Mapping the phase diagram of strongly interacting matter

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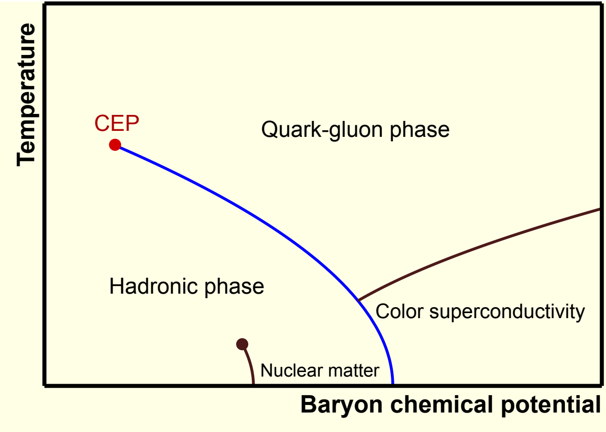
e-Print: arXiv:1008.4549

EMMI "Strongly Coupled Systems"

Outline

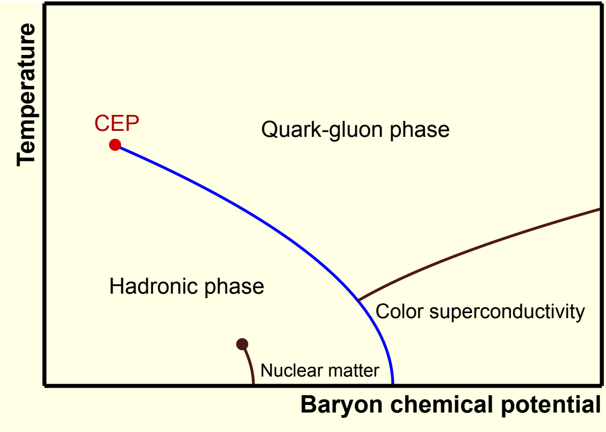
- Motivation
- Simple example of conformal mapping application
- Analytical structure of thermodynamic functions on the complex μ plane (chiral limit)
- Location of the second-order phase transition singularity
- Outlook: Finite size effect

QCD phase diagram



- Experiments
- Model calclulations
- Lattice QCD

QCD PHASE DIAGRAM



- Experiments
- Model calclulations
- Lattice QCD
- Functional methods (talk by C. Fischer)

Lattice QCD and "sign problem" at finite μ

- The partition function of QCD with integrated out quark degrees of freedom

$$Z(\mu) = \int \mathcal{D}A \exp(-S[A]) \det[D(\mu)]$$

with $\det[D(\mu)] \in$ **Complex**.

- The weight function is **not positive** definite. The Monte-Carlo technique fails.
- **Indirect** approaches to sidestep the sign problem
  - Reweighting technique (Z. Fodor and S. Katz)
  - Imaginary baryon chemical potential: $\mathrm{Im}\mu \neq 0$, $\mathrm{Re}\mu = 0$ (Ph. de Forcand and O. Philipsen; M.-P. Lombardo and M. D'Elia)
  - Taylor expansion (hotQCD collaborations, R. V. Gavai and S. Gupta)

- Thermodynamic function is given by its Taylor expansion at $\mu = 0$

$$\frac{P}{T^4} = \sum_k c_{2k}(T) \times \left(\frac{\mu}{T}\right)^{2k}, \quad c_{2k} = \frac{1}{(2k)!} \frac{\partial^{2k}(P/T^4)}{\partial(\mu/T)^{2k}}\Bigg|_{\mu=0}$$

- **Radius of convergence** R and its estimates R_k

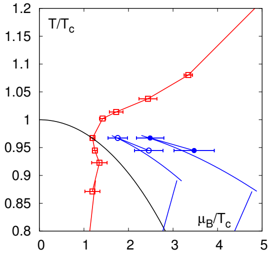
$$R = \lim_{k->\infty} \inf R_{2k} \quad \text{with} \quad R_{2k} = \left|\frac{c_{2k}}{c_{2k+2}}\right|^{1/2} \quad \text{or} \quad R_{2k} = \left|\frac{1}{c_{2k}}\right|^{1/(2k)}$$

- Convergence radius is defined by the closest singularity on the complex μ plane
Conversely, the convergence properties of a power series provides information on the closest singularity of the original function

C. Schmidt(2010)

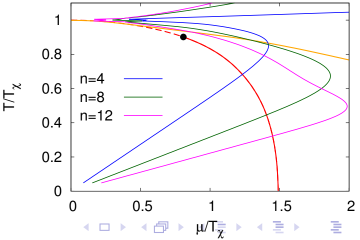
$R_2(P/T^4)$
$R_2(\chi_B)$
$R_4(\chi_B)$ open symb.

freeze-out curve



F. Karsch et. al., arXiv:1009.5211

M. Wagner talk



Ways to improve Taylor series convergence

- Pade approximant – a ratio of two power series
  $P_M^N = \frac{\sum_{i=0}^{N} a_i x^i}{1+\sum_{i=1}^{M} b_i x^i}$
  Pade approximant may be superior to Taylor expansion and may work even beyond a radius of convergence.
  The drawback of Pade approximation: it is uncontrolled.

- Conformal mapping
  Conformal mapping is a transformation $\xi = \xi(z)$ that preserves local angles.
  The main idea is to extend the radius of convergence and to enhance the sensitivity to the properties of the critical point by a non-linear transformation of an original series.
  E.g. by conformal mapping one can move the physical singularities closer to the expansion point, while taking non-physical singularities as far away as possible.

Applications

- In condensed matter physics (3d Ising model and high temperature expansion)
  A. Danielian and K.W. Stevens, Proc. Phys. Soc. B70, 326 (1957).
  C. Domb and M.F. Sykes, J. Math. Phys. 2, 63 (1961).
  C. J. Pearce, Adv. Phys. 27, 89 (1978).

- In scattering theory to extend applicability of low energy approximations
  W. R. Frazer, Phys. Rev. 123, 2180 (1961).
  A. Gasparyan and M. F. M. Lutz, arXiv:1003.3426 [hep-ph].

  a pedagogical example for an exactly solvable theory: I. V. Danilkin, A. Gasparyan, and M. F. M. Lutz, [arXiv:1009.5928 [hep-ph]]

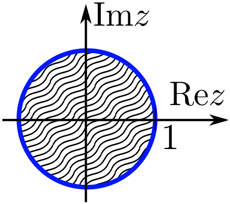
- In quantum field theory for analytic continuation of perturbative results to the strong coupling regime
  D. I. Kazakov *et al.*, Theor. Math. Phys. 38, 9 (1979).

- ...

Simple example: Euler transformation

$$\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots$$

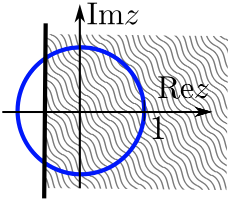
The radius of convergence of the series is $|z| < 1$ owing to a branch point singularity at $z = -1$.



The power series in $z \to$ the power series in the variable $\xi = \frac{z}{1+z}$.
The power series is obtained from the original

$$\ln(1+z) = -\ln(1-\xi) = \xi + \frac{1}{2}\xi^2 + \frac{1}{3}\xi^3 + \cdots$$

The radius of convergence of the series is $|\xi| < 1$.
$|\xi| = \frac{x^2+y^2}{(x+1)^2+y^2} < 1$ or $x > -\frac{1}{2}$.

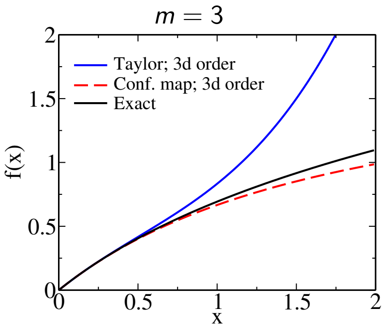


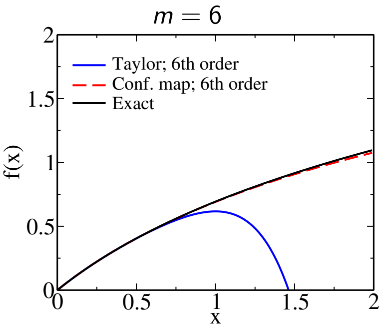
- The original series is known up to m-th order
  $f(x) \approx \sum_{i=0}^{m} c_i x^i$

- Change of the variables: $x \to \xi/(1-\xi)$ and series expansion around $\xi = 0$
  $\sum_{i=0}^{m} c_i' \xi^i$,
  where c_i' are linear combination of c_i with $i \leq m$.

- Back substitution $\xi \to x/(1+x)$
  $\sum_{i=0}^{m} c_i' (\frac{x}{1+x})^i$

$$f(x) = \ln(1-x) \approx \sum_{i=0}^{m} c_i x^i$$

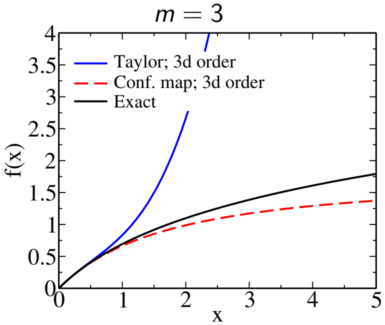


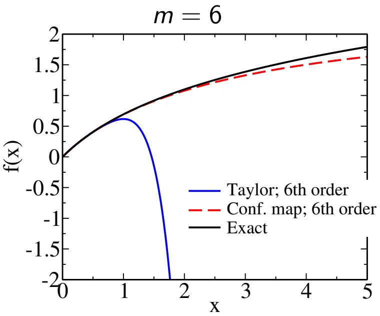


Original Taylor expansion does not describe the exact function for $x > 1$ even at large m.

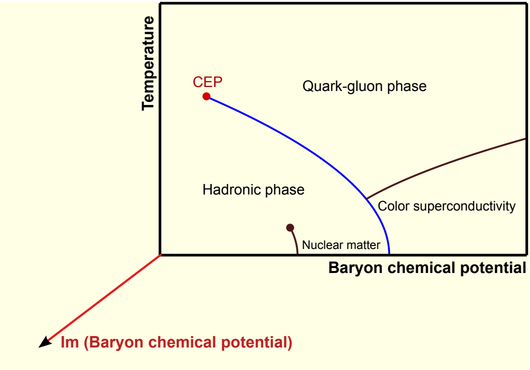
Taylor expansion + conformal mapping provides systematic improvement with increasing m.

Higher values of x :





Radius of convergence and importance of the analytical structure



To understand the dependence of convergence radius on temperature and to invent an appropriate conformal mapping, the analytical structure of thermodynamic functions is to be studied.

Analytic structure of the complex μ plane

general structure illustrated by the quark-meson model ($\approx$ NJL) in the mean-field approximation.

Analytic structure of the complex μ plane

general structure illustrated by the quark-meson model ($\approx$ NJL) in the mean-field approximation.

Chiral limit.

Singular points expected on the complex μ plane

Singularities on the complex plane are related to

- a critical point of a second-order phase transition. The singularity is on the real μ axis.
- to a crossover transition. The singularity is at some complex μ.

  See P. C. Hemmer and E. H. Hauge, Phys. Rev. **133**, A1010 (1964); C. Itzykson, R. B. Pearson and J. B. Zuber, Nucl. Phys. B **220**, 415 (1983); M. A. Stephanov, Phys. Rev. D **73**, 094508 (2006).
- spinodal lines for a first-order phase transition. Singularities are either at the real or complex values of μ.

  See M. A. Stephanov, Phys. Rev. D **73**, 094508 (2006);
- "thermal singularities" associated with zeros of the inverse Fermi-Dirac function

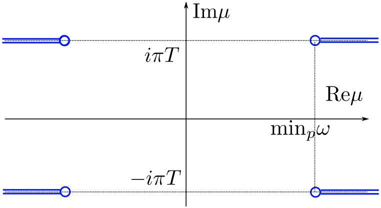
  See F. Karbstein and M. Thies, Phys. Rev. D **75**, 025003 (2007)

"Thermal" singularities

The inverse Fermi-Dirac function has zeros in the complex plane. This leads to "thermal" singularities of thermodynamic functions.

$$[f_F(\omega)]^{-1} = \exp\left(\frac{\omega - \mu}{T}\right) + 1$$

$$\mathrm{Im}\mu = i\pi T \rightsquigarrow -\exp\left(\frac{\omega - Re[\mu]}{T}\right) + 1 = 0$$



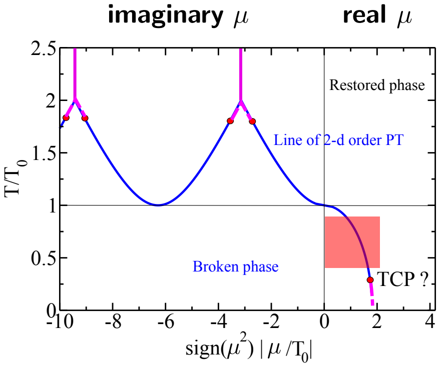
Examples:

- for massless particles zeros of the Fermi-Dirac functions are located on the lines $\mathrm{Re}\mu = p$, $\mathrm{Im}\mu = i\pi T + 2i\pi n T$, $n = 0, \pm 1, \pm 2, \cdots$.
- for particles with mass m, $\mathrm{Re}\mu = \sqrt{m^2 + p^2}$, $\mathrm{Im}\mu = i\pi T + 2i\pi n T$, $n = 0, \pm 1, \pm 2, \cdots$.

See also F. Karbstein and M. Thies, Phys. Rev. D **75**, 025003 (2007)

Analytic structure of the complex μ plane.

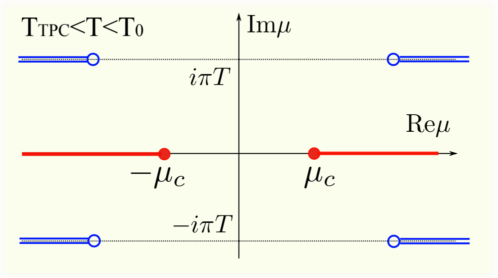
Phase diagram



second-order PT line;

possible TPC

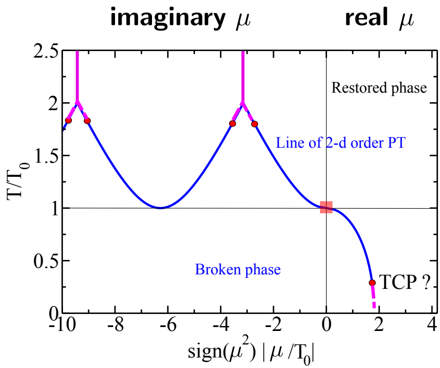
Complex μ plane



thermal cuts and associated singularities (branch points) denoted by open dots;

second-order PT

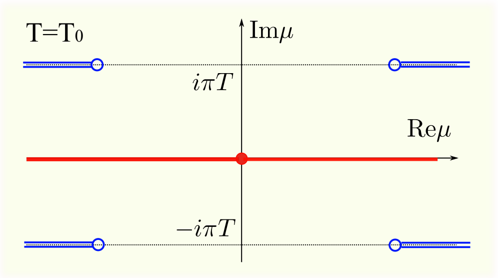
Phase diagram



second-order PT line;

possible TPC

Complex μ plane



thermal cuts and associated singularities (branch points) denoted by open dots;

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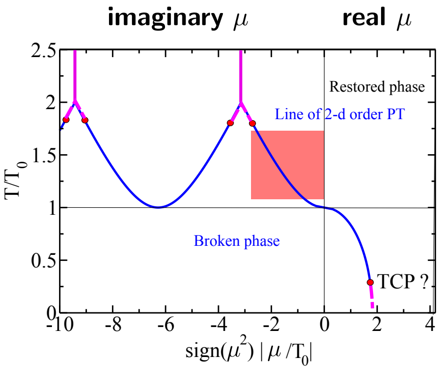
Analytic structure of the complex μ plane.

Phase diagram



second-order PT line;

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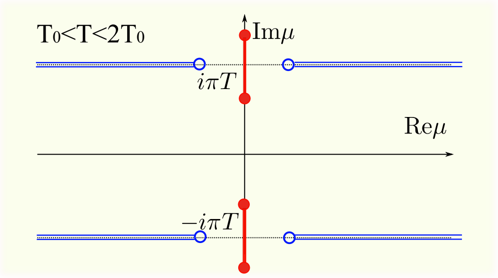
Complex μ plane



thermal cuts and associated singularities (branch points) denoted by open dots;

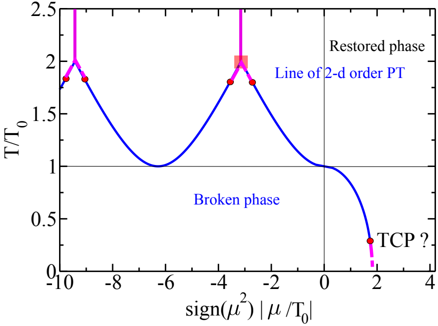
second-order PT

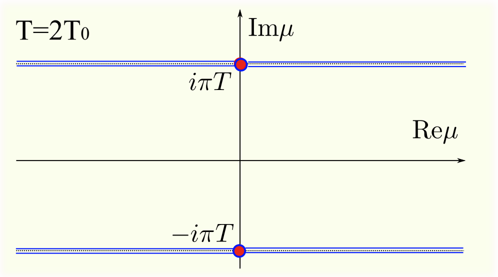
Analytic structure of the complex μ plane.

Phase diagram

Complex μ plane

imaginary μ **real μ**





second-order PT line;

possible TPC

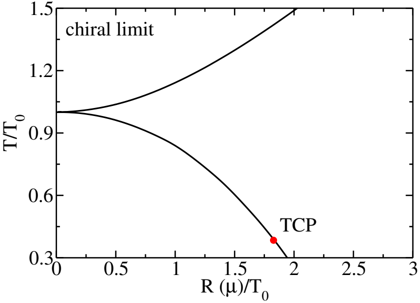
thermal cuts and associated singularities (branch points) denoted by open dots;

second-order PT

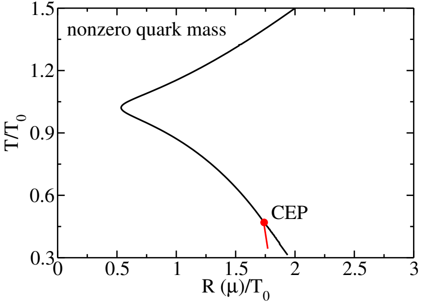
Position of CP and the radius of convergence

Assumption: the critical point defines the closest singularity.
The QM model for finite and zero pion mass:

The radius of convergence (or distance to singularity) in **the chiral limit**



The radius of convergence for **finite pion mass**

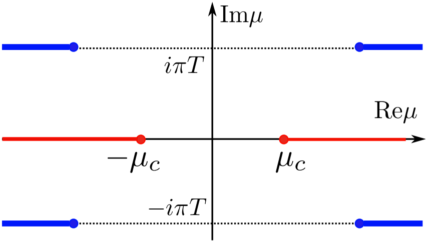


see also RM calculations, M. Stephanov PRD73:094508, 2006

TCP and CEP do not exhibit unique features in R^{μ} as function of T.

Thermal singularities can affect the radius of convergence.

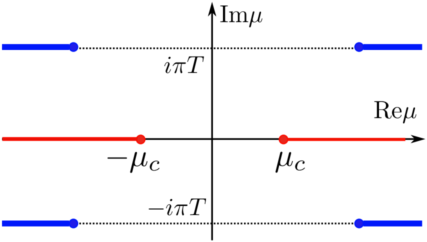
Chemical potential plane for 2nd order PT



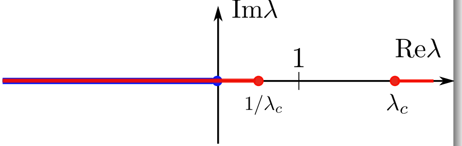
Fugacity plane

Thermal singularities can affect the radius of convergence.

Chemical potential plane for 2nd order PT



Corresponding fugacity plane



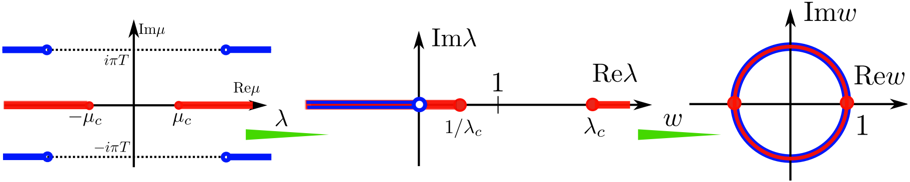
The solution: fugacity $\lambda = \exp(\mu/T)$. In the complex λ plane the thermal branch points are mapped onto the negative real λ-axis. Images of the critical points are located at positive λ, $\lambda_c = e^{\mu_c/T}$ and at $1/\lambda_c$. The closest singularity of the thermal cuts is a branch point at $\lambda_{th} = 0$. Since $\lambda_c > 1$ and hence $0 < 1/\lambda_c < 1$, the singularity closest to $\lambda = 1$ is the one at $1/\lambda_c$.

In the complex λ plane, the radius of convergence, $R^\lambda = 1 - 1/\lambda_c$, is not affected by the thermal singularities. This is the case also for crossover phase transition.

Conformal mapping: maps the cut fugacity plane onto the interior of the unit circle:

$$w(\lambda) = \frac{\sqrt{\lambda\lambda_c - 1} - \sqrt{\lambda_c - \lambda}}{\sqrt{\lambda\lambda_c - 1} + \sqrt{\lambda_c - \lambda}}$$

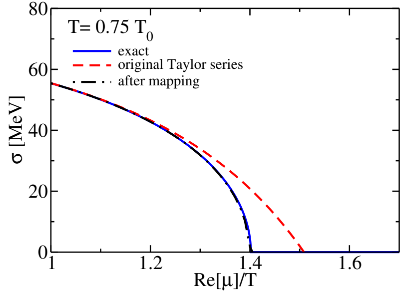
the exact location of the critical point λ_c is used to proceed with the map



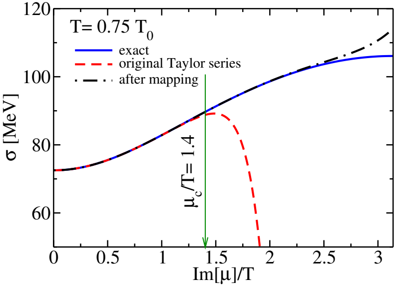
- the branch points at λ_c and $1/\lambda_c$ are mapped onto $w = 1$ and $w = -1$
- the corresponding cuts are mapped onto the circumference of the unit circle
- a Taylor series about $w = 0$, which corresponds to $\lambda = 1$ ($\mu = 0$), converges for all points within the unit circle in w plane and in whole cut μ-plane.
- we obtain an analytic continuation of the Taylor series in λ or μ, which is valid also beyond the radius of convergence in the λ or μ plane, R^λ or R^μ .

Number of terms in original the Taylor expansion 12;
the location of the critical point is provided λ_c .

The order parameter for real chemical potential



the order parameter for imaginary chemical potential



T_0 is the critical T for $\mu = 0$.

We also can inverse this procedure in order to use information of imaginary chemical potential as a supplement for Taylor expansion at $\mu = 0$. Lattice calculations done for both cases with the same lattice parameters are required.

Locating singularity I

In reality we do not know the location of the singularity

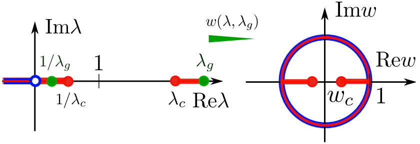
- mapping :

$$w(\lambda, \lambda_g) = \frac{\sqrt{\lambda\lambda_g - 1} - \sqrt{\lambda_g - \lambda}}{\sqrt{\lambda\lambda_g - 1} + \sqrt{\lambda_g - \lambda}}$$

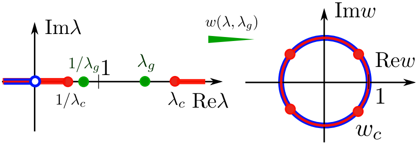
λ_g is the **guessed** value for the location of the singularity

- Imagine: $N \to \infty$, all Taylor coefficients are known
- the analytical structure in w plane depends on λ_g

- $\lambda_g > \lambda_c$: the critical points are mapped inside the unit circle at $w = \pm w_c = \pm w_g(\lambda_c; \lambda_g)$

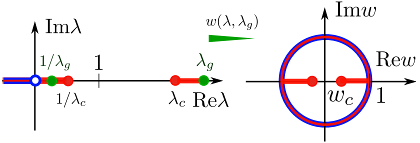


- $\lambda_g < \lambda_c$: the critical points are mapped onto the circumference of the unit circle



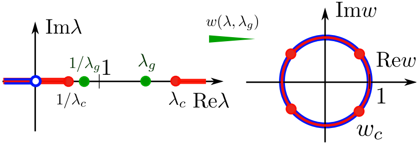
Locating singularity II

- $\lambda_g > \lambda_c$:



The closest singularity is at $\pm w_c$, therefore $R^w = w_c$.

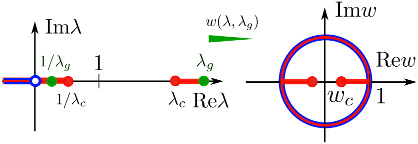
- $\lambda_g < \lambda_c$:



The closest singularity is on the unit circle, therefore $R^w = 1$.

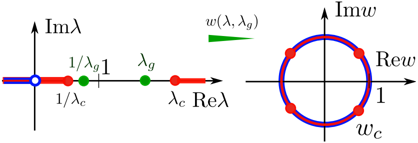
Locating singularity II

- $\lambda_g > \lambda_c$:



The closest singularity is at $\pm w_c$, therefore $R^w = w_c$.

- $\lambda_g < \lambda_c$:



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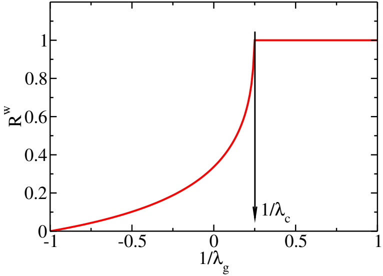
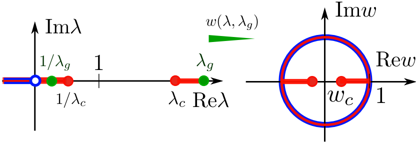
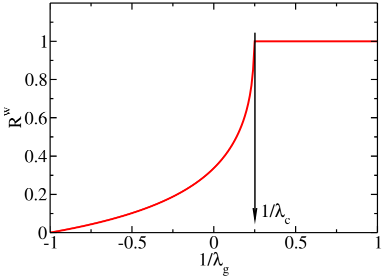


Locating singularity II

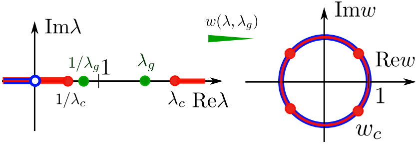
- $\lambda_g > \lambda_c$:



The closest singularity is at $\pm w_c$, therefore $R^w = w_c$.



- $\lambda_g < \lambda_c$:

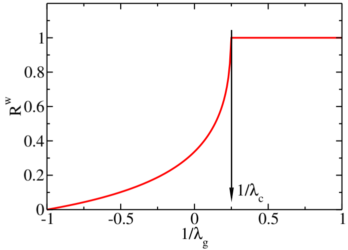


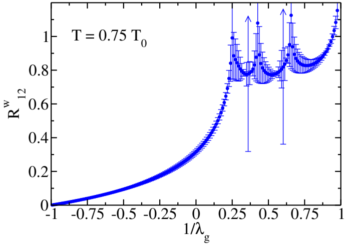
The closest singularity is on the unit circle, therefore $R^w = 1$.

In order to enhance the sensitivity to the location of CP and minimize the influence of other singularities, it is advantageous to use a mapping which leaves the singularity of interest close to the origin and moves all others as far away as possible. This can be achieved by varying λ_g .

Locating singularity III: finite N

In reality only finite number of terms N in the Taylor expansion is known.





$N \to \infty$

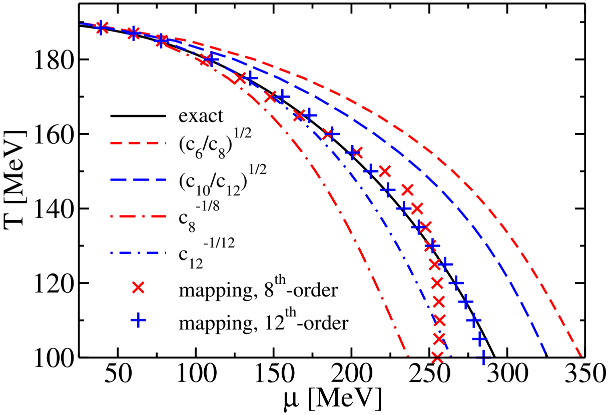
$N = 12$, error bars show the deviation of radius of convergence $R^w_{12} = 1/(c^w_{12})^{1/12}$ from one obtained with $N = 10$

Numerical value for an approximate location of CP is obtained by performing a fit

$$R^\omega = const + w(\lambda_c, \lambda_g),$$

with λ_c as a **fitting parameter**. $\mu_c = T\ln(\lambda_c)$.

Repeating this procedure for different temperatures, we obtain



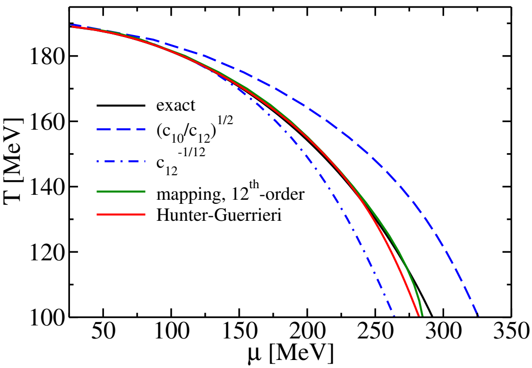
N $= 8, \; 12$

$$\sigma = \sum_{k=0}^{\mathbf{N}} c_k \left(\frac{\mu}{T}\right)^k ,$$

$$c_k = \left.\frac{\partial^k \sigma}{\partial(\mu/T)^k}\right|_{\mu=0} .$$

Information is only provided for the coefficients c_k

Better estimate for transition temperature.



A comparison to the procedure by Hunter and Guerrieri [C. Hunter and B. Guerrieri, SIAM J. Appl. Math. 39, 248 (1980)].

HG's procedure is based on the Darboux theorem for late coefficients of a Taylor series.

if

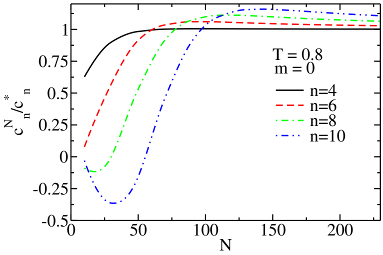
$$f(x) = (1 - x/x_c)^{-\nu} r(x) + a(x)$$

then

$$c_n \sim \frac{n^{\nu-1}}{x_c^n \Gamma(\nu)}$$

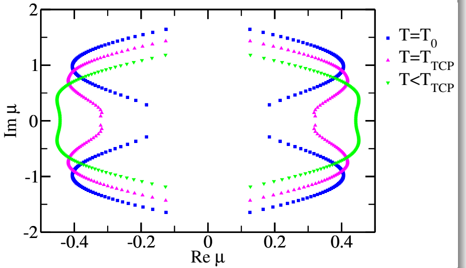
Finite size effects with Random Matrix Model

- Taylor coefficients c_n^N for finite system size can significantly deviate from their thermodynamical limits c_n^*



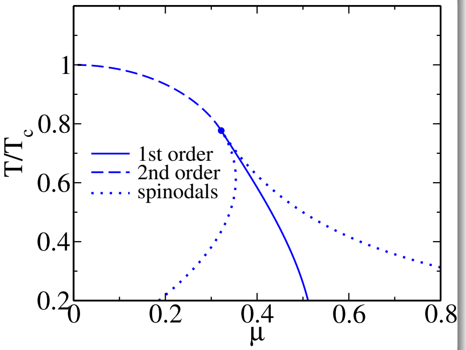
- Cuts in the complex μ plane are transformed to Lee-Yang zeros; $Z(T, \mu) = 0$

- Density of Lee-Yang zeros depends on system size;



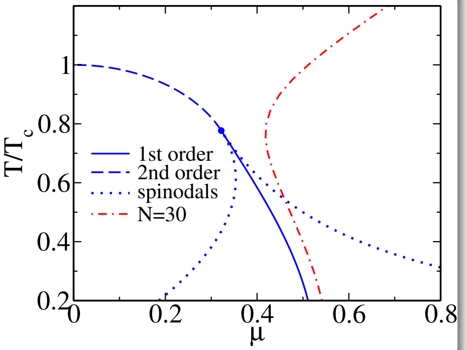
Finite size effects: possible signature of TCP (CEP)

- The phase diagram of the RM model in thermodynamic limit



Finite size effects: possible signature of TCP (CEP)

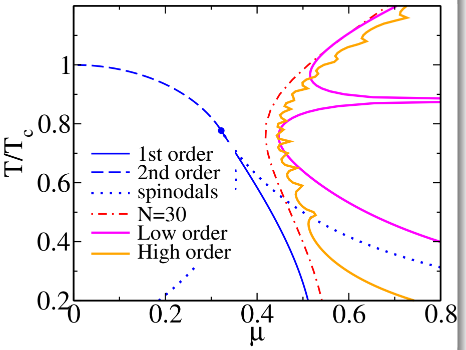
- The phase diagram of the RM model in thermodynamic limit
- The radius of convergence defined exactly by the location of the closest Lee-Yang zero for finite system



Finite size effects: possible signature of TCP (CEP)

- The phase diagram of the RM model in thermodynamic limit
- The radius of convergence defined exactly by the location of the closest Lee-Yang zero for finite system
- The radius of convergence defined by a low order coefficients in the Taylor expansion

The radius of convergence defined by a high order coefficients



Conclusions

- Conformal mapping approach yields better estimate for the phase boundary
- Supplementary information for imaginary chemical potential can be used to further improve the results.
- The extracted phase boundary for finite systems could deviate from the one in thermodynamic limit.
- Non-zero pion mass and location of CEP ???