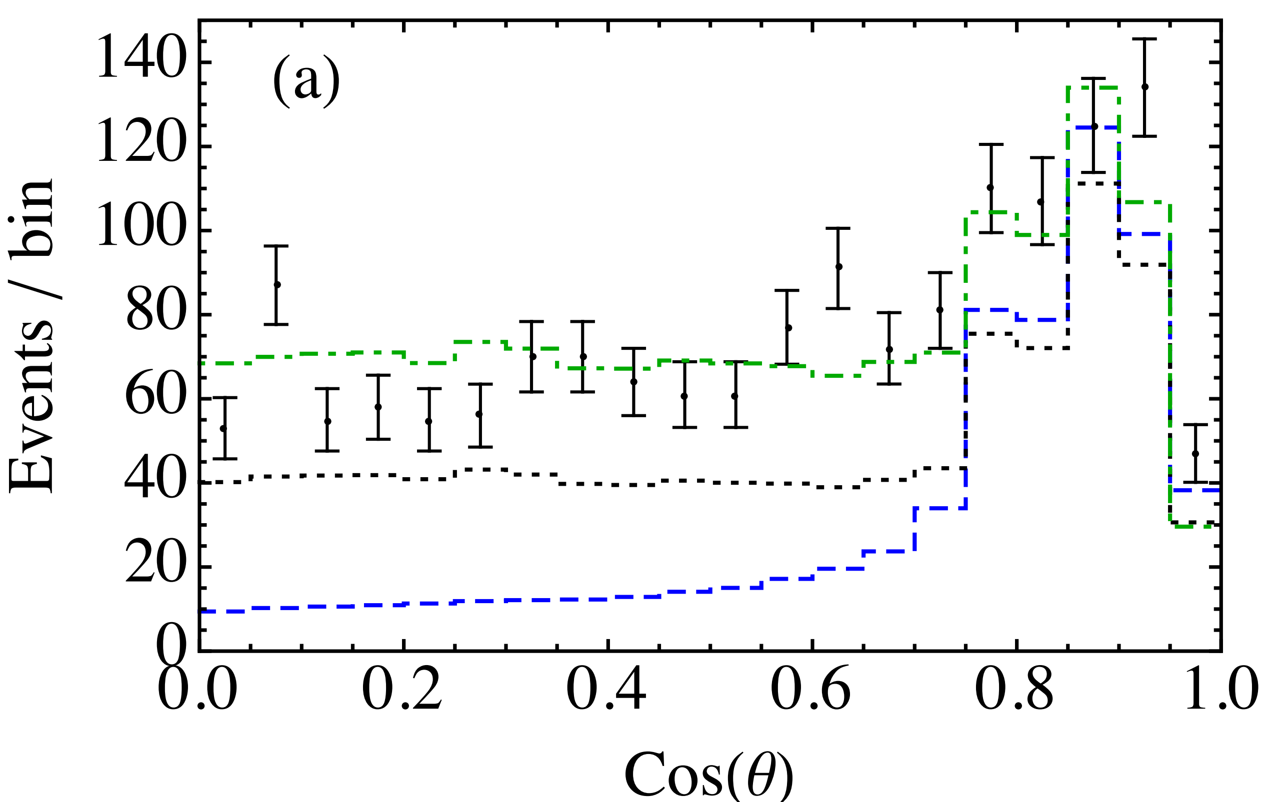






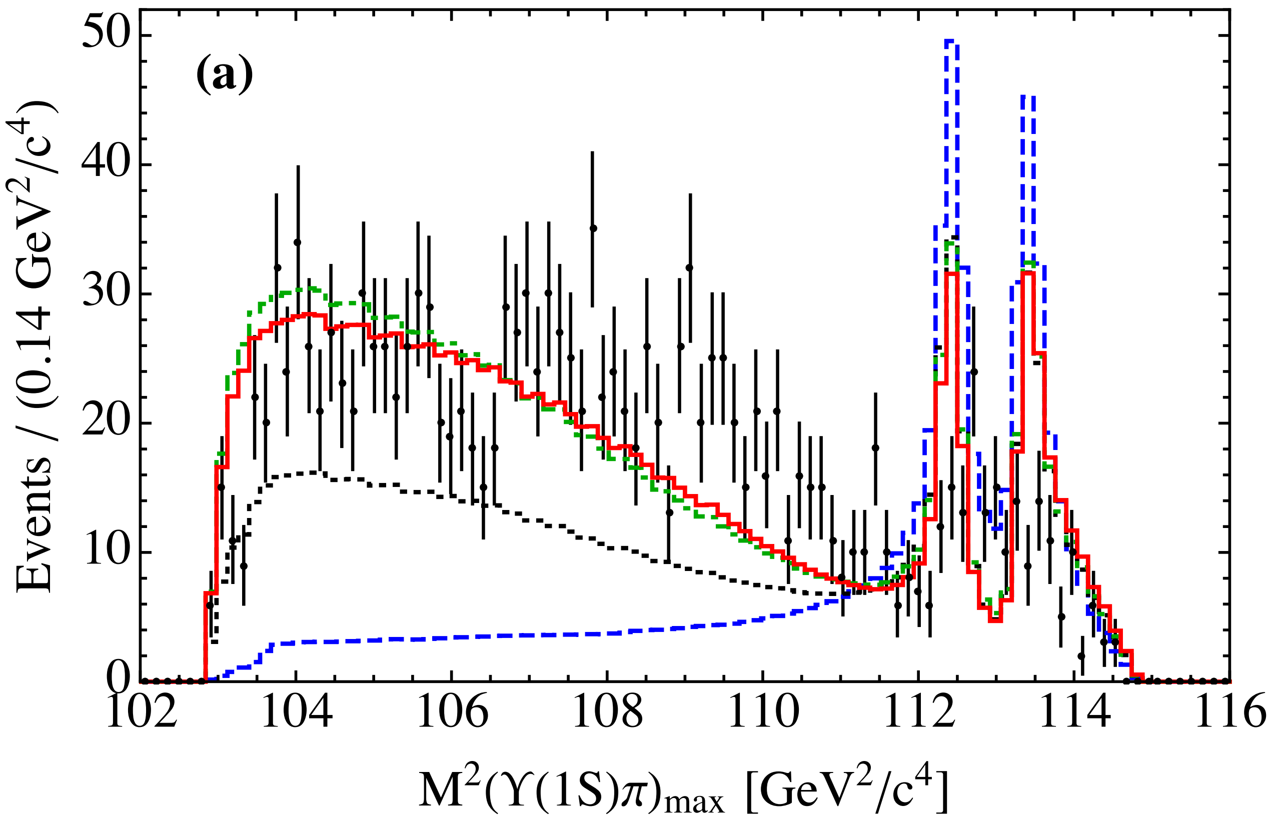
$M^L(t,u) = U(t) + U(u)$ "Z_b"

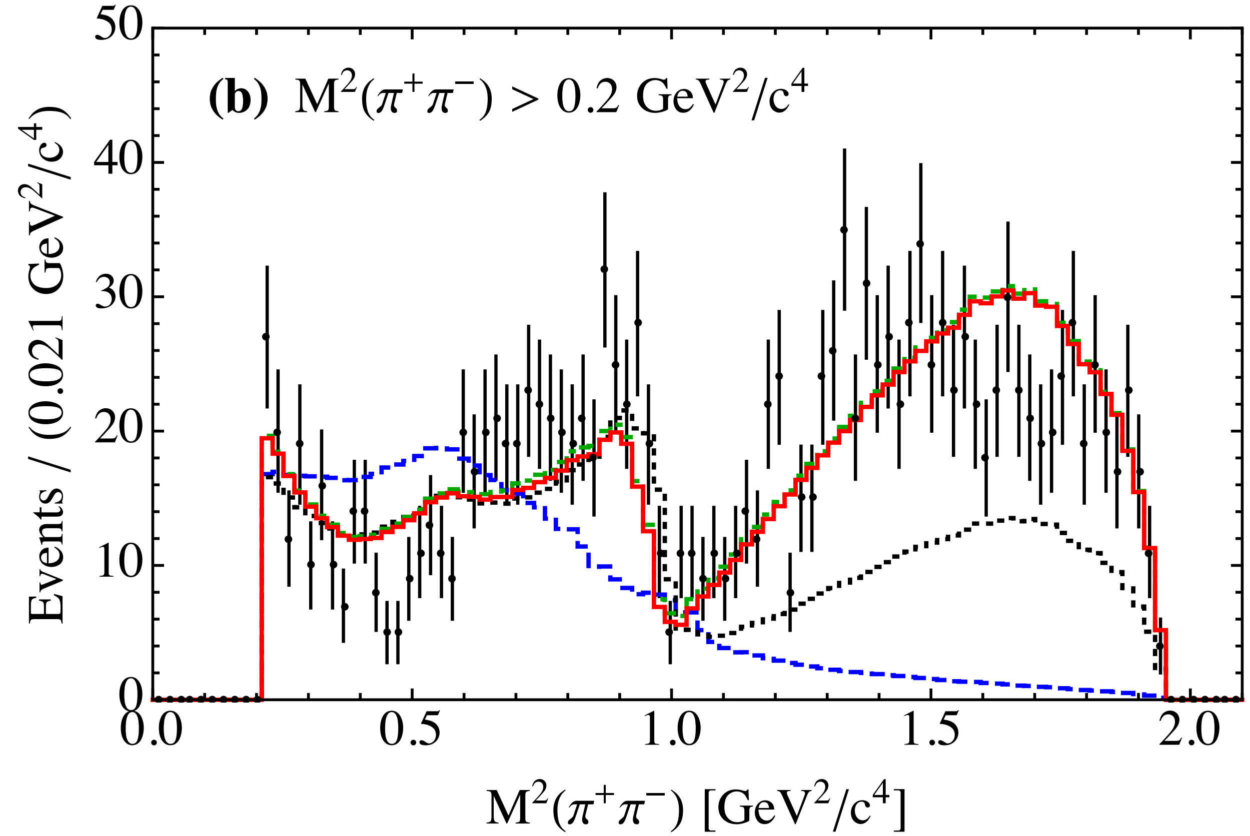
+ dispersive integral

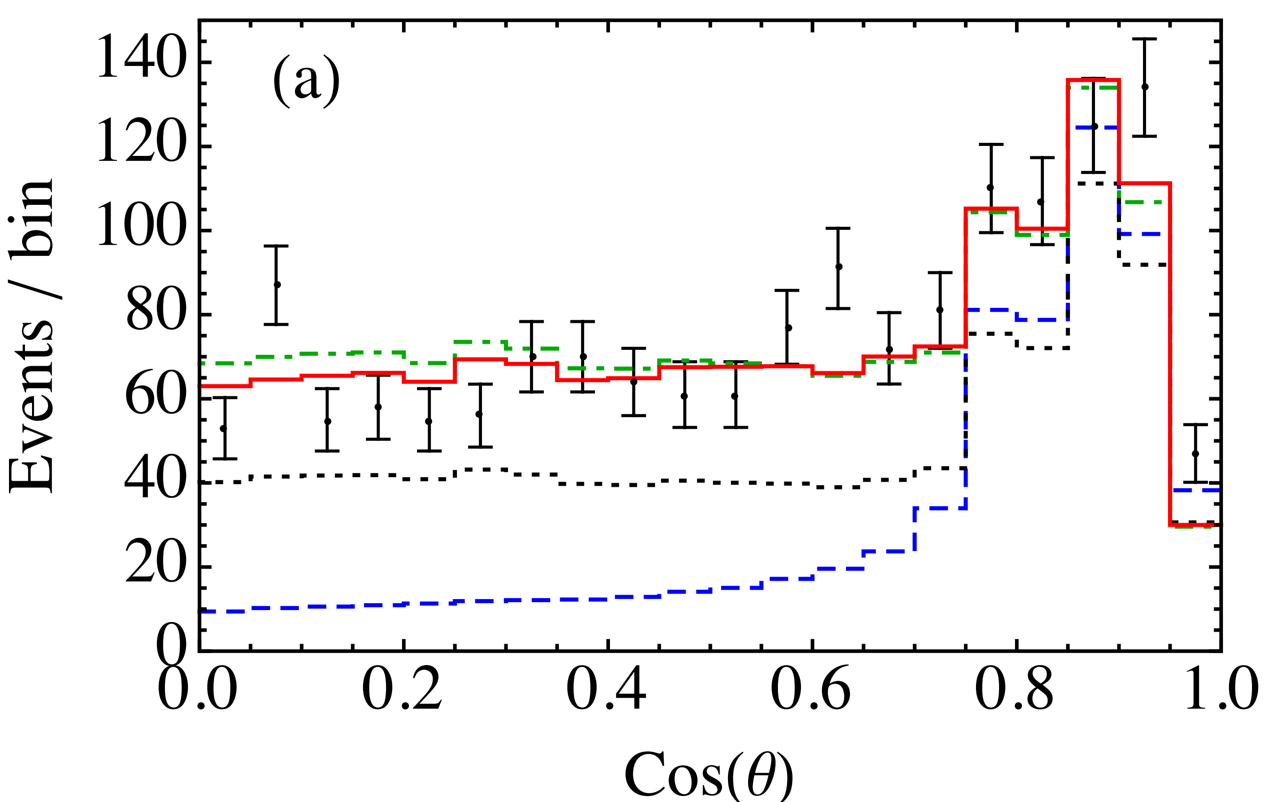
+ chiral CT's

Dalitz plot projections: Individual Contrib's.

$\Upsilon(10860) \rightarrow \Upsilon(1S)\,\pi^+\pi^-$







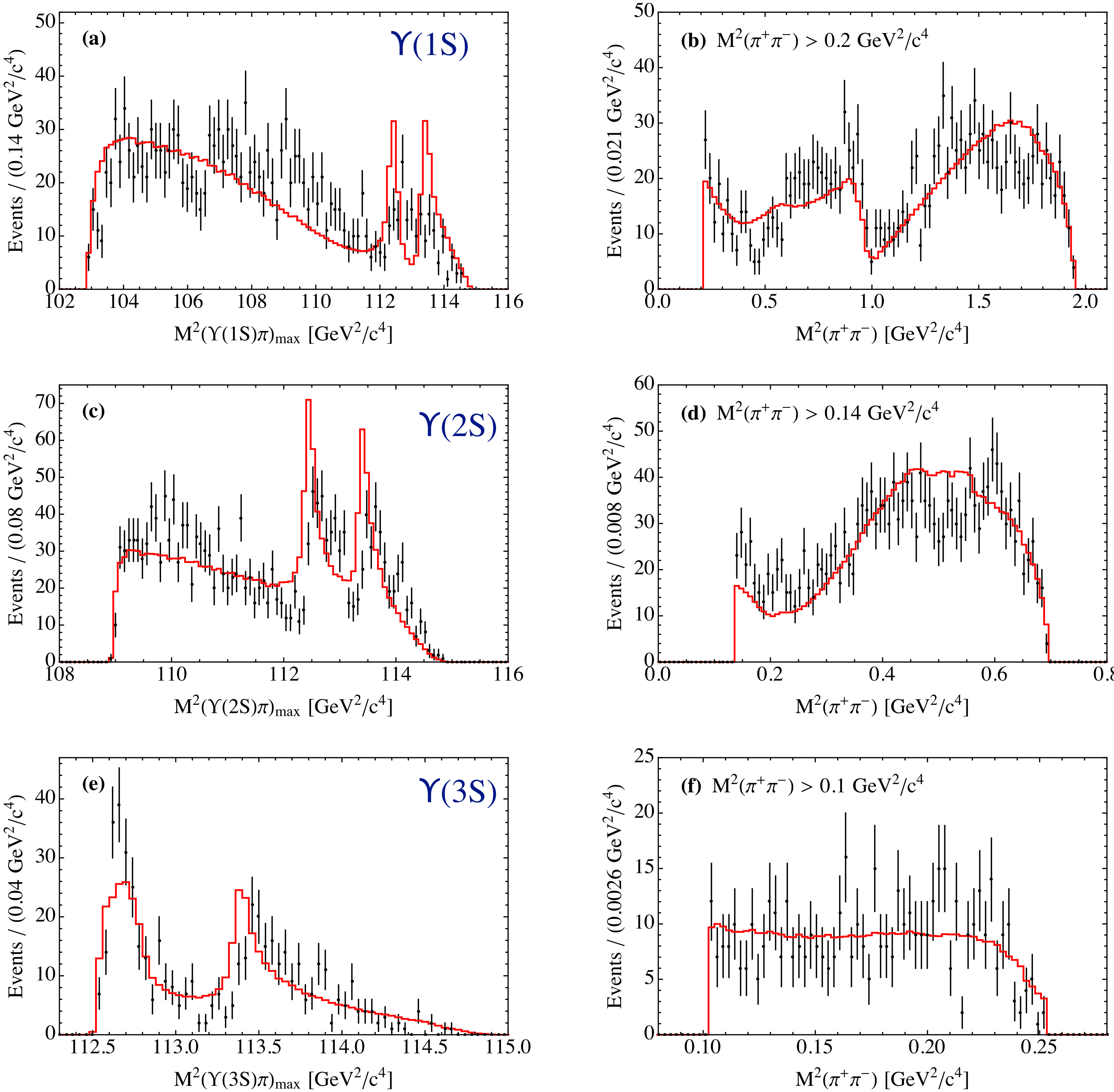
$M^L(t,u) = U(t) + U(u)$ "Z_b"

+ dispersive integral

+ chiral CT's

+ D-wave $\pi\pi$ FSI

Results for $M(\pi\Upsilon(\mathrm{nS}))^2$ and $M(\pi\pi)^2$ projections



$\Upsilon(1S)$ and $\Upsilon(2S)$

☛ $\pi\pi$-KK FSI very important

Key contribution to:

— right shoulder in $M(\pi\pi)^2$

— left shoulder in $M(\pi\Upsilon(\mathrm{nS}))^2$ n=1,2

— dip region ~1 GeV in $M(\pi\pi)^2$

$\Upsilon(3S)$

☛ Completely dominated by $M^L(t,u) = U(t)+U(u)$

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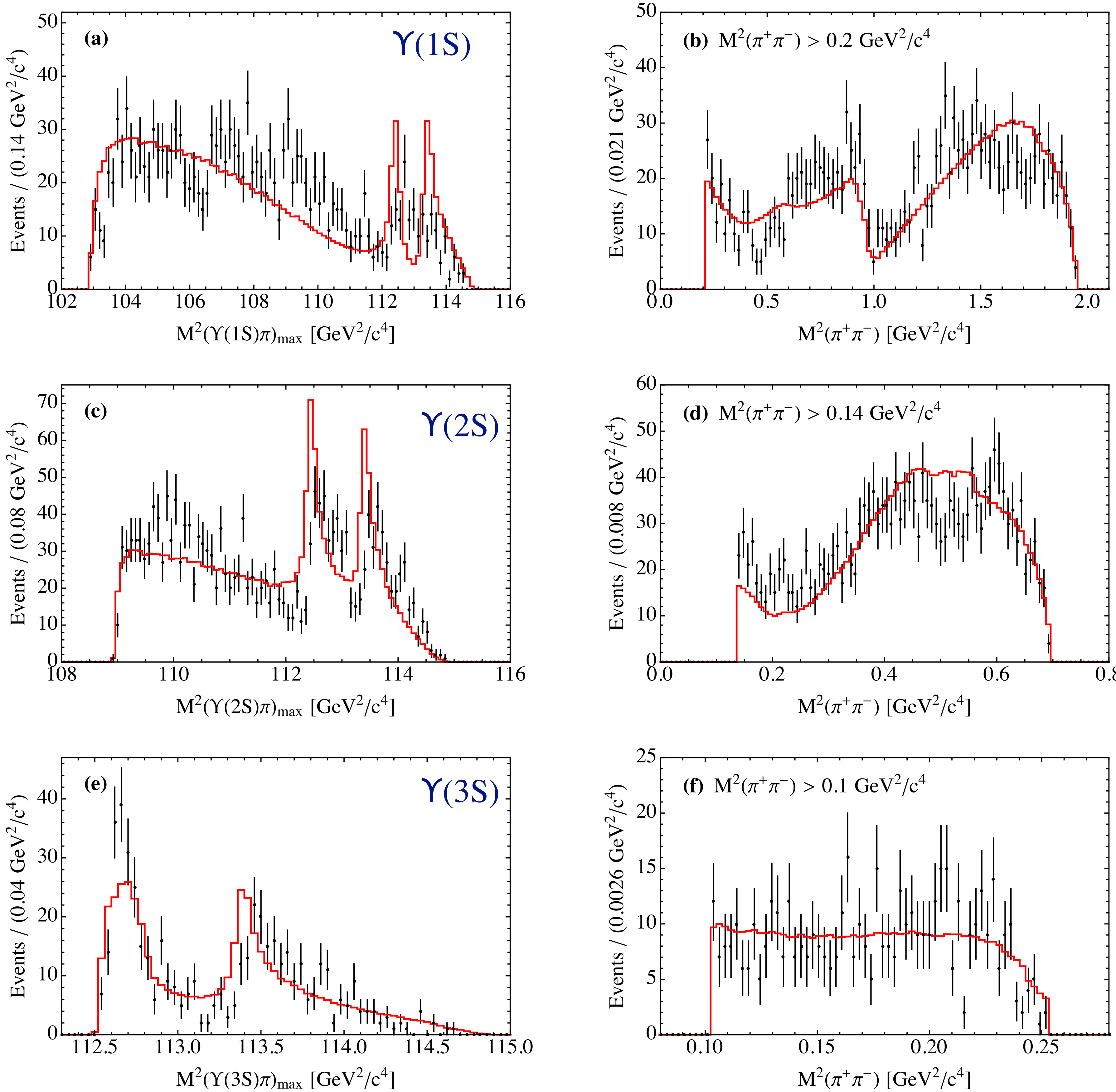
Results for $M(\pi\Upsilon(\mathrm{nS}))^2$ and $M(\pi\pi)^2$ projections



Υ(1S) and Υ(2S)

- $\pi\pi$-KK FSI very important

Key contribution to:

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— left shoulder in $M(\pi\Upsilon(\mathrm{nS}))^2$ n=1,2

— dip region ~1 GeV in $M(\pi\pi)^2$

Υ(3S)

- Completely dominated by $M^L(t,u) = U(t)+U(u)$

- $\pi\pi$ FSI is not important

—Very reasonable overall description

- Peaks of the Z_b's, consistent with $B^{(*)}\overline{B}^*$ and $\pi\pi h_b(mP)$, are not exactly in accord with $\pi\Upsilon$(nS)

Summary

- A chiral EFT based analysis of $\Upsilon(10860) \to \pi Z_b^{(\prime)} \begin{array}{l} \to \pi B^{(*)}\bar{B}^* \\ \to \pi\pi h_b(mP) \end{array}$

  $\Longrightarrow$ poles and residues of the Z_b(10610) and Z_b(10650) are extracted

- An analysis of the Dalitz plots for $\Upsilon(10860) \to \pi Z_b^{(\prime)} \to \pi\pi\Upsilon(nS)$

  — The production amplitude w/o $\pi\pi$ FSI is taken from the EFT approach *parameter free*

  — A dispersive approach to deal with crossed channels is employed: the $\pi\pi$-K$\bar{\text{K}}$ FSI is calculated with the minimal number of parameters and Im parts being under control

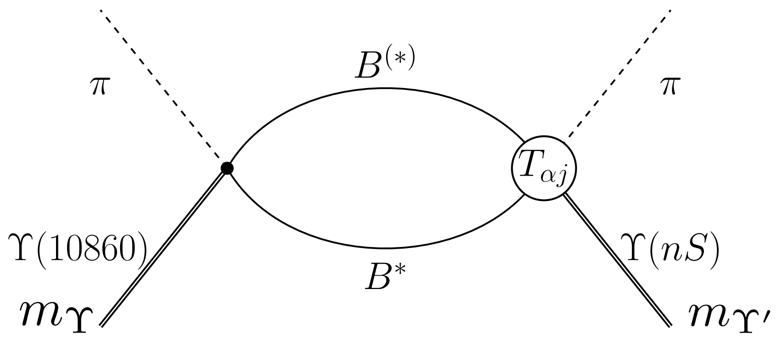
  $\Longrightarrow$ A very reasonable description of the $\pi\pi$ and $\pi\Upsilon$ spectra serves as a good consistency check for the whole approach

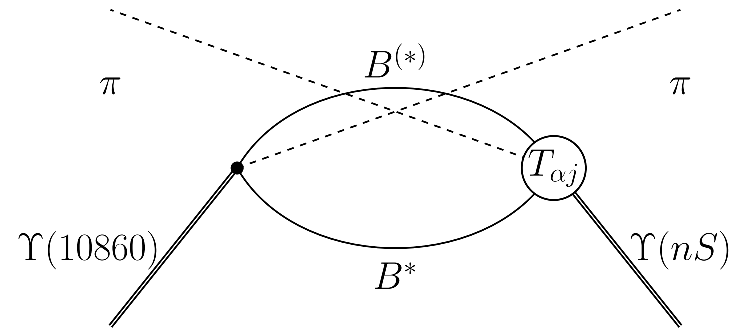
Next steps: — A dispersive analysis including the OPEP

— A combined analysis of data in all channels within the same framework

Backup

Left-hand cut production amplitude

$M^L(t,u) =$ 

+ 

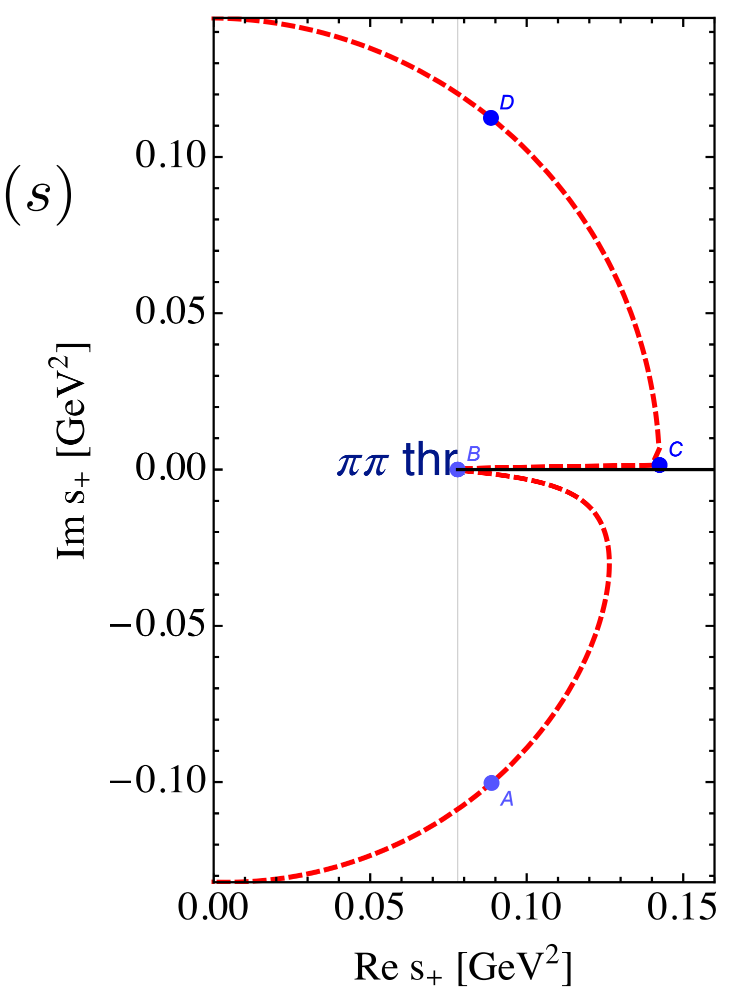
☛ $Z_b(10610)/Z_b(10650)$ are poles in the coupled-channel amplitudes

$$M^L(t,u) = U(t) + U(u) = -\frac{1}{\pi}\int_{(m_\pi+m_{\Upsilon(1S)})^2}^{\infty} d\mu^2\, \mathrm{Im}\, U(\mu^2)\left(\frac{1}{t-\mu^2}+\frac{1}{u-\mu^2}\right) \xrightarrow{\text{S-wave}} M_0^L(s)$$

— Take care about anomalous thresholds = Log branch points of $M_0^L(s)$

$$s_\pm = \frac{(m_\Upsilon^2 - m_{\Upsilon'}^2)^2}{4\mu^2} - \frac{\left(\sqrt{\lambda(m_\Upsilon^2, m_\pi^2, \mu^2)} \pm \sqrt{\lambda(m_{\Upsilon'}^2, m_\pi^2, \mu^2)}\right)^2}{4\mu^2}$$

s+ when μ is varied →

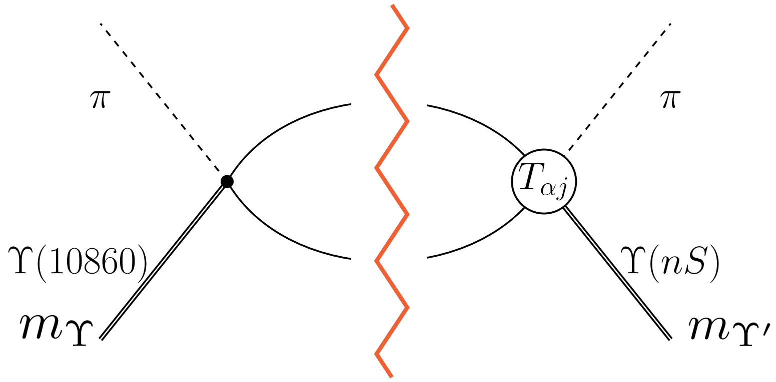
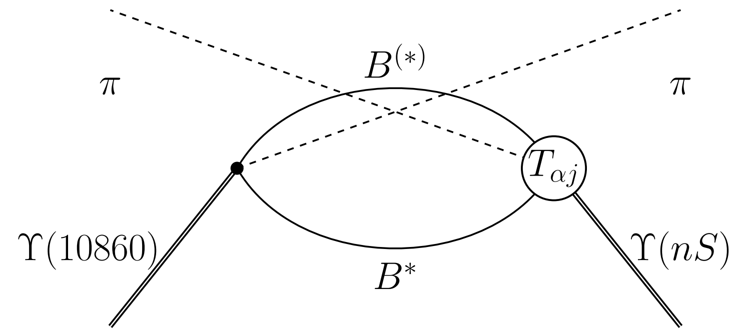


☛ s+ hits the $\pi\pi$ threshold and goes to the 1st sheet at the point B if

$$\mu^2 \le \mu^2_{\text{crit}} \equiv \frac{1}{2}(m^2_{\Upsilon(nS)} + m^2_{\Upsilon(10860)}) - m_\pi^2$$

For ϒ(3S): $\mu_{\text{crit}} = 10.6097\,\text{GeV}$ very close to $Z_b(10610)$ pole

Left-hand cut production amplitude

$M^L(t,u) =$  + 

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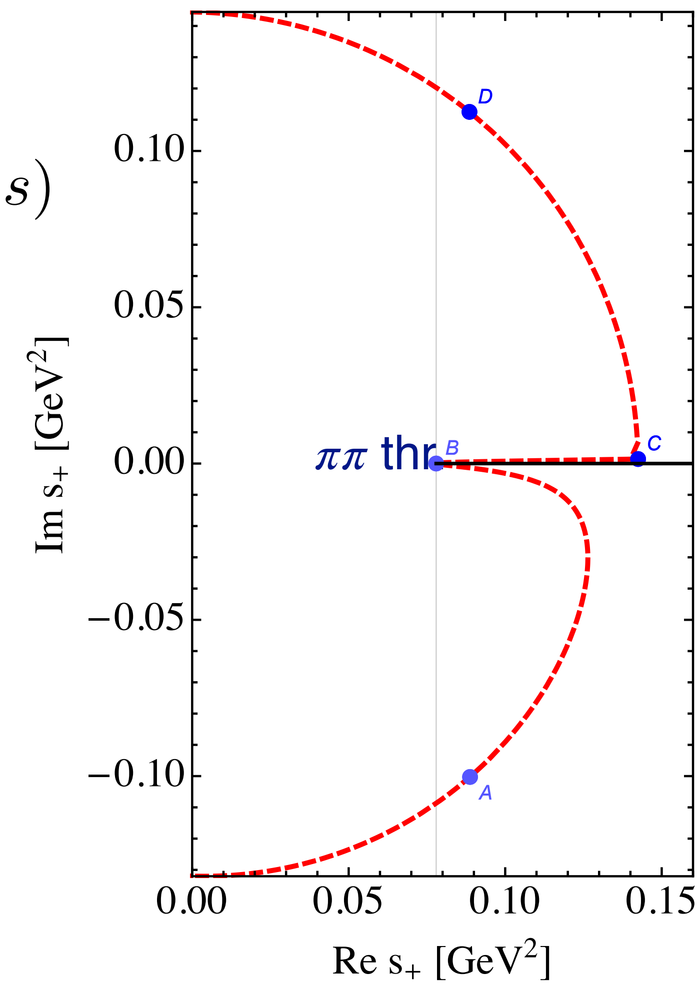

☛ s+ hits the $\pi\pi$ threshold and goes to the 1st sheet at the point B if

$$\mu^2 \le \mu_{\rm crit}^2 \equiv \frac{1}{2}(m_{\Upsilon(nS)}^2 + m_{\Upsilon(10860)}^2) - m_\pi^2$$

For Υ(3S): $\mu_{\rm crit} = 10.6097\,\mathrm{GeV}$ very close to $Z_b(10610)$ pole

— $\mathrm{Im}\,M_0^L(s)$: Leading contribution is from the $B^{(*)}\bar{B}^*$ cuts, these states can be on shell

subleading one — from inelastic channels