Causal Charge Diffusion in Relativistic Heavy-Ion Collisions

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XQCD 2018

Frankfurt

XQCD 2018, Frankfurt, DE
May 22, 2018

Quantifying the strength of Diffusion:

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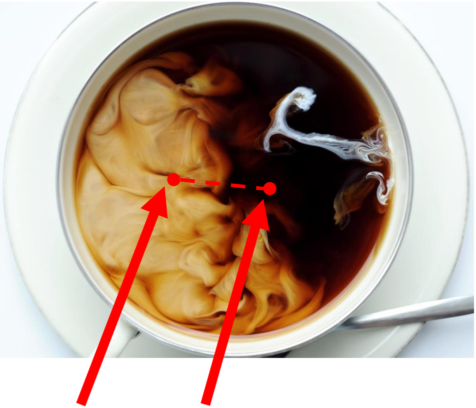
Cups of coffee:



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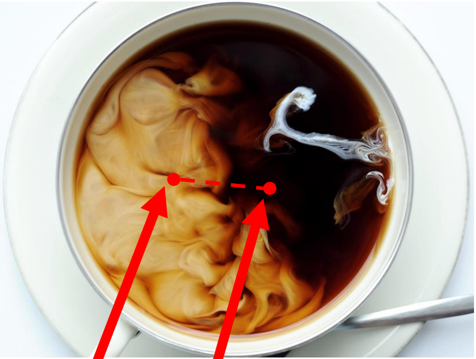

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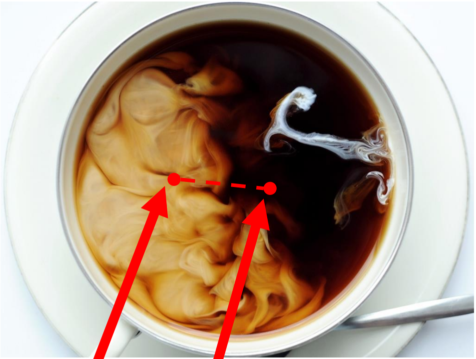

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$$B_{h^+h^-}(y_1 - y_2) \equiv \left\langle \frac{dN}{dy} \right\rangle_{\mathrm{ev}}^{-1} \left\langle \delta\left(\frac{dN}{dy_1}\right) \delta\left(\frac{dN}{dy_2}\right) \right\rangle_{\mathrm{ev}}$$

Charge Diffusion in Fluctuating Hydrodynamics

- Identify charge of interest: Q (electric charge)
- Construct 4-current J_Q^μ:

$$J_Q^\mu = \underbrace{n_Q u^\mu}_{\text{ideal}} + \underbrace{\Delta J_Q^\mu}_{\text{viscous}} + \underbrace{I^\mu}_{\text{stochastic}}$$

- Require J_Q to be conserved: $\partial_\mu J_Q^\mu = 0$ (EOM)
- Self-consistency of hydro $\Longrightarrow \langle I^\mu I^\nu \rangle \propto \sigma_Q$ (FDT)
- Use solution to EOM + FDT to construct interesting quantities, e.g., $\langle \delta n_Q(x)\, \delta n_Q(x') \rangle$

$$\begin{aligned} \Delta J_Q^\mu &= \sigma_Q T \Delta^\mu \left(\frac{\mu_Q}{T} \right) , \\ I^\mu &\equiv s(\tau) f(\xi, \tau) \left(\sinh \xi, \vec{0}_\perp, \cosh \xi \right) \end{aligned}$$

with

$$\tau \equiv \sqrt{t^2 - z^2} \text{ and } \xi = \frac{1}{2} \ln \left| \frac{t+z}{t-z} \right| .$$

Steps to solve $\partial_\mu J_Q^\mu = 0$:

- Conservation of J_Q^μ yields Langevin-type equation of motion for $\delta\tilde{n}(k,\tau)$:

$$\partial_\mu J_Q^\mu = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial\tau}(\tau\delta\tilde{n}) + \frac{D_Q k^2}{\tau^2}(\tau\delta\tilde{n}) = -iks\tilde{f}$$

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- Homogeneous equation (with $\tilde{f} = 0$) describes relaxation of single fluctuation mode $\delta\tilde{n}(k,\tau)$
$\longrightarrow$ solution defines the *response function* $\tilde{G}(k;\tau,\tau')$

Steps to solve $\partial_\mu J^\mu_Q = 0$:

- Conservation of J^μ_Q yields Langevin-type equation of motion for $\delta\tilde{n}(k,\tau)$:

$$\partial_\mu J^\mu_Q = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial\tau}(\tau\delta\tilde{n}) + \frac{D_Q k^2}{\tau^2}(\tau\delta\tilde{n}) = -iks\tilde{f}$$

- Homogeneous equation (with $\tilde{f} = 0$) describes relaxation of single fluctuation mode $\delta\tilde{n}(k,\tau)$
$\longrightarrow$ solution defines the *response function* $\tilde{G}(k;\tau,\tau')$
- Solve inhomogeneous equation (with $\tilde{f} \neq 0$) with:

$$\delta\tilde{n}(k,\tau) = -\frac{1}{\tau}\int_{\tau_0}^{\tau} d\tau' s(\tau')\tilde{G}(k;\tau,\tau')\tilde{f}(k,\tau')\,.$$

Use 2-point functions to study fluctuations systematically:

$$\begin{aligned}\langle \delta\tilde{n}(k_1,\tau_1)\,\delta\tilde{n}(k_2,\tau_2)\rangle &= \frac{1}{\tau_1\tau_2}\int_{\tau_0}^{\tau_1} d\tau_1' s(\tau_1')\int_{\tau_0}^{\tau_2} d\tau_2' s(\tau_2') \\ &\times \tilde{G}(k_1;\tau_1,\tau_1')\tilde{G}(k_2;\tau_2,\tau_2') \\ &\times \langle \tilde{f}(k_1,\tau_1')\tilde{f}(k_2,\tau_2')\rangle\end{aligned}$$

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Problem:

$$\langle \tilde{f}(k_1,\tau_1')\tilde{f}(k_2,\tau_2')\rangle = \text{ ?}$$

Interlude:

White noise vs. **Colored** noise

Noise in a static medium

- White noise: $\langle f(x_1)f(x_2)\rangle = N\delta^4(x_1 - x_2)$
  - Pro: simple to implement
  - Con: acausal signal propagation ($v_Q^2 \to \infty$)

[1] Aarts *et al.*, JHEP **1502**, 186 (2015)

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- **Colored** noise: $\langle f(x_1) f(x_2) \rangle = \frac{N}{2\tau_Q} \mathrm{e}^{-|t_1 - t_2|/\tau_Q} \delta^3(\vec{x}_1 - \vec{x}_2)$
  - Pro: causal ($v_Q^2 = D_Q/\tau_Q$)

$$D_Q \sim \tau_Q \sim \frac{1}{2\pi T} \Longrightarrow v_Q^2 \sim 1^{1}$$

$$\Delta J^\mu \to \sigma_Q T \Delta^\mu (1 + \tau_Q u \cdot \partial)^{-1} \left(\frac{\mu_Q}{T} \right)$$

  - Con: more complicated implementation

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Complications:

$\rightarrow$ Colored two-point function is *non-local* on scales $\sim \mathcal{O}(\tau_Q)$

$\rightarrow$ $\partial \cdot u \sim \frac{1}{\tau}$, $\tau \sim \mathcal{O}\left(1\,\mathrm{fm}/c\right)$

$\rightarrow$ Static approach breaks down at $\tau \sim \tau_Q$!

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Noise in a dynamical medium

Solution: rewrite two-point functions *locally*

- White noise:

$$\left\langle \tilde{f}(k_1,\tau_1)\tilde{f}(k_2,\tau_2)\right\rangle \quad\propto\quad \delta(k_1+k_2)\delta(\tau_1-\tau_2)$$

- **Colored** noise:

$$\Big\langle (1+\tau_Q\partial/\partial\tau_1)\,\tilde{f}(k_1,\tau_1) \times (1+\tau_Q\partial/\partial\tau_2)\,\tilde{f}(k_2,\tau_2)\Big\rangle \quad\propto\quad \delta(k_1+k_2)\delta(\tau_1-\tau_2)$$

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Checklist:

✓ Solve $\partial_\mu J^\mu_Q = 0$ with $\tau_Q \neq 0$ for $\delta\tilde{n}(k,\tau)$

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✓ Fix $\Big\langle \tilde{f}(k_1,\tau_1)\tilde{f}(k_2,\tau_2)\Big\rangle$

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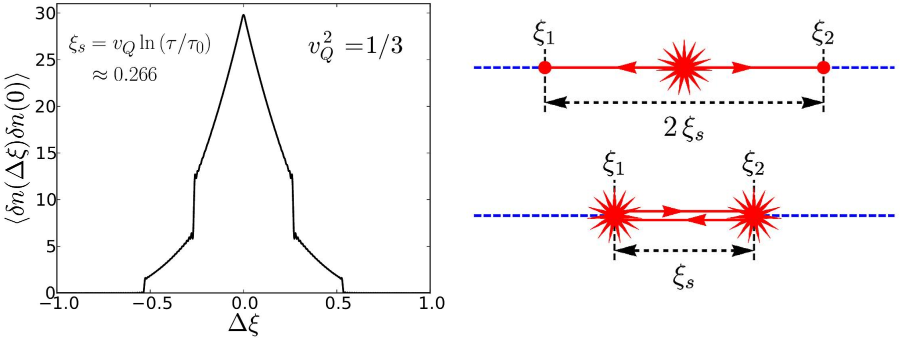
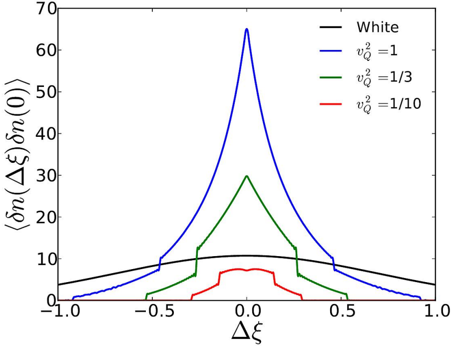
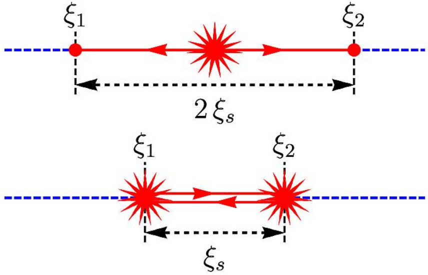
- ✓ Get $\langle \delta\tilde{n}(k_1,\tau_1)\delta\tilde{n}(k_2,\tau_2)\rangle$
- ✓ Get $B_{h^+h^-}\,(y_1-y_2)$

Results

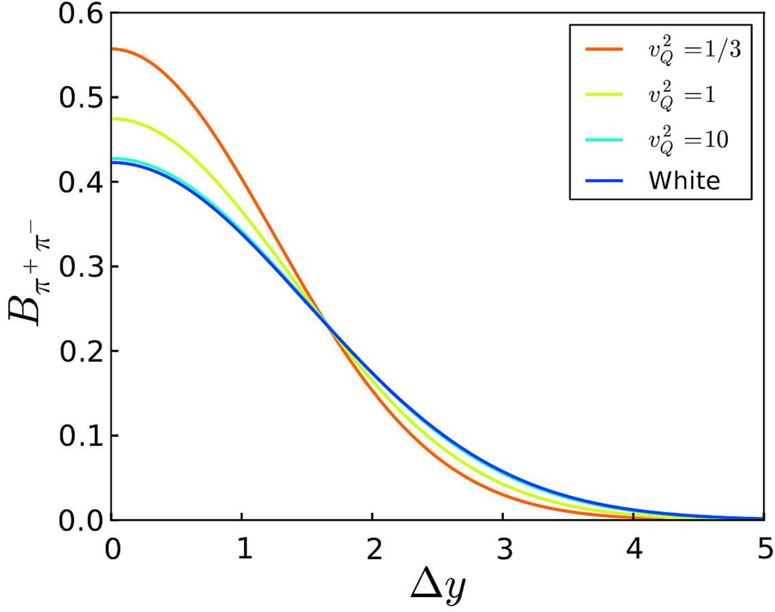


$\xi_s = v_Q \ln(\tau/\tau_0)$
≈ 0.266
$v_Q^2 = 1/3$
$\langle \delta n(\Delta\xi)\delta n(0)\rangle$
$\Delta\xi$
ξ_1
ξ_2
$2\,\xi_s$
ξ_1
ξ_2
ξ_s

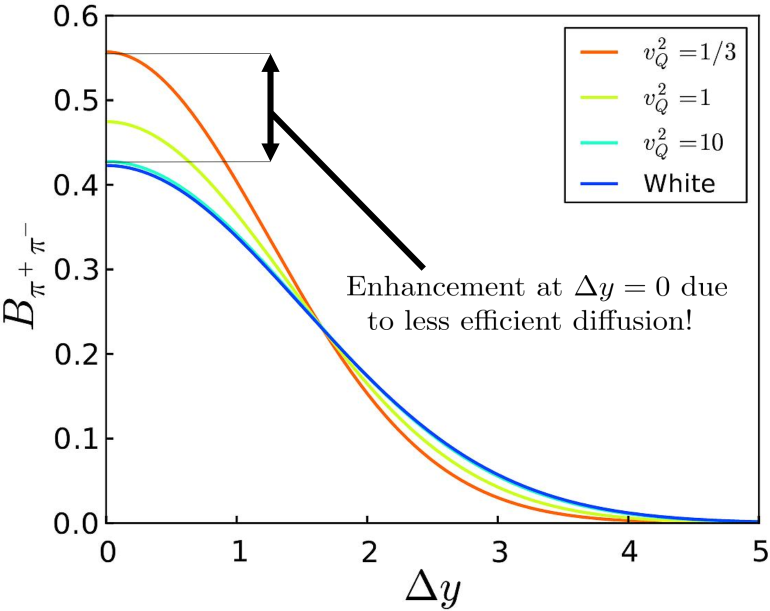




- Both sets of wavefronts reflect essential aspects of causal signal propagation
- Wavefronts travel farther with larger v_Q^2
- *No* wavefronts for $v_Q^2 \to \infty$!



0.6
0.5
0.4
0.3
0.2
0.1
0.0
$B_{\pi^+\pi^-}$
0
1
2
3
4
5
Δy
$v_Q^2 = 1/3$
$v_Q^2 = 1$
$v_Q^2 = 10$
White


$v_Q^2 = 1/3$
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White
Enhancement at $\Delta y = 0$ due to less efficient diffusion!
$B_{\pi^+\pi^-}$
Δy
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0
1
2
3
4
5

Summary:

- Diffusion is an essential aspect of heavy-ion collisions
- Hydrodynamic fluctuations offer a natural framework for modeling diffusion
- White noise leads to violations of relativistic causality
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Future/ongoing work:

- More sophisticated models of colored noise (e.g., correlations in space and time)
- Match onto holographic dispersion relations
- Incorporate into 3+1D hydrodynamic simulations
- Subtract/project out self-correlations
- ...

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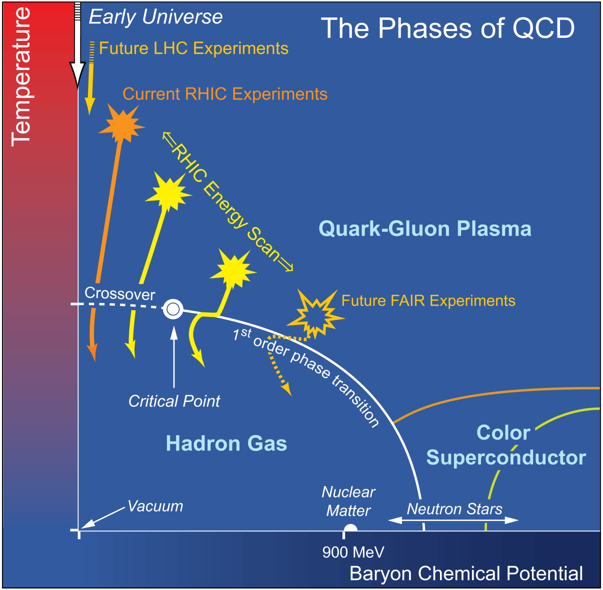
Thanks for your attention!

Backup slides


The Phases of QCD
Temperature
Early Universe
Future LHC Experiments
Current RHIC Experiments
⇐RHIC Energy Scan⇒
Quark-Gluon Plasma
Future FAIR Experiments
Crossover
1st order phase transition
Critical Point
Hadron Gas
Color Superconductor
Vacuum
Nuclear Matter
Neutron Stars
900 MeV
Baryon Chemical Potential

Steps to solve $\partial_\mu J^\mu_Q = 0$ (with $\tau_Q \neq 0$):

$$\partial_\mu J^\mu_Q = 0 \quad \Longrightarrow$$

$$\frac{\partial^2}{\partial\tau^2}(\tau\delta\tilde{n}) + \left[\frac{1}{\tau_Q} - \frac{\partial}{\partial\tau}\ln\left(\frac{\chi_Q T D_Q}{\tau}\right)\right]\frac{\partial}{\partial\tau}(\tau\delta\tilde{n}) + \frac{D_Q k^2}{\tau_Q\tau^2}(\tau\delta\tilde{n})$$
$$= -iks\left[\frac{\partial\tilde{f}}{\partial\tau} + \left(\frac{1}{\tau_Q} - \frac{1}{\tau} - \frac{\partial}{\partial\tau}\ln\left(\frac{\chi_Q T D_Q}{\tau}\right)\right)\tilde{f}\right].$$

T is the temperature, D_Q is the electric charge diffusion coefficient, σ_Q is the electric charge conductivity, and $\chi_Q = \sigma_Q/D_Q$ is the electric charge susceptibility. Finally, the quantity k is Fourier-conjugate to the spatial rapidity ξ; for any quantity X, we define

$$X(\xi,\tau) \equiv \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik\xi}\tilde{X}(k,\tau)$$

Subtracting self-correlations

White noise density correlator:

$$\langle \delta n\left(\xi_1, \tau_f\right) \delta n\left(\xi_2, \tau_f\right)\rangle = \frac{\chi_Q\left(\tau_f\right) T_f}{A\tau_f}\left[\delta\left(\xi_1 - \xi_2\right) - \frac{1}{\sqrt{\pi w^2}} e^{-\left(\xi_1 - \xi_2\right)^2 / w^2}\right]$$

- First term: “self-correlations”
  - Represent trivial correlations of a particle with itself
  - Not measured experimentally
- Second term: diffusive correlations
  - Represent physical, non-trivial correlations of distinct particles
  - Are actually what we care about

$\rightarrow$ Self-correlations need to be subtracted out to compare with experiment!

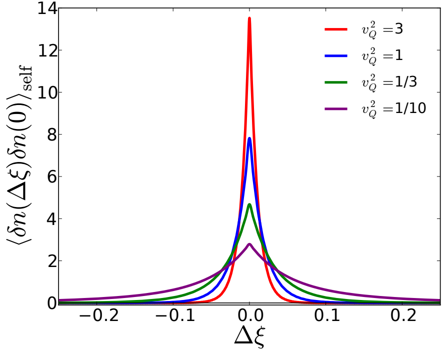
$\rightarrow$ Not so hard to do for white noise...

$\rightarrow$ ...but highly non-trivial for colored noise!

Subtracting self-correlations

Colored noise density self-correlations ($v_Q \gg 1$):

$$\langle \delta n(\xi_1, \tau_f)\, \delta n(\xi_2, \tau_f) \rangle_{\text{self}} \approx \frac{\chi_Q(\tau_f)\, T_f}{A \tau_f} \frac{v_Q \tau_f}{2 D_Q} \exp\left(-\frac{v_Q \tau_f}{D_Q} |\xi_1 - \xi_2| \right)$$



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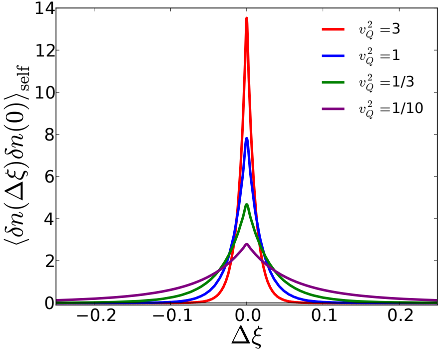

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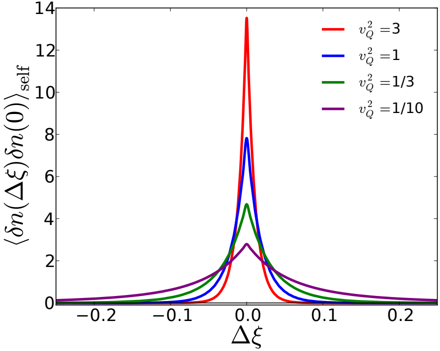


- "Adiabatic limit" ($v_Q \gg 1$) reduces to exponential form on quasi-static background
- "Instantaneous limit" ($v_Q \ll 1$) just takes all correlations to zero

See these references for more detail:

- Ling, Springer, and Stephanov [PRC **89**, 064901 (2014)]
- Kapusta and CP [PRC **97**, 014906 (2018)]

Holographic considerations

Key idea: matching colored noise to holographic dispersion relations yields estimates for D_Q, τ_Q
Gurtin-Pipkin noise:

$$\begin{aligned}
\frac{\partial}{\partial t} - D_Q \nabla^2 + \tau_1 \frac{\partial^2}{\partial t^2} + \tau_2^2 \frac{\partial^3}{\partial t^3} - \tau_3 D_Q \frac{\partial}{\partial t} \nabla^2 &= 0 \\
\Rightarrow \tau_2^2 \omega^3 + i\tau_1 \omega^2 - \left(1 + \tau_3 D_Q k^2\right) \omega - i D_Q k^2 &= 0
\end{aligned}$$

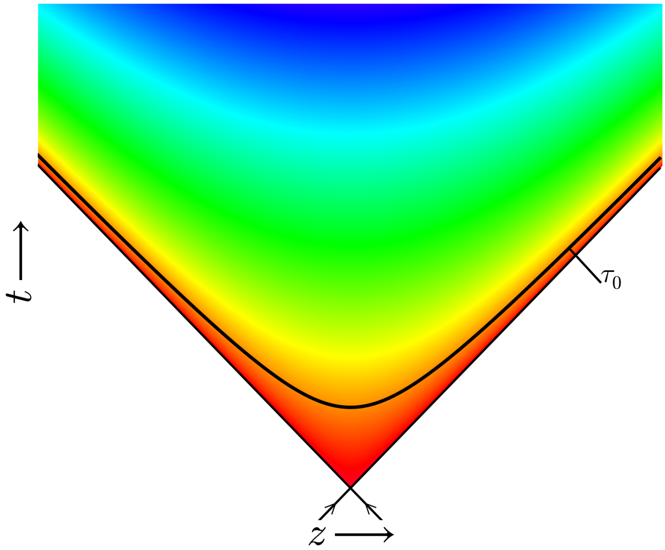
Holography[2,3] yields Kaluza-Klein-type tower of poles in holographic dispersion relation:

$$\omega(k=0) = (\pm n - in) 2\pi T,\, n = 0, 1, 2, \ldots$$

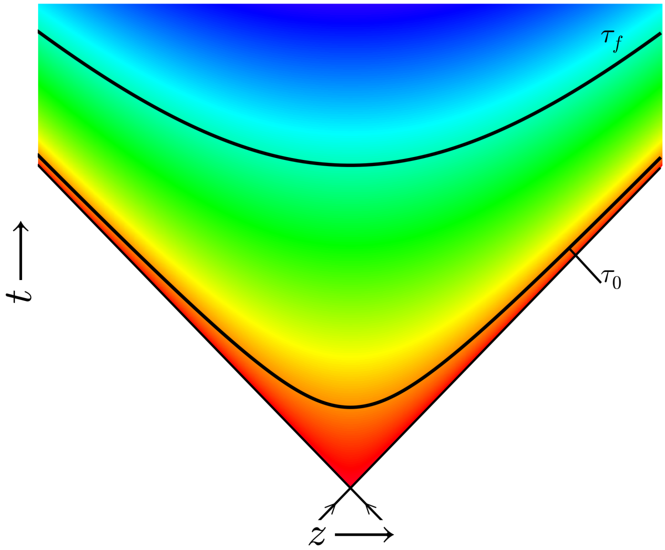
Match GP noise onto three lowest frequency poles

$$\Rightarrow D_Q = \tau_Q = \tau_1 = \frac{1}{2\pi T},\, \tau_2 = \tau_1/\sqrt{2}, \tau_3 = \tau_1/2$$

[2] Nunez and Starinets [PRD **67**, 124013 (2003)]
[3] Policastro, Son and Starinets [JHEP **0209**, 043 (2002)]



$$\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle = 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}N\left(\tau'\right)\tilde{G}_X(k_1;\tau_f,\tau')\tilde{G}_Y(k_2;\tau_f,\tau')$$



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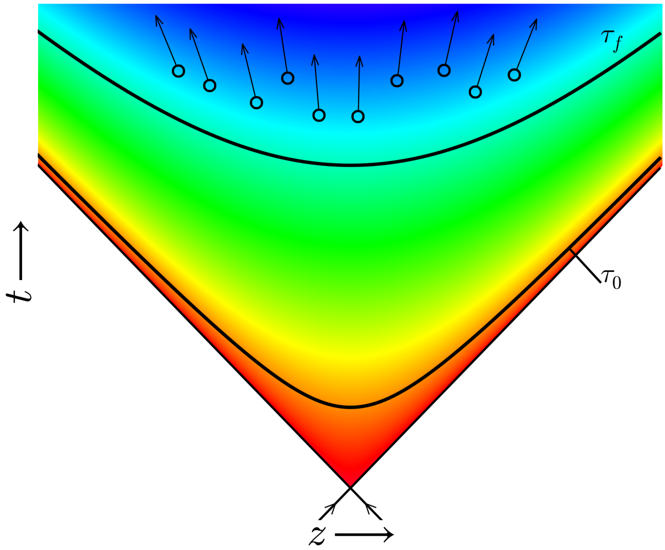
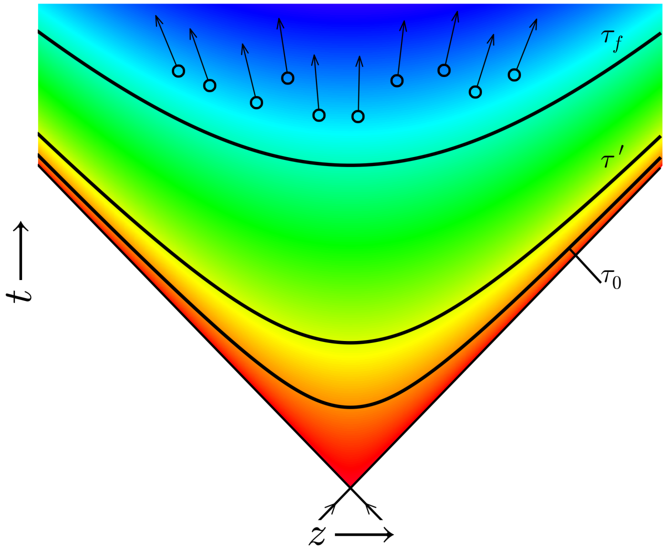
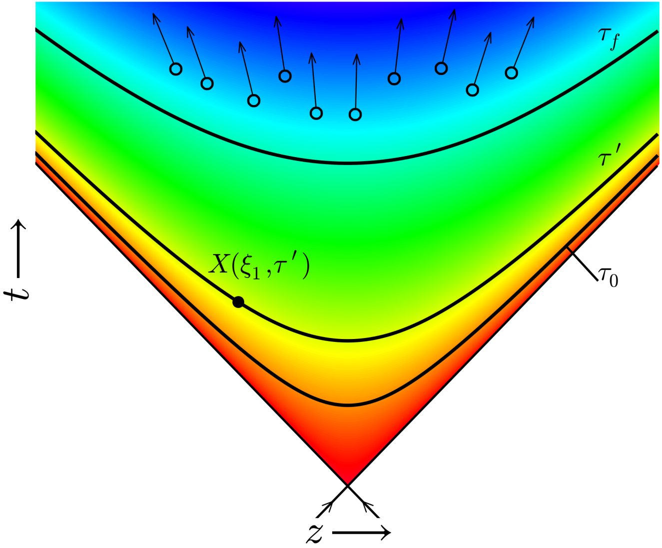


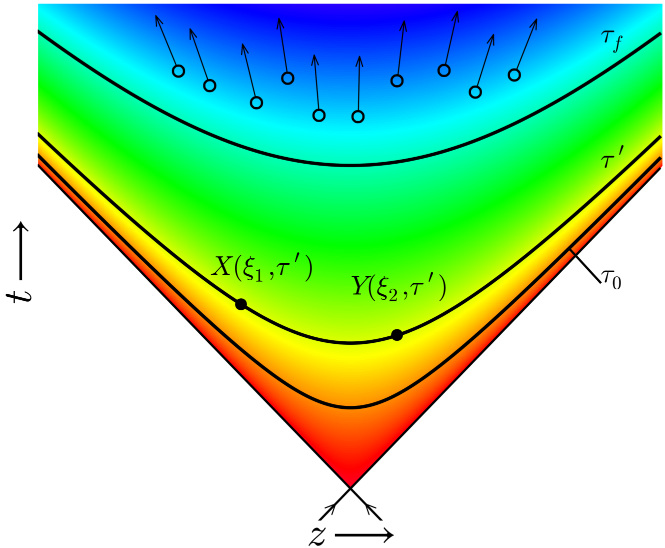
$$\langle \tilde{\mathbf{X}}(\mathbf{k_1}, \tau_\mathbf{f}) \tilde{\mathbf{Y}}(\mathbf{k_2}, \tau_\mathbf{f}) \rangle = 4\pi\delta(k_1 + k_2) \int_{\tau_0}^{\tau_f} \frac{d\tau'}{\tau'} N\left(\tau'\right) \tilde{G}_X(k_1; \tau_f, \tau') \tilde{G}_Y(k_2; \tau_f, \tau')$$



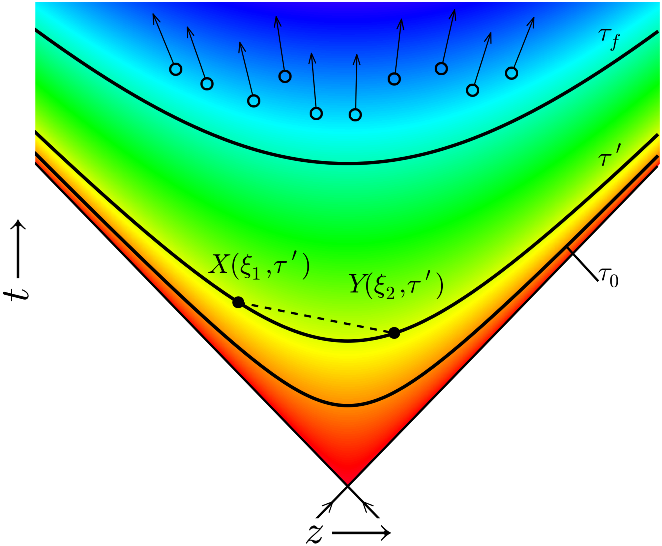
$$\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle = 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}N\left(\tau'\right)\tilde{G}_X(k_1;\tau_f,\tau')\tilde{G}_Y(k_2;\tau_f,\tau')$$



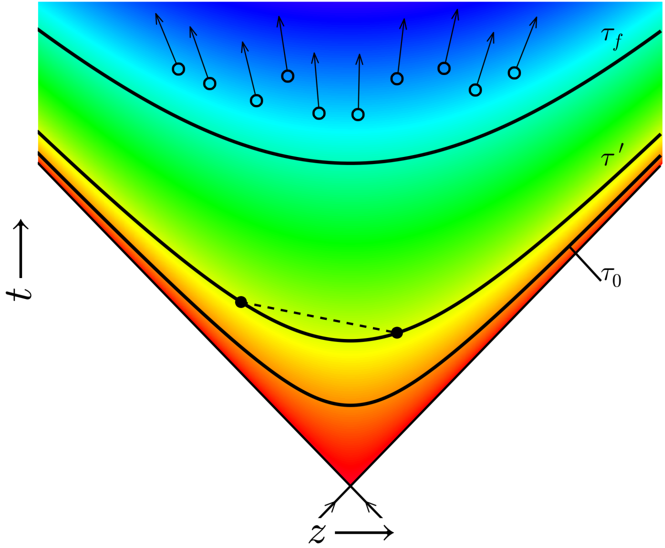
$$\left\langle \tilde{X}(k_1, \tau_f)\tilde{Y}(k_2, \tau_f)\right\rangle = 4\pi\delta(k_1 + k_2)\int_{\tau_0}^{\tau_f} \frac{d\tau'}{\tau'} N\left(\tau'\right) \mathbf{\tilde{G}_X(k_1; \tau_f, \tau')} \tilde{G}_Y(k_2; \tau_f, \tau')$$



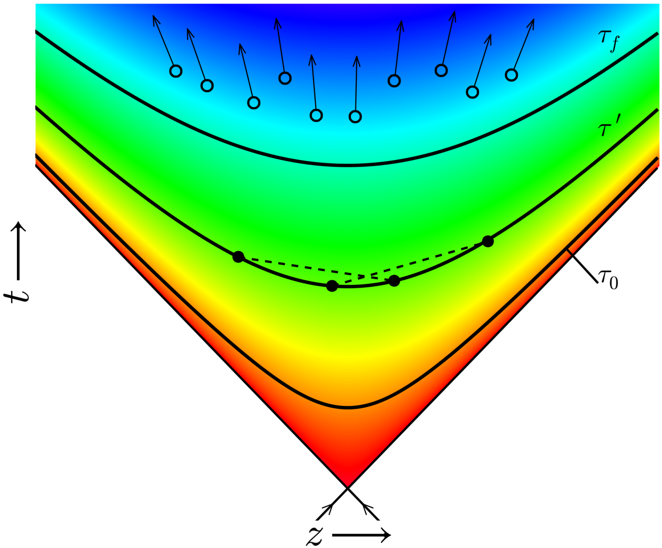
$$\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle = 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}N\left(\tau'\right)\mathbf{\tilde{G}_X(k_1;\tau_f,\tau')\tilde{G}_Y(k_2;\tau_f,\tau')}$$



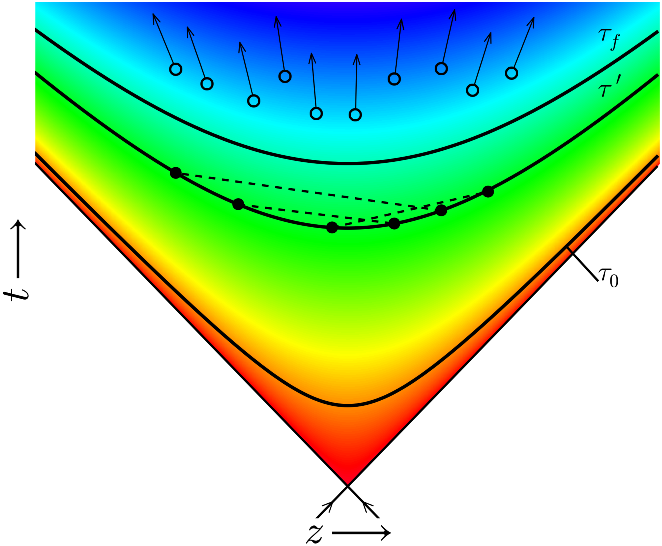
$$\left\langle \tilde{X}(k_1, \tau_f)\tilde{Y}(k_2, \tau_f)\right\rangle = 4\pi\delta(k_1 + k_2)\int_{\tau_0}^{\tau_f} \frac{d\tau'}{\tau'} \mathbf{N}\left(\tau'\right)\tilde{\mathbf{G}}_{\mathbf{X}}(\mathbf{k_1}; \tau_\mathbf{f}, \tau')\tilde{\mathbf{G}}_{\mathbf{Y}}(\mathbf{k_2}; \tau_\mathbf{f}, \tau')$$



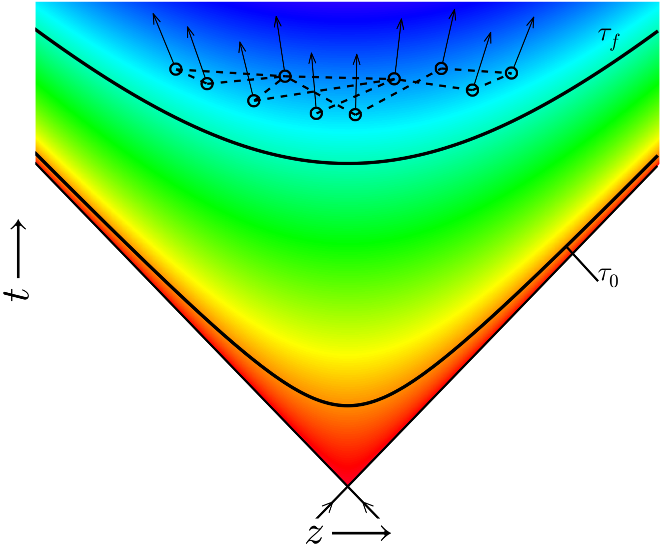
$$\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle = 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}\mathbf{N}\left(\tau'\right)\tilde{\mathbf{G}}_{\mathbf{X}}(\mathbf{k_1};\tau_{\mathbf{f}},\tau')\tilde{\mathbf{G}}_{\mathbf{Y}}(\mathbf{k_2};\tau_{\mathbf{f}},\tau')$$



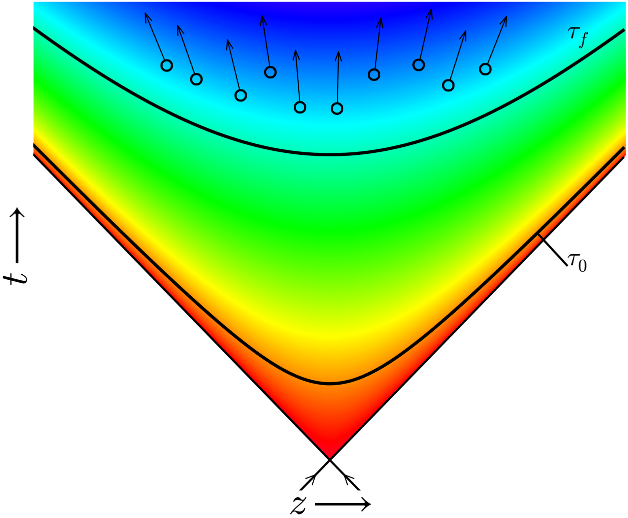
$$\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle = 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}\mathbf{N}\left(\tau'\right)\tilde{\mathbf{G}}_{\mathbf{X}}(\mathbf{k_1};\tau_{\mathbf{f}},\tau')\tilde{\mathbf{G}}_{\mathbf{Y}}(\mathbf{k_2};\tau_{\mathbf{f}},\tau')$$



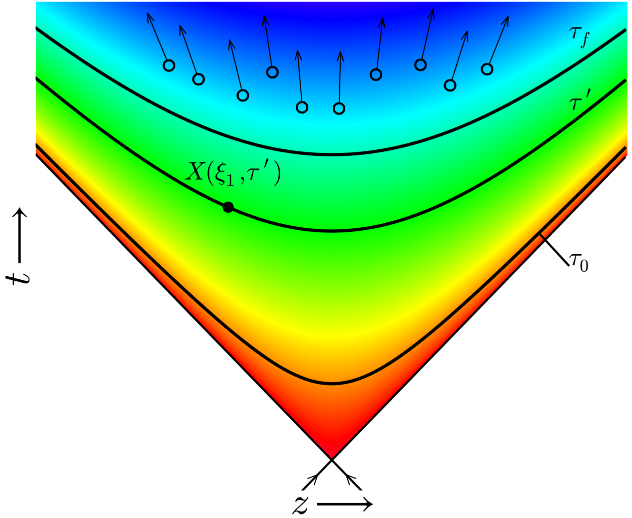
$$\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle = 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}\mathbf{N}\left(\tau'\right)\tilde{\mathbf{G}}_{\mathbf{X}}(\mathbf{k_1};\tau_f,\tau')\tilde{\mathbf{G}}_{\mathbf{Y}}(\mathbf{k_2};\tau_f,\tau')$$



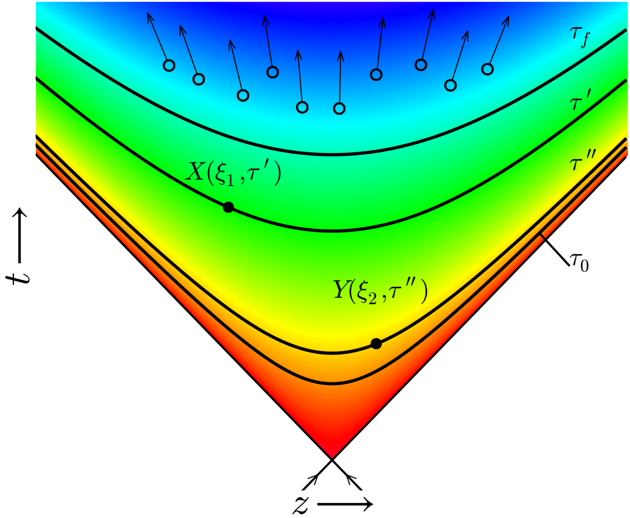
$$\left\langle \tilde{\mathbf{X}}(\mathbf{k_1},\tau_\mathbf{f})\tilde{\mathbf{Y}}(\mathbf{k_2},\tau_\mathbf{f})\right\rangle = 4\pi\delta(k_1+k_2)\int_{\tau_\mathbf{0}}^{\tau_\mathbf{f}} \frac{\mathbf{d}\tau'}{\tau'}\mathbf{N}\left(\tau'\right)\tilde{\mathbf{G}}_\mathbf{X}(\mathbf{k_1};\tau_\mathbf{f},\tau')\tilde{\mathbf{G}}_\mathbf{Y}(\mathbf{k_2};\tau_\mathbf{f},\tau')$$



$$\color{red}{\langle \tilde{\mathbf{X}}(\mathbf{k_1}, \tau_\mathbf{f}) \tilde{\mathbf{Y}}(\mathbf{k_2}, \tau_\mathbf{f}) \rangle} = 4\pi\delta(k_1 + k_2) \int_{\tau_0}^{\tau_f} \frac{d\tau'}{\tau'} \tilde{G}_X(k_1; \tau_f, \tau') \int_{\tau_0}^{\tau_f} \frac{d\tau''}{\tau''} \tilde{G}_Y(k_2; \tau_f, \tau'')$$
$$\times \left(\frac{1}{2\tau_C} \mathsf{e}^{-\frac{|\tau' - \tau''|}{\tau_C}} \right) \int_0^\infty \frac{d\tau}{\tau_C} \mathsf{e}^{-\frac{\tau}{\tau_C}} N\left(\min\left(\tau', \tau''\right) - \tau\right)$$

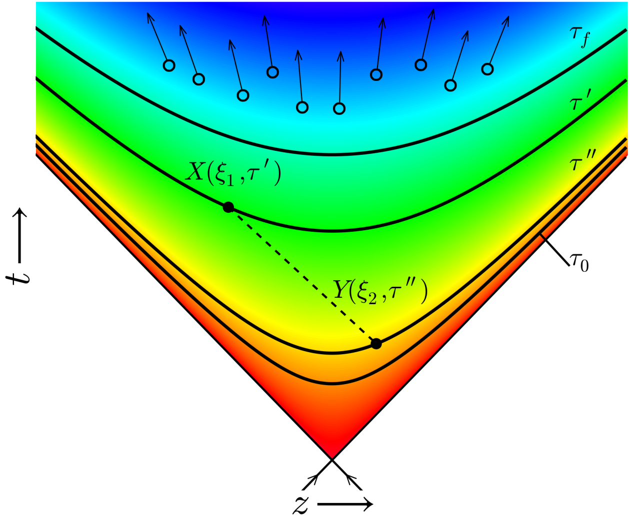


$$
\begin{aligned}
\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle \quad &= \quad 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}\,\tilde{\mathbf{G}}_{\mathbf{X}}(\mathbf{k_1};\tau_{\mathbf{f}},\tau')\int_{\tau_0}^{\tau_f}\frac{d\tau''}{\tau''}\,\tilde{G}_Y(k_2;\tau_f,\tau'') \\
&\times \quad \left(\frac{1}{2\tau_C}\mathsf{e}^{-\frac{|\tau'-\tau''|}{\tau_C}}\right)\int_0^{\infty}\frac{d\tau}{\tau_C}\mathsf{e}^{-\frac{\tau}{\tau_C}}\,N\left(\min\left(\tau',\tau''\right)-\tau\right)
\end{aligned}
$$

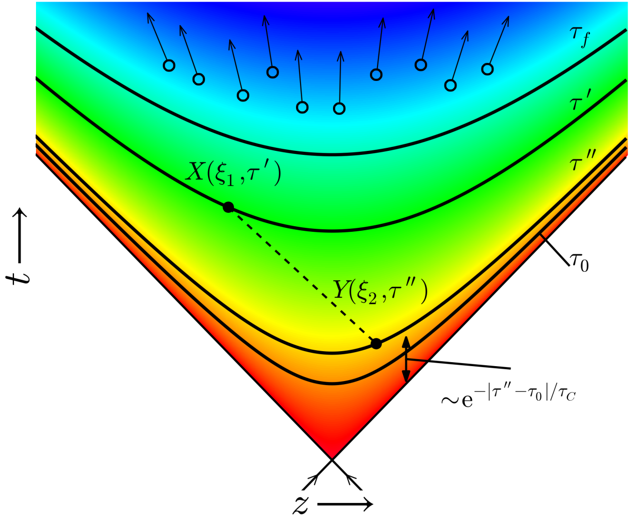


$$\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle = 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}\tilde{\mathbf{G}}_{\mathbf{X}}(\mathbf{k_1};\tau_{\mathbf{f}},\tau')\int_{\tau_0}^{\tau_f}\frac{d\tau''}{\tau''}\tilde{\mathbf{G}}_{\mathbf{Y}}(\mathbf{k_2};\tau_{\mathbf{f}},\tau'')$$

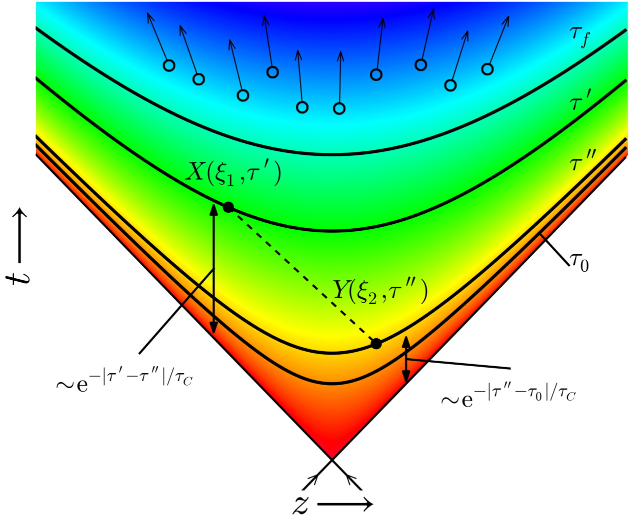
$$\times \left(\frac{1}{2\tau_C}\mathrm{e}^{-\frac{|\tau'-\tau''|}{\tau_C}}\right)\int_0^\infty \frac{d\tau}{\tau_C}\mathrm{e}^{-\frac{\tau}{\tau_C}}N\left(\min\left(\tau',\tau''\right)-\tau\right)$$



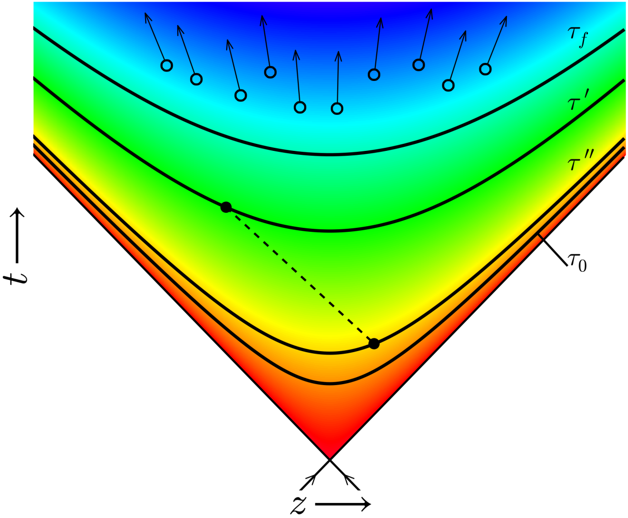
$$
\begin{aligned}
\langle \tilde{X}(k_1, \tau_f)\tilde{Y}(k_2, \tau_f)\rangle \quad &= \quad 4\pi\delta(k_1 + k_2)\int_{\tau_0}^{\tau_f} \frac{d\tau'}{\tau'} \tilde{\mathbf{G}}_{\mathbf{X}}(\mathbf{k_1}; \tau_{\mathbf{f}}, \tau') \int_{\tau_0}^{\tau_f} \frac{d\tau''}{\tau''} \tilde{\mathbf{G}}_{\mathbf{Y}}(\mathbf{k_2}; \tau_{\mathbf{f}}, \tau'') \\
&\times \quad \left(\frac{1}{2\tau_C} \mathrm{e}^{-\frac{|\tau' - \tau''|}{\tau_C}} \right) \int_0^\infty \frac{d\tau}{\tau_C} \mathrm{e}^{-\frac{\tau}{\tau_C}} N\left(\min\left(\tau', \tau''\right) - \tau\right)
\end{aligned}
$$



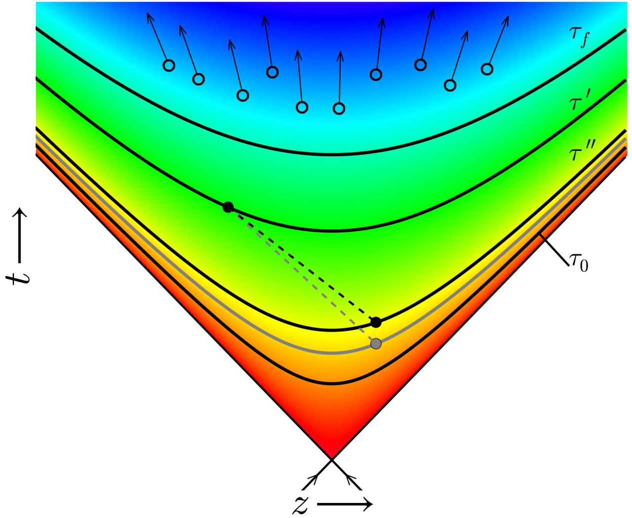
$$
\begin{aligned}
\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\rangle \quad &= \quad 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}\tilde{\mathbf{G}}_{\mathbf{X}}(\mathbf{k_1};\tau_{\mathbf{f}},\tau')\int_{\tau_0}^{\tau_f}\frac{d\tau''}{\tau''}\tilde{\mathbf{G}}_{\mathbf{Y}}(\mathbf{k_2};\tau_{\mathbf{f}},\tau'') \\
&\times \quad \left(\frac{1}{2\tau_C}\mathrm{e}^{-\frac{|\tau'-\tau''|}{\tau_C}}\right)\int_{\mathbf{0}}^{\infty}\frac{\mathbf{d}\tau}{\tau_{\mathbf{C}}}\mathrm{e}^{-\frac{\tau}{\tau_{\mathbf{C}}}}\mathbf{N}\left(\min\left(\tau',\tau''\right)-\tau\right)
\end{aligned}
$$



$$
\begin{aligned}
\left\langle \tilde{X}(k_1, \tau_f)\tilde{Y}(k_2, \tau_f)\right\rangle \quad = \quad & 4\pi\delta(k_1 + k_2)\int_{\tau_0}^{\tau_f} \frac{d\tau'}{\tau'}\,\tilde{\mathbf{G}}_{\mathbf{X}}(\mathbf{k_1}; \boldsymbol{\tau}_{\mathbf{f}}, \boldsymbol{\tau}')\int_{\tau_0}^{\tau_f} \frac{d\tau''}{\tau''}\,\tilde{\mathbf{G}}_{\mathbf{Y}}(\mathbf{k_2}; \boldsymbol{\tau}_{\mathbf{f}}, \boldsymbol{\tau}'') \\
\times \quad & \left(\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\tau}_{\mathbf{C}}}\mathsf{e}^{-\frac{|\tau' - \tau''|}{\tau_{\mathbf{C}}}}\right)\int_0^{\infty} \frac{d\tau}{\tau_C}\,\mathsf{e}^{-\frac{\tau}{\tau_C}}\, N\left(\min\left(\tau', \tau''\right) - \tau\right)
\end{aligned}
$$

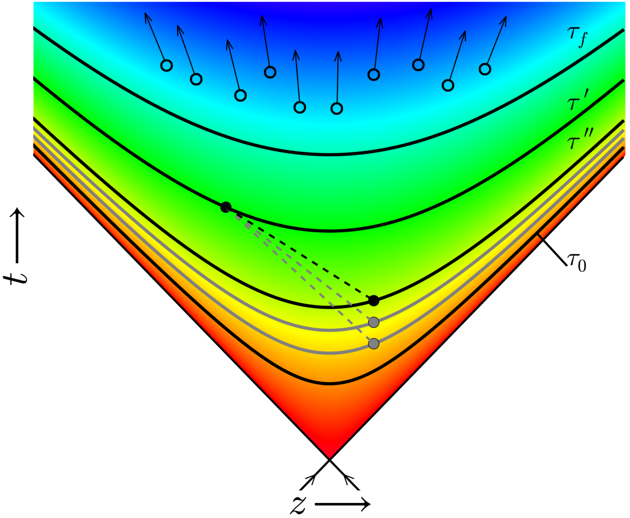


$$\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle = 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}\mathbf{\tilde{G}_X(k_1;\tau_f,\tau')}\int_{\tau_0}^{\tau_f}\frac{d\tau''}{\tau''}\mathbf{\tilde{G}_Y(k_2;\tau_f,\tau'')}$$
$$\times \left(\frac{1}{2\tau_C}\mathsf{e}^{-\frac{|\tau'-\tau''|}{\tau_C}}\right)\int_0^\infty \frac{d\tau}{\tau_C}\mathsf{e}^{-\frac{\tau}{\tau_C}}N\left(\min\left(\tau',\tau''\right)-\tau\right)$$



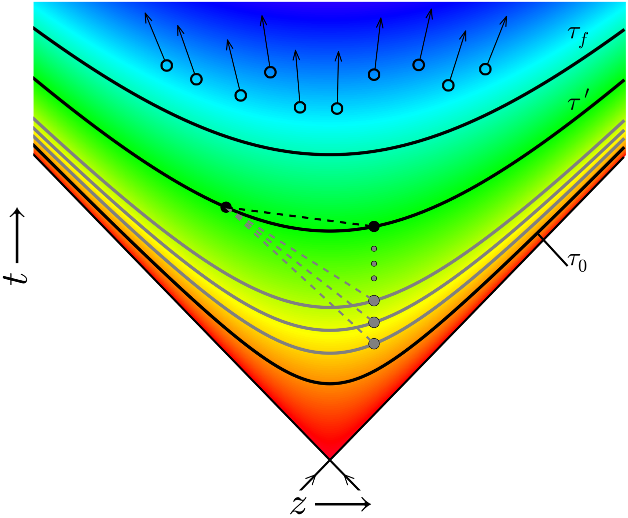
$$\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle = 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}\tilde{\mathbf{G}}_{\mathbf{X}}(\mathbf{k_1};\tau_{\mathbf{f}},\tau')\int_{\tau_{\mathbf{0}}}^{\tau_{\mathbf{f}}}\frac{\mathbf{d}\tau''}{\tau''}\tilde{\mathbf{G}}_{\mathbf{Y}}(\mathbf{k_2};\tau_{\mathbf{f}},\tau'')$$

$$\times \left(\frac{1}{2\tau_C}\mathrm{e}^{-\frac{|\tau'-\tau''|}{\tau_C}}\right)\int_0^\infty \frac{d\tau}{\tau_C}\mathrm{e}^{-\frac{\tau}{\tau_C}}N\left(\min\left(\tau',\tau''\right)-\tau\right)$$

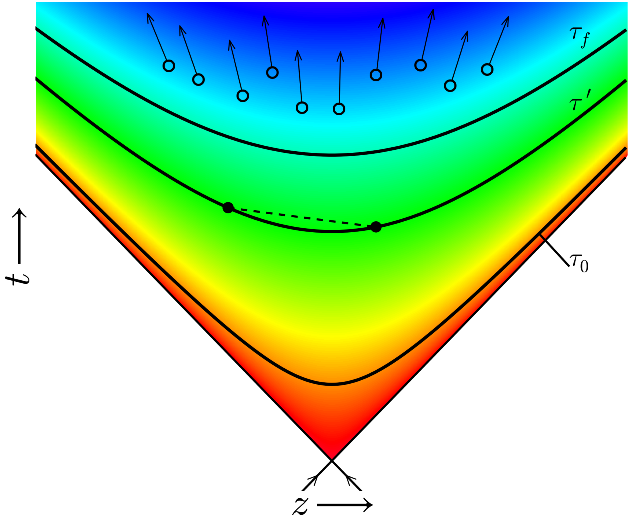


$$\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle = 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}\mathbf{\tilde{G}_X(k_1;\tau_f,\tau')}\int_{\tau_0}^{\tau_f}\frac{\mathbf{d}\tau''}{\tau''}\mathbf{\tilde{G}_Y(k_2;\tau_f,\tau'')}$$

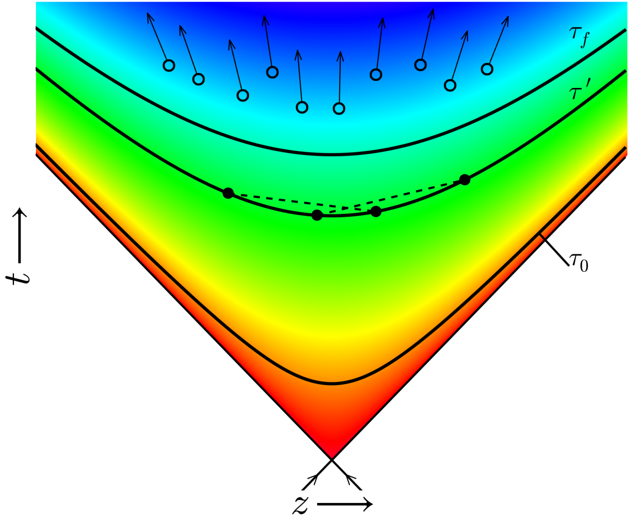
$$\times \left(\frac{1}{2\tau_C}\mathrm{e}^{-\frac{|\tau'-\tau''|}{\tau_C}}\right)\int_0^\infty \frac{d\tau}{\tau_C}\mathrm{e}^{-\frac{\tau}{\tau_C}}N\left(\min\left(\tau',\tau''\right)-\tau\right)$$



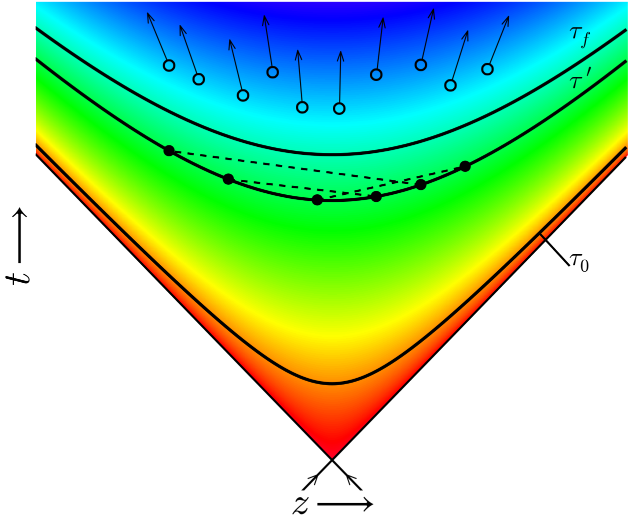
$$
\begin{aligned}
\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle \quad &= \quad 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}\,\color{red}{\mathbf{\tilde{G}_X(k_1;\tau_f,\tau')}\int_{\tau_0}^{\tau_f}\frac{\mathbf{d}\tau''}{\tau''}\,\mathbf{\tilde{G}_Y(k_2;\tau_f,\tau'')}} \\
&\times \quad \left(\frac{1}{2\tau_C}\mathsf{e}^{-\frac{|\tau'-\tau''|}{\tau_C}}\right)\int_0^\infty \frac{d\tau}{\tau_C}\mathsf{e}^{-\frac{\tau}{\tau_C}}N\left(\min\left(\tau',\tau''\right)-\tau\right)
\end{aligned}
$$



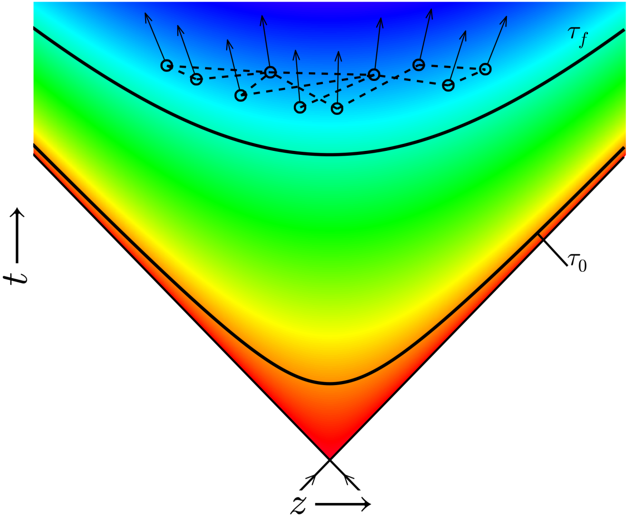
$$
\begin{aligned}
\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle \quad &= \quad 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}\tilde{\mathbf{G}}_{\mathbf{X}}(\mathbf{k_1};\tau_f,\tau')\int_{\tau_0}^{\tau_f}\frac{d\tau''}{\tau''}\tilde{\mathbf{G}}_{\mathbf{Y}}(\mathbf{k_2};\tau_f,\tau'') \\
&\times \quad \left(\frac{1}{2\tau_C}\mathrm{e}^{-\frac{|\tau'-\tau''|}{\tau_C}}\right)\int_0^\infty \frac{d\tau}{\tau_C}\mathrm{e}^{-\frac{\tau}{\tau_C}}N\left(\min\left(\tau',\tau''\right)-\tau\right)
\end{aligned}
$$



$$\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle = 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}\tilde{G}_X(k_1;\tau_f,\tau')\int_{\tau_0}^{\tau_f}\frac{d\tau''}{\tau''}\tilde{G}_Y(k_2;\tau_f,\tau'')$$
$$\times \left(\frac{1}{2\tau_C}\mathrm{e}^{-\frac{|\tau'-\tau''|}{\tau_C}}\right)\int_0^\infty \frac{d\tau}{\tau_C}\mathrm{e}^{-\frac{\tau}{\tau_C}}N\left(\min\left(\tau',\tau''\right)-\tau\right)$$



$$
\begin{aligned}
\left\langle \tilde{X}(k_1,\tau_f)\tilde{Y}(k_2,\tau_f)\right\rangle \quad &= \quad 4\pi\delta(k_1+k_2)\int_{\tau_0}^{\tau_f}\frac{d\tau'}{\tau'}\tilde{G}_X(k_1;\tau_f,\tau')\int_{\tau_0}^{\tau_f}\frac{d\tau''}{\tau''}\tilde{G}_Y(k_2;\tau_f,\tau'') \\
&\times \quad \left(\frac{1}{2\tau_C}\mathrm{e}^{-\frac{|\tau'-\tau''|}{\tau_C}}\right)\int_0^\infty \frac{d\tau}{\tau_C}\mathrm{e}^{-\frac{\tau}{\tau_C}}N\left(\min\left(\tau',\tau''\right)-\tau\right)
\end{aligned}
$$



$$
\begin{aligned}
\left\langle \tilde{\mathbf{X}}(\mathbf{k_1},\tau_\mathbf{f})\tilde{\mathbf{Y}}(\mathbf{k_2},\tau_\mathbf{f})\right\rangle \quad &= \quad 4\pi\delta(k_1+k_2)\int_{\tau_\mathbf{0}}^{\tau_\mathbf{f}}\frac{\mathbf{d}\tau'}{\tau'}\tilde{\mathbf{G}}_\mathbf{X}(\mathbf{k_1};\tau_\mathbf{f},\tau')\int_{\tau_\mathbf{0}}^{\tau_\mathbf{f}}\frac{\mathbf{d}\tau''}{\tau''}\tilde{\mathbf{G}}_\mathbf{Y}(\mathbf{k_2};\tau_\mathbf{f},\tau'') \\
&\times \quad \left(\frac{1}{2\tau_C}\mathsf{e}^{-\frac{\left|\tau'-\tau''\right|}{\tau_C}}\right)\int_0^\infty \frac{d\tau}{\tau_C}\mathsf{e}^{-\frac{\tau}{\tau_C}} N\left(\min\left(\tau',\tau''\right)-\tau\right)
\end{aligned}
$$