Infrared structure of QED as a many-body theory of worldlines

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arXiv:2206.04188

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Outline:

1. Motivation

2. From fields to worldlines: a many-body & point-like particle picture of QED.

3. QED perturbation theory without Feynman diagrams: the cusp anomalous dimension

4. IR structure of QED worldline amplitudes: Dyson vs Faddeev-Kulish S-matrices.

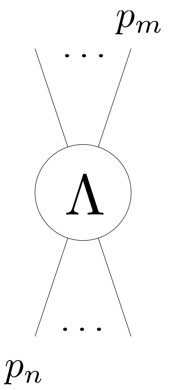
5. Brief summary

1. Motivation.

Understanding the IR

In QED, **UV** divergences are removed by renormalization. **IR** divergences are perhaps not so well understood conceptually.

Conventional wisdom: **observables are IR finite** but **amplitudes are not** *Bloch-Nordsieck/Yennie-Frautschi-Suura*



Cross section with arbitrary number of virtual photons attached

$$\underbrace{\sigma^{virt}_{\alpha\beta}(\leq \Lambda)} = \exp\left\{-\Gamma_{\alpha\beta}\int_\lambda^\Lambda \frac{d\omega_k}{\omega_k}\right\}\sigma^0_{\alpha\beta} = \exp\left\{-\Gamma_{\alpha\beta}\log\Big(\Lambda/\lambda\Big)\right\}\sigma^0_{\alpha\beta} = \Big(\lambda/\Lambda\Big)^{\Gamma_{\alpha\beta}}\sigma^0_{\alpha\beta}$$

$$\underbrace{\sigma^{real}_{\alpha\beta}(\leq E)} = \exp\left\{+\Gamma_{\alpha\beta}\int_\lambda^E \frac{d\omega_k}{\omega_k}\right\}\sigma^0_{\alpha\beta} = \exp\left\{+\Gamma_{\alpha\beta}\log\Big(E/\lambda\Big)\right\}\sigma^0_{\alpha\beta} = \Big(E/\lambda\Big)^{\Gamma_{\alpha\beta}}\underbrace{\sigma^0_{\alpha\beta}}$$

Cross section with arbitrary number of real photons attached

Transition α→ β without any real or virtual photons attached

$$\Gamma_{\alpha\beta} = -\frac{\alpha}{2\pi}\sum_{nm} Q_n\eta_n Q_m\eta_m\Big(\gamma_{nm}\coth\gamma_{nm} - 1\Big)$$

↓ Cusp anomalous dimension: depending on the angles of charges at infinity

↓ $\cosh\gamma_{nm} = \dfrac{p_n\cdot p_m}{\sqrt{p_n^2 p_m^2}}$

$\eta = +1$ Outgoing charges

$\eta = -1$ Incoming charges

Universal behavior of IR divergences in **abelian theories** with massless bosons. In **Yang-Mills theories**, a long standing problem: infinitely massless IR bosons in the cascade contribute to the IR divergence.

IR renormalization of amplitudes

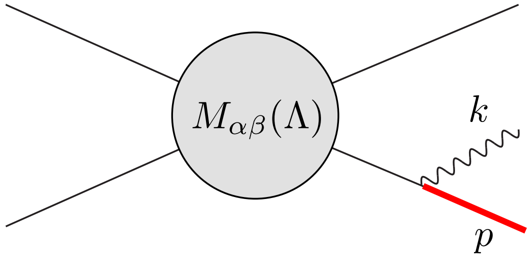
Dollard J. Math. Phys. 5 (1964) 729
Chung PR 140 (1965) 1110
Kibble J. Math. Phys 9 (1968) 315
Kibble PR 173 (1968) 1527
Kibble PR 174 (1968) 1882
Kibble PR 175 (1968) 1624
Kulish, Faddeev Theor. Math. Phys. 4 (1970) 745

Observables always IR finite in nature, but amplitudes not.

In Abelian theories with massless bosons **IR divergences exponentiate** and set **S-matrix elements to zero.**

Faddeev and Kulish:
Gauge interactions have infinite range so isolated charges/free asymptotic states do not really exist.
Free asymptotic states create the IR singularities.

Example:



$$\lim_{k\to 0} M^{\mu}_{\alpha\beta}(k,\Lambda) = \lim_{k\to 0}\left\{\frac{ep^{\mu}}{(p+k)^2 - m^2 - i\epsilon}\right\} M_{\alpha\beta}(\Lambda) = \lim_{k\to 0}\left\{\frac{ep^{\mu}}{p\cdot k - i\epsilon}\right\} M_{\alpha\beta}(\Lambda)$$

Because $p^2 - m^2 = 0$

→ **Key idea:** IR divergences **depend only on the directions of charges at infinity and signal residual asymptotic interactions,** so it is possible to capture them as a **coherent clouds** of infinite IR bosons **dressing the states** to construct **IR finite amplitudes.**

Why revisiting the IR

There is **non-perturbative physics** in the **IR**

New symmetries of QED:
IR photons as Goldstone bosons
of spontaneously broken large gauge transformations

Arkani-Hamed, Pate, Raclariu, Strominger JHEP 08 (2021) 062
Kapec, Perry, Raclariu, Strominger PRD 96 (2017) 085002
Strominger arXiv:1703.05448

IR singularities and **cusp anomalous dimensions:**
very active field

Hannesdottir, Schwartz PRD 101 (2020) 10, 105001
Anastasiou, Sterman JHEP 07 (2019) 056
Henn, Korchemsky, Mistlberger JHEP 04 (2020) 018
Bechert, Neubert JHEP 01 (2020) 025

(and even a very recent paper by
Weinberg in QED, PRD 99 (2019) 076018)

Weinberg's "Infrared Photons and Gravitons":

> But these remarks do not apply to theories involving charged massless particles. In such theories (including the Yang-Mills theory) a soft photon emitted from an external line can itself emit a pair of soft charged massless particles, which themselves emit soft photons, and so on, building up a cascade of soft massless particles each of which contributes an infrared divergence. The elimination of such complicated interlocking infrared divergences would certainly be a Herculean task, and might even not be possible.
>
> We may be thankful that the zero charge of soft photons and the zero gravitational mass of soft gravitons saves the real world from this mess. Perhaps it would not be too much to suggest that it is the infrared divergences that prohibit the existence of Yang-Mills quanta or other charged massless particles.

Gluon saturation

Small-x QCD:
Regge physics

Why worldlines

Renewed interest in HEP: potential advantages in wide range of practical QCD computations, chiral kinetic theory, DIS in the Regee limit, Schwinger pair production, the role of the chiral anomaly in the proton's spin, asymptotics ...

Mueller, Venugopalan PRD97 (2018) 051901
Mueller, Venugopalan PRD96 (2017) 016023
Gould, Rajantie, Xie PRD98 (2018) 056022

Mueller, Tarasov, Venugopalan PRD102 (2020) 016007
Tarasov, Venugopalan PRD100 (2019) 054007
Tarasov, Venugopalan hep.ph/2109.10370
Bonocore JHEP02 (2021) 007

IR semi-classical behavior suggests formulating the problem using particle like descriptions.

Wilson operators are main building blocks in form factors, TMDs, Drell-Yan processes, Higgs production, re-summation of large Sudakov logarithms, eikonal or next-to-eikonal exponentiation theorems of "*webs*", SCETs ...

→ **Bosonic approximations** are fine, but sometimes problematic. Worldlines provide the **exact exponentiation** of **spin**, **helicity**, **color** d.o.f. in **background** or **dynamical fields.**

Common understanding of **Wilson loop renormalization program,** the **Faddeev-Kulish dressings** and the **Bloch-Nordsieck/Yennie-Frautschi-Suura** standpoints on the IR problem.

→ Perhaps go beyond: connect **renormalization program of the worldline** with the **calculation of vertex functions**.

Non-perturbative & first-quantized Hamiltonian formulation of gauge theories, natural for the quantum computer

2. From fields to worldlines: a many-body & point-like particle picture of QED.

We want to construct worldline amplitudes/S-matrix elements in QED to all loop orders & free of soft singularities.

Non-perturbative amplitudes can be obtained in the worldline by integrating out all the fields. For instance:

$$\mathrm{Z}=\underbrace{\int \mathcal{D}A \exp\Big[-\frac{1}{4}\int d^4x F_{\mu\nu}^2-\frac{1}{2\zeta}\int d^4x(\partial_\mu A_\mu)^2}_{\text{}}+\underbrace{\ln\det(\not{D}+m)}_{\text{}}\Big]$$

Integration in the gauge field A_μ configurations
→ A_μ is a dynamical field

One-loop effective action = Exponential of the amplitude of 1-fermion to perform a loop in the presence of A_μ

Write the one-loop fermion determinant in the exponential in worldline form,

$$\underbrace{x_\mu(\tau),\tau\in[0,1],x_\mu(1)=x_\mu(0)} \quad \underbrace{\psi_\mu(\tau),\tau\in[0,1],\psi_\mu(1)=-\psi_\mu(0)} \quad \underbrace{\epsilon(\tau),\tau\in[0,1],\epsilon(0)=\epsilon_0}$$

commuting worldline = bosonic d.o.f.

anti-commuting worldline = fermionic d.o.f.

commuting einbein = Schwinger parameter of bosonic d.o.f. = classical proper time

$$\ln\det(\not{D}+m)=\mathrm{Tr}\int_0^\infty\frac{d\epsilon_0}{2\epsilon_0}e^{-\epsilon_0}-\int_0^\infty\frac{d\epsilon_0}{2\epsilon_0}e^{-m^2\epsilon_0}\int_P\mathcal{D}^4x\int_{AP}\mathcal{D}^4\psi$$

$$\times\exp\Big\{-\frac{1}{4\epsilon_0}\int_0^1 d\tau\Big(\frac{dx_\mu}{d\tau}\Big)^2-\frac{1}{4}\int_0^1 d\tau\psi_\mu(\tau)\frac{d\psi_\mu}{d\tau}+i\underbrace{g\int_0^1 d\tau\frac{dx_\mu}{d\tau}A_\mu(x(\tau))}-i\underbrace{\frac{g\epsilon_0}{2}\int_0^1 d\tau\psi_\mu(\tau)\psi_\nu(\tau)F_{\mu\nu}(x(\tau))}\Big\}$$

Wilson loop term = charged current inter. with A_μ

Extra term = The spin precession in A_μ

Expand in the number of virtual fermions and refer to the pure gauge sea of disconnected photon loops

$$\frac{\mathrm{Z}}{\mathrm{Z}_{\mathrm{MW}}}=\frac{1}{\mathrm{Z}_{\mathrm{MW}}}\int\mathcal{D}A\exp\Big\{-\frac{1}{4}\int d^4xF_{\mu\nu}^2-\frac{1}{2\zeta}\int d^4x(\partial_\mu A_\mu)^2\Big\}\times\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\Big(\ln\det\left(D_\mu\gamma_\mu+m\right)\Big)^\ell$$

Hence $\frac{\mathrm{Z}}{\mathrm{Z}_{\mathrm{MW}}} = \sum_{\ell=0}^{\infty} \mathrm{Z}^{(\ell)} = \sum_{l=0}^{\infty} \underbrace{\frac{(-1)^\ell}{\ell!}}_{\text{Loop Parity}} \mathrm{W}^{(\ell)}$ where

$$\mathrm{W}^{(\ell)} = \left[\prod_{i=1}^{\ell} \int_0^\infty \frac{d\epsilon_0^i}{2\epsilon_0^i} \int_{\mathrm{P}} \mathcal{D}x_i \int_{\mathrm{AP}} \mathcal{D}\psi_i \exp\left\{ -m^2\epsilon_0^i - \frac{1}{4\epsilon_0^i}\int_0^1 d\tau \left(\frac{dx_i}{d\tau}\right)^2 - \frac{1}{4}\int_0^1 d\tau \psi_\mu^i \frac{d\psi_\mu^i}{d\tau} \right\} \right]$$

$$\frac{1}{\mathrm{Z}_{\mathrm{MW}}} \int \mathcal{D}A \exp\left\{ -\frac{1}{4}\int d^4x F_{\mu\nu}^2(x) - \frac{1}{2\zeta}\int d^4x \left(\partial_\mu A_\mu(x)\right)^2 \right.$$

$$\left. + i\sum_{i=1}^{\ell} g \int_0^1 d\tau \frac{dx_\mu^i}{d\tau} A_\mu\left(x_i(\tau)\right) - i\sum_{i=1}^{\ell} \frac{g\epsilon_0^i}{2}\int_0^1 d\tau \psi_\mu^i(\tau)\psi_\nu^i(\tau) F_{\mu\nu}\left(x_i(\tau)\right) \right]$$

Integrating out the gauge field one gets (*d=4, Feynman gauge and fixed einbein*)

$$\frac{\mathrm{Z}}{\mathrm{Z}_{\mathrm{MW}}} = \sum_{\ell=0}^{\infty} \underbrace{\frac{(-1)^\ell}{\ell!}}_{\substack{\text{Loop Parity and}\\ \text{symmetry factors}}} \left\langle \exp\left\{ -\frac{g^2}{8\pi^2} \sum_{ij=1}^{\ell} \int_0^1 d\tau_i \left(\underbrace{\frac{dx_i^\mu}{d\tau_i}}_{\substack{\text{Charged}\\ \text{local current}\\ \text{of particle “i”}}} - i\epsilon_0^i \underbrace{\sigma_{\mu\rho}^i}_{\substack{\text{Particle “i”}\\ \text{local}\\ \text{spin tensor}}}(\tau_i) \frac{\partial}{\partial x_\rho^i} \right) \times \underbrace{\int_0^1 d\tau_i \left(\frac{dx_\mu^j}{d\tau_j} - i\epsilon_0^j \sigma_{\mu\eta}^j(t_j) \frac{\partial}{\partial x_\eta^j} \right) \frac{1}{(x_i - x_j)^2}}_{\substack{\text{Dynamical gauge field at point x\_i created by charged}\\ \text{particle “j” current and spin precession}}} \right\} \right\rangle$$

$\langle \star \rangle \equiv$ Sum/path integrate over all possible closed worldline configurations (loops) and sum over all possible proper times

→ **Lorentz forces** between “l” charges including self-exchanges and sum over all possible paths of the commuting and anti-commuting worldlines, encoding the bosonic and fermionic d.o.f.

Multi-particle theory of worldlines living in loops with proper times ~ ϵ_0^i and where no final or initial particles exist

Feynman Physical Review 80 3 (1950) 440

in velocity. When there are several particles (other than the virtual pairs already included) one use a separate u for each, and writes the amplitude for each set of trajectories as the exponental of $-i$ times

$$\frac{1}{2}\sum_n \int_0^{u_0^{(n)}} \left(\frac{dx_\mu^{(n)}}{du}\right)^2 du+\sum_n \int_0^{u_0^{(n)}} \frac{dx_\mu^{(n)}}{du} B_\mu(x_\mu^{(n)}(u))du$$
$$+\frac{e^2}{2}\sum_n\sum_m \int_0^{u_0^{(n)}}\int_0^{u_0^{(m)}} \frac{dx_\nu^{(n)}(u)}{du}\frac{dx_\nu^{(m)}(u')}{du'}$$
$$\times\delta_+((x_\mu^{(n)}(u)-x_\mu^{(m)}(u'))^2)du du', \quad (11A)$$

where $x_\mu^{(n)}(u)$ are the coordinates of the trajectory of the nth particle.[22] The solution should depend on the $u_0^{(n)}$ as $\exp(-\frac{1}{2}im^2\Sigma_n u_0^{(n)})$.

Feynman Physical Review 84 1 (1950) 108

I have expended considerable effort to obtain an equally simple word description of the quantum mechanics of the Dirac equation. Very many modes of description have been found, but none are thoroughly satisfactory. For example, that of Eq. (32-a) is incomplete, even aside from the geometrical mysteries involved in the superposition of hypercomplex numbers. For in (32-a) the

GAUGE FIELDS AS RINGS OF GLUE

A.M. POLYAKOV
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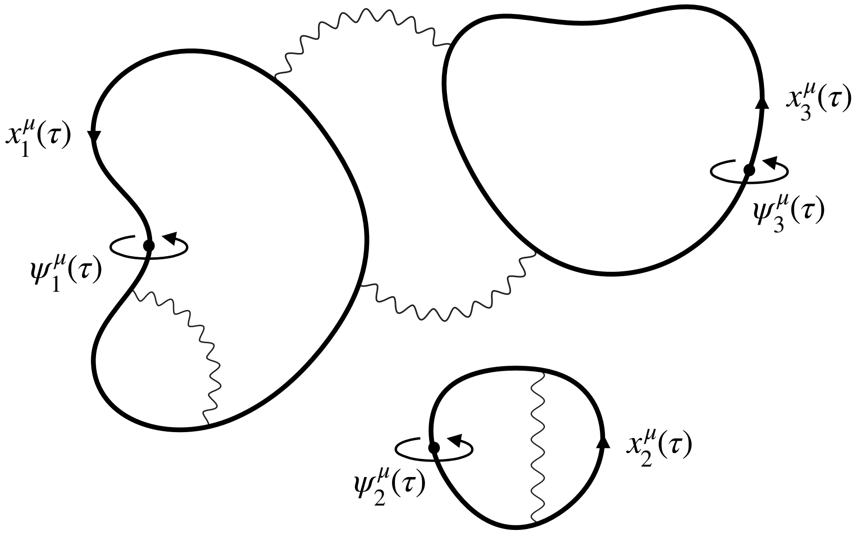
Received 24 September 1979

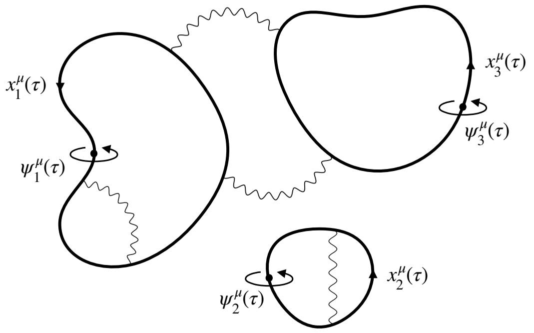
The basic idea is that gauge fields can be considered as chiral fields, defined on the space of all possible contours (the loop space) [1]. The origin of the idea lies in the expectation that, in the confining phase of a gauge theory, *closed* strings should play the role of *elementary* excitations [2,3]. In contrast, in conventional field theories, the elementary excitations are just point-like particles. This observa-



A 3-loop contribution. Each 0+1-dimensional point particle is fully described by a super-pair of closed worldlines in propertime, created at $\tau = 0$ and destroyed at $\tau = 1$; these emit, reabsorb, and exchange an arbitrary number photons that transmit the Lorentz forces between spin-1/2 charges.

Along the same lines for scalar particles:
Schubert, Phys.Rept. 355 (2001) 73
Affleck, Alvarez, Manton Nucl.Phys.B 197 (1982) 509
Gies, Sanchez-Guillen, Vazquez JHEP 08 (2005) 067

The QED vacuum in terms of virtual (0+1)-dimensional worldlines



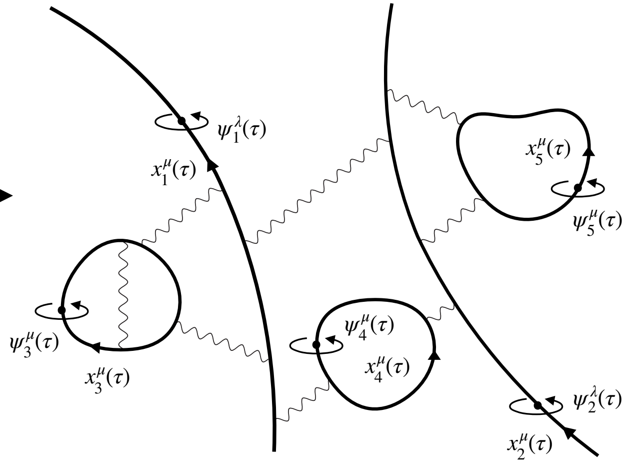
$$\frac{\mathrm{Z}}{\mathrm{Z}_{\mathrm{MW}}} = \sum_{\ell=0}^{\infty} \mathrm{Z}^{(\ell)} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \underbrace{\mathrm{W}^{(\ell)}}$$

Amplitude of l virtual fermions to describe a loop, exchanging an arbitrary number of photons.

An S-matrix on equal footing:

Fradkin and Gitman, PRD 44 (1991) 3230.

Using Fradkin-Gitman powerful result for the gauge-invariant (reparametrization and supergauge-invariant) dressed fermion propagator, one gets, following same steps



real fermions

$$\mathcal{S}_{fi}^{(r)} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \mathrm{S}_{fi}^{(r,\ell)} = \frac{\mathrm{Z}_{\mathrm{MW}}}{\mathrm{Z}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \left\{ \lim_{\substack{x^0_{f,n}\to+\infty \\ x^0_{i,n}\to-\infty}} \int d^3\vec{x}_f^n \int d^3\vec{x}_i^n \Psi^{(+)\dagger}_{f_n}(x_f^n) \exp\left\{\bar{\gamma}_\lambda \frac{\partial}{\partial\theta^n_\lambda}\right\} \bar{\gamma}_0 \Psi^{(+)}_{i_n}(x_i^n) \right\}$$

virtual fermions

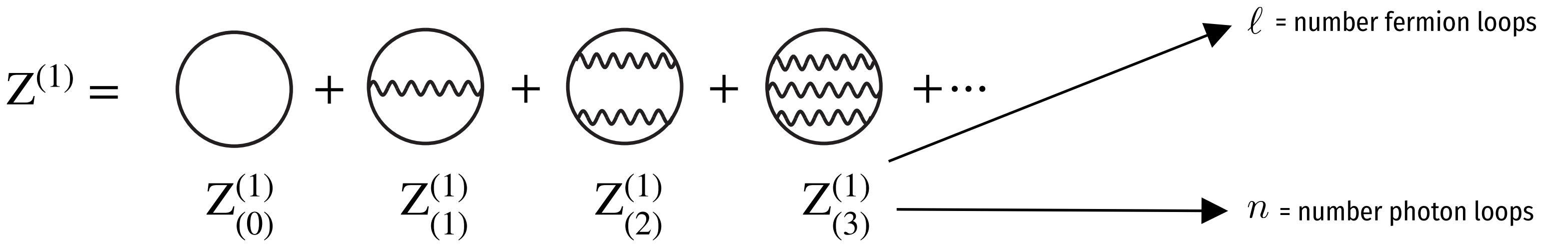
$$\times \underbrace{\mathrm{W}^{(r,\ell)}(x_f^r, x_i^r, \theta^r, \ldots, x_f^1, x_i^1, \theta^1)} + \text{final state permutations}\,.$$

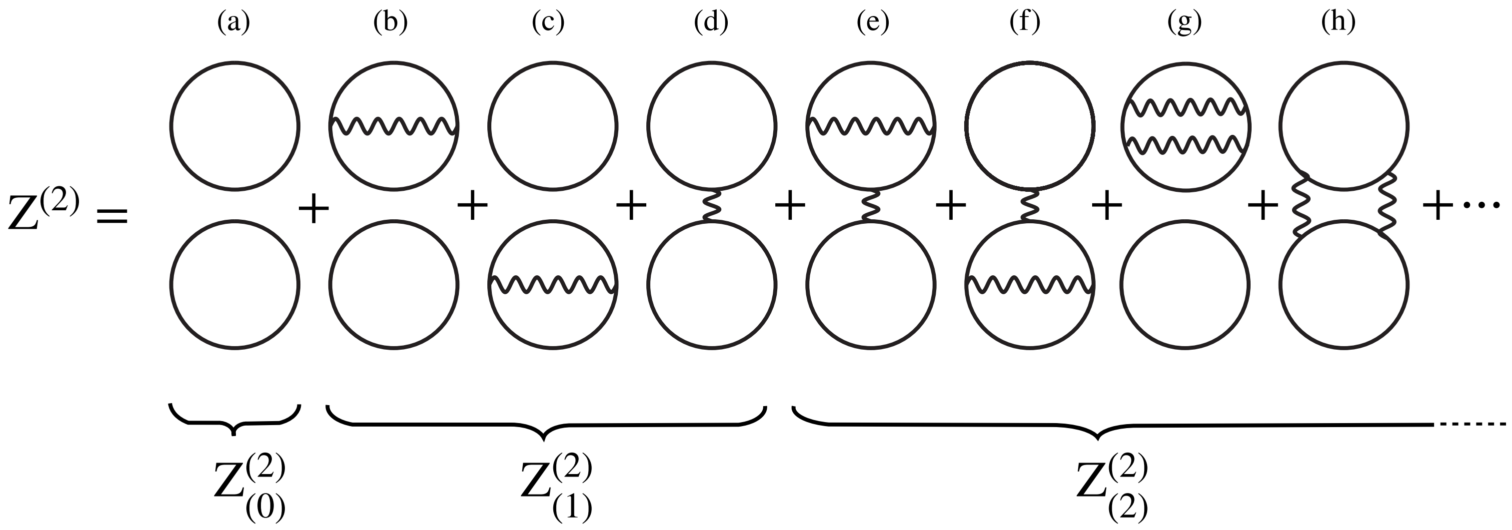
Amplitude of the r real particles to go from $\{x_i^n\}$ to $\{x_f^n\}$ with l virtual fermions describing loops present, and exchanging an arbitrary number of photons amongst eachother.

3. QED perturbation theory without Feynman diagrams.

QED IN THE WORLDLINE: EFFICIENCY IN HIGHER ORDER PERTURBATIVE CALCULATIONS

Expanding in loops:





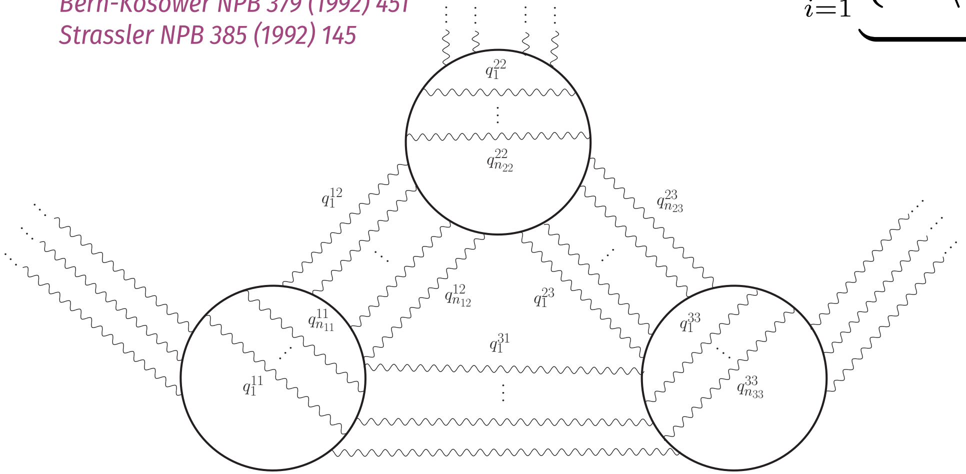
Specifically:

$$\frac{1}{Z^{(0)}}\frac{Z}{Z_{\rm MW}} = 1 + \underbrace{\left(Z^{(1)}_{(0)} + Z^{(1)}_{(1)} + \cdots\right)}_{\text{multi-photon, 1-fermion loop}} + \underbrace{\left(Z^{(2)}_{(0)} + Z^{(2)}_{(1)} + \cdots\right)}_{\text{multi-photon, 2-fermion loops}} + \underbrace{\left(Z^{(3)}_{(0)} + Z^{(3)}_{(1)} + \cdots\right)}_{\text{multi-photon, 3-fermion loops}} + \cdots$$

General **l-loop n-photon amplitudes** are reduced to computing a **product of "l" one-loop n-rank vacuum polarization tensors:**

$$Z^{(\ell)}_{(n)} = \frac{1}{\ell!}\sum_{n_{11}+n_{12}+\cdots+n_{\ell\ell}=n}\prod_{i,j=1}^{\ell}\left\{\frac{1}{2^{n_{ij}}}\frac{1}{n_{ij}!}\prod_{k=1}^{n_{ij}}\left\{\int\frac{d^4q_k^{ij}}{(2\pi)^4}\tilde{D}^B_{\mu_k^{ij}\nu_k^{ij}}(q_k^{ij})\right\}\right\}$$
$$\times\underbrace{\prod_{i=1}^{\ell}\left\{-\left\langle\prod_{j=1}^{\ell}\prod_{k=1}^{n_{ij}}\left(iJ_{\mu_k^{ij}}(-q_k^{ij})\right)\prod_{k=1}^{n_{ji}}\left(iJ_{\nu_k^{ji}}(+q_k^{ji})\right)\right\rangle\right\}}_{\text{Conventional n-rank photon polarization tensor}}$$

l=1:
Bern-Kosower NPB 379 (1992) 451
Strassler NPB 385 (1992) 145



which in the worldline reduce to evaluating path-integral **expectation values of products of currents**

$$J_\mu(q) = \int_0^1 d\tau e^{+iq\cdot x(\tau)}\left(\dot{x}_\mu(\tau) + i\varepsilon_0 q_\nu\psi_\mu(\tau)\psi_\nu(\tau)\right)$$

Worldline advantage:

Nice discussion in Cvitanovic's https://cns.gatech.edu/papers/finiteQED

1. All orders in PT (arbitrary "l" and "n") generated from an **universal and compact one-loop** l=1 **expression**:

$$\left\langle iJ_{\mu_1}(k_1)\cdots iJ_{\mu_n}(k_n)\right\rangle = 2g^n(2\pi)^d\delta^d\left(\sum_{i=1}^n k_i\right)\int\frac{d^dp}{(2\pi)^d}\int_0^\infty\frac{d\epsilon_0}{\epsilon_0}e^{-\epsilon_0(p^2+m^2)}$$
$$\times\prod_{i=1}^n\underbrace{\int_0^1 d\tau_i\int d\bar\theta_i d\theta_i\exp\left\{\frac{1}{2}\sum_{ij=1}^n\int_0^1 d\tau\int_0^1 d\tau'\left(\epsilon_0 j^{B,i}_{\mu_i\rho}(\tau)G_B(\tau,\tau')j^{B,j}_{\mu_j\rho}(\tau')-j^{F,i}_{\mu_i\rho}(\tau)G_F(\tau,\tau')j^{F,j}_{\mu_j\rho}(\tau)\right)\right\}}$$

(Unordered) Proper time integrals of the worldline green functions:

$$G_B(\tau,\tau')=|\tau-\tau'|-(\tau-\tau')^2\,,\quad G_F(\tau,\tau')=\mathrm{sign}(\tau-\tau')$$

2. One obtains **dimensionally regularized** amplitudes in Schwinger/Feynman parameters. **For instance, n=2**:

$$\Pi_{\mu\nu}(k_1,k_2) = -\langle i\tilde J_\mu(k_1)i\tilde J_\nu(k_2)\rangle = (2\pi)^d\delta^d(k_1+k_2)\left(\eta_{\mu\nu}k_1^2-k^1_\mu k^1_\nu\right)$$
$$\times\left\{-8\frac{g^2\mu^{4-d}}{(4\pi)^{d/2}}\int_0^1 d\tau\tau(1-\tau)\int_0^\infty d\epsilon_0\epsilon_0^{1-d/2}e^{-\epsilon_0\left(m^2+k_1^2\tau(1-\tau)\right)}\right\}$$

In the conventional construction one goes through lengthy algebraic manipulations
(*manipulation spin dependent terms, introduce Feynman parameters in momentum loop integrals, Wick rotate, dimensional regularize, drop terms odd in momentum, replace terms even in momentum using Lorentz invariance and symmetry, finally perform the loop momentum integrals, take care of symmetry factors*)

3. Each Feynman diagram in **conventional PT** corresponds to **one particular permutation of photon insertions:** avoid the factorial growth of diagrams in PT.

4. **Non-Abelian case**: extra gluon quadratic term introduces additional structures that can be extracted in a systematic way using the pinching rules of Bern and Kosower.

Bern-Kosower NPB 379 (1992) 451
Strassler NPB 385 (1992) 145

Example: the cusp anomalous dimension without Feynman diagrams

Amplitudes contain **UVs** from photons attached to short distances and **IRs** from photons attached to long distances in the worldlines.

Consider the **self-interaction problem of a classical charge** moving along

$$x_\mu^{\rm cl}(t) = p_\mu t, \quad t<0, \quad x_\mu^{\rm cl}(t) = q_\mu t, \quad t>0$$

Polyakov, NPB164 (1979) 171
Brandt, Neri, Sato, PRD 24 (1981) 879
Korchemsky and Radyushkin NPB 283 (1987) 342
Grozin, Henn, Korchemsky, Marquard JHEP 01 (2016) 140
Brüser, Dlapa, Henn, Yan PRL 126 (2021) 02160
Henn, Korchemsky, Mistlberger, JHEP 04 (2020) 018

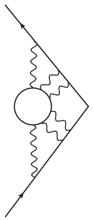
The dressings in the gauge field to all orders in PT are given by:

$$\mathcal{W} = \left(\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} Z_{(n)}^{(\ell)}\right)^{-1} \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^\ell}{\ell!}\frac{1}{2^n n!}\left\langle \prod_{i=1}^{n}\int d^d x_i \int d^d y_i \left(iJ_{\mu_i}^{(\ell)}(x_i) + iJ_{\mu_i}^{\rm cl}(x_i)\right) iD_B^{\mu_i\nu_i}(x_i-y_i)\left(iJ_{\nu_i}^{(\ell)}(y_i) + iJ_{\nu_i}^{\rm cl}(y_i)\right)\right\rangle$$

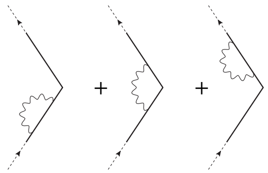
One gets:

$$\mathcal{W} = 1 + \underbrace{\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{D}^B_{\mu\nu}(k) i\tilde{J}^{\rm cl}_\mu(-k) i\tilde{J}^{\rm cl}_\nu(+k)}_{\text{Tree-level self-energy diagrams}} + \underbrace{\frac{1}{2}\left\{\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{D}^B_{\mu\nu}(k) i\tilde{J}^{\rm cl}_\mu(-k) i\tilde{J}^{\rm cl}_\nu(+k)\right\}^2}_{\text{Tree-level self-energy diagrams squared}}$$

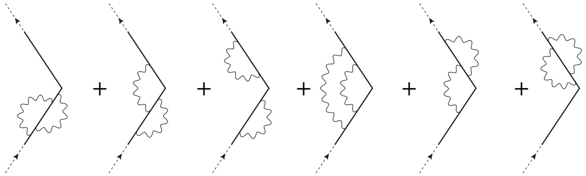
$$-\frac{1}{2}\int \frac{d^dk_1}{(2\pi)^d}\tilde{D}^B_{\mu_1\nu_1}(k_1)\int \frac{d^dk_2}{(2\pi)^d}\tilde{D}^B_{\mu_2\nu_2}(k_2) i\tilde{J}^{\rm cl}_{\mu_1}(-k_1)\underbrace{\left\{-\left\langle iJ_{\nu_1}(+k_1) iJ_{\mu_2}(-k_2)\right\rangle\right\}}_{\text{First genuine fermion worldline loop}} i\tilde{J}^{\rm cl}_{\nu_2}(+k_2) + \cdots$$



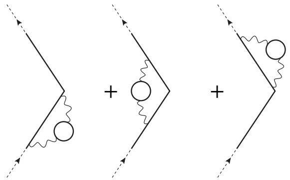
n=6 l=1



n=1 l=0



n=2 l=0



n=2 l=1

To **one-loop**:

$$\left\{\mu\frac{d}{d\mu} + \Gamma(\gamma, g_R)\right\}\mathcal{W}_R = 0\,, \quad \Gamma(\gamma, g_R) = \frac{\alpha}{\pi}(\gamma\coth\gamma - 1) + \mathcal{O}(\alpha^2)$$

4. IR structure of QED

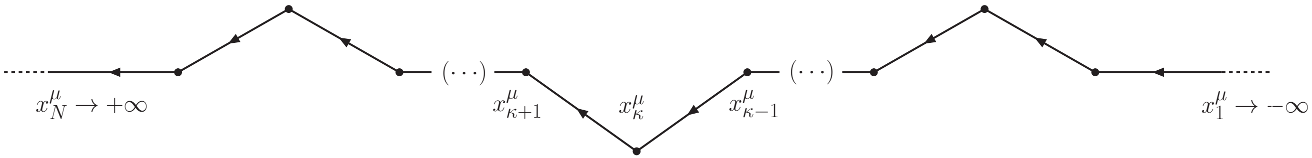
IR STRUCTURE

$$\mathcal{S}^{(r)}_{fi} = \frac{Z_{\rm MW}}{Z}\prod_{n=1}^{r}\left\{\lim_{\substack{t_f^n\to+\infty\\ t_i^n\to-\infty}}\int d^3\vec{x}_f^n\int d^3\vec{x}_i^n u^\dagger_{s_f^n}(p_f^n)e^{+ip_f^n\cdot x_f^n}\exp\left\{\bar{\gamma}_\lambda\frac{\partial}{\partial\theta^n_\lambda}\right\}\bar{\gamma}_0 u_{s_i^n}(p_i^n)e^{-ip_i^n\cdot x_i^n}\right\}\times\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\mathrm{W}^{(r,\ell)}(x_f^r,x_i^r,\theta^r,\ldots,x_f^1,x_i^1,\theta^1)\Big|_{\theta=0}$$

A theory of pairwise interactions between charged currents (virtual and real particles on equal footing):

$$\mathrm{W}^{(r,\ell)}(x_f^r,x_i^r,\theta^r,\ldots,x_f^1,x_i^1,\theta^1)=\left\langle\exp\left\{-i\sum_{ab=1}^{r+\ell}S_{ab}\right\}\right\rangle,\quad S_{ab}=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}J^a_\mu(-k)\frac{g_{\mu\nu}}{k^2+i\epsilon}J^b_\nu(+k),\quad \lim_{k\to0}\tilde{J}^a_\mu(k)\sim g\int_0^1 d\tau\,\dot{x}^a_\mu(\tau)e^{+ik\cdot x^a(\tau)}.$$

The interaction (a,b) will introduce an **IR** when **both currents** contain **1/k terms** surviving when k→ 0**.**



(Virtual charges = closed loops, do not contain surviving 1/k contributions)

$$\tilde{J}^\mu(k)\simeq g\int_0^1 d\tau\frac{dx^\mu}{d\tau}e^{+ik\cdot x(\tau)}=g\sum_{\kappa=1}^{N-1}\int_{\tau_k}^{\tau_{k+1}}d\tau\frac{dx^\mu}{d\tau}e^{+ik\cdot x(\tau)}=\frac{g}{i}\sum_{\kappa=1}^{N-1}\frac{\delta x^\mu_\kappa}{k\cdot\delta x_\kappa}\left(e^{ik\cdot x_{\kappa+1}}-e^{ik\cdot x_\kappa}\right),\qquad m\frac{\delta x_k}{\delta\tau}\equiv p_k$$

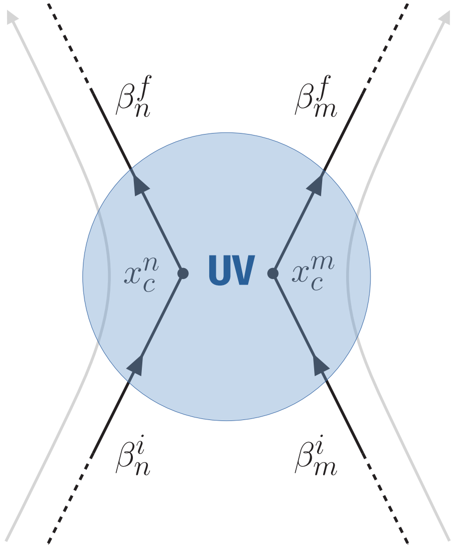
Soft photon pinched to $x_{\kappa+1}$ — Soft photon pinched to x_κ

When $t_f^n\to+\infty$ and $t_i^n\to-\infty$ the virtual soft photons pinched at infinity are dropped: the charged current of any real charge has always two **1/k** terms, corresponding to the soft virtual photons attached to the two external legs in the directions of p_f^n and p_i^n

→ Worldline reformulation, to all orders in PT, of Low's soft theorem.

ABELIAN EXPONENTIATION OF IRs = LONG-DISTANCE CLASSICAL INTERACTIONS

Example: Möller scattering



Separate the scales defining an **IR** region

$$S^{ab} = S^{ab}_{\rm IR} + S^{ab}_{\rm UV}$$

$$S^{ab}_{\rm IR} = -\frac{1}{2}\int_0^{\Lambda} \frac{d^4k}{(2\pi)^4} \tilde{J}^a_\mu(-k)\frac{g^{\mu\nu}}{k^2+i\epsilon}\tilde{J}^b_\nu(+k), \quad S^{ab}_{\rm UV} = -\frac{1}{2}\int_{\Lambda}^{\infty} \frac{d^4k}{(2\pi)^4} \tilde{J}^a_\mu(-k)\frac{g^{\mu\nu}}{k^2+i\epsilon}\tilde{J}^b_\nu(+k)$$

In the **IR** contribution

$$x^n_\mu(t) = x^n_{i,\mu} + \beta^n_{i,\mu}(t-t^n_i)\,, \qquad t\in(t^n_i, t^n_c)\,,$$
$$x^n_\mu(t) = x^n_{c,\mu} + \beta^n_{f,\mu}(t-t^n_c)\,, \qquad t\in(t^n_c, t^n_f)$$

Then

$$\tilde{J}^n_{\mu,\rm IR}(k) = g\int_{-\infty}^{+\infty} dt\dot{x}^n_\mu(t)e^{+ik\cdot x(t)} = \frac{g}{i}\left\{\frac{\beta^n_{i,\mu}}{k\cdot\beta^n_i - i\epsilon} - \frac{\beta^n_{f,\mu}}{k\cdot\beta^n_f + i\epsilon}\right\}e^{+ik\cdot x^n_c}$$

and

$$S^{nm}_{\rm IR} \simeq \frac{1}{2}\int_0^{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i\epsilon}\left\{\frac{\beta^n_{i,\mu}}{-k\cdot\beta^n_i - i\epsilon} - \frac{\beta^n_{f,\mu}}{-k\cdot\beta^n_f + i\epsilon}\right\}\left\{\frac{\beta^m_{i,\mu}}{k\cdot\beta^m_i - i\epsilon} - \frac{\beta^m_{f,\mu}}{k\cdot\beta^m_f + i\epsilon}\right\}$$

For any other worldlines approaching asymptotia with identical angles, the **IR** contribution is the same, so this is the final answer for small enough Λ
→ well known **exponentiation of virtual IR divergences**

Weinberg Phys. Rev. 140 (1965) B520

CURING THE IRs:

Keep the *in* and *out* asymptotic currents

Standard Dyson S-matrix:

$$\mathcal{S}^{(r)}_{fi} = \frac{\mathrm{Z_{MW}}}{\mathrm{Z}} \prod_{n=1}^{r} \left\{ \lim_{\substack{t_f^n \to +\infty \\ t_i^n \to -\infty}} \int d^3\vec{x}_f^n \int d^3\vec{x}_i^n u^\dagger_{s_f^n}(p_f^n) e^{+ip_f^n\cdot x_f^n} \exp\left\{\bar{\gamma}_\lambda \frac{\partial}{\partial \theta_\lambda^n}\right\} \bar{\gamma}_0 u_{s_i^n}(p_i^n) e^{-ip_i^n\cdot x_i^n} \right\}$$

$$\times \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \mathrm{W}^{(r,\ell)}(x_f^r, x_i^r, \theta^r, \dots, x_f^1, x_i^1, \theta^1)\Big|_{\theta=0}$$

Faddeev-Kulish S-matrix:

$$\bar{\mathcal{S}}^{(r)}_{fi} = \frac{\mathrm{Z_{MW}}}{\mathrm{Z}} \prod_{n=1}^{r} \left\{ \int d^3\vec{x}_f^n \int d^3\vec{x}_i^n u^\dagger_{s_f^n}(p_f^n) e^{+ip_f^n\cdot x_f^n} \exp\left\{\bar{\gamma}_\lambda \frac{\partial}{\partial \theta_\lambda^n}\right\} \bar{\gamma}_0 u_{s_i^n}(p_i^n) e^{-ip_i^n\cdot x_i^n} \right\}$$

$$\times \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \mathrm{W}^{(r,\ell)}(x_f^r, x_i^r, \theta^r, \dots, x_f^1, x_i^1, \theta^1)\Big|_{\theta=0}$$

The limits $t^n_{f,i} \to \pm\infty$ taken only after all the IR divergences of the diagrams generated by $\bar{\mathcal{S}}^{(r)}_{fi}$ get canceled in k→0.

Cancellation k→0 = Not an external line

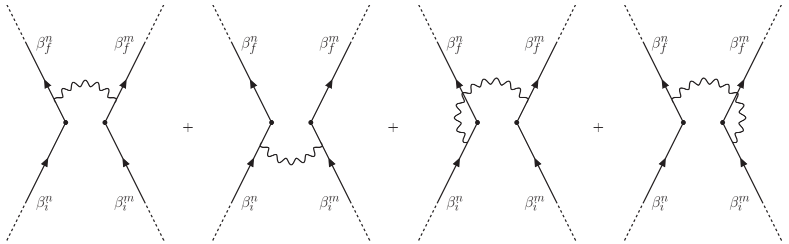
Cancellation k→0 = Not an external line

$$\tilde{J}^\mu_n(k) = \underbrace{\frac{g}{i}\frac{p^\mu_{f,n}}{k\cdot p_{f,n}} e^{ik\cdot x_N}}_{\text{Final asymptotic current}} \underbrace{- \frac{g}{i}\frac{p^\mu_{f,n}}{k\cdot p_{f,n}} e^{ik\cdot x_{N-1}} + (\cdots) + \frac{g}{i}\frac{p^\mu_{i,n}}{k\cdot p_{i,n}} e^{ik\cdot x_2}}_{\text{Current in the Dyson S-matrix}} - \underbrace{\frac{g}{i}\frac{p^\mu_{i,n}}{k\cdot p_{i,n}} e^{ik\cdot x_1}}_{\text{Initial asymptotic current}}$$

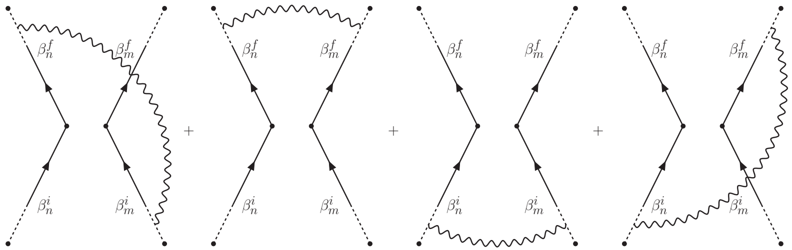
The three currents are of order 1/k, but after taking k→0, they cancel exactly amongst eachother

→ **No IR divergences in the FK S-matrix, to all orders in PT.**

Diagrams in the exponentiation of virtual IR divergences in the traditional Dyson S-matrix



New asymptotic diagrams appearing in the IR finite Faddeev-Kulish S-matrix



(+ 8 others)

Diagrams in the asymptotic region, unconventional but not unfamilar

BREMSSTRAHLUNG FROM MULTIPLE SCATTERING

J. S. BELL
Atomic Energy Research Establishment, Harwell, Didcot, Berks.

Received 11 July 1958

where **r** and **v** are position and velocity at time t and **k** is related to the unit vector $\hat{\mathbf{k}}$ by $\mathbf{k} = \omega\hat{\mathbf{k}}$. We shall start instead with

$$dI = \left(\frac{e}{2\pi}\right)^2 d\omega\, d\hat{\mathbf{k}} \left| \int_{-\infty}^{\infty} dt\, e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} \frac{d}{dt} \frac{\hat{\mathbf{k}}\times\mathbf{v}}{1-\hat{\mathbf{k}}\cdot\mathbf{v}} \right|^2, \qquad (2)$$

which is obtained from (1) by partial integration, ignoring a contribution from $t = -\infty$ on the ground that we are not interested in radiation emitted while the electron was initially accelerated. The use of (2) forces us to be

Infrared divergences in QED revisited

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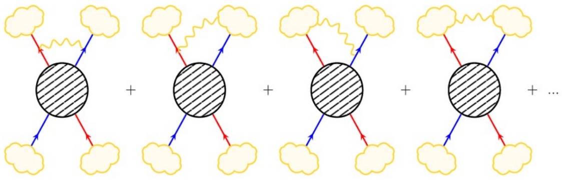


The exponentiation of soft dressings in the F-K S-matrix leads to:

$$-i\sum_{nm}^{r} \tilde{S}_{\rm IR}^{nm} = \left(R + R_{as}\right) + \left(i\Phi + i\Phi_{as}\right) = \frac{g^2}{8\pi^2}\sum_{nm=1}^{\ell} \eta_n\eta_m\gamma_{nm}\coth\gamma_{nm}\log\frac{\Lambda}{\Lambda'} - i\frac{g^2}{8\pi}\sum_{nm}^{\ell}\eta_i\eta_j\coth\gamma_{nm}\log\frac{\Lambda}{\Lambda'}$$

independently of λ the IR cut-off, as anticipated by construction.

Summarizing:

- Worldlines are exact; they provide the **exponentiation of non-scalar d.o.f. and fully dynamical fields**, going well beyond the Wilson loop/line picture and high energy approximations.

- Importantly, one avoids the well-known issues of light-cone kinematics.

- We challenged the frequent misconception of worldlines as “one-loop” formulations.

- They open a clear path to **efficient higher order computations without Feynman diagrams:** cusp anomalous dimension, g-2? ...

- They are **first-quantized, particle-like descriptions** of field theories: we got an insightful view of the IR problem and, by the same token, we can now explore non-perturbative aspects with **semi-classical expansions**, **Monte Carlo methods**, or **quantum computer** implementations (a fully covariant Hamiltonian of a Coulomb system of relativistic and spinning charges).

- **Future directions**: real photons, efficient calculation of higher orders of the cusp anomalous dimension, Yang-Mills implementation, celestial amplitudes & asymptotic symmetries ...

Thanks!

Backup

Integrating the gauge fields:

We got $\frac{\mathrm{Z}}{\mathrm{Z}_{\mathrm{MW}}} = \sum_{\ell=0}^{\infty} \mathrm{Z}^{(\ell)} = \sum_{l=0}^{\infty} \frac{(-1)^\ell}{\ell!} \mathrm{W}^{(\ell)}$ where

$$\mathrm{W}^{(\ell)} = \left\{ \prod_{i=1}^{\ell} \int_0^\infty \frac{d\epsilon_0^i}{2\epsilon_0^i} \int_{\mathrm{P}} \mathcal{D}x_i \int_{\mathrm{AP}} \mathcal{D}\psi_i \exp\left\{ -m^2\epsilon_0^i - \frac{1}{4\epsilon_0^i}\int_0^1 d\tau \dot{x}_i^2(\tau) - \frac{1}{4}\int_0^1 d\tau \psi_\mu^i(\tau)\dot{\psi}_\mu^i(\tau)\Big)\Big]\right\}$$

$$\frac{1}{\mathrm{Z}_{\mathrm{MW}}} \int \mathcal{D}A \exp\Big\{ -\frac{1}{4}\int d^4x F_{\mu\nu}^2(x) - \frac{1}{2\zeta}\int d^4x \big(\partial_\mu A_\mu(x)\big)^2$$

$$+i\sum_{i=1}^{\ell} g \int_0^1 d\tau \dot{x}_\mu^i(\tau) A_\mu\big(x_i(\tau)\big) - i\sum_{i=1}^{\ell} \frac{g\epsilon_0^i}{2}\int_0^1 d\tau \psi_\mu^i(\tau)\psi_\nu^i(\tau) F_{\mu\nu}\big(x_i(\tau)\big)\Big]$$

For each $\mathrm{W}^{(\ell)}$ the integration in $A_\mu(x)$ is quadratic. Rewrite:

$$-\frac{1}{4}\int d^4x F_{\mu\nu}^2(x) - \frac{1}{2\zeta}\int d^4x (\partial_\mu A_\mu(x))^2 = -\frac{1}{2}\int d^4x \int d^4y A_\mu(x) D_{\mu\nu}^{-1}(x-y) A_\nu(y)$$

and

$$\sum_{i=1}^{\ell} g \int_0^1 d\tau \dot{x}_\mu^i(\tau) A_\mu\big(x_i(\tau)\big) - \sum_{i=1}^{\ell} \frac{g\epsilon_0^i}{2}\int_0^1 d\tau \psi_\mu^i(\tau)\psi_\nu^i(\tau) F_{\mu\nu}\big(x_i(\tau)\big) := \int d^4x A_\mu(x) J_\mu(x)$$

One can reintroduce the gauge fields in a first quantized interpretation

$$\mathrm{W}^{(\ell)} := \left\langle \exp\left\{ -\frac{1}{2}\sum_{ij=1}^{\ell}\int_0^1 d\tau\, \underbrace{g\frac{dx^i_\mu}{d\tau}}_{\text{Charged local current of particle ``i''}} \left[\underbrace{A^{x_j}_\mu(x_i)}_{\text{Field created by particle ``j'' charged current}} + \underbrace{A^{\psi_j}_\mu(x_i)}_{\text{Field created by particle ``j'' spin precession}}\right] - \frac{i}{4}\sum_{ij=1}^{\ell}\int_0^1 d\tau\, \underbrace{\epsilon^i_0 g\sigma^i_{\mu\nu}(\tau)}_{\text{Particle ``i'' local spin tensor}} \left[\underbrace{F^{x_j}_{\mu\nu}(x_i)}_{\text{Field strength created by particle ``j'' charged current}} + \underbrace{F^{\psi_j}_{\mu\nu}(x_i)}_{\text{Field strength created by particle ``j'' spin precession}}\right]\right\}\right\rangle$$

$\langle\star\rangle \equiv$ Finally sum over all closed worldline configurations (loops) and over all possible proper times
→ **Quantum fluctuations in coordinate space**

This expression is equivalent to:

$$\frac{\mathrm{Z}}{\mathrm{Z_{MW}}} = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\left\langle \exp\left\{ -\frac{g^2}{8\pi^2}\sum_{ij=1}^{\ell}\int_0^1 d\tau_i \left(\underbrace{\frac{dx^\mu_i}{d\tau_i}}_{\text{Wilson loop term (bosonic d.o.f)}} - \underbrace{i\epsilon^i_0\sigma^i_{\mu\rho}(\tau_1)\frac{\partial}{\partial x_\rho i}}_{\text{Exponentiation fermionic d.o.f}}\right) \times \int_0^1 d\tau_i \left(\underbrace{\frac{dx^j_\mu}{d\tau_j}}_{\text{Wilson loop term (bosonic d.o.f)}} - \underbrace{i\epsilon^j_0\sigma^j_{\mu\eta}(t_2)\frac{\partial}{\partial x^j_\eta}}_{\text{Exponentiation fermionic d.o.f}}\right)\frac{1}{(x_i - x_j)^2}\right\}\right\rangle$$

→ **Lorentz forces** between "l" charges including self-exchanges and sum over all possible paths of the commuting and anti-commuting worldlines, encoding the bosonic and fermionic d.o.f.

More generally, in arbitrary gauge, *einbein* and d-dimensions one gets for the interaction between charges (a) and (b)

$$S_{ab} = S_{ab}^{\mathrm{BB}} + S_{ab}^{\mathrm{BF}} + S_{ab}^{\mathrm{FB}} + S_{ab}^{\mathrm{FF}}$$

$$S_{ab}^{\mathrm{BB}} = -\frac{g^2\mu^{4-d}}{8\pi^{\frac{d}{2}}}\Gamma\left(\frac{d-2}{2}\right)\left(\frac{1+\zeta}{2}\right)\int_0^1 d\tau_a \int_0^1 d\tau_b \frac{\dot{x}_\mu^a \dot{x}_\mu^b}{\left[\left(x_\mu^a - x_\mu^b\right)^2\right]^{\frac{d}{2}-1}} - \frac{g^2\mu^{4-d}}{8\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}\right)(1-\zeta)\int_0^1 d\tau_a \int_0^1 d\tau_b \frac{\dot{x}_\mu^a (x_\mu^a - x_\mu^b)\dot{x}_\nu^b (x_\nu^a - x_\nu^b)}{\left[\left(x_\mu^a - x_\mu^b\right)^2\right]^{\frac{d}{2}}}$$

$$S_{ab}^{\mathrm{BF}} = -\frac{g^2\mu^{4-d}}{4\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}\right)\int_0^1 d\tau_a \int_0^1 d\tau_b \varepsilon_b \frac{\dot{x}_\mu^a \psi_\mu^b \psi_\nu^b (x_\nu^a - x_\nu^b)}{\left[\left(x_\mu^a - x_\mu^b\right)^2\right]^{\frac{d}{2}}} - \frac{g^2\mu^{4-d}}{4\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}+1\right)(1-\zeta)\int_0^1 d\tau_a \int_0^1 d\tau_b \varepsilon_b \frac{\dot{x}_\mu^a (x_\mu^a - x_\mu^b)\psi_\nu^b (x_\nu^a - x_\nu^b)\psi_\rho^b (x_\rho^a - x_\rho^b)}{\left[\left(x_\mu^a - x_\mu^b\right)^2\right]^{\frac{d}{2}+1}}$$

$$S_{ab}^{\mathrm{FF}} = -\frac{g^2\mu^{4-d}}{4\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}\right)\int_0^1 d\tau_a \varepsilon_a \int_0^1 d\tau_b \varepsilon_b \frac{\psi_\mu^a \psi_\nu^a \psi_\mu^b \psi_\nu^b}{\left[\left(x_\mu^a - x_\mu^b\right)^2\right]^{\frac{d}{2}}} + \frac{g^2\mu^{4-d}}{2\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}+2\right)(1-\zeta)\int_0^1 d\tau_a \varepsilon_a \int_0^1 d\tau_b \varepsilon_b \frac{\left[\psi_\mu^a (x_\mu^a - x_\mu^b)\right]^2 \left[\psi_\nu^b (x_\nu^a - x_\nu^b)\right]^2}{\left[\left(x_\mu^a - x_\mu^b\right)^2\right]^{\frac{d}{2}+2}}$$

$$-\frac{g^2\mu^{4-d}}{2\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}+1\right)\int_0^1 d\tau_a \varepsilon_a \int_0^1 d\tau_b \varepsilon_b \frac{\psi_\mu^a \psi_\nu^a (x_\nu^a - x_\nu^b)\psi_\rho^b (x_\rho^a - x_\rho^b)\psi_\mu^b}{\left[\left(x_\mu^a - x_\mu^b\right)^2\right]^{\frac{d}{2}+1}}$$

Also generated from the application of the 0+1 dimensional N=1 SUSY algebra to the scalar term. *C. Schubert, Phys. Rept. 355 (2001) 73.*

Computing amplitudes with external particles

$$Z := Z[0,\bar\eta,\eta] = \underbrace{\int \mathcal{D}A \exp\Big\{ -\frac{1}{4}\int d^4x F_{\mu\nu}^2 - \frac{1}{2\zeta}\int d^4x (\partial_\mu A_\mu)^2}_{} + \underbrace{\ln\det\left(D_\mu\gamma_\mu + m\right)}_{} + \underbrace{\int d^4x \int d^4y \bar\eta(x) D_F^A(x,y)\eta(y)}_{}\Big\}$$

Integration in the background gauge field A configurations
→ A is a dynamical field again

Exponential 1-fermion loop in A
→Virtual fermion loop dressings

Exponential amplitude 1-fermion to go from y to x in the presence of A

Examples:

1 →1 = self-interacting problem of a single charge

$$\frac{1}{Z}\frac{\delta Z[\bar\eta,\eta]}{\delta\eta(x_i)\delta\bar\eta(x_f)}\bigg|_{\bar\eta=\eta=0} = \frac{1}{Z}\int \mathcal{D}A \exp\left[-\frac{1}{4}\int d^4x F_{\mu\nu}^2 - \frac{1}{2\zeta}\int d^4x (\partial_\mu A_\mu)^2 + \ln\det\left(D_\mu\gamma_\nu + m\right)\right\} D_F^A(x_f,x_i) \equiv \left\langle D_F^A(x_f,x_i)\right\rangle_A$$

2→2 = interaction and self-interaction problem of two real charges

$$\frac{1}{Z}\frac{\delta Z[\bar\eta,\eta]}{\delta\eta(x_i^1)\delta\eta(x_i^2)\delta\bar\eta(x_f^2)\delta\bar\eta(x_f^1)}\bigg|_{\bar\eta=\eta=0} = \left\langle D_F^A(x_f^1,x_i^1) D_F^A(x_f^2,x_i^2)\right\rangle_A - \left\langle D_F^A(x_f^2,x_i^1) D_F^A(x_f^1,x_i^2)\right\rangle_A$$

Identical procedure: integrate out the gauge field expressing the fermion dressed Green functions in worldline form.

Fradkin and Gitman, PRD 44 (1991) 3230.
Van Holten, NPB 457 (1995) 457.
Reuter, Schmidt, and Schubert, Ann. Phys. 259 (1997) 313.
Schubert, Phys. Rept. 355 (2001) 73.
Ahmadiniaz, Banda Guzman, Bastianelli, Corradini, Edwards, Schubert, JHEP 08 (2020) 08, 049.

By the same token: recover the gauge-invariant (reparametrization and supergauge-invariant) action of the relativistic spinning particle, and generalize then the Bargman-Michel-Telegdi equations to fully dynamical fields.

Berezin and Marinov, JETP Lett. 21, 320 (1975)
Barducci, Casalbuoni and Lusanna, Nuov. Cim. A 33 (1976) 115

Open worldline formulation is more involved:

E.S. Fradkin and D.M. Gitman, PRD 44 (1991) 3230.
J.W. Van Holten, NPB 457 (1995) 457.
M. Reuter, M. Schmidt, and C. Schubert, Annals Phys., 259 (1997) 313.
C. Schubert, Phys. Rept. 355 (2001) 73.
N. Ahmadiniaz, V. M. Banda Guzman, F. Bastianelli, O. Corradini, J. P. Edwards, and C. Schubert, JHEP 08 (2020) 08, 049.

Amplitude of a single spinning-charge of going from x_i^α to x_f^α

$$\bar{D}_F^A(x_f^\alpha, x_i^\alpha) = \frac{i}{N_5} \exp\left[\bar{\gamma}^n \frac{\partial}{\partial \theta_\alpha^n}\right] \int_0^\infty \underbrace{d\epsilon_\alpha^0}_{\text{Bosonic d.o.f proper time}} \int \underbrace{d\chi_\alpha^0}_{\text{Fermionic d.o.f proper time}} \int \underbrace{\mathcal{D}\epsilon_\alpha \frac{\mathcal{D}\pi_\alpha}{2\pi}}_{\text{Gauge-fixing term bosonic einbein}} \int \underbrace{\mathcal{D}\chi_\alpha \mathcal{D}\nu_\alpha}_{\text{Gauge-fixing term fermionic einbein}} \int \mathcal{D}x_\alpha \int \mathcal{D}\psi_\alpha$$

$$\times \exp\left[- iS_{R,0} + ig \int_0^1 d\tau \dot{x}_\mu^\alpha(\tau) A^\mu\left(x_\alpha(\tau)\right) - \frac{g}{2} \int_0^1 d\tau \epsilon_\alpha(\tau) \psi_\mu^\alpha(\tau) \psi_\nu^\alpha(\tau) F^{\mu\nu}\left(x_\alpha(\tau)\right)\right]\Bigg|_{\theta_\alpha = 0}$$

$$S_\alpha^{\text{free}} = -\frac{i}{4}\psi_n^\alpha(1)\psi_\alpha^n(0) + \int_0^1 dt \left[\pi_\alpha(t)\dot{\epsilon}_\alpha(t) + \epsilon_\alpha(t) m^2 + \frac{\dot{x}_\alpha^2(t)}{4\epsilon_\alpha(t)}\right]$$
$$+ i \int_0^1 dt \left[\nu_\alpha(t)\dot{\chi}_\alpha(t) + \chi_\alpha(t)\left(\frac{\dot{x}_\mu^\alpha(t)\psi_\alpha^\mu(t)}{2\epsilon_\alpha(t)} - m\psi_\alpha^5(t)\right) - \frac{1}{4}\psi_n^\alpha(t)\dot{\psi}_\alpha^n(t)\right]$$

Bosonic worldline & einbein coordinates: $x_\alpha^\mu(\tau), \quad \tau \in [0,1], \quad x_\alpha^\mu(0) := x_\alpha^{i,\mu}, \quad x_\alpha^\mu(1) := x_\alpha^{f,\mu}, \quad \epsilon_\alpha(0) = \epsilon_\alpha^0$

Fermionic worldline & einbein coordinates: $\psi_\alpha^\mu(\tau), \quad \tau \in [0,1], \quad \psi_\alpha^\mu(1) := -\psi_\alpha^\mu(0) + 2\theta_\alpha^n, \quad \chi_\alpha(0) = \chi_\alpha^0$

Same features as with the vacuum, except for the new fermionic d.o.f in the propagator

Normalization removes disconnected loops of the sea

$$\left\langle \bar{D}_F(x_f^\alpha, x_i^\alpha) \right\rangle_A = \overbrace{\frac{Z_{\rm MW}}{Z}} \frac{i}{N_5} \exp\left[\bar{\gamma}^n \frac{\partial}{\partial \theta_\alpha^n}\right] \sum_{\ell=0}^{\infty} \underbrace{\frac{(-1)^\ell}{\ell!}}_{\text{Loop Parity Factor}} \mathrm{W}^{(1,\ell)}(x_f^\alpha, x_i^\alpha, \theta^\alpha)$$

The interactions are encoded in one generalized **Wilson line** with **l- loops attached**:

$\langle \star \rangle \equiv$ Sum over all worldline configurations of the "l" particles describing loops, and The external "α" particle with open boundaries

$$\mathrm{W}^{(1,\ell)}(x_f^\alpha, x_i^\alpha, \theta^\alpha) = \Bigg\langle \exp\Bigg\{ -\frac{i}{2}\int_0^1 d\tau_i \Bigg(\overbrace{g\dot{x}_\mu^i(\tau_i) + ig\epsilon_i(\tau_i)\psi_\mu^j(\tau_i)\psi_\rho^i(\tau_i)\frac{\partial}{\partial x_\rho^i}}^{\text{Bosonic and fermionic currents of particle “i”}} \Bigg)$$
$$\times \int_0^1 d\tau_j \Bigg(\underbrace{g\dot{x}_\nu^j(\tau_j) + ig\epsilon_j(\tau_j)\psi_\nu^j(\tau_j)\psi_\sigma^j(\tau_j)\frac{\partial}{\partial x_\sigma^j}}_{\text{Bosonic and fermionic currents of particle “j”}} \Bigg) D_F^{\mu\nu}(x_i(\tau_i) - x_j(\tau_j)) \Bigg\} \Bigg\rangle$$

One can construct any matrix element. For instance, **n→n** particle scattering

$$\mathrm{S}_{fi}^{(r,\ell)} = \frac{Z_{\rm MW}}{Z} \prod_{\alpha=1}^{n} \left\{ \left[u_{\alpha_f}^\dagger \exp\left(\bar{\gamma}^n \frac{\partial}{\partial \theta_\alpha^n}\right) \gamma^5 \gamma^0 u_{\alpha_i} \right] \int d^3\vec{x}_f^\alpha e^{+ip_f^\alpha \cdot x_f^\alpha} \int d^3\vec{x}_i^\alpha e^{-ip_i^\alpha \cdot x_i^\alpha} \right\} \mathrm{W}^{(r,\ell)}(x_f^1, x_i^1, \theta^1, \ldots, x_f^r, x_i^r, \theta^r) \Big\rangle$$

NON-PERTURBATIVE RULES TO CONSTRUCT AN AMPLITUDE: A THEORY OF CURRENTS

1. **Introduce a super-pair for any real and virtual charge present** $\big(x(\tau), \psi(\tau)\big)\,, \quad \tau \in [0,1].$

Real fermions go from $x^n_\mu(0) := x^{n,\mathrm{i}}_\mu$ to $x^n_\mu(1) := x^{n,\mathrm{f}}_\mu$; and $\psi^n_\lambda(1) = -\psi^n_\lambda(0) + 2\theta^n_\lambda.$

Virtual fermions describe loops $x^i_\mu(0) := x^i_\mu(1), \quad \psi^i_\mu(1) = -\psi^i_\mu(0).$

2. **Write the current created by each charge, and the total current of the system** $(a = 1, \ldots, r+\ell)$

$$J^a_\mu(x) = g\int_0^1 d\tau \left\{\frac{dx^a_\mu}{d\tau} - \varepsilon^a_0 \psi^a_\mu(\tau)\psi^a_\nu(\tau)\frac{\partial}{\partial x_\nu}\right\}\delta^4\big(x - x^a(\tau)\big)\,, \qquad J^{(r,\ell)}_\mu(x) = \sum_{a=1}^{r+\ell} J^a_\mu(x)$$

3. **Write the free action of each real (R) and virtual (V) charge, and the total free action of the system** $(n = 1, \ldots, r, \quad i = r+1, \ldots, r+\ell)$

$$S^n_{R,0} = \frac{1}{4}\psi^n_\lambda(1)\psi^n_\lambda(0) + \int_0^1 d\tau \left\{\varepsilon^n_0 m^2 + \frac{1}{4\epsilon^n_0}\left(\frac{dx^n_\mu}{d\tau}\right)^2 + \frac{1}{4}\psi^n_\lambda \frac{d\psi^n_\lambda}{d\tau} - \chi^n_0\left(m\psi^n_5 + \frac{i}{2\epsilon^n_0}\frac{dx^n_\mu}{d\tau}\psi^n_\mu\right)\right\},$$

$$S^i_{V,0} = \int_0^1 d\tau \left\{\varepsilon^i_0 m^2 + \frac{1}{4\epsilon^i_0}\left(\frac{dx^i_\mu}{d\tau}\right)^2 + \frac{1}{4}\psi^i_\mu \frac{d\psi^i_\mu}{d\tau}\right\} \qquad S^{(r,\ell)}_0 = \sum_{n=1}^{r} S^n_{R,0} + \sum_{i=r+1}^{r+\ell} S^i_{V,0}$$

4. **Write the total action of this Coulomb system**

$$S^{(r,\ell)} = S^{(r,\ell)}_0 + \frac{1}{2}\int d^4x \int d^4y J^{(r,\ell)}_\mu(x) D^B_{\mu\nu}(x-y) J^{(r,\ell)}_\nu(y)$$

5. **Finally, sum over all possible trajectories and all possible proper times, for each real and virtual charge present**

$$\underbrace{\mathrm{W}^{(r,\ell)}(x^n_f, x^n_i, \theta^n, \ldots, x^1_f, x^1_i, \theta^1)}_{\text{Amplitude of the system to go from } \{x^n_i\} \text{ to } \{x^n_f\}} = \prod_{n=1}^{r}\left\{\int \mathcal{D}x^n_\mu \int \mathcal{D}\psi^n_\lambda \int_0^\infty d\varepsilon^n_0 \int d\chi^n_0\right\} \times \prod_{i=r+1}^{r+\ell}\left\{\int_{\mathrm{P}} \mathcal{D}x^i_\mu \int_{\mathrm{AP}} \mathcal{D}\psi^i_\mu \int_0^\infty \frac{d\varepsilon_0}{2\varepsilon_0}\right\} e^{-S^{(r,\ell)}}$$

QED instantons: a first-quantized Lagrangian/Hamiltonian picture of gauge theories

$$\mathcal{L} = p^\mu \dot{x}_\mu + \pi\dot{\varepsilon} + i\nu\dot{\chi} - \frac{i}{4}\psi^n\dot{\psi}_n - H(p,x,\psi,\varepsilon,\nu)\,, \qquad \tau\in[0,1]$$

$$H = -\varepsilon\Big(\underbrace{m^2 - (p_\mu + gA_\mu)^2 - i\frac{g}{2}\psi_\mu\psi_\nu F^{\mu\nu}}_{\text{Energy-momentum constraint}}\Big) + i\chi\Big(\underbrace{m\psi_5 - (p_\mu + gA_\mu)\psi^\mu}_{\text{Helicity momentum-constraint}}\Big)$$

Fradkin, Gitman PRD44 (1991) 3230
Berezin, Marinov Ann. Phys. 104 (1977) 336
Barducci, Casalbuoni, Lusanna, Nuov. Cim. A 25 (377) 1976
Wong, Nuov. Cim. A 65 (1970) 689
Papapetrou, Proc. Roy. Soc. A 209 (1951) 248
Mathisson, Act. Phys. Pol. 6 (1937) 163
Dixon, Proc. Roy. Soc. A 314 (1970) 499

More generally, in the full many-body picture, **dynamical fields**:

$$\delta S = \sum_{\alpha=1}^{r}\int_0^1 d\tau\Bigg\{\left[m_R^2 - \frac{1}{4\varepsilon_\alpha^2}(\dot{x}_\alpha^\mu + i\chi_\alpha\psi_\alpha^\mu)^2\right]\delta\varepsilon_\alpha + i\left[\frac{\dot{x}_\mu^\alpha\psi_\alpha^\mu}{2\varepsilon_\alpha} - m\psi_\alpha^5\right]\delta\chi_\alpha + i\left[\chi_\alpha m - \frac{\psi_\alpha^5}{2}\right]\delta\psi_5^\alpha$$

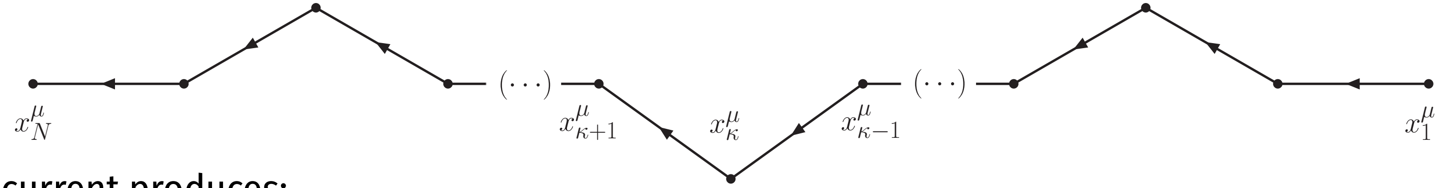
$$-i\left[\frac{\dot{\psi}_\alpha^\rho}{2} + \chi_\alpha\frac{\dot{x}_\alpha^\rho}{2\varepsilon_\alpha} - g\psi_\nu F_{all}^{\rho\nu}\right]\delta\psi_\rho^\alpha - \left[\frac{d}{d\tau}\left(\frac{1}{2\varepsilon_\alpha}(\dot{x}_\alpha^\rho + i\chi_\alpha\psi_\alpha^\rho)\right) - g\dot{x}_\mu^\alpha F_{all}^{\rho\mu} - i\frac{g}{2}\varepsilon_\alpha\psi_\mu^\alpha\psi_\nu^\alpha\frac{\partial F_{all}^{\mu\nu}}{\partial x_\rho^\alpha}\right]\delta x_\rho^\alpha\Bigg\}$$

$$+\sum_{\alpha=r+1}^{r+\ell}\int_0^1 d\tau\Bigg\{\left[m_R^2 - \frac{\dot{x}_\alpha^2}{4\varepsilon_\alpha^2}\right]\delta\varepsilon_\alpha - i\left[\frac{\dot{\psi}_\alpha^\rho}{2} - g\psi_\nu F_{all}^{\rho\nu}\right]\delta\psi_\rho^\alpha - \left[\frac{d}{d\tau}\left(\frac{\dot{x}_\alpha^\rho}{2\varepsilon_\alpha}\right) - g\dot{x}_\mu^\alpha F_{all}^{\rho\mu} - i\frac{g}{2}\varepsilon_\alpha\psi_\mu^\alpha\psi_\nu^\alpha\frac{\partial F_{all}^{\mu\nu}}{\partial x_\rho^\alpha}\right]\delta x_\rho^\alpha\Bigg\}$$

The condition $\delta S = 0$ yields the classical e.o.m. of a set of spinning charges with back reaction:
covariant, full generalization of the Bargman-Michel-Telegdi equations.

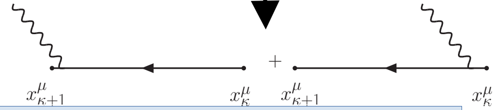
QCD → Wong equations, **Gravity** → Papapetrou-Mathisson-Dixon equations

Examining the currents



The current produces:

$$\tilde{J}^\mu(k) \simeq g\int_0^1 d\tau \frac{dx^\mu}{d\tau} e^{+ik\cdot x(\tau)} = g\sum_{\kappa=1}^{N-1}\int_{\tau_k}^{\tau_{k+1}} d\tau \frac{dx^\mu}{d\tau} e^{+ik\cdot x(\tau)} = \frac{g}{i}\sum_{\kappa=1}^{N-1}\frac{\delta x_\kappa^\mu}{k\cdot\delta x_\kappa}\left(e^{ik\cdot x_{\kappa+1}} - e^{ik\cdot x_\kappa}\right)$$



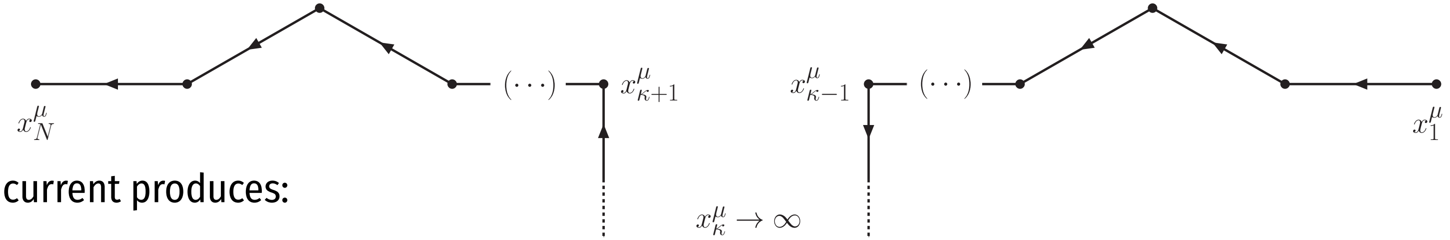
Fixed x_κ and $x_{\kappa+1}$: term in parenthesis of order k, cancels the 1/k when k→ 0.

Internal point x_κ at **infinity**: photon pinched to x_κ shall be drop, photons pinched to $x_{\kappa+1}$ and $x_{\kappa-1}$ can be paired, canceling their 1/k poles when k→0.

External points x_N or x_1 at **infinity**: photons pinched to x_{N-1} and x_2 left unpaired, introducing two different 1/k contributions surviving when k→0.

- **Virtual charges** → all points internal → **no 1/k contributions**
- **Real charges** → two external points at infinity → **two 1/k contributions**.

Examining the currents



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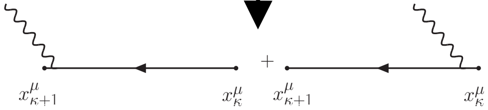

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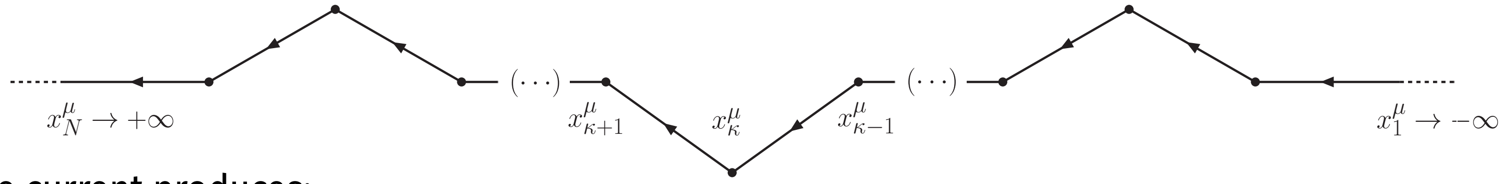

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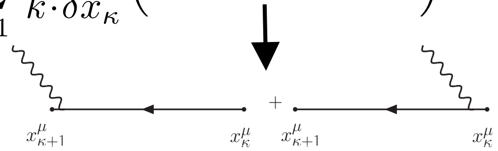

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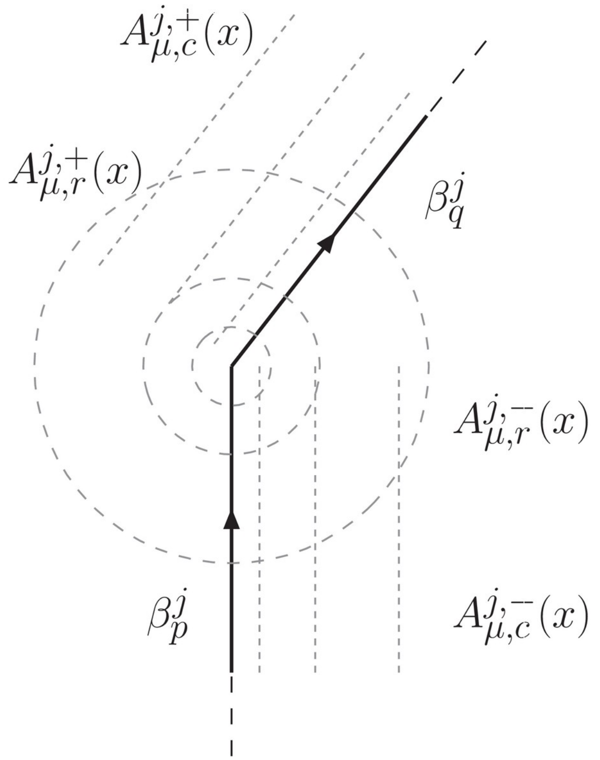
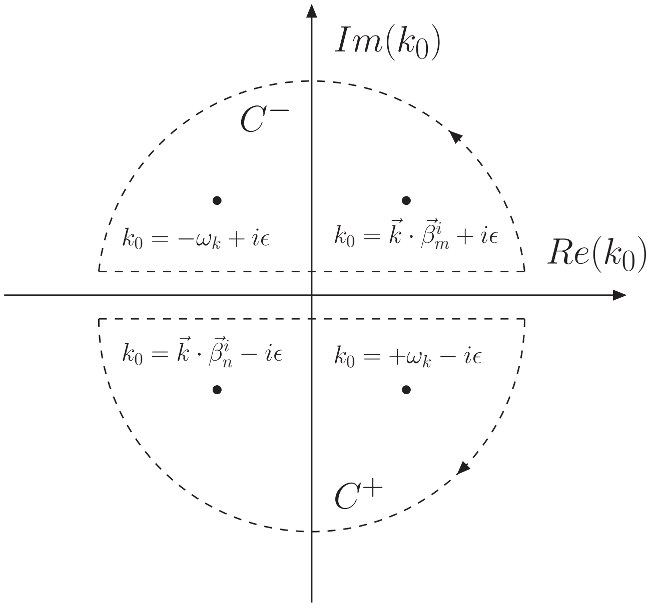
Computing the IR dressings and physical content

Real parts contain the radiative modes of the interactions. Imaginary parts, interactions with the **Coulomb/Liénard-Wiechert** fields of al charges approaching infinity **(Dalitz phase)**

$$S_{nm} = \frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i\epsilon}\left(\frac{\beta_n^i}{-k\cdot\beta_n^i - i\epsilon} - \frac{\beta_n^f}{-k\cdot\beta_n^f + i\epsilon}\right)e^{-ik\cdot x_n^0}\cdot\left(\frac{\beta_m^i}{k\cdot\beta_m^i - i\epsilon} - \frac{\beta_m^f}{k\cdot\beta_m^f + i\epsilon}\right)e^{+ik\cdot x_m^0}$$





$\underbrace{k_0 = \omega_k - i\epsilon,}$ Action of "n" in the radiative photon emitted by "m"

$\underbrace{k_0 = -\omega_k + i\epsilon,}$ Action of "m" in the radiative photon emitted by "n"

$\underbrace{k_0 = \vec{k}\cdot\vec{\beta}_n^i - i\epsilon,}$ Charge m action in the Lienard-Weichert field of n coming from infinity in uniform motion

$\underbrace{k_0 = \vec{k}\cdot\vec{\beta}_m^i + i\epsilon}$ Charge n action in the Lienard-Weichert field of m coming from infinity in uniform motion

Conventional approach to the Faddeev-Kulish S-matrix

Kulish, Faddeev Theor. Math. Phys. 4 (1970) 745

ASYMPTOTIC CONDITIONS AND INFRARED DIVERGENCES IN QUANTUM ELECTRODYNAMICS

P. P. Kulish and L. D. Faddeev

A definition which is free of infrared divergences is proposed for the S matrix of a relativistic theory of interacting charged particles. This is achieved by a modification of the asymptotic condition and the introduction of a new space of asymptotic states. This state differs from the Fok space, but is separable and relativistically and gauge invariant. The mass operator has no nonvanishing discrete eigenvalues.

$$\underbrace{\hat{\mathrm{S}} = \mathrm{T}\left[\exp\left(-ig\int d^4x\hat{\bar{\Psi}}(x)\gamma^\mu\hat{A}_\mu(x)\hat{\Psi}(x)\right)\right]}_{\text{Dyson S-matrix}}$$

$$\hat{\mathrm{S}}_{fk} = \underbrace{\exp\left\{-\hat{R}(t_+) - i\hat{\phi}(t_+)\right\}}_{\text{Dressing at plus infinity}}\hat{\mathrm{S}}\underbrace{\exp\left\{+i\hat{\phi}(t_-) + \hat{R}(t_-)\right\}}_{\text{Dressing at minus infinity}}$$

Action of coherent state operator R (spanning the Hilbert space of asymptotic charged particle states) leads to the asymptotic radiative (real) dressings, order by order in PT

$$\hat{R}(t) = e\sum_\lambda \underbrace{\int\frac{d^3\vec{p}}{(2\pi)^3}\rho_p}\underbrace{\int\frac{d^3\vec{k}}{(2\pi)^3}}\frac{1}{\sqrt{2\omega_k}}\frac{\beta_p^\mu}{k\cdot\beta_p}\left\{\underbrace{a_\lambda^{k\dagger}\epsilon_{\lambda,\mu}^{k*}}e^{ik\cdot\beta_p t} - \underbrace{a_\lambda^k\epsilon_{\lambda,\mu}^k}e^{-ik\cdot\beta_p t}\right\}$$

Picks momenta of virtual fermions

Picks momenta and helicity of real and virtual photons

$$\rho_p := \underbrace{\sum_s\left(b_s^{p\dagger}b_s^p - d_s^{p\dagger}d_s^p\right)}_{\text{Fermion number op.}}$$

Action of coherent state operator ϕ, to the asymptotic Coulomb (imaginary) dressings, order by order in PT

$$\hat{\phi}(t) = \frac{e^2}{8\pi^2}\int\frac{d^3\vec{p}_1}{(2\pi)^3}\int\frac{d^3\vec{p}_2}{(2\pi)^3}:\hat{\rho}_{p_1}\hat{\rho}_{p_2}:\frac{p_1\cdot p_2}{\sqrt{(p_1\cdot p_2)^2 - m^4}}\log t$$

Recall that the conventional form of the FK dressings,

$$\bar{\mathcal{S}}_{fk} = \underbrace{\exp\left\{-\hat{R}(t_+) - i\hat{\phi}(t_+)\right\}}_{\text{Dressing at plus infinity}} \hat{\mathcal{S}} \underbrace{\exp\left\{+i\hat{\phi}(t_-) + \hat{R}(t_-)\right\}}_{\text{Dressing at minus infinity}}$$

Kulish, Faddeev Theor. Math. Phys. 4 (1970) 745

$$\bar{\mathcal{S}} = \lim_{t_{f,i}\to\pm\infty} \underbrace{\mathrm{U}^\dagger_{as}(t_f,t)\mathrm{U}_0(t_f,t)}_{\text{Asymptotic region}} \mathcal{S} \underbrace{\mathrm{U}^\dagger_0(t_i,t)\mathrm{U}_{as}(t_i,t)}_{\text{Asymptotic region}}$$

Hannesdottir, Schwartz PRD 101 (2020) 10, 105001

In the worldline fields have been integrated out. Back-reaction prevents us from having local interaction functionals linear in the gauge field, that allow a factorization of the asymptotic dressings.

However, it presents potential advantages:

- It allows an **all-order proof of the cancellation of IR divergences in QED** (as anticipated by explicit order-by-order evaluations)

- It makes manifest that asymptotically dressed states cannot be thought as direct products of dressed single particle states: **soft photon clouds depend on the rest of charges & exchanges between the central and the asymptotic regions**.

- Efficiency in higher order perturbative calculations: **universal compact worldline formulation of l-loop n-rank polarization tensors.**

- **Vacuum transitions** (related to conservation laws of some fundamental symmetries) are implemented within the definition of the amplitude itself on equal footing as the rest of interactions.

Real photons and Bloch-Nordsieck approach

To compute the amplitude of emission of n_γ^r real photons from n_q charges define

$$\mathrm{S}[\mathcal{J}] = \frac{1}{\mathrm{Z}[0]} \prod_{\alpha=1}^{n_e} \left\{ \left[u^\dagger_{\alpha_f} \exp\left(\bar{\gamma}^n \frac{\partial}{\partial \theta^n_\alpha} \right) \gamma^5 \gamma^0 u_{\alpha_i} \right] \int d^3\vec{x}^\alpha_f e^{+ip^\alpha_f \cdot x^\alpha_f} \int d^3\vec{x}^\alpha_i e^{-ip^\alpha_i \cdot x^\alpha_i} \right\} \mathrm{W}^{(0)}[\mathcal{J}]$$

$$\mathrm{S}^0_{n_q} := \underbrace{\mathrm{S}[0]}$$

Matrix element without real photons

Then in the IR limit, after cutting the lines

$$\mathrm{S}^{n_\gamma}_{n_q} = \prod_{i=1}^{n_\gamma} \left\{ i\epsilon^*_{\mu_i}(k_i, \lambda_i) \int d^4x_i e^{ik_i \cdot x_i} (-k_i^2) \frac{\delta}{\delta \mathcal{J}(x_i)} \right\} \mathrm{S}[\mathcal{J}] \Big|_{\mathcal{J}=0} \longrightarrow$$

$$\mathrm{S}^{n_\gamma}_{n_q} = \left\{ \prod_{i=1}^{n_\gamma} \left(-g\epsilon^*_{\mu_i}(k_i, \lambda_i) \sum_{j=1}^{n_q} \frac{\eta_j \beta_j^{\mu_i}}{k_i \cdot \beta_j} \right) \right\} \mathrm{S}^0_{n_q}$$

Low PR 110 (1958) 974, Weinberg PR 140 (1965) 516

Squaring the matrix element with real photons and summing in n_γ^r

$$\sigma^{\gamma^r}_{n_q} = \underbrace{\exp\left\{ -g^2 \int_\lambda^{\Lambda'} \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_k} \sum_{j,k=1}^{n_q} \eta_j \eta_k \frac{\beta_j}{k \cdot \beta_j} \cdot \frac{\beta_j}{k \cdot \beta_j} \right\}}_{\text{Exponentiation of real IR divergences}} \sigma^0_{n_q}$$

And this cancellation is independent of the cancellations presumed in the amplitude

Squaring the matrix element with virtual photons and summing in n_γ^v

$$\sigma^{\gamma^v}_{n_q} = \underbrace{\exp\left\{ +g^2 \int_\lambda^{\Lambda} \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_k} \sum_{j,k=1}^{n_q} \eta_j \eta_k \frac{\beta_j}{k \cdot \beta_j} \cdot \frac{\beta_j}{k \cdot \beta_j} \right\}}_{\text{Exponentiation of virtual IR divergences}} \sigma^0_{n_q} \longrightarrow \sigma^{\gamma^{v+r}}_{n_q} = \overbrace{\underbrace{\exp\left\{ +g^2 \int_\Lambda^{\Lambda'} \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_k} \sum_{j,k=1}^{n_q} \eta_j \eta_k \frac{\beta_j}{k \cdot \beta_j} \cdot \frac{\beta_j}{k \cdot \beta_j} \right\}}_{\text{Finite independently of } \lambda \text{ the photon mass}}}^{\text{Bloch-Nordsieck cancellation of IR divergences}} \sigma^0_{n_q}$$