Kinematic Refit

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Concept

Least Square Minimization

$$\chi^2 = \sum_{i=1}^{N} (\frac{y_i - f_i}{\sigma_i})^2$$

$$f_i = f(\theta_L, x_N)$$

And in matrix notation:

$$\chi^2 = (y - \theta)^T V^{-1} (y - \theta)$$

Kinematic refit "Improved Measurements"

If the unknowns (model parameter) are the observables themselves, least square minimization can be used to find a set of **improved measurements** or **fitted variables**.

$$\chi^2 = (y - \eta)^T V^{-1} (y - \eta)$$

Theory

Suppose that there are N measurable and J unmeasurable variables related by K constraint equations:

$$f_K(\eta_1, \eta_2, ..., \eta_N, \xi_1, \xi_2, ..., \xi_J) = 0$$

$$\chi^2(\eta) = (y - \eta)^T V(y)(y - \eta) = minimum$$

under the constraint

$$f(\eta, \xi) = 0$$

Lagrange Multipliers: We introduce K additional unknowns $\lambda(\lambda_1, ..., \lambda_K)$, then our **Lagrangian** becomes:

$$\chi^2(\eta, \xi, \lambda) = (y - \eta)^T V(y)(y - \eta) + 2\lambda^T f(\eta, \xi) = minimum$$

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Theory

Minimizing will result in N+J+K equations:

$$\nabla_\eta \chi^2 = -2V^{-1}(y-\eta) + 2F_\eta^T \lambda = 0,$$

$$\nabla_\xi \chi^2 = 2F_\xi^T \lambda = 0,$$

$$\nabla_\lambda \chi^2 = 2f(\eta,\xi) = 0,$$

where the matrices $F_\eta(KxN)$ and $F_\xi(KxJ)$ are defined as:

$$(F_\eta)_{ki} = \frac{\partial f_k}{\partial \eta_i}, \qquad (F_\xi)_{kj} = \frac{\partial f_k}{\partial \xi_j}$$

The solution of these set of equations in the general case can be found by iterations, producing successively better approximations.

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Assuming that we start at iteration ν, *which gives a good approximation of the solution*. The solution at iteration $\nu + 1$ can be found as follows:
Introduce the notations:

$$r = f^{\nu} + F_{\eta}^{\nu}(y - \eta^{\nu}), \qquad S = F_{\eta}^{\nu} V (F_{\eta}^{T})^{\nu}$$

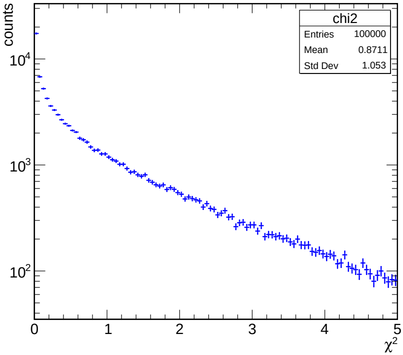
The algorithm defined as follows:

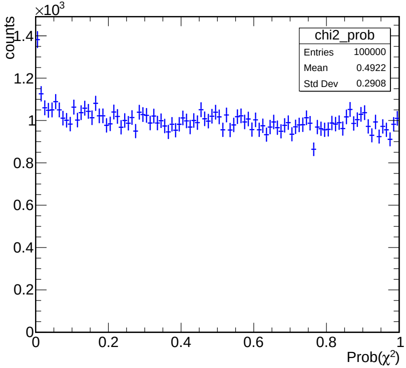
1. $\xi^{\nu+1} = \xi^{\nu} - \mathbf{lr}(F_{\xi}^{T} S^{-1} F_{\xi})^{-1} F_{\xi}^{T} S^{-1} r$
2. $\lambda^{\nu+1} = S^{-1}[r + F_{\xi}(\xi^{\nu+1} - \xi^{\nu})]$
3. $\eta^{\nu+1} = y - \mathbf{lr} V F_{\eta}^{T} \lambda^{\nu+1}$
4. $V^{\nu+1} =$
   $V^{\nu} - \mathbf{lr} V^{\nu} \left[F_{\eta}^{T} S^{-1} F_{\eta} - ((F_{\eta}^{T} S^{-1} F_{\xi})(F_{\xi}^{T} S^{-1} F_{\xi})^{-1} (F_{\eta}^{T} S^{-1} F_{\xi})^{T}) \right] V^{\nu}$
5. calculate the updated χ^2

where **lr** is a learning rate parameter takes values from 0 to 1.

A simple example

Pions of 500 MeV in z direction, each photon energy smeared by 12 MeV generated by ROOT TGenPhaseSpace $\pi^0 \rightarrow \gamma\gamma$.
The constraint equation is $f(E_1, E_2) = 2E_1E_2(1 - cos(\theta)) - m_\pi^2 = 0$

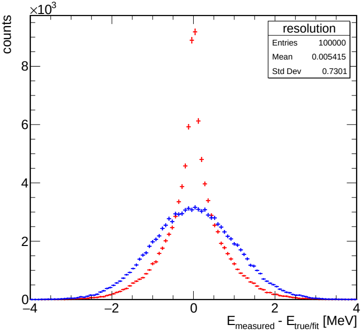


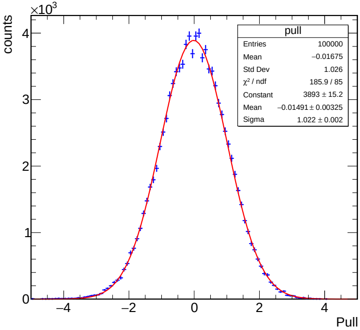


A simple example (Quality of the fit)

Stretch function "Pull distribution"

$$z_i = \frac{y_i - \eta_i}{\sqrt{\sigma^2(y_i) - \sigma^2(\eta_i)}} \qquad \textit{ideally} \qquad N(0,1)$$





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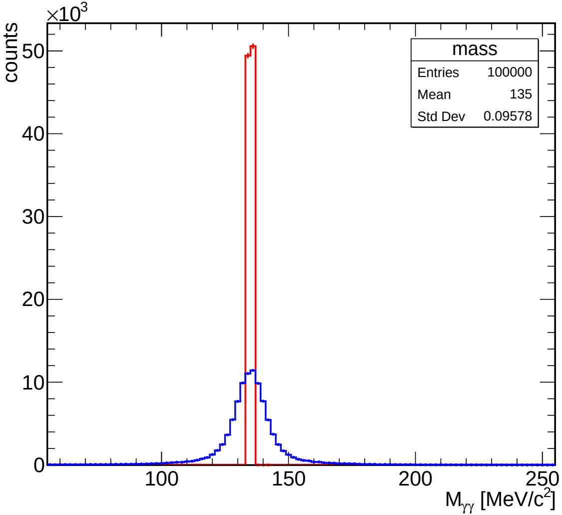
A simple example



Kinematic refit for Hydra

Track Representation

5 parameter track representation is used (from `HParticleCand`)

$$\begin{pmatrix} 1/P & \theta & \phi & R & Z \end{pmatrix}$$

Covariance Matrix

$$V = \begin{bmatrix} \sigma^2_{1/P} & 0 & \dots & 0 \\ \vdots & \sigma^2_{\theta} & \dots & \vdots \\ 0 & \dots & \sigma^2_{\phi} & 0 \end{bmatrix}$$

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Kinematic refit for Hydra (Implementing constraints)

Vertex constraint 1C fit

$(d_1 \times d_2) \cdot (b_1 - b_2)$

Invariant mass constraint 1C fit

$d = E^2 - P_x^2 - P_y^2 - P_z^2 - M^2$

Missing mass constraint 1C fit

$d = (E_t + E_b - \sum_{i=1}^{n} E_i)^2 - (\vec{p}_t + \vec{p}_b - \sum_{i=1}^{n} \vec{p}_i)^2 - M_{miss}^2$

$$b_x = R \cdot cos(\phi + \frac{\pi}{2})$$
$$b_y = R \cdot sin(\phi + \frac{\pi}{2})$$
$$b_z = z$$

$$d_x = sin\theta \cdot cos\phi$$
$$d_y = sin\theta \cdot sin\phi$$
$$d_z = cos\theta$$

$$P_x = P \cdot sin\theta \cdot cos\phi$$
$$P_y = P \cdot sin\theta \cdot sin\phi$$
$$P_z = P \cdot cos\theta$$
$$E = \sqrt{P^2 + m^2}$$

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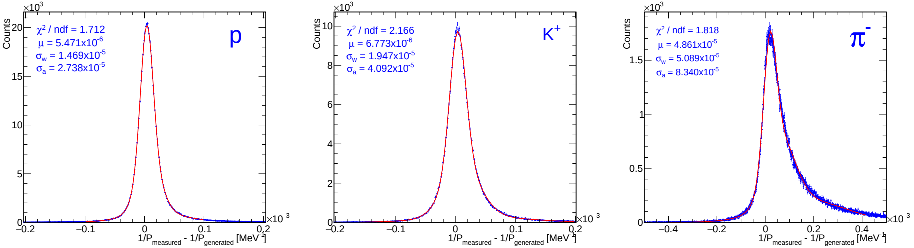
Kinematic refit for Hydra (Estimating Covariance Matrix)
Resolution Plots for **Momentum**



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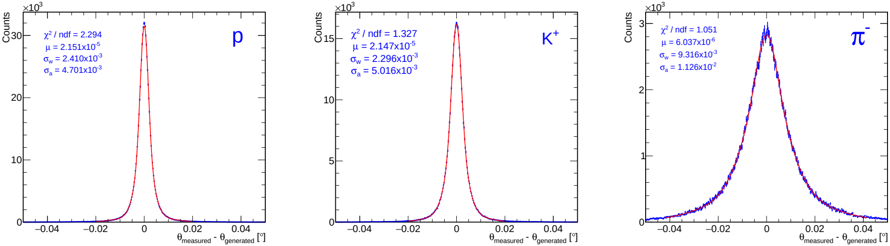
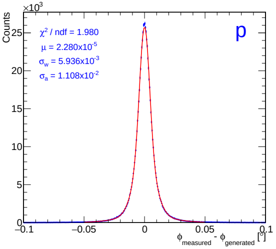
Kinematic refit for Hydra (Estimating Covariance Matrix)
Resolution Plots for **Polar Angle**



Kinematic refit for Hydra (Estimating Covariance Matrix) Resolution Plots for **Azimuthal Angle**



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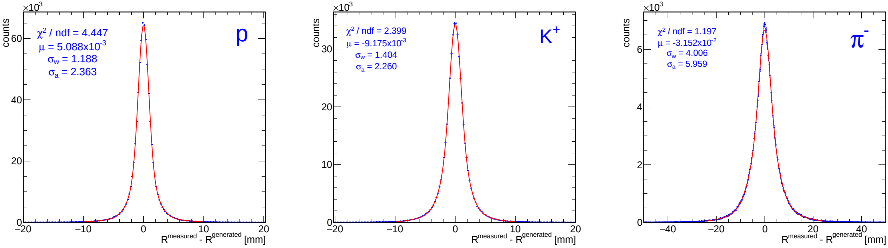
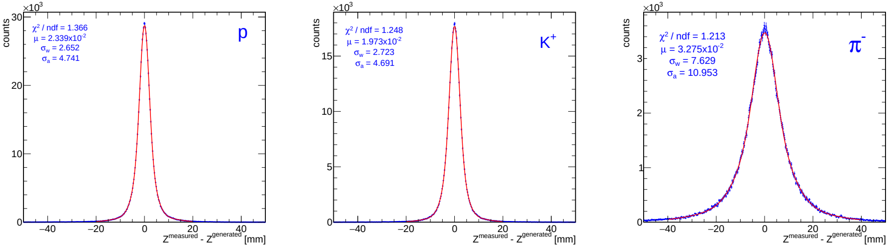
Kinematic refit for Hydra (Estimating Covariance Matrix)
Resolution Plots for **R**



Kinematic refit for Hydra (Estimating Covariance Matrix) Resolution Plots for **Z**



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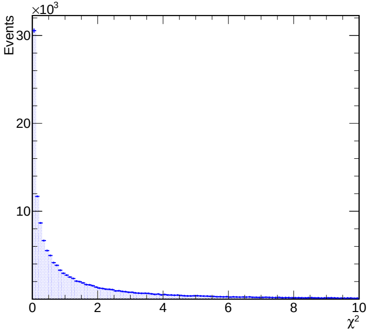
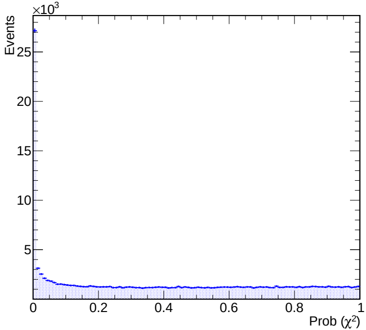
Reaction Channel

All plots shown from $p(4.5GeV)p \rightarrow pK^{+}Y^{*} \rightarrow pK^{+}\Lambda\gamma \rightarrow pK^{+}p\pi^{-}\gamma$

Vertex Constraint (Primary Vertex)

$p(4.5GeV)p \rightarrow pK^{+}p\pi^{-}\gamma$



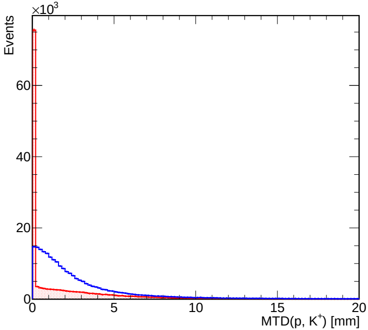


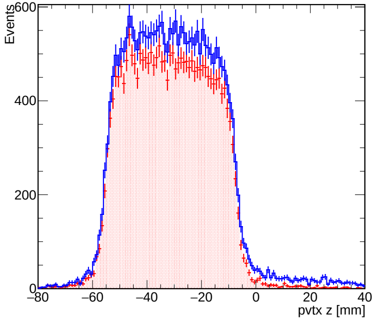
Kinematic refit for Hydra

Vertex Constraint (Primary Vertex)

$p(4.5 GeV)p \rightarrow pK^{+}p\pi^{-}\gamma$

After confidence level cut $P(\chi^2) > 1\%$





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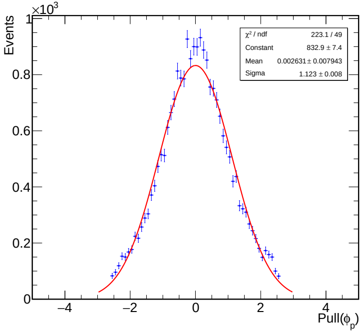
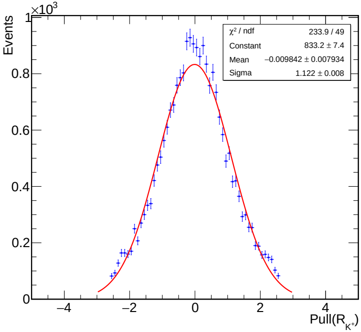
Kinematic refit for Hydra

Vertex Constraint (Primary Vertex)

$p(4.5GeV)p \rightarrow pK^{+}p\pi^{-}\gamma$

Pull example after confidence level cut $P(\chi^2) > 1\%$





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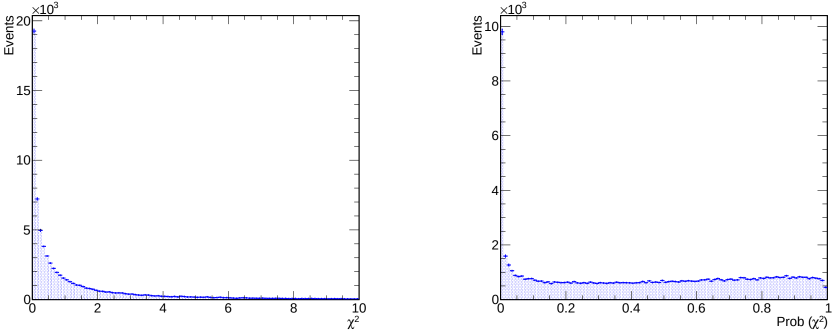
Invariant Mass Constraint

$p(4.5GeV)p \rightarrow pK^+ p\pi^- \gamma$



Kinematic refit for Hydra

Invariant Mass Constraint

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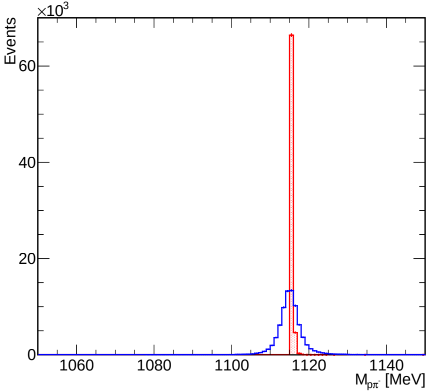
After confidence level cut $P(\chi^2) > 1\%$



Kinematic refit for Hydra

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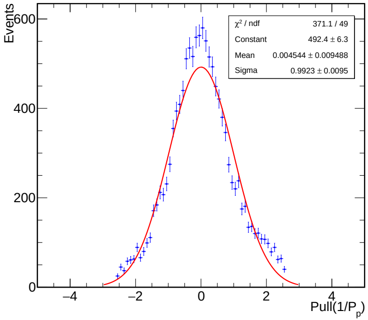

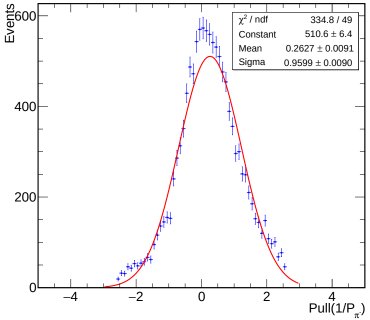

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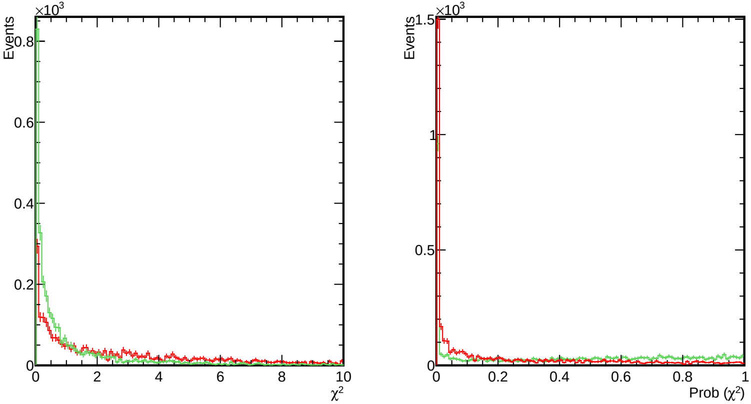
Kinematic refit for Hydra

Missing Mass Constraint (rejecting dominant background)

$p(4.5GeV)p \rightarrow pK^+p\pi^-\gamma$ Signal Channel

$p(4.5GeV)p \rightarrow pK^+p\pi^-\pi^0$ Background Channel



Kinematic refit for Hydra (code)

FParticleCand

Introduce a new data structure (inherited from `TLorentzVector`) whose sole purpose is to store the covariance matrix which is represented as `ROOT TMatrix`

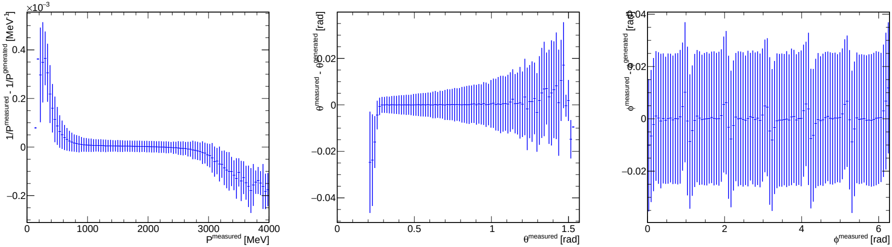
HFitter.h

A class that implementing the fitting procedure. It has a universal `fit` method that returns `bool`

in the analysis code

```
HFitter fitter(n, std::vector<FParticleCand>);
fitter.addMissingMassConstraint(134.9766, ppsys)
bool ok = fitter.fit();
FParticleCand cand0 = fitter.getDaughter(0);
```

Kinematic refit for Hydra (Improving Covariance Matrix)
Work in progress ...



Also Introducing off-diagonal terms (e.g. Momentum is correlated with the polar angle).

Conclusion

- A procedure for kinematic refit is introduced
- A reasonable preliminary results, but needs deeper study
- A 2C-fit (Mass+Vertex) has issues.
- A `code` for measurable variables (all plots shown) and also for unmeasurable variables (only one constraint)
- Hopefully a core for a future kinematic fit package for `Hydra`

Main Resource

Probability and Statistics in Particle Physics, A.G.Frodesen, 1979.

Thank You!