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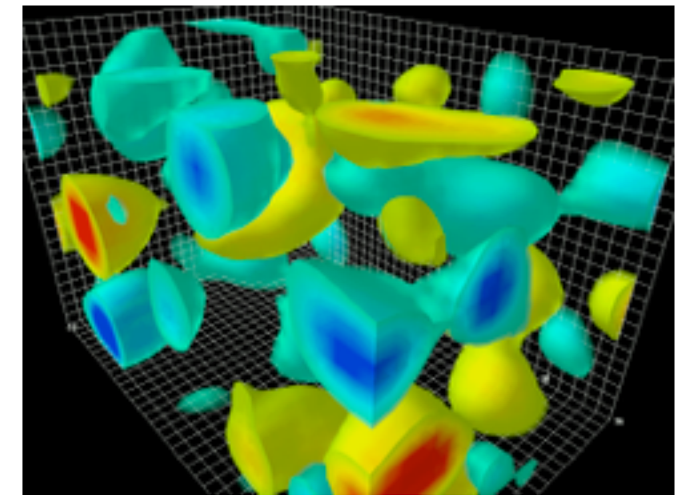

Basic Introduction to Lattice Quantum Chromodynamics

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CBM school, Wuhan, China
22 Sep 2017 to 23 September 2017

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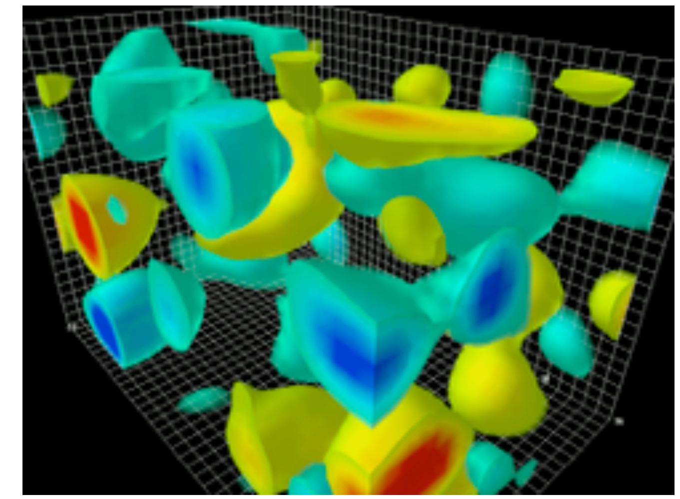

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Books & literatures

✤ "Quantum Chromodynamics on the Lattice",
C. Gattringer and C. B. Lang, Springer 2010

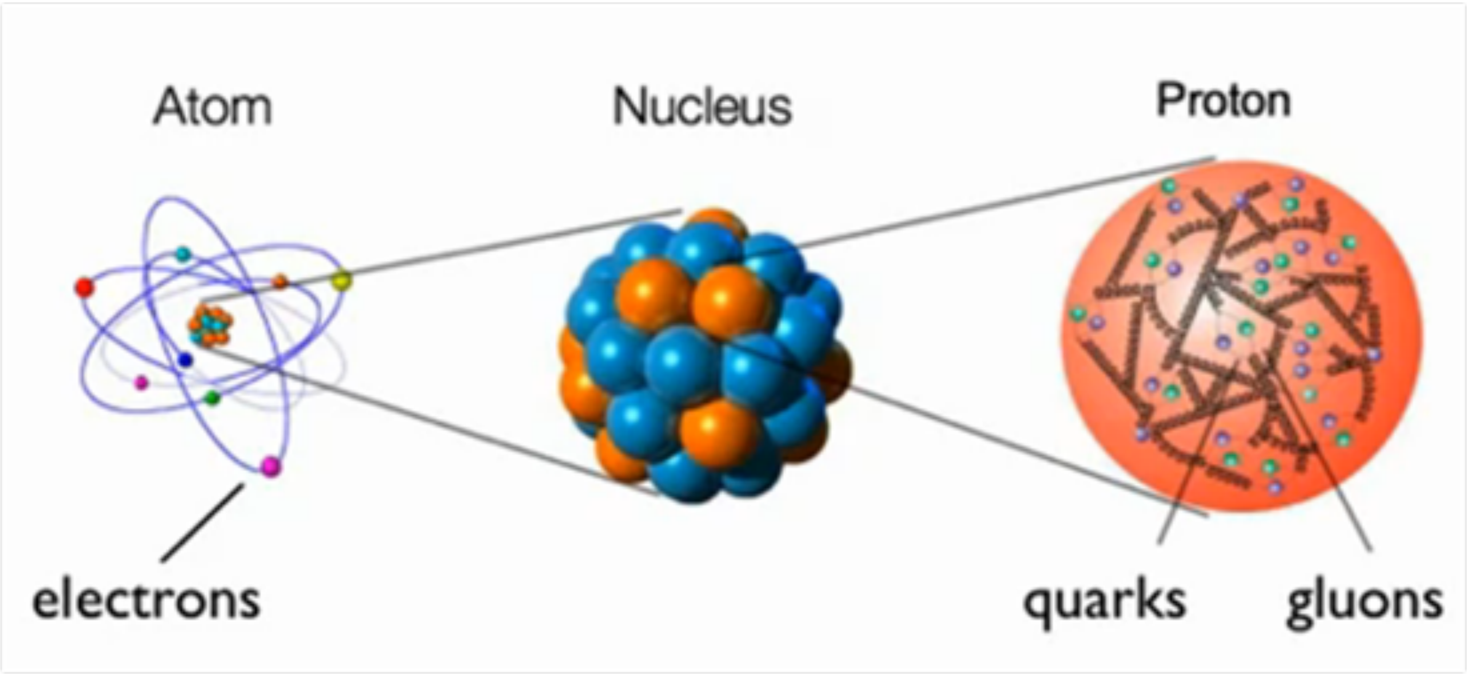
✤ "Lattice QCD for Novices",
G. Peter Lepage, arXiv:hep-lat/0506036

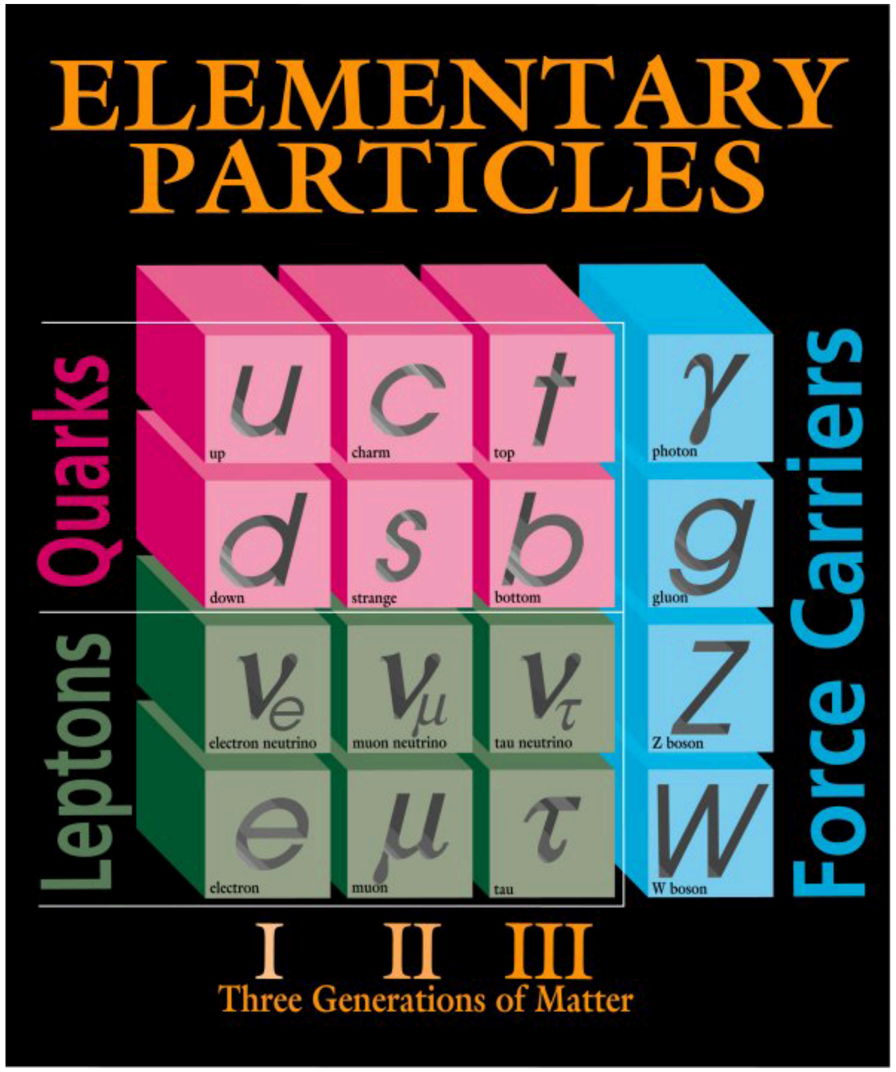
✤ "Thermodynamics of strong-interaction matter from Lattice QCD",
HTD, F. Karsch, S. Mukherjee, arXiv:1504.05274

✤ Conference proceedings in the annual "lattice conference"

- Lattice 2017, Granda, Spain
- Lattice 2018, Michigan, USA
- Lattice 2019, CCNU, Wuhan, China
- …

quarks, gluons & strong force





mass of proton ~ 938 MeV

mass of u(d) quarks ~ 3 MeV

$m=E/c^2$

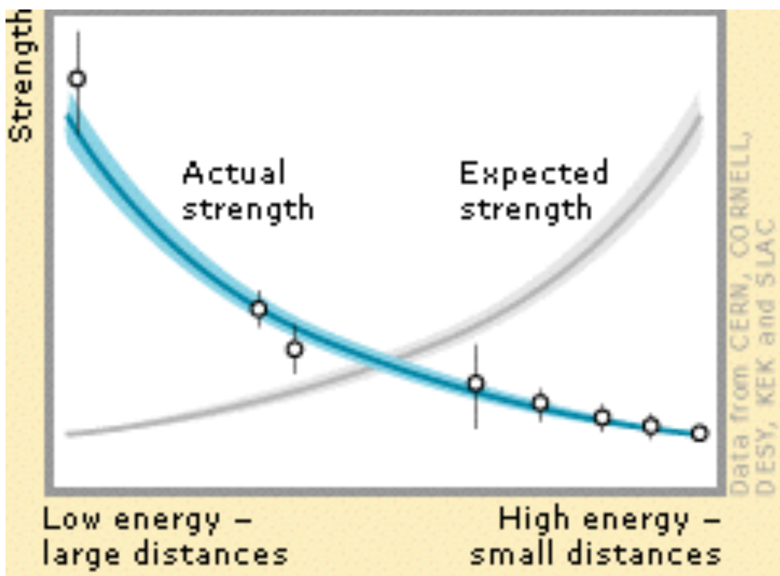
99% of the proton mass comes from the strong force

Quantum ChromoDynamics

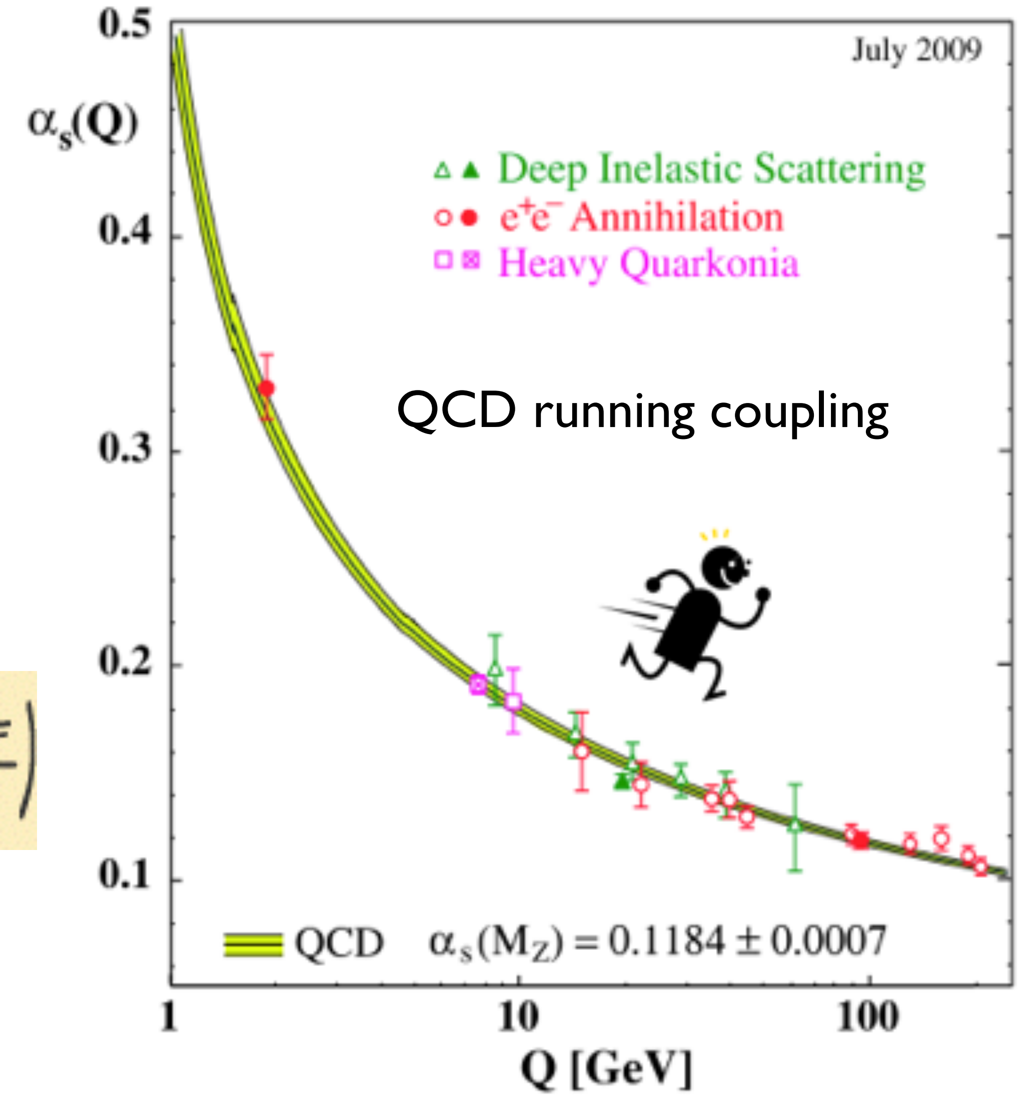
$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \bar{q}\gamma^\mu (i\partial_\mu - g t^a A^a_\mu) q - m\bar{q}q$$

two peculiar features of QCD:

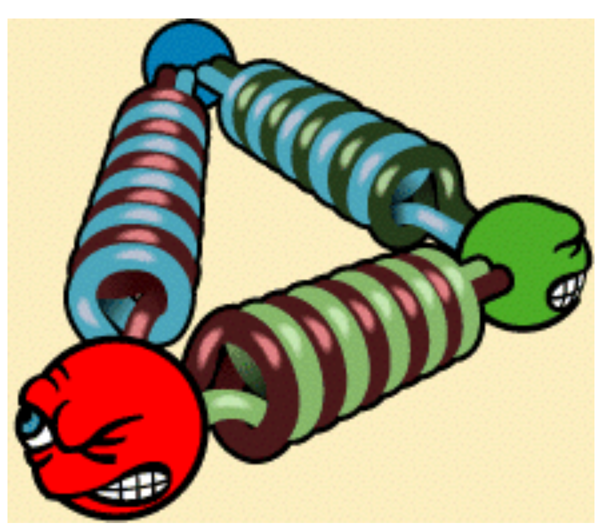
asymptotic freedom



$$\beta(g) = -\frac{g^4}{16\pi^2}\left(\frac{11}{3}N_c - \frac{4}{3}\frac{N_F}{2}\right)$$



confinement

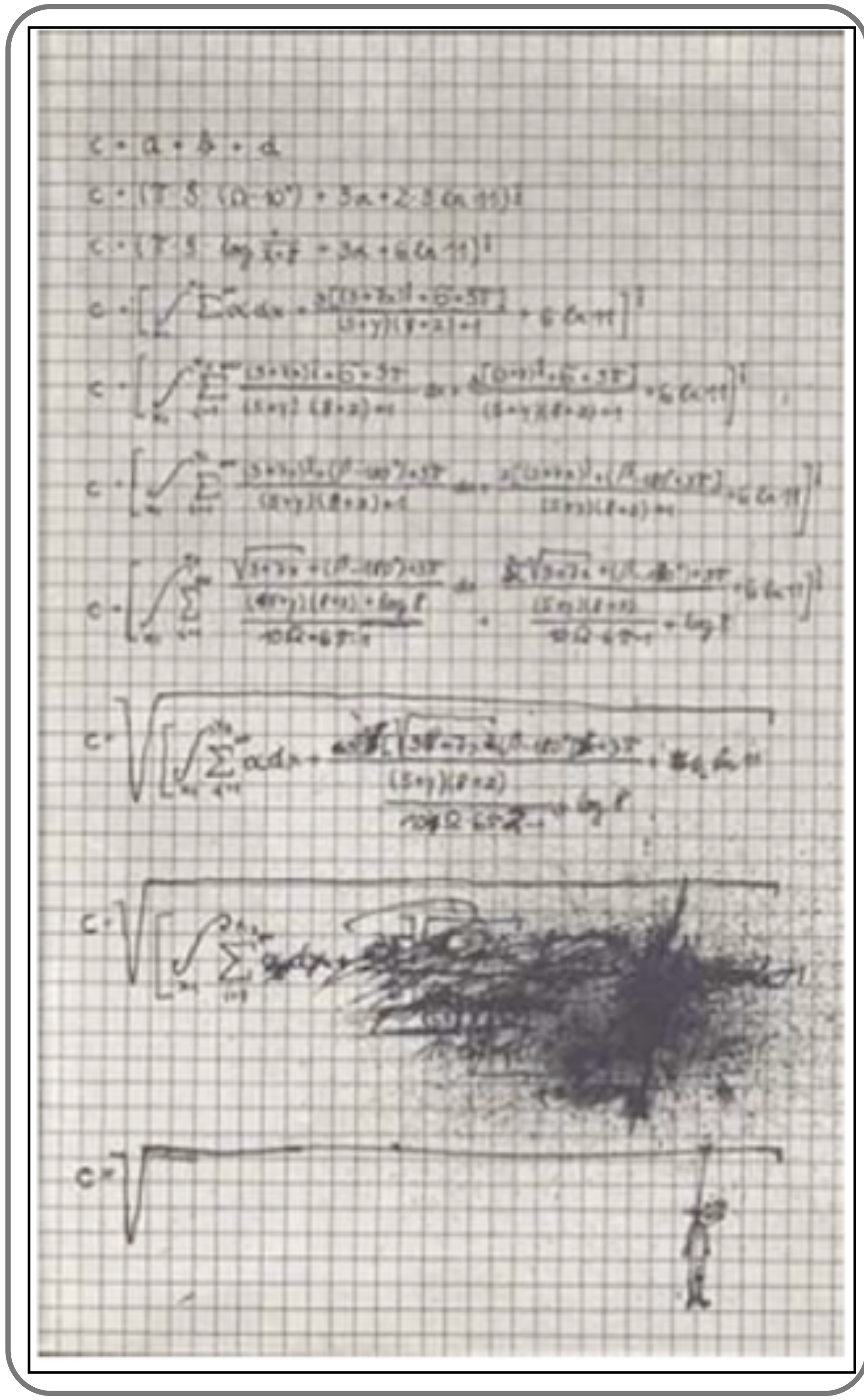


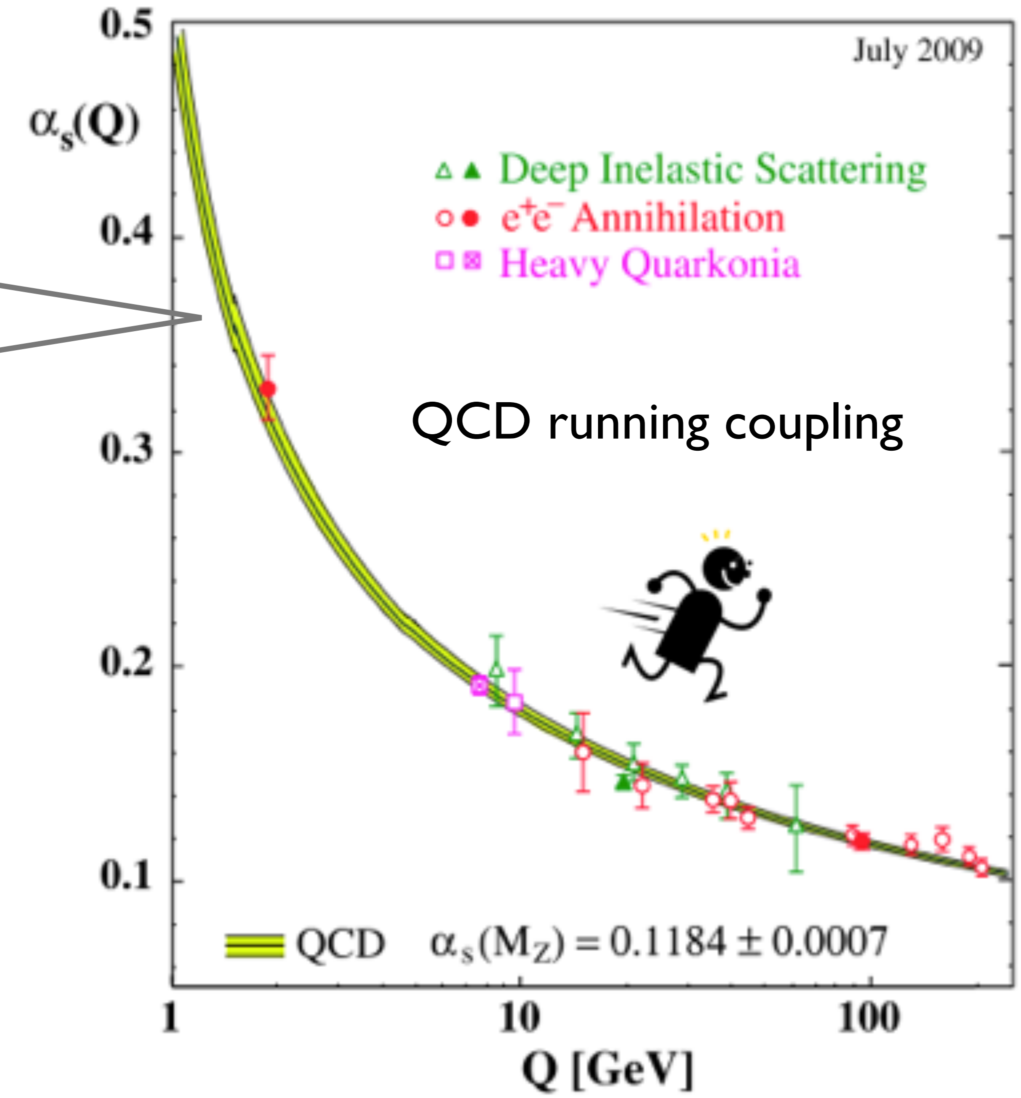
David J. Gross H. David Politzer Frank Wilczek

2004

for the discovery of asymptotic freedom in the theory of the strong interaction

Non-perturbative physics





perturbative methods not able to
describe low-energy & long distance physics

first principle calculations?

Lattice gauge theory

PHYSICAL REVIEW D VOLUME 10, NUMBER 8 15 OCTOBER 1974

Confinement of quarks*

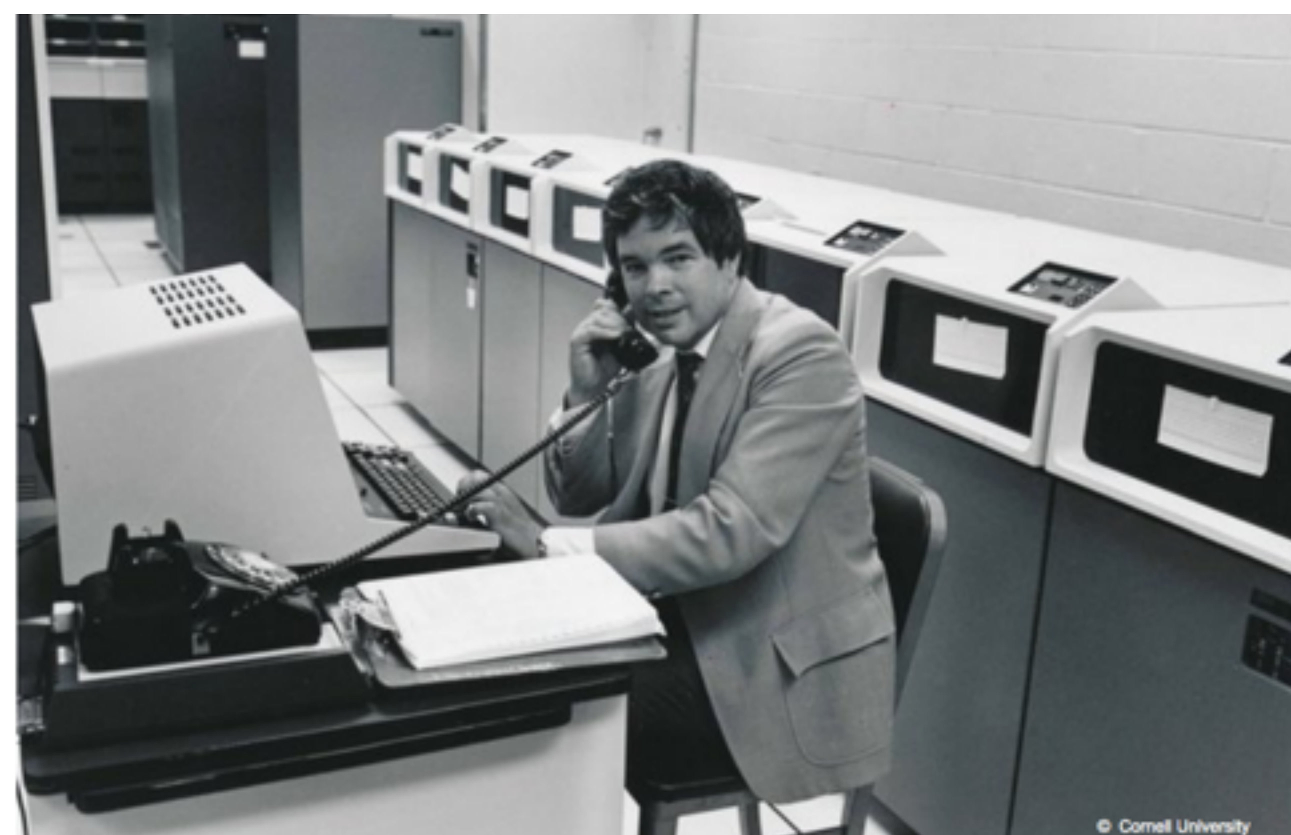
Kenneth G. Wilson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
(Received 12 June 1974)

A mechanism for total confinement of quarks, similar to that of Schwinger, is defined which requires the existence of Abelian or non-Abelian gauge fields. It is shown how to quantize a gauge field theory on a discrete lattice in Euclidean space-time, preserving exact gauge invariance and treating the gauge fields as angular variables (which makes a gauge-fixing term unnecessary). The lattice gauge theory has a computable strong-coupling limit; in this limit the binding mechanism applies and there are no free quarks. There is unfortunately no Lorentz (or Euclidean) invariance in the strong-coupling limit. The strong-coupling expansion involves sums over all quark paths and sums over all surfaces (on the lattice) joining quark paths. This structure is reminiscent of relativistic string models of hadrons.

Kenneth G. Wilson
June 8, 1936 - June 15, 2013

for his theory for critical phenomena in connection with phase transitions

1982



first numerical lattice QCD study

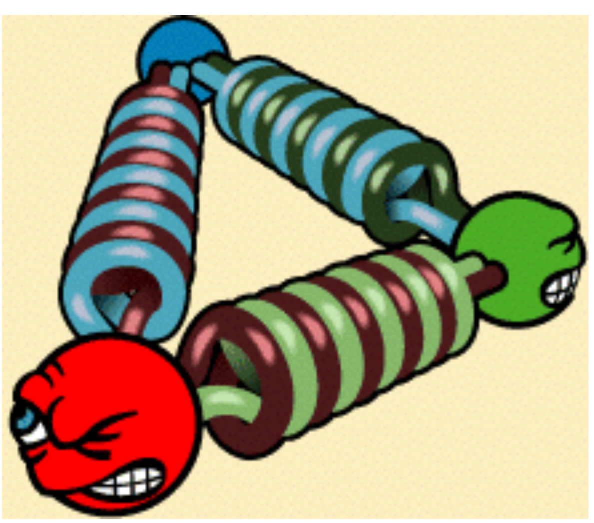
PHYSICAL REVIEW D VOLUME 21, NUMBER 8 15 APRIL 1980

Monte Carlo study of quantized SU(2) gauge theory

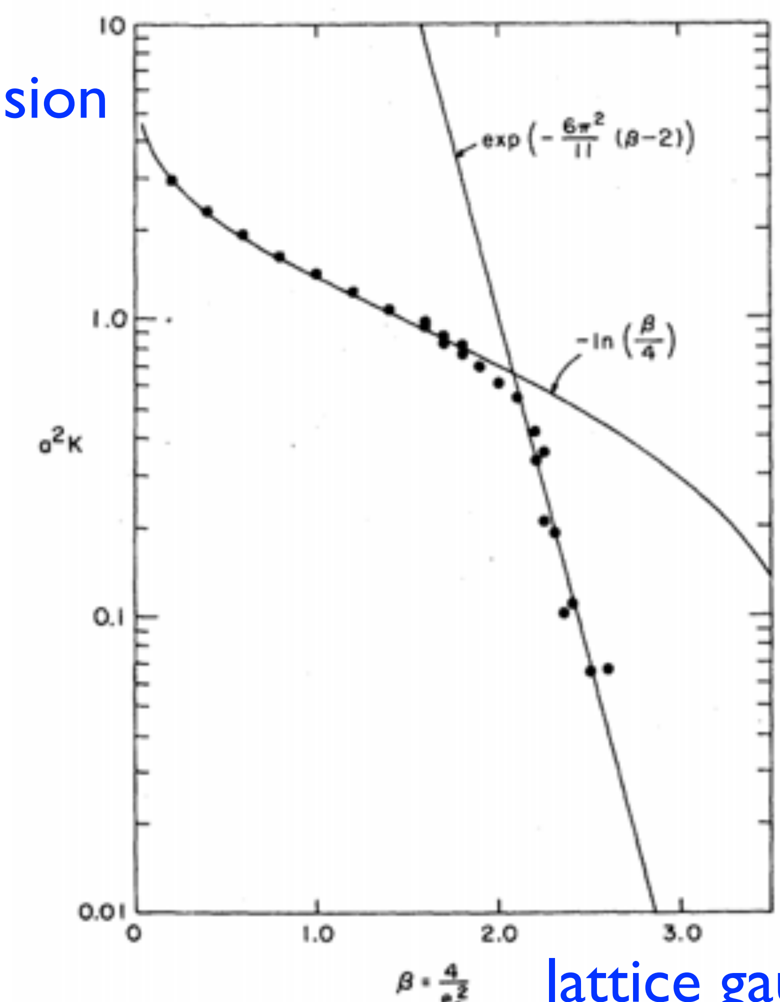
Michael Creutz
Department of Physics, Brookhaven National Laboratory, Upton, New York 11973

(Received 24 October 1979)

Using Monte Carlo techniques, we evaluate path integrals for pure SU(2) gauge fields. Wilson's regularization procedure on a lattice of up to 10^4 sites controls ultraviolet divergences. Our renormalization prescription, based on confinement, is to hold fixed the string tension, the coefficient of the asymptotic linear potential between sources in the fundamental representation of the gauge group. Upon reducing the cutoff, we observe a logarithmic decrease of the bare coupling constant in a manner consistent with the perturbative renormalization-group prediction. This supports the coexistence of confinement and asymptotic freedom for quantized non-Abelian gauge fields.

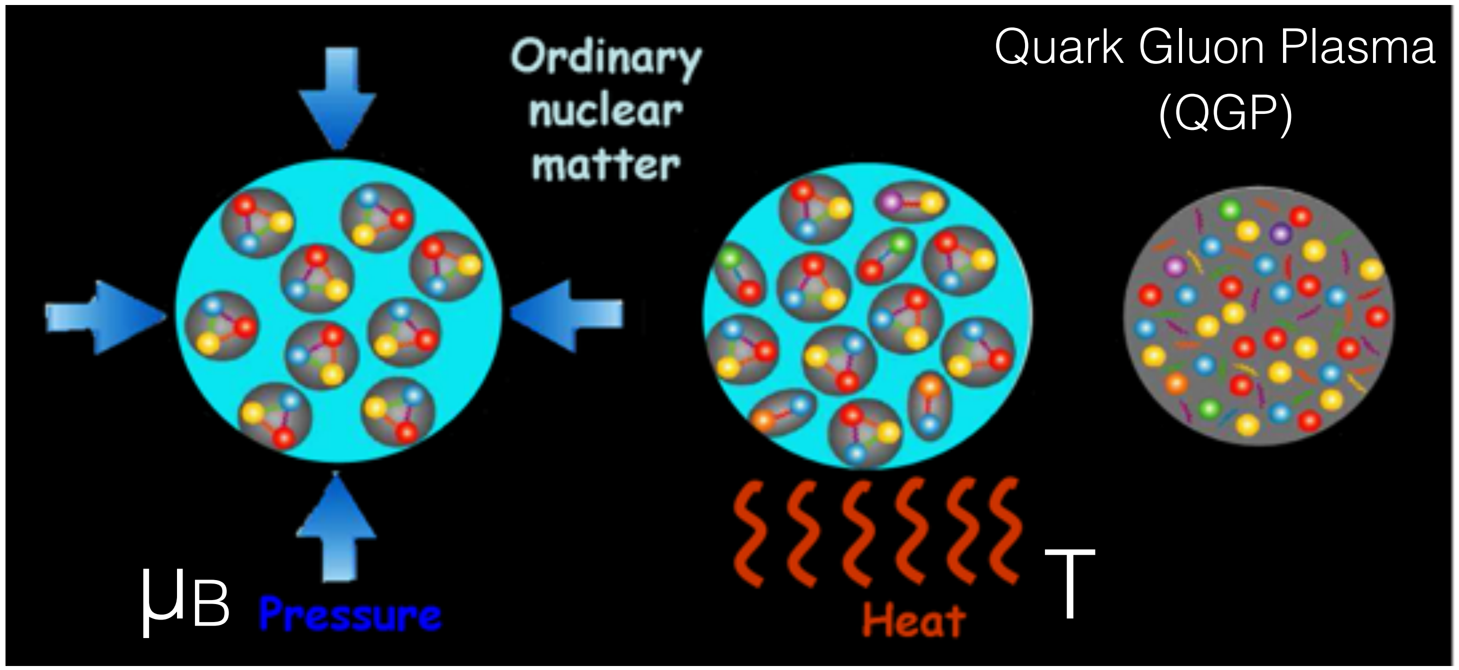


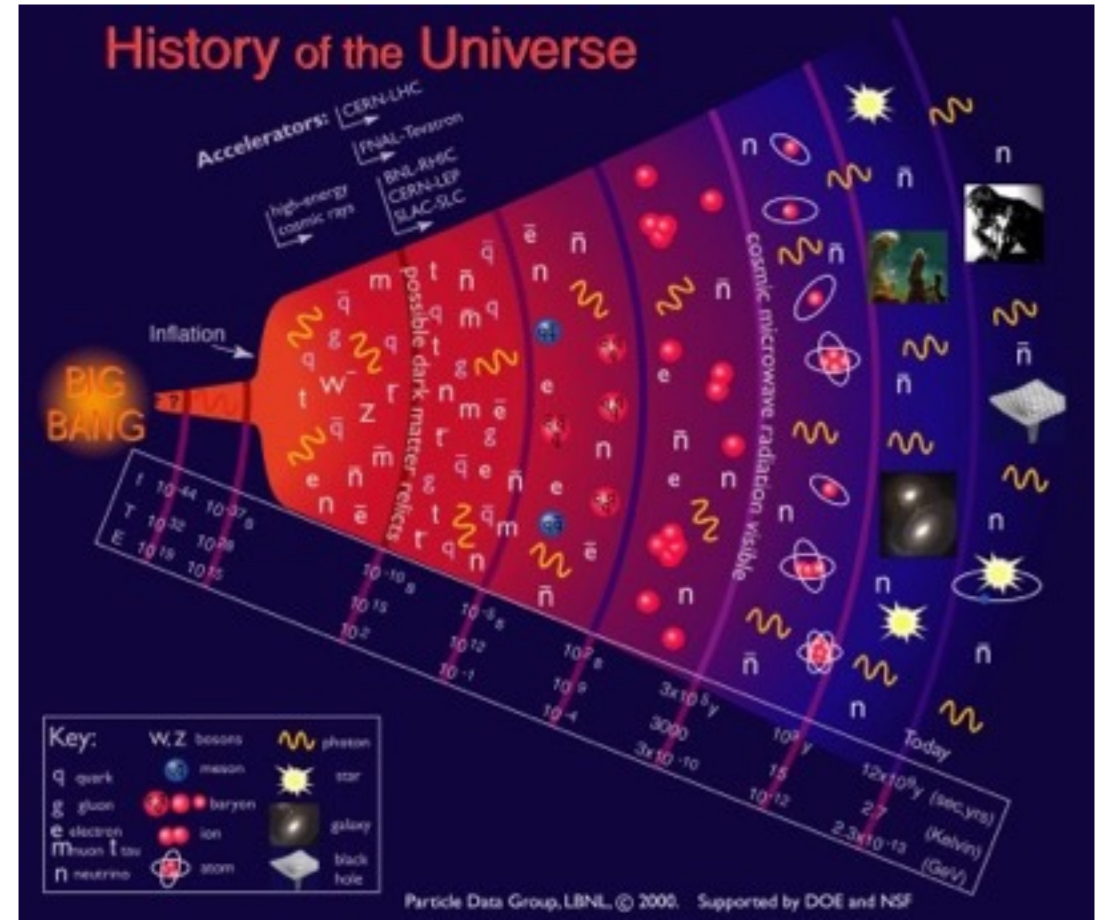
string tension



lattice gauge coupling

Symmetry restoration in extreme conditions: QCD phase transitions





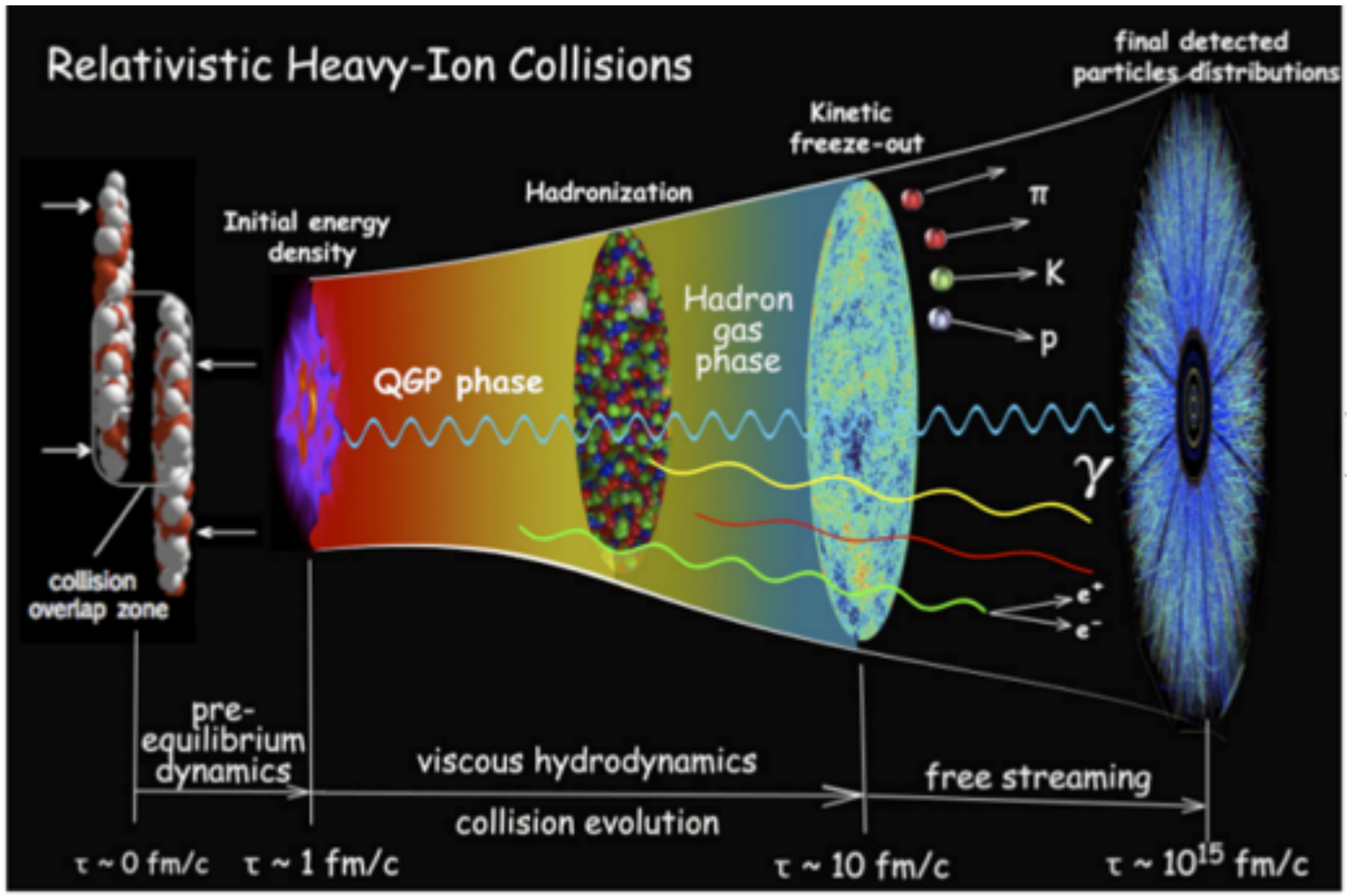
"The whole is more than sum of its parts."
Aristotle, Metaphysica 10f-1045a

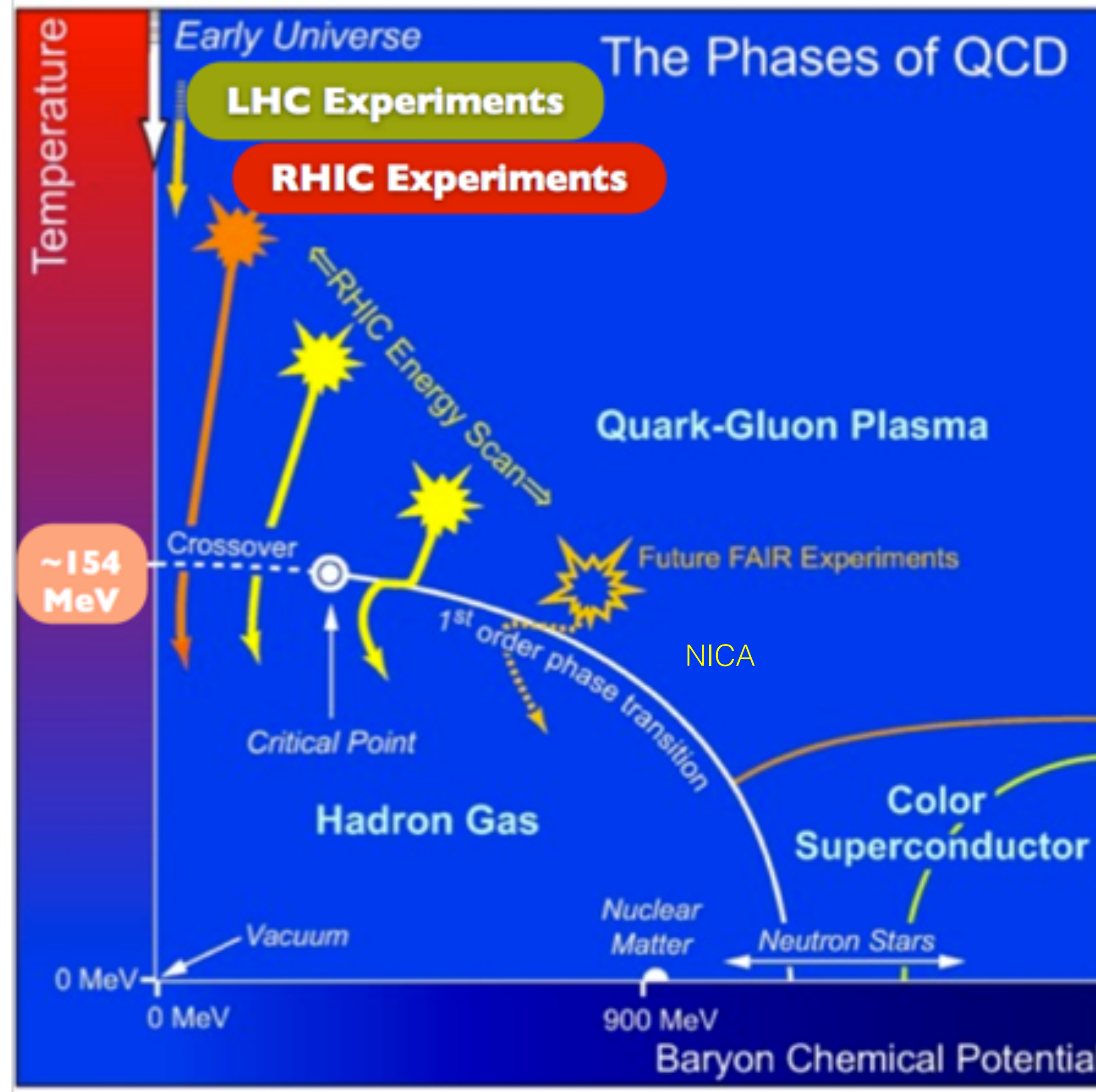
What are the phases of strong-interaction matter and what roles do they play in cosmos?

What are the T_c, orders and universality classes of (chiral & deconfinement) phase transitions?

What does QCD predict for the properties of the strong-interaction matter in extreme conditions?

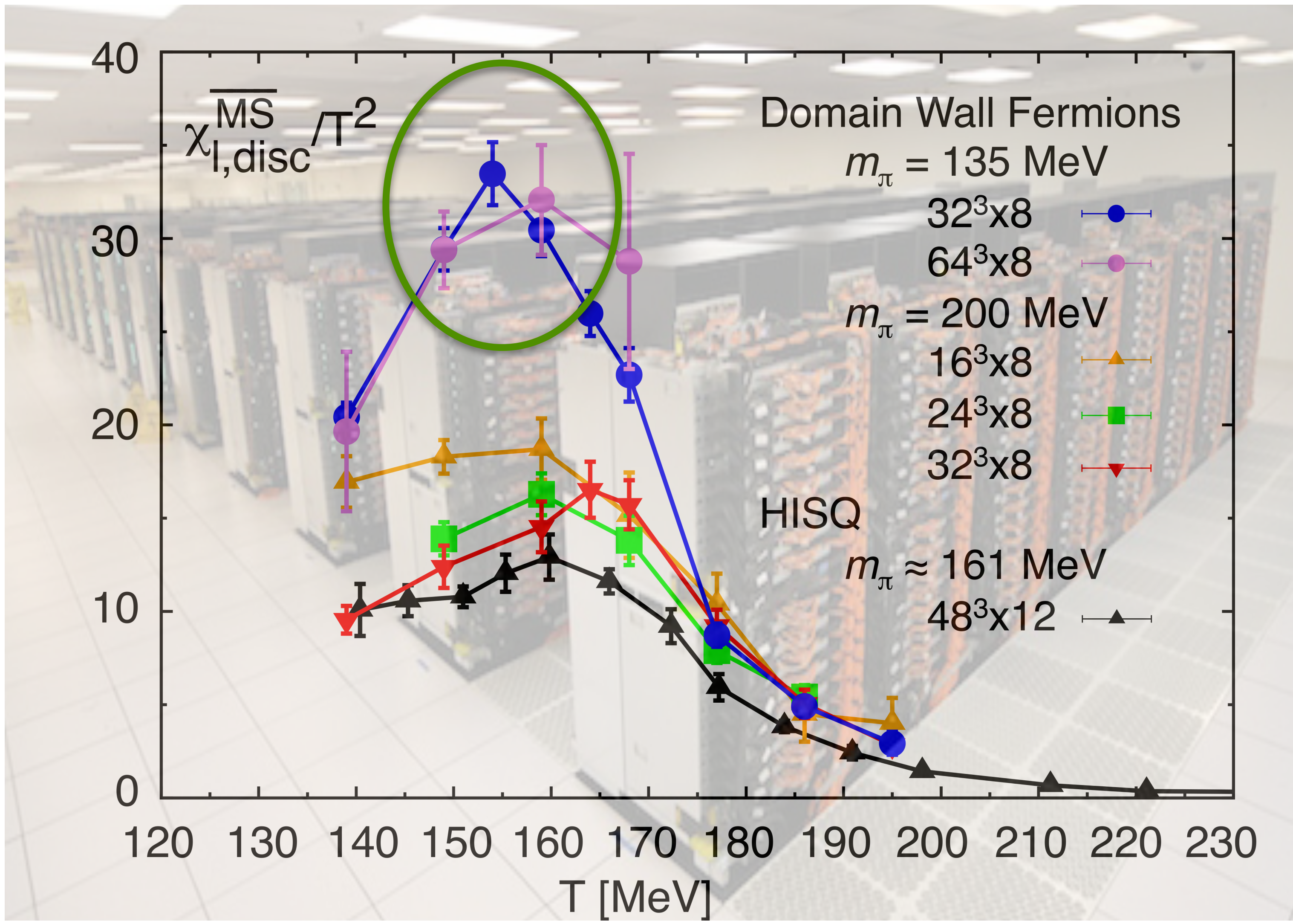
Recreate QGP in Heavy Ion Collisions (HIC)...





Transition from hadronic phase to QGP phase at $\mu_B = 0$

disconnected chiral susceptibility



HotQCD: PRL 113 (2014) 082001

- Consistent results with 3 discretization schemes with m_π=135 MeV:
  Domain wall, HISQ, stout

  T_{pc} =155(1)(8) MeV

- Not a true (chiral or deconfinement) phase transition but a rapid chiral crossover

See also the consistent **continuum extrapolated results** of HISQ, stout, and overlap in:
Wuppertal-Budapest: Nature 443(2006)675, JHEP 1009 (2010) 073 , HotQCD: PRD 85 (2012)054503

Borsanyi et al., [WB collaboration], arXiv: 1510.03376, Phys.Lett. B713 (2012) 342

Two key equations

- Useful to extract matrix elements of operators and the energy spectrum of the theory

$$\lim_{T\to\infty} \frac{1}{Z_T} \mathrm{Tr}[e^{-(T-t)\hat{H}}\, \hat{O}_2\, e^{-t\hat{H}}\, \hat{O}_1] = \sum_n \langle 0|\hat{O}_2|n\rangle\langle n|\hat{O}_1|0\rangle\, e^{-tE_n}$$

partition function: $Z_T = \mathrm{Tr}[e^{-T\hat{H}}]$

- Path integral formalism used to be evaluated numerically on the lattice

$$\frac{1}{Z_T} \mathrm{Tr}[e^{-(T-t)\hat{H}}\, \hat{O}_2\, e^{-t\hat{H}}\, \hat{O}_1] = \frac{1}{Z_T} \int \mathcal{D}[\Phi] e^{-S_E[\Phi]}\, O_2[\Phi(.,t)]\, O_1[\Phi(.,0)]$$

Path integral for a scalar field theory

Lagrangian $$L(\Phi, \partial_\mu \Phi) = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{m^2}{2}\Phi^2 - V(\Phi)$$

Hamiltonian operator $$\widehat{H} = \int \mathrm{d}^3x \left(\underbrace{\frac{1}{2}\widehat{\Pi}(\boldsymbol{x})^2}_{H_0} + \underbrace{\frac{1}{2}\left(\nabla\widehat{\Phi}(\boldsymbol{x})\right)^2 + \frac{m^2}{2}\widehat{\Phi}(\boldsymbol{x})^2 + V\left(\widehat{\Phi}(\boldsymbol{x})\right)}_{U} \right)$$

Discretization in time:

$$Z_T = \int \mathcal{D}\Phi_0 \, \langle \Phi_0 | e^{-T\widehat{H}} | \Phi_0 \rangle = \lim_{N_T \to \infty} \int \mathcal{D}\Phi_0 \, \langle \Phi_0 | \widehat{W}_\varepsilon{}^{N_T} | \Phi_0 \rangle$$

$$\hat{W}_\epsilon = e^{-\epsilon \hat{U}/2} e^{-\epsilon \hat{H}_0} e^{-\epsilon \hat{U}/2} \ , \qquad T = N_T \, \epsilon$$

ε: spacing in the temporal direction

Discretization in 3D-space: $\boldsymbol{x} \;\Rightarrow\; a\,\boldsymbol{n}\,, \quad n_i = 0, 1, \ldots, N-1 \quad \text{for} \quad i = 1, 2, 3$

$$\partial_j \widehat{\Phi}(\boldsymbol{x}) = \frac{\widehat{\Phi}(\boldsymbol{n}+\hat{j}) - \widehat{\Phi}(\boldsymbol{n}-\hat{j})}{2a} + \mathcal{O}(a^2)$$

a: spacing in the spatial direction

$$\widehat{H}_0 = a^3 \sum_{\boldsymbol{n}\in\Lambda_3} \frac{1}{2}\left(-\frac{\mathrm{i}}{a^3}\frac{\partial}{\partial\Phi(\boldsymbol{n})}\right)^2 = -\frac{1}{2\,a^3}\sum_{\boldsymbol{n}\in\Lambda_3}\frac{\partial^2}{\partial\Phi(\boldsymbol{n})^2}$$

$$\widehat{U} = a^3 \sum_{\boldsymbol{n}\in\Lambda_3}\left(\frac{1}{2}\sum_{j=1}^{3}\left(\frac{\widehat{\Phi}(\boldsymbol{n}+\hat{j}) - \widehat{\Phi}(\boldsymbol{n}-\hat{j})}{2a}\right)^2 + \frac{m^2}{2}\widehat{\Phi}(\boldsymbol{n})^2 + V\left(\widehat{\Phi}(\boldsymbol{n})\right)\right)$$

$$Z_T = \int \mathcal{D}\Phi_0 \, \langle \Phi_0 | e^{-T\hat{H}} | \Phi_0 \rangle = \lim_{N_T \to \infty} \int \mathcal{D}\Phi_0 \, \langle \Phi_0 | \widehat{W}_\varepsilon{}^{N_T} | \Phi_0 \rangle$$

$$= \int \mathcal{D}\Phi_0 \ldots \mathcal{D}\Phi_{N_T-1} \, \langle \Phi_0 | \widehat{W}_\varepsilon | \Phi_{N_T-1} \rangle \langle \Phi_{N_T-1} | \widehat{W}_\varepsilon | \Phi_{N_T-2} \rangle \ldots \langle \Phi_1 | \widehat{W}_\varepsilon | \Phi_0 \rangle$$

$$= C^{N^3 N_T} \int \mathcal{D}\Phi_0 \ldots \mathcal{D}\Phi_{N_T-1} \, \mathrm{e}^{-S_E[\Phi]}$$

periodic boundary condition is used

$$S_E[\Phi] = \frac{1}{2} \sum_{j=0}^{N_T-1} a^3 \sum_{\boldsymbol{n} \in \Lambda_3} \frac{1}{\varepsilon} \left(\Phi(\boldsymbol{n})_{j+1} - \Phi(\boldsymbol{n})_j \right)^2 + \varepsilon \sum_{j=0}^{N_T-1} U[\Phi_j]$$

In a compact 4D-space

$$S_E[\Phi] = \varepsilon a^3 \sum_{(\boldsymbol{n}, n_4) \in \Lambda} \left(\frac{1}{2} \left(\frac{\Phi(\boldsymbol{n}, n_4+1) - \Phi(\boldsymbol{n}, n_4)}{\varepsilon} \right)^2 + \right.$$

$$\left. \frac{1}{2} \sum_{j=1}^{3} \left(\frac{\Phi(\boldsymbol{n}+\hat{j}, n_4) - \Phi(\boldsymbol{n}-\hat{j}, n_4)}{2a} \right)^2 + \frac{m^2}{2} \Phi(\boldsymbol{n}, n_4)^2 + V\left(\Phi(\boldsymbol{n}, n_4)\right) \right)$$

Partition function & correlators

Partition function:

$$Z_T^\varepsilon = C^{N^3 N_T} \int \mathcal{D}[\Phi]\, \mathrm{e}^{-S_E[\Phi]} \quad , \quad \mathcal{D}[\Phi] = \prod_{(\boldsymbol{n},n_4)\in\Lambda} \mathrm{d}\,\Phi(\boldsymbol{n}, n_4)$$

Correlation function:

$$\langle O_2(t)\, O_1(0)\rangle_T^\varepsilon = \frac{C^{N^3 N_T}}{Z_T^\varepsilon} \int \mathcal{D}[\Phi]\, \mathrm{e}^{-S_E[\Phi]}\, O_2[\Phi(.\,, n_t)]\, O_1[\Phi(.\,, 0)]$$

- The field theory can be defined by integrals over all possible configurations of fields weighted by the Euclidean action

- The lattice procedure we went through provides a way to regulate the formally infinite functional integrals

- The a->0 limit provides a definite of the theory beyond perturbation theory

Brief review of QCD

$$S_F[\psi,\overline{\psi},A] = \sum_{f=1}^{N_f} \int \mathrm{d}^4x\, \overline{\psi}^{(f)}(x) \left(\gamma_\mu \left(\partial_\mu + \mathrm{i}\, A_\mu(x)\right) + m^{(f)}\right) \psi^{(f)}(x)$$

$$= \sum_{f=1}^{N_f} \int \mathrm{d}^4x\, \overline{\psi}^{(f)}(x)_{\substack{\alpha\\c}} \left((\gamma_\mu)_{\alpha\beta} \left(\delta_{cd}\partial_\mu + \mathrm{i}\, A_\mu(x)_{cd}\right) + m^{(f)} \delta_{\alpha\beta}\delta_{cd} \right) \psi^{(f)}(x)_{\substack{\beta\\d}}$$

$\boldsymbol{\alpha}$, β: Dirac index, 1,2,3,4   μ: Lorentz index, 1,2,3,4   c,d: color index, 1,2,3

$$S_G[A] = \frac{1}{2\,g^2} \int \mathrm{d}^4x\; \mathrm{tr}\left[F_{\mu\nu}(x) F_{\mu\nu}(x)\right] = \frac{1}{4\,g^2} \sum_{i=1}^{8} \int \mathrm{d}^4x\, F^{(i)}_{\mu\nu}(x) F^{(i)}_{\mu\nu}(x)$$

$$F^{(i)}_{\mu\nu}(x) = \partial_\mu A^{(i)}_\nu(x) - \partial_\nu A^{(i)}_\mu(x) - f_{ijk} A^{(j)}_\mu(x) A^{(k)}_\nu(x)$$

Invariant under gauge transformations:

$$\psi(x) \to \psi'(x) = \Omega(x)\psi(x), \quad \overline{\psi}(x) \to \overline{\psi}'(x) = \overline{\psi}(x)\Omega(x)^\dagger$$

$$A_\mu(x) \to A'_\mu(x) = \Omega(x) A_\mu(x) \Omega(x)^\dagger + \mathrm{i}\left(\partial_\mu \Omega(x)\right)\Omega(x)^\dagger$$

SU(3) matrix:

$$\Omega(x)^\dagger = \Omega(x)^{-1}$$

$$\det \Omega(x) = 1$$

Discretization of the fermion action

- Free fermion action (A =0):

$$S_F^0[\psi,\overline{\psi}] = \int \mathrm{d}^4x\, \overline{\psi}(x)\left(\gamma_\mu\partial_\mu + m\right)\psi(x) \qquad \boxed{\partial_\mu\psi(x) \;\to\; \frac{1}{2a}\left(\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})\right)}$$

$$S_F^0[\psi,\overline{\psi}] = a^4 \sum_{n\in\Lambda} \overline{\psi}(n)\left(\sum_{\mu=1}^{4}\gamma_\mu \frac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a} + m\,\psi(n)\right)$$

- Not gauge invariant:

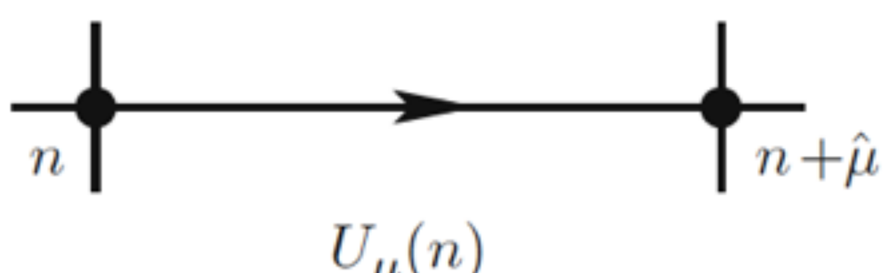
$$\psi(x) \to \psi'(x) = \Omega(x)\psi(x) \qquad \overline{\psi}(x) \to \overline{\psi}'(x) = \overline{\psi}(x)\Omega(x)^\dagger$$

$$\overline{\psi}(n)\psi(n+\hat{\mu}) \to \overline{\psi}'(n)\,\psi'(n+\hat{\mu}) = \overline{\psi}(n)\,\Omega(n)^\dagger\,\Omega(n+\hat{\mu})\,\psi(n+\hat{\mu})$$

- Introduction of a gauge link：

$$\overline{\psi}'(n)\,U'_\mu(n)\,\psi'(n+\hat{\mu}) \;=\; \overline{\psi}(n)\,\Omega(n)^\dagger\,U'_\mu(n)\,\Omega(n+\hat{\mu})\,\psi(n+\hat{\mu})$$

$$U_\mu(n) \;\to\; U'_\mu(n) = \Omega(n)\,U_\mu(n)\,\Omega(n+\hat{\mu})^\dagger$$



$$U_\mu(n) = \exp\left(\mathrm{i}\,aA_\mu(n)\right)$$

Doubler problem & Wilson fermion action

Naïve fermion action: $$S_F[\psi,\overline{\psi},U] = a^4 \sum_{n\in\Lambda} \overline{\psi}(n)\left(\sum_{\mu=1}^{4} \gamma_\mu \frac{U_\mu(n)\psi(n+\hat{\mu}) - U_{-\mu}(n)\psi(n-\hat{\mu})}{2a} + m\,\psi(n)\right)$$

$$S_F[\psi,\overline{\psi},U] = a^4 \sum_{n,m\in\Lambda} \sum_{a,b,\alpha,\beta} \overline{\psi}(n)_{\substack{\alpha\\a}} D(n|m)_{\substack{\alpha\beta\\ab}} \psi(m)_{\substack{\beta\\b}}$$

$$D(n|m)_{\substack{\alpha\beta\\ab}} = \sum_{\mu=1}^{4} (\gamma_\mu)_{\alpha\beta} \frac{U_\mu(n)_{ab}\,\delta_{n+\hat{\mu},m} - U_{-\mu}(n)_{ab}\,\delta_{n-\hat{\mu},m}}{2a} + m\,\delta_{\alpha\beta}\,\delta_{ab}\,\delta_{n,m}$$

Propagator: $$\widetilde{D}(p)^{-1}\Big|_{m=0} = \frac{-\mathrm{i}a^{-1}\sum_\mu \gamma_\mu \sin(p_\mu a)}{a^{-2}\sum_\mu \sin(p_\mu a)^2} \xrightarrow{a\to 0} \frac{-\mathrm{i}\sum_\mu \gamma_\mu p_\mu}{p^2}$$

$$\frac{sin(p_\mu a)}{a} \to p_\mu$$

physical poles: $p = (0,\,0,\,0,\,0)$

unwanted poles,doublers: $p = (\pi/a, 0, 0, 0)\ ,\ (0, \pi/a, 0, 0)\ ,\ \ldots\ ,(\pi/a, \pi/a, \pi/a, \pi/a)$

Wilson fermion matrix: $$\widetilde{D}(p) = m\mathbb{1} + \frac{\mathrm{i}}{a}\sum_{\mu=1}^{4} \gamma_\mu \sin(p_\mu a) + \underbrace{\mathbb{1}\frac{1}{a}\sum_{\mu=1}^{4}(1-\cos(p_\mu a))}_{\text{Wilson term}}$$

Wilson term vanishes when p_μ =0 and gives an extra mass l/a (infinity at a=0)

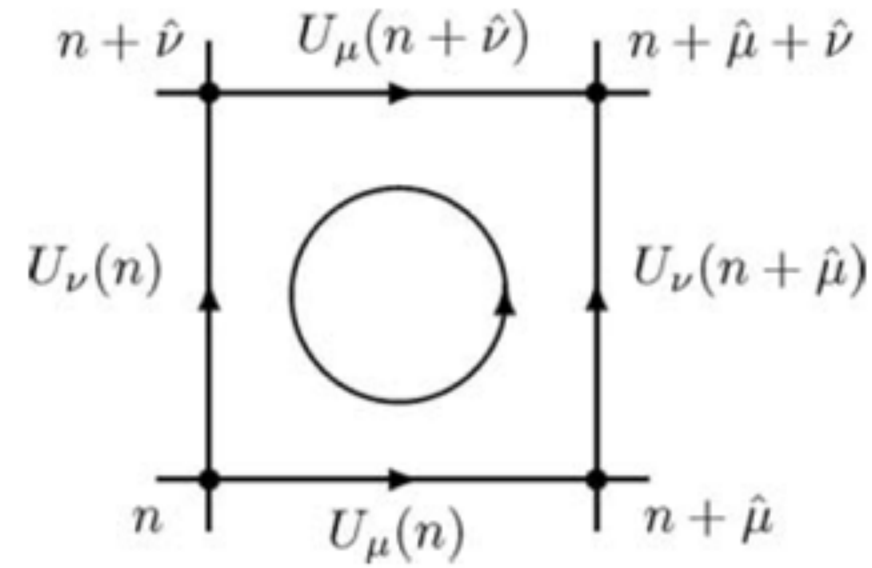
Wilson fermion action: $$S_F[\psi,\overline{\psi},U] = \sum_{f=1}^{N_f} a^4 \sum_{n,m\in\Lambda} \overline{\psi}^{(f)}(n)\,D^{(f)}(n|m)\,\psi^{(f)}(m)$$

$$D^{(f)}(n|m)_{\substack{\alpha\beta\\ab}} = \left(m^{(f)} + \frac{4}{a}\right)\delta_{\alpha\beta}\,\delta_{ab}\,\delta_{n,m} - \frac{1}{2a}\sum_{\mu=\pm1}^{\pm4}(\mathbb{1}-\gamma_\mu)_{\alpha\beta}\,U_\mu(n)_{ab}\,\delta_{n+\hat{\mu},m}$$

Wilson gauge action

Plaquette: $$U_{\mu\nu}(n) = U_\mu(n)\,U_\nu(n+\hat\mu)\,U_{-\mu}(n+\hat\mu+\hat\nu)\,U_{-\nu}(n+\hat\nu)$$

smallest Wilson loop that is gauge invariant



Wilson gauge action:

$$S_G[U] = \frac{2}{g^2}\sum_{n\in\Lambda}\sum_{\mu<\nu}\mathrm{Re}\,\mathrm{tr}\left[\mathbb{1} - U_{\mu\nu}(n)\right]$$

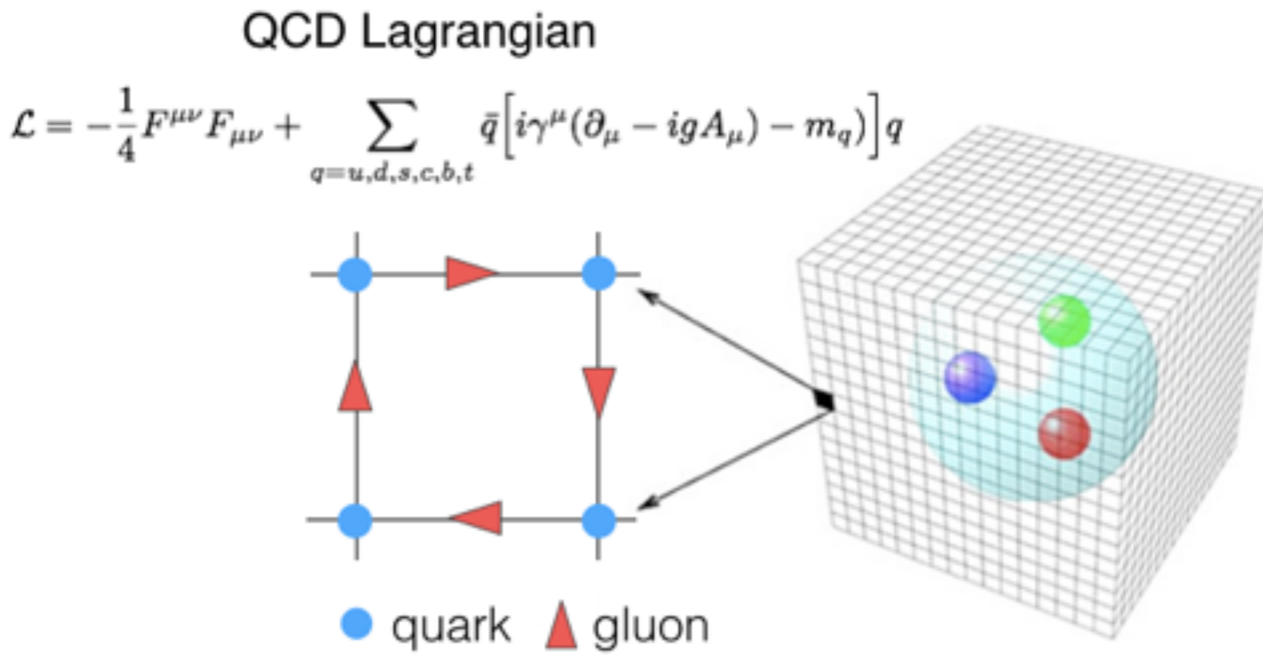
Reproduce the gauge action in the continuum limit with an order a^2 correction

$$S_G[U] = \frac{a^4}{2\,g^2}\sum_{n\in\Lambda}\sum_{\mu,\nu}\mathrm{tr}[F_{\mu\nu}(n)^2] + \mathcal{O}(a^2)$$

The above the above equation can be obtained with the help of :

$$U_\mu(n) = \exp\left(\mathrm{i}\,aA_\mu(n)\right), \quad \exp(A)\,\exp(B) = \exp\left(A + B + \frac{1}{2}[A,B] + \cdots\right)$$

The lattice QCD Path integral



Discretization in
Euclidean space

quarks: lattice sites
gluons: lattice links

Supercomputing the QCD matter:

structural equivalence
between
statistical mechanics
& QFT on the lattice

$$< \mathcal{O} >= \frac{1}{Z}\int \mathcal{D}\mathcal{U}\ \mathcal{D}\psi\ \mathcal{D}\bar{\psi}\ \mathcal{O}\ e^{-S_{lat}}$$

$$S_{lat} = S_g + S_f$$

$$Z = \int \mathcal{D}\mathcal{U}\ \mathcal{D}\psi\ \mathcal{D}\bar{\psi}\ e^{-S_{lat}} = \int \mathcal{D}\mathcal{U}\ e^{-S_g}\ \det M_f$$

$N_c \otimes N_f \otimes N_{spin} \otimes N_d \otimes N_\sigma \otimes N_T^3 \gtrsim 10^6$

det M_f = 1: Quenched approximation

det M_f =/= 1: dynamic/full QCD simulation

Monte Carlo simulation

Expectation value: $\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[U]\, e^{-S_G[U]} \det[D_u] \det[D_d]\; O$

Partition function: $Z = \int \mathcal{D}[U]\; \mathrm{e}^{-S_G[U]}\; \det[D_u]\; \det[D_d]$

- Treat the fermion determinant as a weight factor

U_n is distributed according to: $\frac{1}{Z}\, \mathrm{e}^{-S_G[U]}\; \det[D_u]\; \det[D_d]$

Should be real and nonnegative as a probability

$$\langle O \rangle \approx \frac{1}{N} \sum O[U_n]$$

- γ_5-hermiticity: $(\gamma_5 D)^\dagger = \gamma_5 D$ or $D^\dagger = \gamma_5 D \gamma_5$

$$\det[D]^* = \det[D^\dagger] = \det[\gamma_5\, D\, \gamma_5] = \det[D] \quad \Rightarrow \quad \det D \in \mathbb{R}$$

$$0 \leq \det[D]\; \det[D] = \det[D]\; \det[D^\dagger] = \det[D\, D^\dagger]$$

Wilson fermion matrix (page 18) satisfy γ_5-hermiticity

Sign problem at μ_B =/=0

QCD: $Z = \mathrm{Tr}\left[e^{-(H-\mu N)/T}\right] = \int [\mathrm{d}A] \underbrace{\det[D + m_q + i\mu\gamma_4]}_{\det D[\mu]} e^{-S(A)}$

- γ_5-hermiticity does not hold and instead: $D^\dagger(-\mu) = \gamma_5 D(\mu) \gamma_5$

det D[μ] is a complex number

- Toy model for demonstration of the sign problem

$$Z = \sum_{\{\phi(x)=\pm 1\}} \mathrm{sign}(\phi)\, e^{-S(\phi)}; \quad Z_0 = \sum_{\{\phi(x)=\pm 1\}} e^{-S(\phi)}$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \sum_{\{\phi(x)=\pm 1\}} \mathcal{O}(\phi)\mathrm{sign}(\phi)\, e^{-S(\phi)} = \frac{\langle \mathcal{O}(\phi)\, \mathrm{sign}(\phi) \rangle_0}{\langle \mathrm{sign}(\phi) \rangle_0}$$

$$\langle \mathrm{sign}(\phi) \rangle_0 = \frac{Z}{Z_0} = e^{-(f-f_0)V/T} \ll 1$$

f(f_0): free energy density corresponding to Z(Z_0)

$$\frac{\Delta \mathrm{sign}(\phi)}{\langle \mathrm{sign}(\phi) \rangle_0} = \frac{\sqrt{\langle \mathrm{sign}^2 \rangle_0 - \langle \mathrm{sign} \rangle_0^2}}{\sqrt{N} \langle \mathrm{sign} \rangle_0} \simeq \frac{e^{(f-f_0)V/T}}{\sqrt{N}} \ll 1 \quad \Rightarrow \quad N \gg e^{2(f-f_0)V/T}$$

Chiral symmetry of QCD

$$S_F[\psi,\overline{\psi},A] = \int \mathrm{d}^4x\, L\left(\psi,\overline{\psi},A\right),\ \ L\left(\psi,\overline{\psi},A\right) = \overline{\psi}\,\gamma_\mu\left(\partial_\mu + \mathrm{i}\,A_\mu\right)\psi = \overline{\psi}\,D\psi$$

D: massless Dirac operator

Chiral rotation: $\psi \to \psi' = \mathrm{e}^{\mathrm{i}\alpha\gamma_5}\psi\ ,\quad \overline{\psi} \to \overline{\psi}' = \overline{\psi}\,\mathrm{e}^{\mathrm{i}\alpha\gamma_5}$

- Lagrangian density is invariant under the chiral rotation:

$$\begin{aligned} L\left(\psi',\overline{\psi}',A\right) &= \overline{\psi}'\,\gamma_\mu\left(\partial_\mu + \mathrm{i}\,A_\mu\right)\psi' = \overline{\psi}\,\mathrm{e}^{\mathrm{i}\alpha\gamma_5}\,\gamma_\mu\left(\partial_\mu + \mathrm{i}\,A_\mu\right)\mathrm{e}^{\mathrm{i}\alpha\gamma_5}\,\psi \\ &= \overline{\psi}\,\mathrm{e}^{\mathrm{i}\alpha\gamma_5}\mathrm{e}^{-\mathrm{i}\alpha\gamma_5}\,\gamma_\mu\left(\partial_\mu + \mathrm{i}\,A_\mu\right)\psi = L\left(\psi,\overline{\psi},A\right) \end{aligned}$$

- A mass term explicitly breaks the chiral symmetry: $m\,\overline{\psi}'\psi' = m\,\overline{\psi}\,\mathrm{e}^{i2\alpha\gamma_5}\,\psi$

$$P_R = \frac{\mathbb{1}+\gamma_5}{2}\,,\quad P_L = \frac{\mathbb{1}-\gamma_5}{2}$$
$$\psi_R = P_R\,\psi\,,\quad \psi_L = P_L\,\psi$$
$$\overline{\psi}_R = \overline{\psi}\,P_L\,,\quad \overline{\psi}_L = \overline{\psi}\,P_R$$

$$\Rightarrow$$

$$L\left(\psi,\overline{\psi},A\right) = \overline{\psi}_L\,D\,\psi_L + \overline{\psi}_R\,D\,\psi_R$$
$$m\,\overline{\psi}\,\psi = m\,\left(\overline{\psi}_R\,\psi_L + \overline{\psi}_L\,\psi_R\right)$$

- Essence of chiral symmetry: $D\,\gamma_5 + \gamma_5\,D = 0$

chiral symmetry on the lattice

- Massless Wilson Dirac operator breaks chiral symmetry

$$D^f(n|m)_{\alpha\beta,ab} = \frac{4}{a}\,\delta_{\alpha\beta}\delta_{ab}\delta_{n,m} - \frac{1}{2a}\sum_{\mu=\pm1}^{\pm4}(1-\gamma_\mu)_{\alpha\beta}\,U_\mu(n)_{ab}\,\delta_{n+\hat{\mu},m}$$

$$\gamma_5^\dagger = \gamma_5,\ \ \gamma_5^2 = 1\,,\ \{\gamma_5,\gamma_\mu\} = 0$$

- The Ginsparg-Wilson equation

$$D\,\gamma_5 + \gamma_5\,D = a\,D\,\gamma_5\,D \xrightarrow{\text{lattice space a ->0}} D\,\gamma_5 + \gamma_5\,D = 0$$

- Lattice fermion satisfy the Ginsparg-Wilson equation preserve the chiral symmetry at nonzero lattice spacing

chiral rotation on the lattice

$$\psi' = \exp\left(\mathrm{i}\,\alpha\,\gamma_5\left(\mathbb{1} - \frac{a}{2}D\right)\right)\psi\,,\quad \overline{\psi}' = \overline{\psi}\exp\left(\mathrm{i}\,\alpha\left(\mathbb{1} - \frac{a}{2}D\right)\gamma_5\right)$$

$$\begin{aligned} L\left(\psi',\overline{\psi}'\right) = \overline{\psi}'\,D\,\psi' &= \overline{\psi}\,\exp\left(\mathrm{i}\,\alpha\,\left(\mathbb{1} - \frac{a}{2}D\right)\,\gamma_5\right)\,D\,\exp\left(\mathrm{i}\,\alpha\,\gamma_5\,\left(\mathbb{1} - \frac{a}{2}D\right)\right)\,\psi \\ &= \overline{\psi}\,\exp\left(\mathrm{i}\,\alpha\,\left(\mathbb{1} - \frac{a}{2}D\right)\,\gamma_5\right)\,\exp\left(-\mathrm{i}\,\alpha\,\left(\mathbb{1} - \frac{a}{2}D\right)\,\gamma_5\right)\,D\,\psi \\ &= \overline{\psi}\,D\,\psi = L\left(\psi,\,\overline{\psi}\right) \end{aligned}$$

chiral symmetry on the lattice

- Massless Wilson Dirac operator breaks chiral symmetry

$$D^f(n|m)_{\alpha\beta,ab} = \frac{4}{a}\,\delta_{\alpha\beta}\delta_{ab}\delta_{n,m} - \frac{1}{2a}\sum_{\mu=\pm1}^{\pm4}(1-\gamma_\mu)_{\alpha\beta}\,U_\mu(n)_{ab}\,\delta_{n+\hat{\mu},m}$$

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$$\widehat{P}_R = \frac{\mathbb{1} + \widehat{\gamma}_5}{2}\,, \quad \widehat{P}_L = \frac{\mathbb{1} - \widehat{\gamma}_5}{2}\,, \quad \widehat{\gamma}_5 = \gamma_5\,(\mathbb{1} - a\,D)$$

$$\widehat{P}_R^{\,2} = \widehat{P}_R\,, \quad \widehat{P}_L^{\,2} = \widehat{P}_L\,, \quad \widehat{P}_R\,\widehat{P}_L = \widehat{P}_L\,\widehat{P}_R = 0\,, \quad \widehat{P}_R + \widehat{P}_L = \mathbb{1}$$

$$\psi_R = \widehat{P}_R\,\psi\,, \quad \psi_L = \widehat{P}_L\,\psi\,, \quad \overline{\psi}_R = \overline{\psi}\,P_L\,, \quad \overline{\psi}_L = \overline{\psi}\,P_R$$

$$\overline{\psi} D\,\psi = \overline{\psi}_L\,D\,\psi_L + \overline{\psi}_R\,D\,\psi_R$$

chiral fermions on the lattice

- Overlap fermion operator D_{ov} : only operator that satisfies the Ginsparg-Wilson equation

$$D_{\rm ov} = \frac{1}{a}\left(\mathbb{1} + \gamma_5\, {\rm sign}[H]\right)\,,\ {\rm sign}(H) = H|H|^{-1} = H(H^2)^{-\frac{1}{2}}\,,\ H = \gamma_5\, A$$

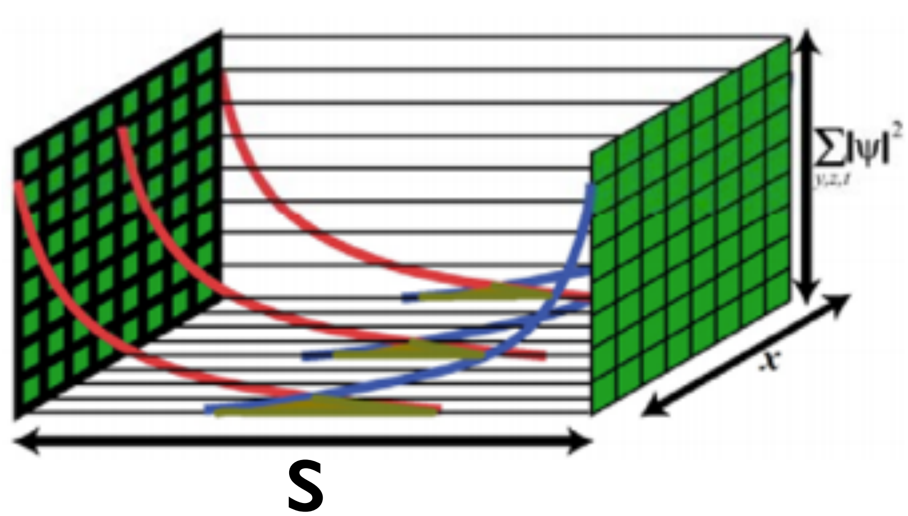
A denotes some suitable γ_5-hermitian “kernel” Dirac operator

large numerical cost due to the evaluation of $(HH^{+})^{-1/2}$

costs > 100 x costs of Wilson formulation

- Domain Wall fermions: introduce the fictitious 5^{th} dimension of extent N_5

preserve exact chiral symmetry N_5. Residual symmetry breaking is quantified by the additive renormalization factor m_{res} to the quark mass



costs > N_5 x costs of Wilson formulation

Ns = 16-64

Staggered fermions

Naïve fermions: $$S_F[\psi,\overline{\psi}] = a^4 \sum_{n\in\Lambda} \overline{\psi}(n) \left(\sum_{\mu=1}^{4} \gamma_\mu \frac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a} + m\,\psi(n) \right)$$

staggered transformation:

$$\psi(n) = \gamma_1^{n_1}\,\gamma_2^{n_2}\,\gamma_3^{n_3}\,\gamma_4^{n_4}\,\psi(n)'\,, \quad \overline{\psi}(n) = \overline{\psi}(n)'\,\gamma_4^{n_4}\,\gamma_3^{n_3}\,\gamma_2^{n_2}\,\gamma_1^{n_1}$$

$$\overline{\psi}(n)\,\gamma_3\,\psi(n\pm\hat{3}) = (-1)^{n_1+n_2}\,\overline{\psi}(n)'\mathbb{1}\,\psi(n\pm\hat{3})'$$

$$S_F\left[\psi',\overline{\psi}'\right] = a^4 \sum_{n\in\Lambda} \overline{\psi}(n)'\mathbb{1} \left(\sum_{\mu=1}^{4} \eta_\mu(x) \frac{\psi(n+\hat{\mu})' - \psi(n-\hat{\mu})'}{2\,a} + m\,\psi(n)' \right)$$

$$\eta_1(n) = 1\,,\ \eta_2(n) = (-1)^{n_1}\,,\ \eta_3(n) = (-1)^{n_1+n_2}\,,\ \eta_4(n) = (-1)^{n_1+n_2+n_3}$$

staggered fermions:

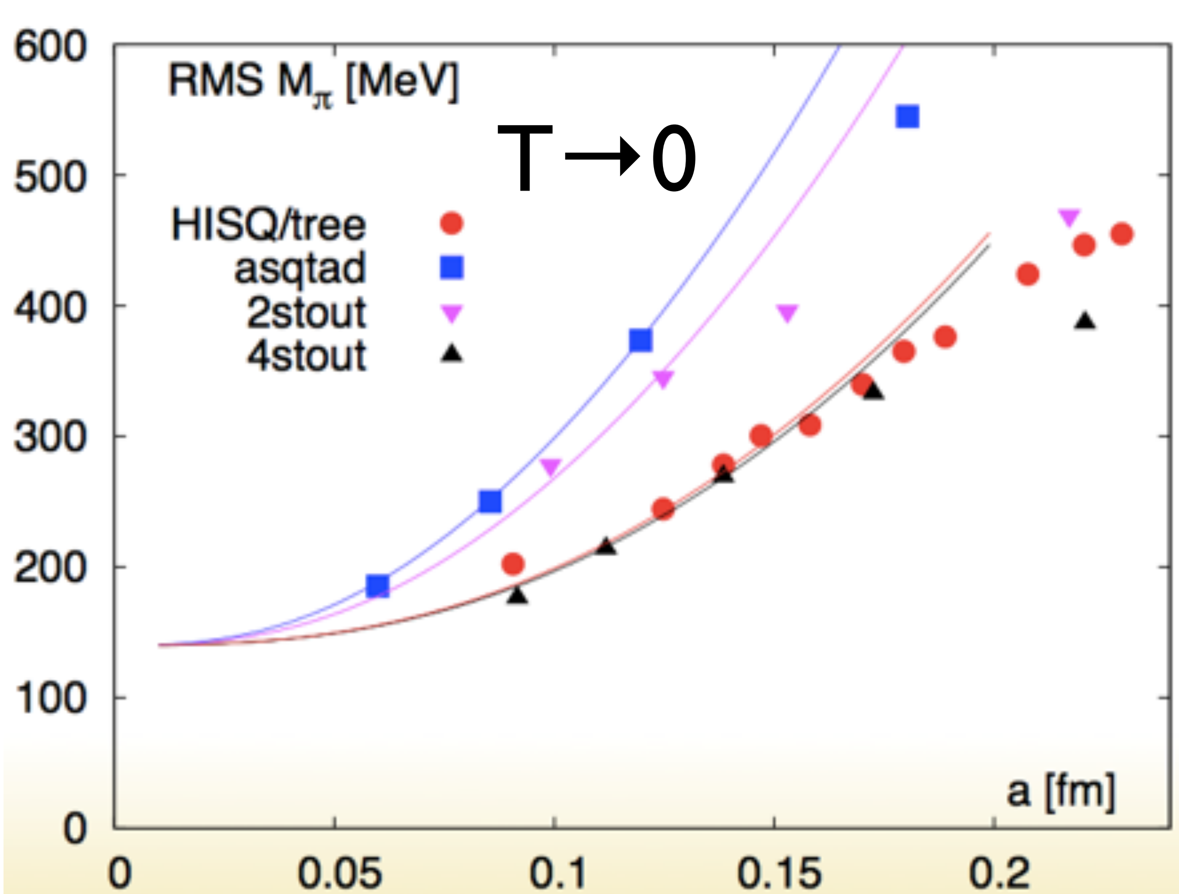
$$S_F[\chi,\overline{\chi}] = a^4 \sum_{n\in\Lambda} \overline{\chi}(n) \left(\sum_{\mu=1}^{4} \eta_\mu(x) \frac{U_\mu(n)\chi(n+\hat{\mu}) - U_\mu^\dagger(n-\hat{\mu})\chi(n-\hat{\mu})}{2\,a} + m\chi(n) \right)$$

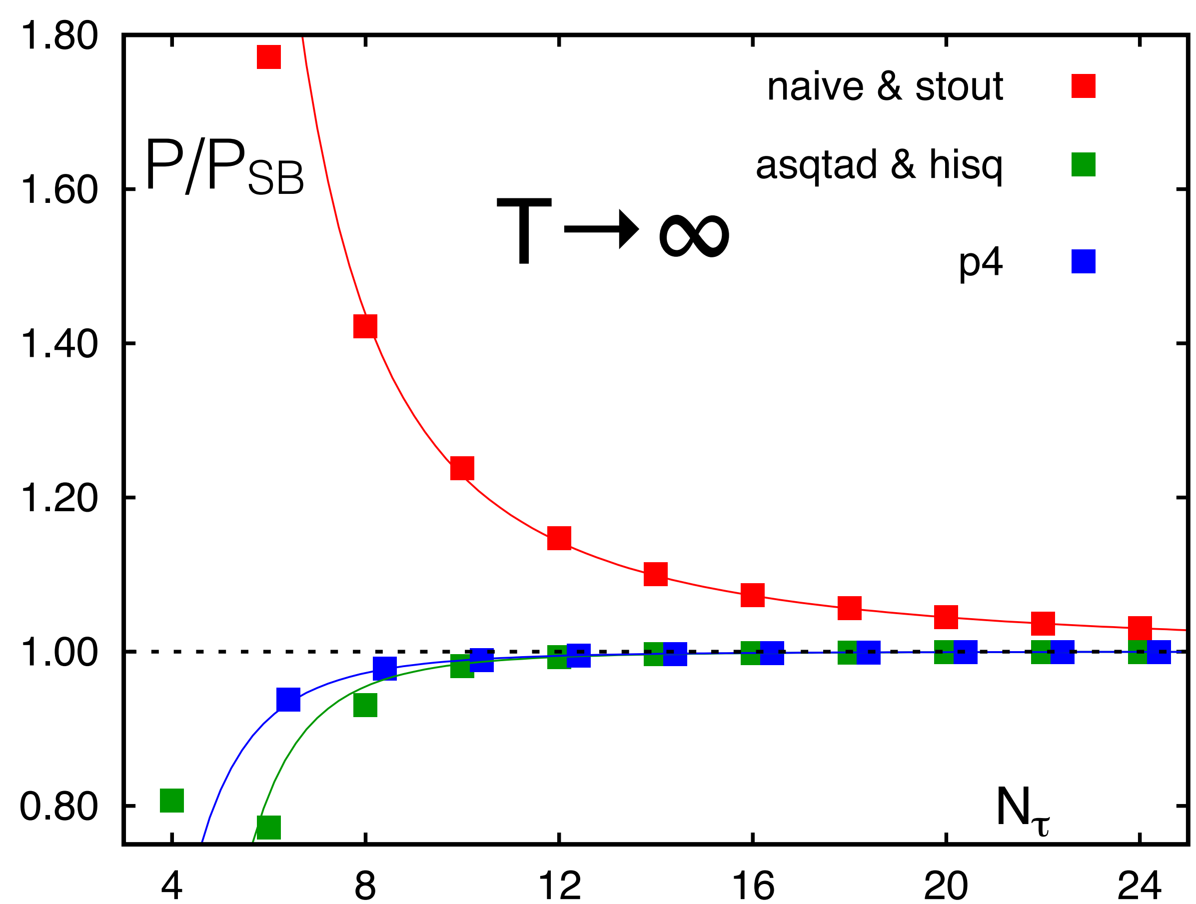
$\chi(n)$:Grassmann-valued fields with only color indices but without Dirac structure

16 -> 4 tastes (doublers)

Taste symmetry breaking of staggered fermions

action(group)	improvements at T→0	improvements at T→∞
naïve (Mumbai)	none	none
p4(BNL-Bi)	poor	very good
asqtad(hotQCD)	ok	good
2stout(WB)	good	none
4stout(WB)	very good	none
HISQ(hotQCD)	very good	good





Current hot & dense lattice QCD simulations

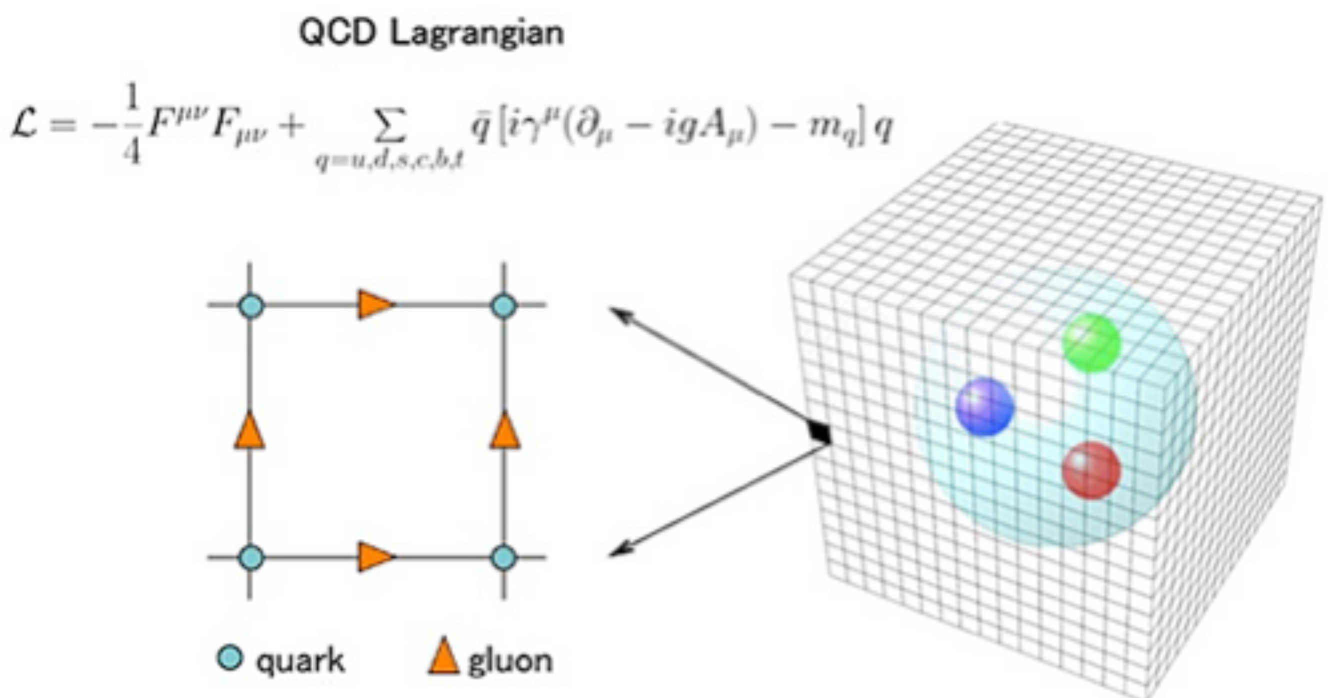
Lattice QCD: discretized version of QCD on a Euclidean space-time lattice, reproduces QCD when lattice spacing $a \to 0$ (continuum limit)

Mostly dynamical QCD with $N_f=2+1$ and physical pion mass

- Staggered actions at $a \neq 0$: taste symmetry breaking
  - 1 physical Goldstone pion +15 heavier unphysical pions
  - averaged pion mass, i.e. Root Mean Squared (RMS) pion mass
  - Smaller RMS pion mass → Better improved action: HISQ, stout
- Chiral fermions(Domain Wall/Overlap) at $a \neq 0$
  - preserves full flavor symmetry and chiral symmetries
  - computationally expensive to simulate, currently starts to produce interesting results on QCD thermodynamics

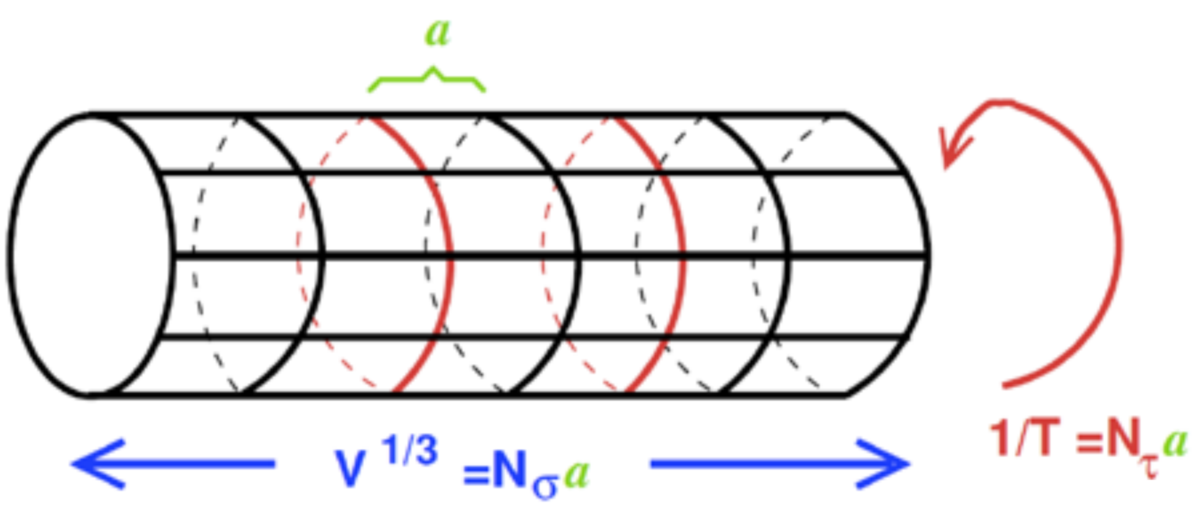
Formulation of lattice gauge theory

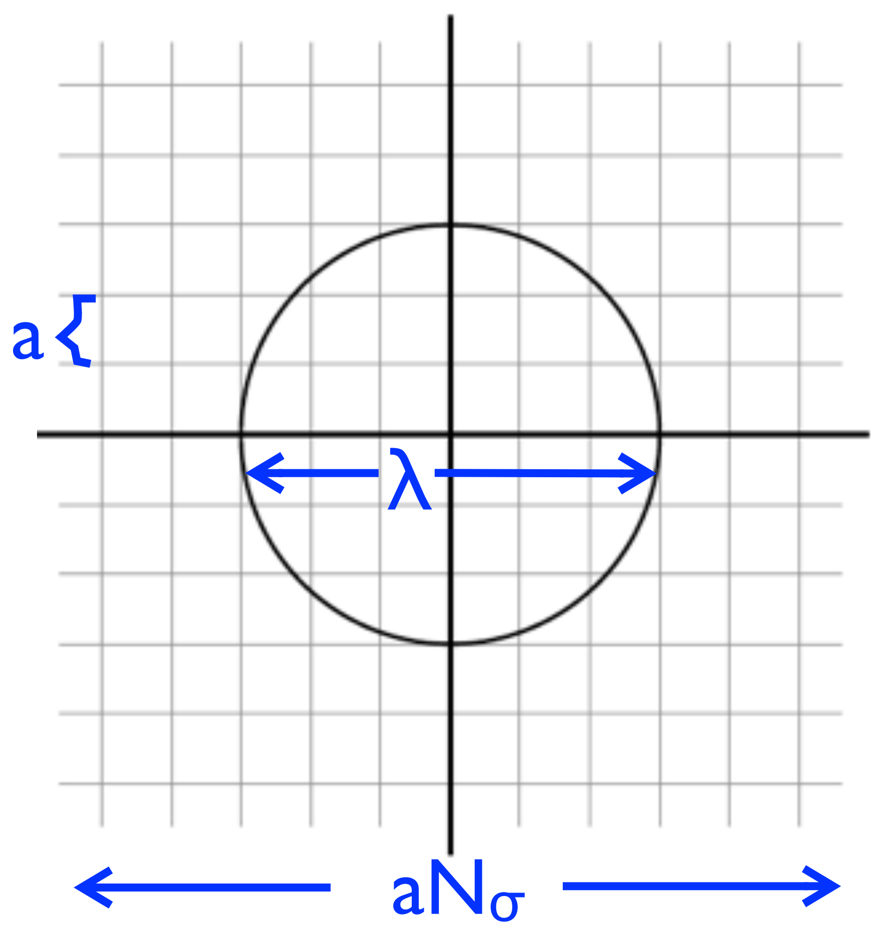
Lattice QCD calculation is a non-perturbative implementation of field theory using the Feynman path integral approach



- discretization of space time
- the transcription of the gauge and fermions degree of freedom
- construction of the action
- definition of the measure of integration in the path integral
- the transcription of the operators used to probe the physics

Basics of Lattice QCD





- Four dim. Euclidean lattice $N_\sigma^3 \times N_\tau$
- Temperature $T = 1/(N_\tau a)$
- $a \ll \lambda \ll N_\sigma a$
- To get continuum physics, make $a \to 0$ at constant V and T

Input parameters

- lattice gauge coupling: $\beta (= 6/g^2)$
- quark masses
- lattice size: $N_\tau, \; N_\sigma$

No free parameters
input bare parameters of QCD Lagrangian
fixed by reproducing physics at T=0

Basics of Lattice QCD (cont.)

Expectation value of QCD observables on the lattice

$$< \mathcal{O} >= \frac{1}{Z} \int \mathcal{D}\mathcal{U}\ \mathcal{D}\psi\ \mathcal{D}\bar{\psi}\ \mathcal{O}\ e^{-S_{lat}}$$

$$S_{lat} = S_g + S_f$$

$$Z = \int \mathcal{D}\mathcal{U}\ \mathcal{D}\psi\ \mathcal{D}\bar{\psi}\ e^{-S_{lat}} = \int \mathcal{D}\mathcal{U}\ e^{-S_g}\ \mathrm{det}M_f$$

- S_f: staggered, Wilson, Domain Wall fermions...
- Operator with each configuration is summed up with weight exp($-S_{lat}$)
- Average over configurations with huge degree of freedoms

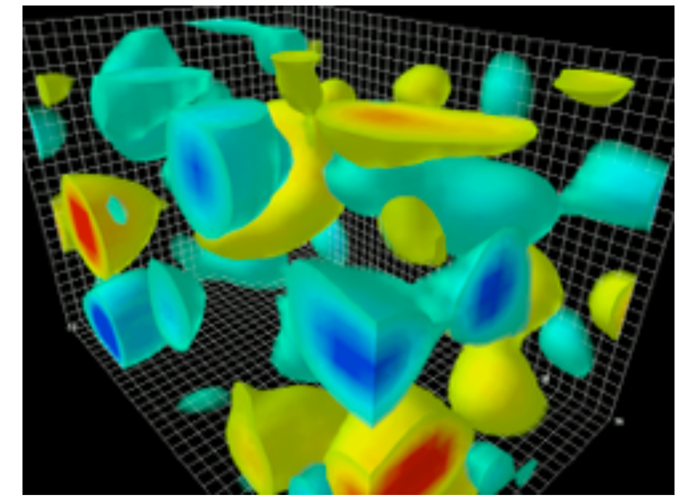
$N_{deg.} \otimes N_c \otimes N_f \otimes N_{spin} \otimes N_d \otimes N_\sigma^3 \otimes N_T \gtrsim 10^6$

- Monte Carlo simulations: generate gauge field configurations with weight exp($-S_g+\log(\det M_f)$)
  - $\det M_f$=constant: quenched approximation
  - $\det M_f \neq$ constant: dynamical full QCD simulation
- O: chiral condensates, susceptibilities, correlation functions

Questions?

Nuclear Science Computing Center at CCNU

QCD phase structure from Lattice Quantum Chromodynamics



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CBM school, Wuhan, China
22 Sep 2017 to 23 September 2017

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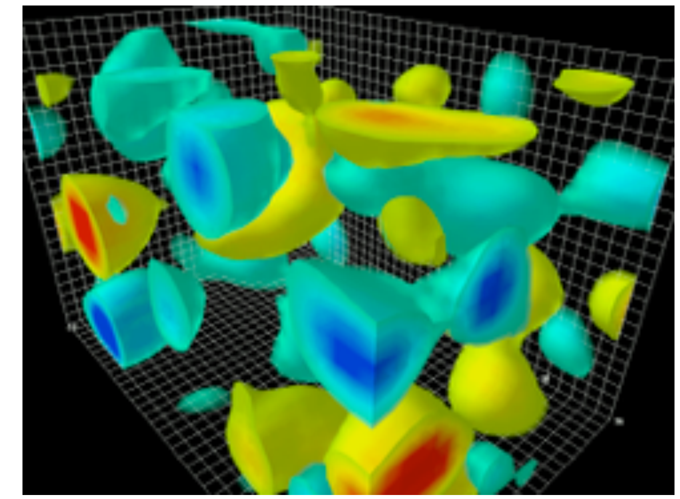
QCD phase structure from Lattice Quantum Chromodynamics



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Symmetries of QCD in the vacuum

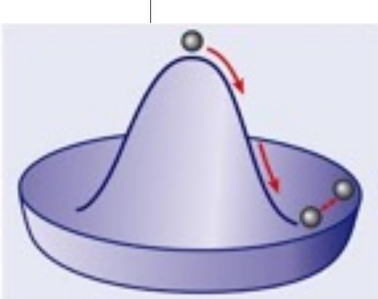
$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \sum_{q=u,d,s,c,b,t} \bar{q}\left[i\gamma^{\mu}(\partial_{\mu} - igA_{\mu}) - m_q\right]q$$

Classical QCD symmetry (m_q=0)

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$$

Quantum QCD vacuum (m_q=0)

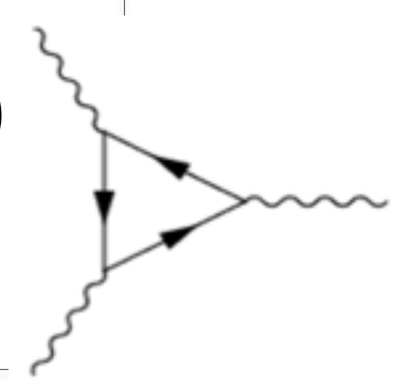
Chiral condensate:
spontaneous mass generation



$$\langle \bar{q}_R q_L \rangle \neq 0$$

Axial anomaly:
quantum violation of $U(1)_A$

$$\partial_{\mu} j_5^{\mu} = \frac{g^2 N_f}{16\pi^2} tr(\tilde{F}_{\mu\nu}F^{\mu\nu})$$

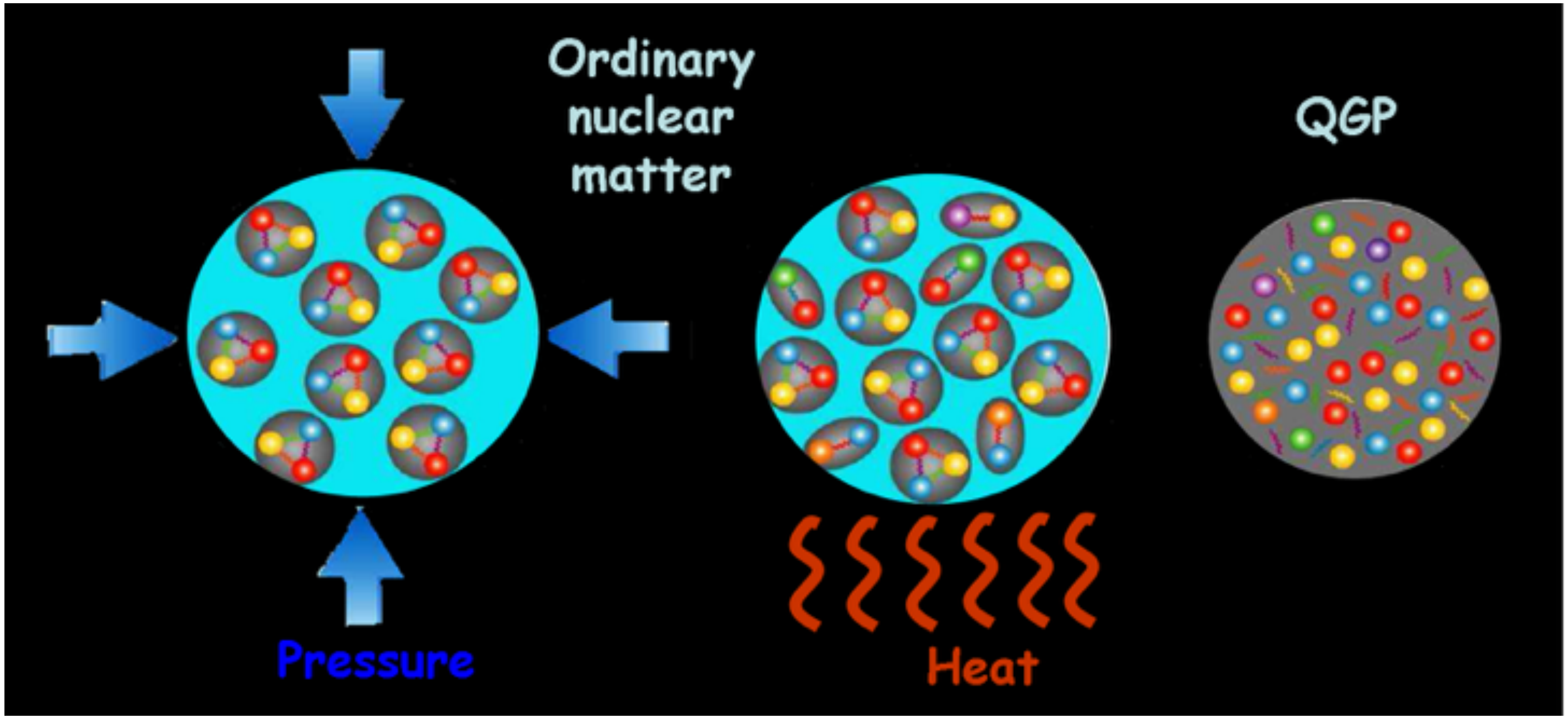


$$SU(N_f)_V \times U(1)_V$$

Symmetry restoration in extreme conditions?

"The whole is more than sum of its parts."
Aristotle, Metaphysica 10f-1045a

$\mu_B \gg \Lambda_{QCD}$ or $T \gg \Lambda_{QCD}$

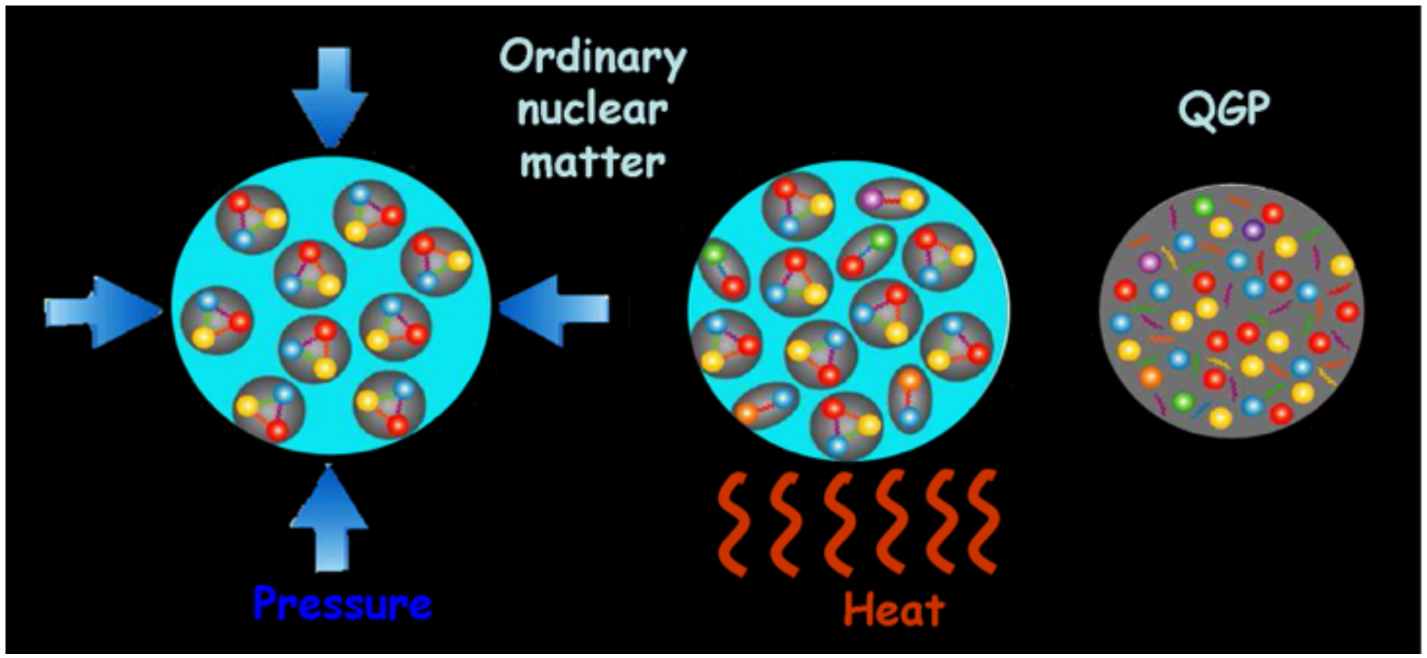


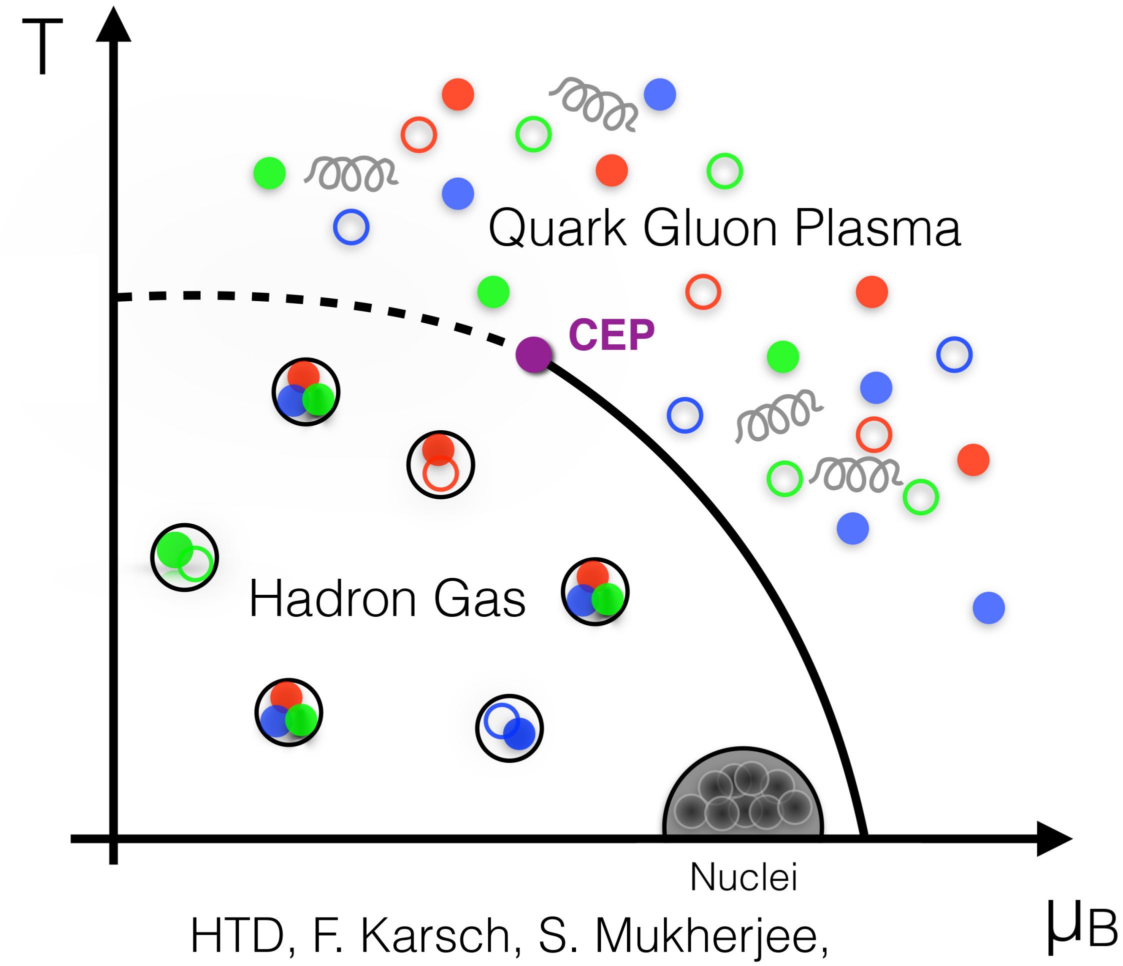
Quarks & Gluons get liberated from nucleons

From hadronic phase to

A new state of matter: Quark Gluon Plasma (QGP)

QCD phase transitions





HTD, F. Karsch, S. Mukherjee, arXiv:1504.0527

What are the orders of QCD phase transitions?

What are the T_c , critical temperatures of these transitions?

What will be the observable phenomena associated with the transitions?

Ginzburg-Landau-Wilson approach

Partition function: $$Z = \int [\mathrm{d}\sigma] \exp\Big(- \int \mathrm{dx}\, \mathcal{L}_{eff}\,(\sigma(\mathrm{x}); K) \Big)$$

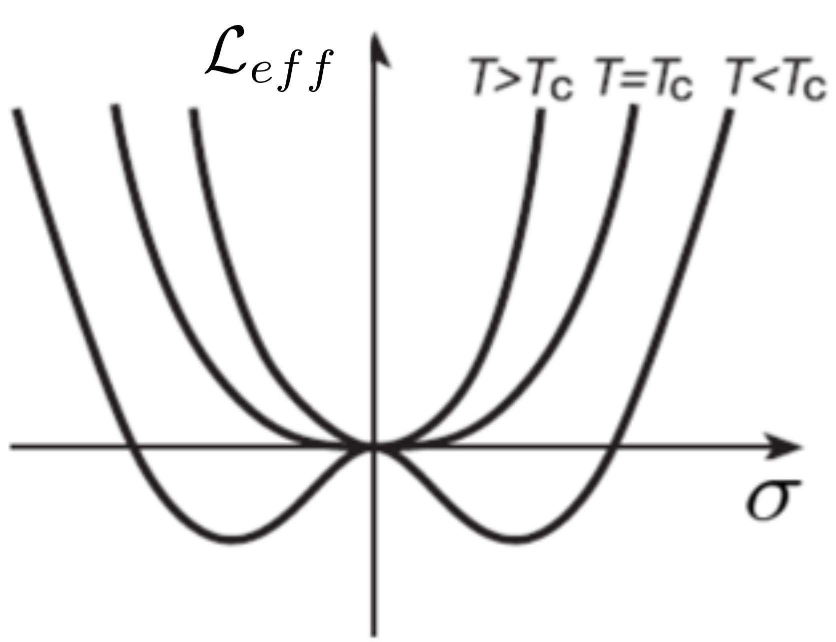
Landau function: $$\mathcal{L}_{eff} = \frac{1}{2}(\nabla\sigma)^2 + \sum_n a_n(K)\sigma^n$$ Same symmetry with the underlying theory

σ(x): order parameter field; K={m,μ}: external parameters

2nd order phase transition

Z(2) Ising model, Nf=2 QCD

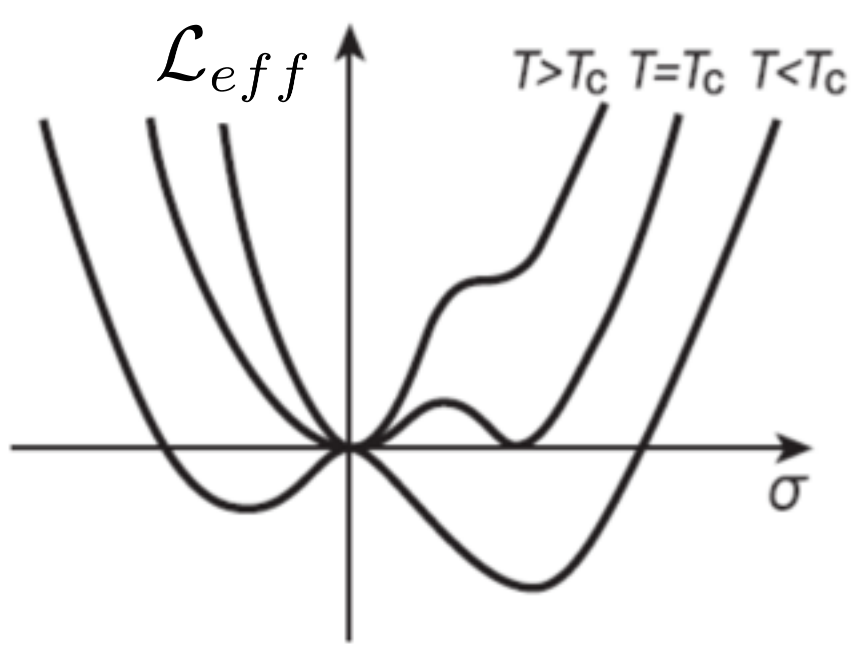
$$\mathcal{L}_{eff} = \frac{1}{2}a\sigma^2 + \frac{1}{4}b\sigma^4$$



1st order phase transition

Z(3) Potts model, Nf=3 QCD

$$\mathcal{L}_{eff} = \frac{1}{2}a\sigma^2 - \frac{1}{3}c\sigma^3 + \frac{1}{4}b\sigma^4$$



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σ(x): order parameter field; K={m,μ}: external parameters

2nd order phase transition

order parameter M:
continuous in T

fluctuations of M:

$$\chi(T) = \frac{T}{V}(\langle M^2 \rangle - \langle M \rangle^2)$$

$$\chi(T_c) \sim V^{(2-\eta)/3}$$

1st order phase transition

M:
discontinuous in T

fluctuations of M:

$$\chi(T_c) \sim V$$

Landau functional of QCD

Pisarski & Wilczek (84)

Symmetry: $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$

Chiral field: $\Phi_{ij} \sim \frac{1}{2}\bar{q}^j(1-\gamma_5)q^i = \bar{q}^j_R q^i_L$

Chiral transformation: $\Phi \to e^{-2i\alpha_A} V_L \, \Phi \, V_R^\dagger$

$$\mathcal{L}_{eff} = \frac{1}{2}\,\mathrm{tr}\,\partial\Phi^\dagger\partial\Phi + \frac{a}{2}\,\mathrm{tr}\,\Phi^\dagger\Phi + \frac{b_1}{4!}\left(\mathrm{tr}\,\Phi^\dagger\Phi\right)^2 + \frac{b_2}{4!}\,\mathrm{tr}\left(\Phi^\dagger\Phi\right)^2$$

$\longrightarrow$ SU(N$_f$)$_L$xSU(N$_f$)$_R$xU(1)$_A$

$$-\frac{c}{2}\left(\det\Phi + \det\Phi^\dagger\right)$$

$\longrightarrow$ SU(N$_f$)$_L$xSU(N$_f$)$_R$

$$-\frac{d}{2}\mathrm{tr}\,h\left(\Phi + \Phi^\dagger\right).$$

$\longrightarrow$ Quark mass term

Results on phase transitions should be eventually checked by Lattice QCD

Pure gauge theory (N_f=0)

center transformation: $A_4(\mathbf{x}, x_4) \to z\, A_4(\mathbf{x}, x_4)$, $z \in Z(N_c)$

- The gauge action is invariant under the center transformation
- Polyakov loop: $\ell = \frac{1}{N_c} \mathrm{Tr} \left[\mathcal{P} \exp \left(-ig \int_0^\beta \mathrm{d}x_4\, A_4(\mathbf{x}, x_4) \right) \right]$

$$\ell \to z\ell \quad \Longrightarrow \quad \langle \ell \rangle = \frac{1}{3} \langle \ell + z\ell + z^2 \ell \rangle = 0$$

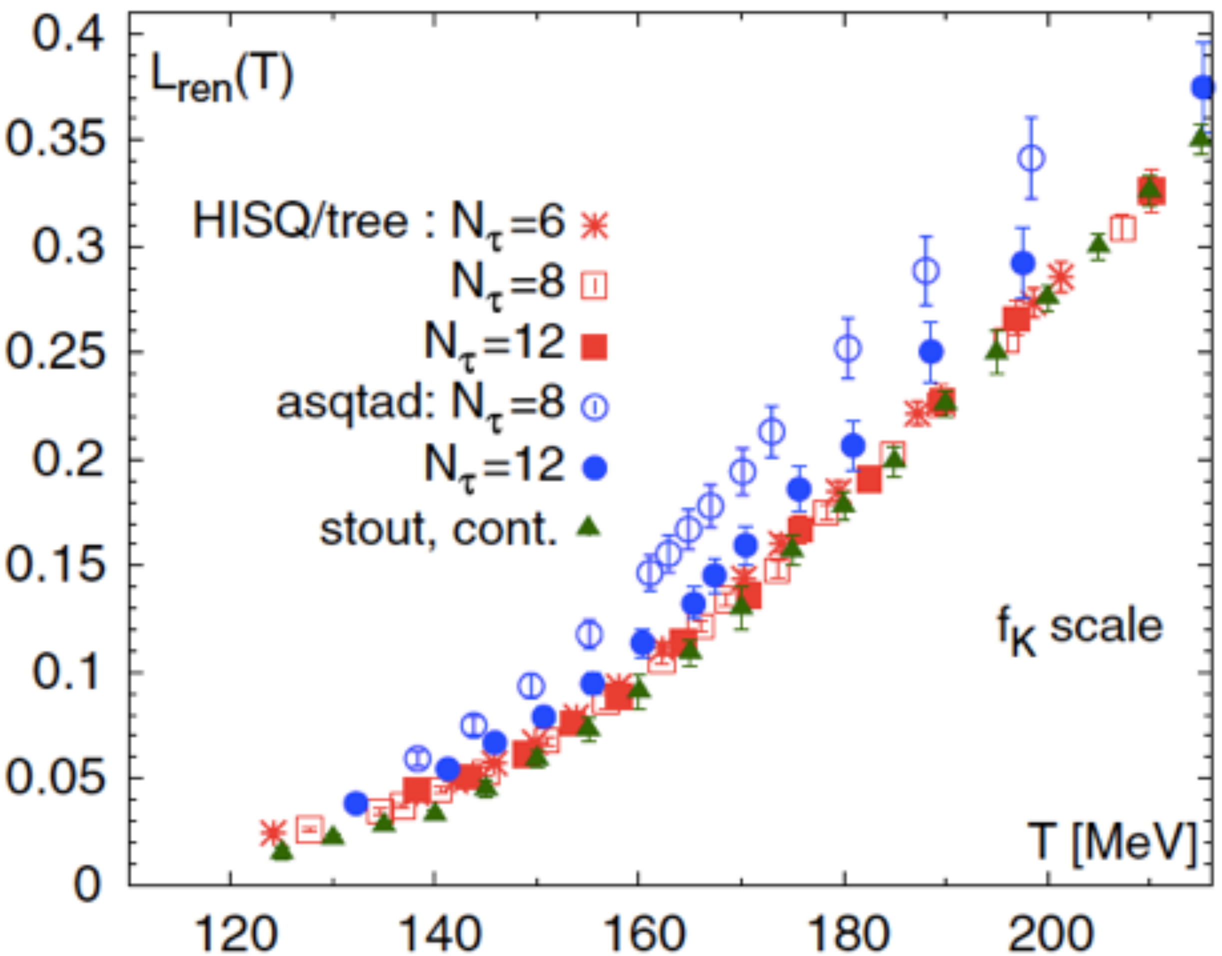
- Polyakov loop is related to the heavy quark (pair) potential:

$$|\langle \ell \rangle| \propto e^{-f_q/T}, \quad \langle \ell^\dagger(r) \ell(0) \rangle \propto e^{-f_{q\bar{q}}(r)/T}$$

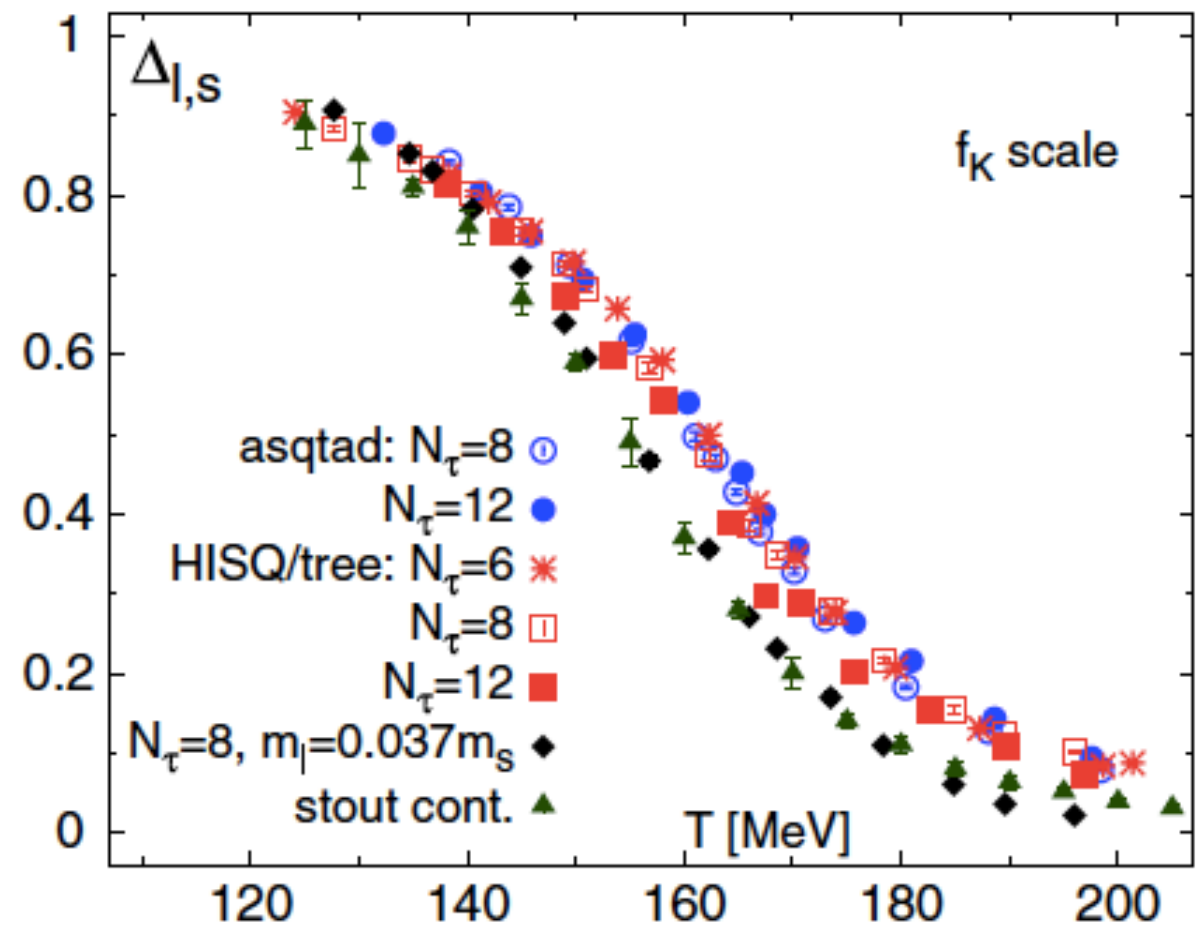
	Confined (Disordered) Phase	Deconfined (Ordered) Phase
Free Energy	$f_\mathrm{q} = \infty$	$f_\mathrm{q} < \infty$
	$f_{\bar{\mathrm{q}}\mathrm{q}} \sim \sigma r$	$f_{\bar{\mathrm{q}}\mathrm{q}} \sim f_\mathrm{q} + f_{\bar{\mathrm{q}}} + \alpha \frac{\mathrm{e}^{-m_\mathrm{M} r}}{r}$
Polyakov Loop	$\langle \ell \rangle = 0$	$\langle \ell \rangle \neq 0$
$(r \to \infty)$	$\langle \ell^\dagger(r) \ell(0) \rangle \to 0$	$\langle \ell^\dagger(r) \ell(0) \rangle \to |\langle \ell \rangle|^2 \neq 0$

Polyakov Loop and chiral condensate in N_f=2+1 QCD with $m_\pi \approx$140 MeV

renormalized Polyakov loop



subtracted chiral condensate

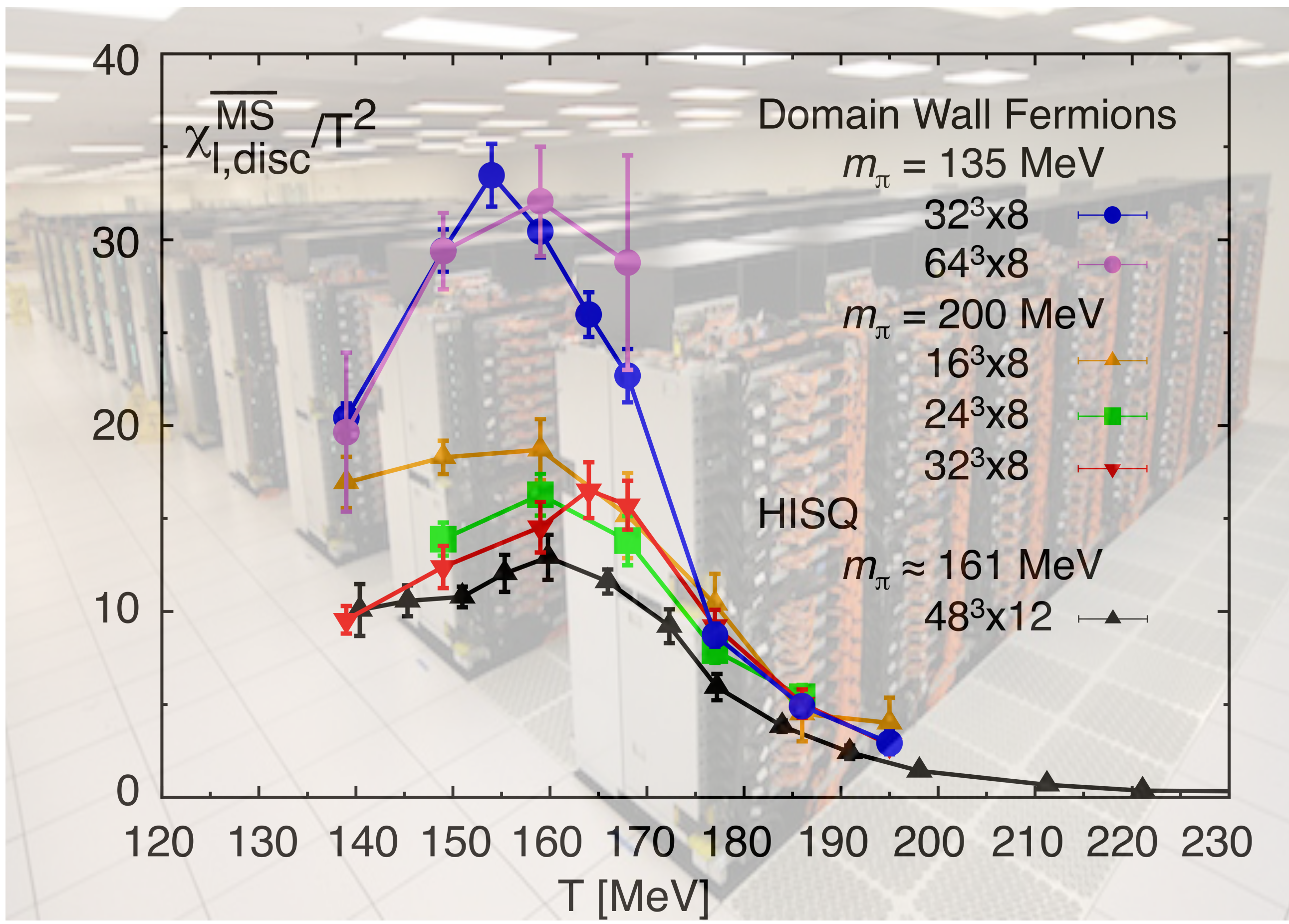


No evidence of a first order phase transition

chiral crossover T_{pc} =155(1)(8) MeV

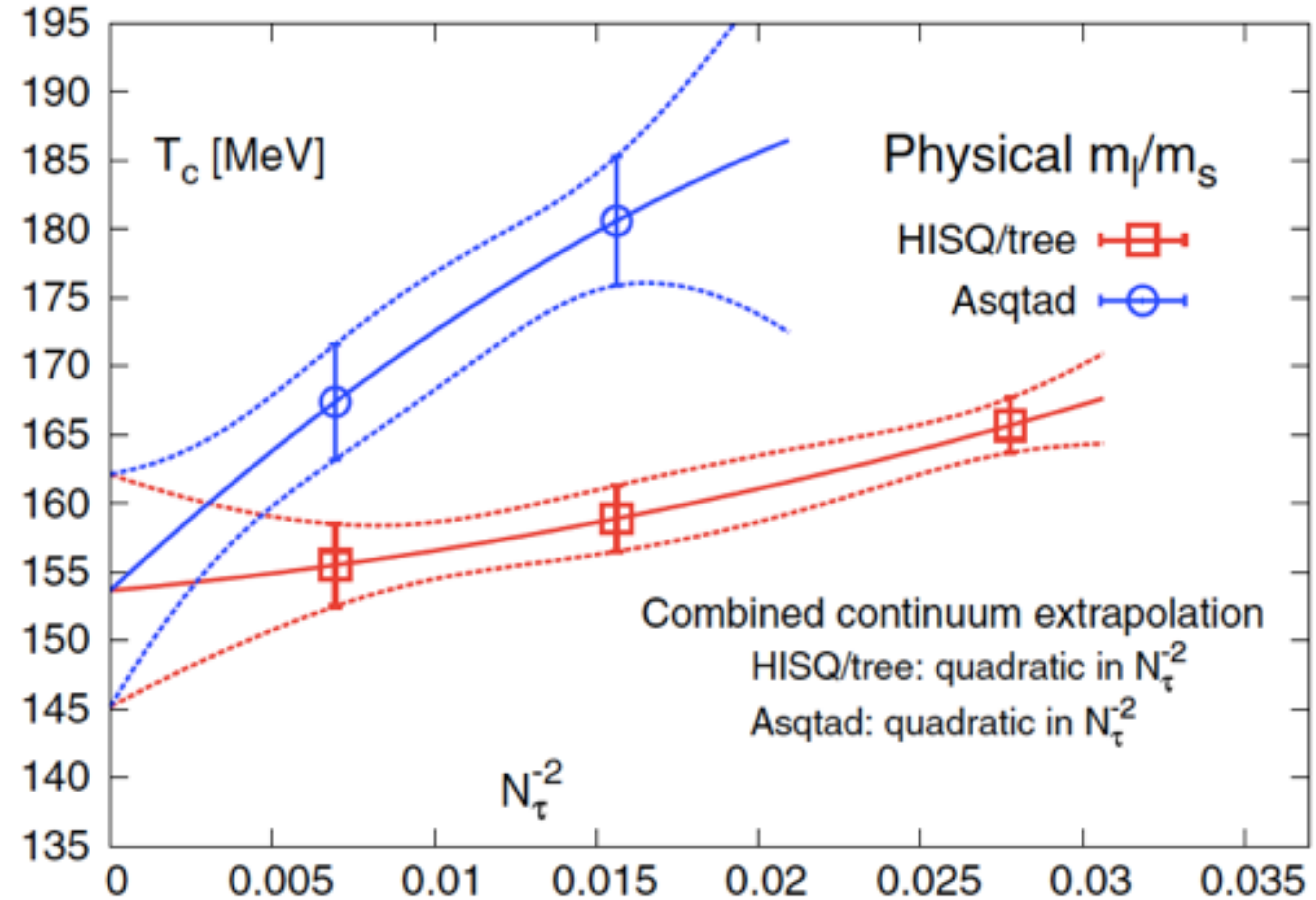
N_f=2+1 QCD with $m_\pi \approx$140 MeV

chiral sus obtained with chiral fermions



HotQCD: PRL 113 (2014) 082001

continuum extrapolation of Tc with HISQ

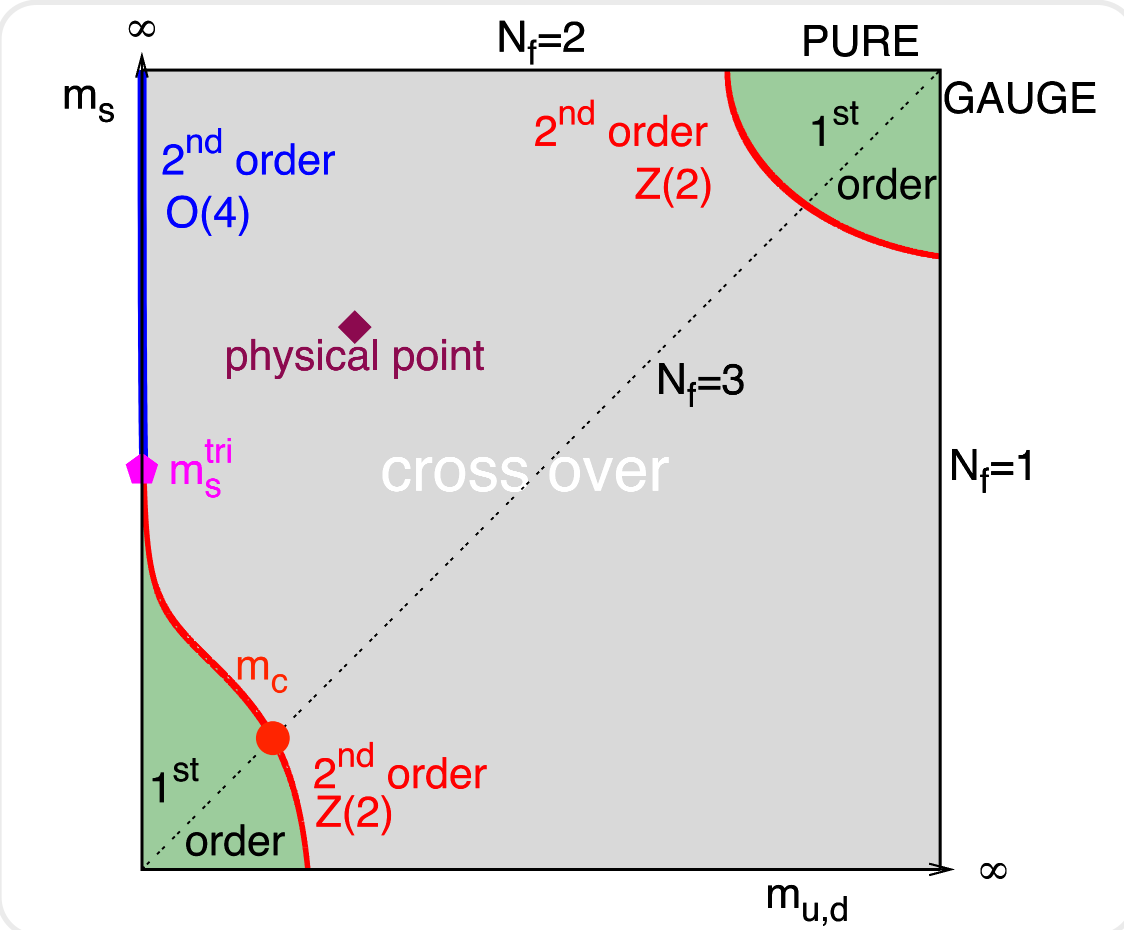


HotQCD: PRD 85 (2012)054503

- Not a true (chiral or deconfinement) phase transition but a rapid chiral crossover See also [WB collaboration], Phys.Lett. B713 (2012) 342, Nature 443(2006)675, JHEP 1009 (2010) 073
- Consistent results obtained from 3 discretization schemes (Domain wall, HISQ, stout)

QCD phase structure in the quark mass plane

columbia plot, PRL 65(1990)2491



HTD, F. Karsch, S. Mukherjee, 1504.05274

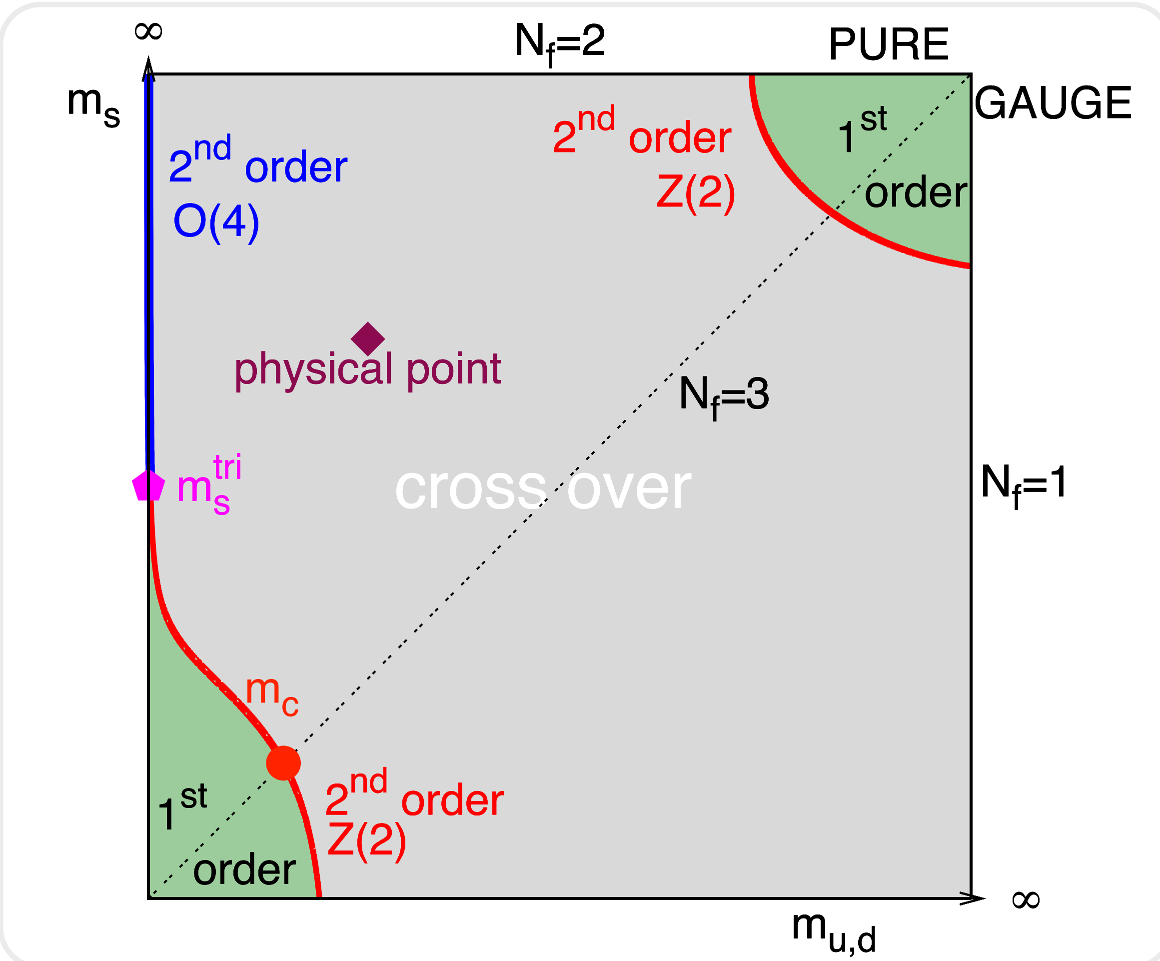
RG arguments:

- m_q=0 or ∞ with N_f=3: a first order phase transition R. Pisarski & F. Wilczek, PRD29 (1984) 338
- Critical lines of second order transition
  - N_f=2: O(4) universality class
  - N_f=3: Z(2) universality class
- $U_A(1)$ symmetry on chiral phase transition restored: 1st or 2nd order ($U(2)_L \otimes U(2)_R / U(2)_V$); broken: 2nd O(4)

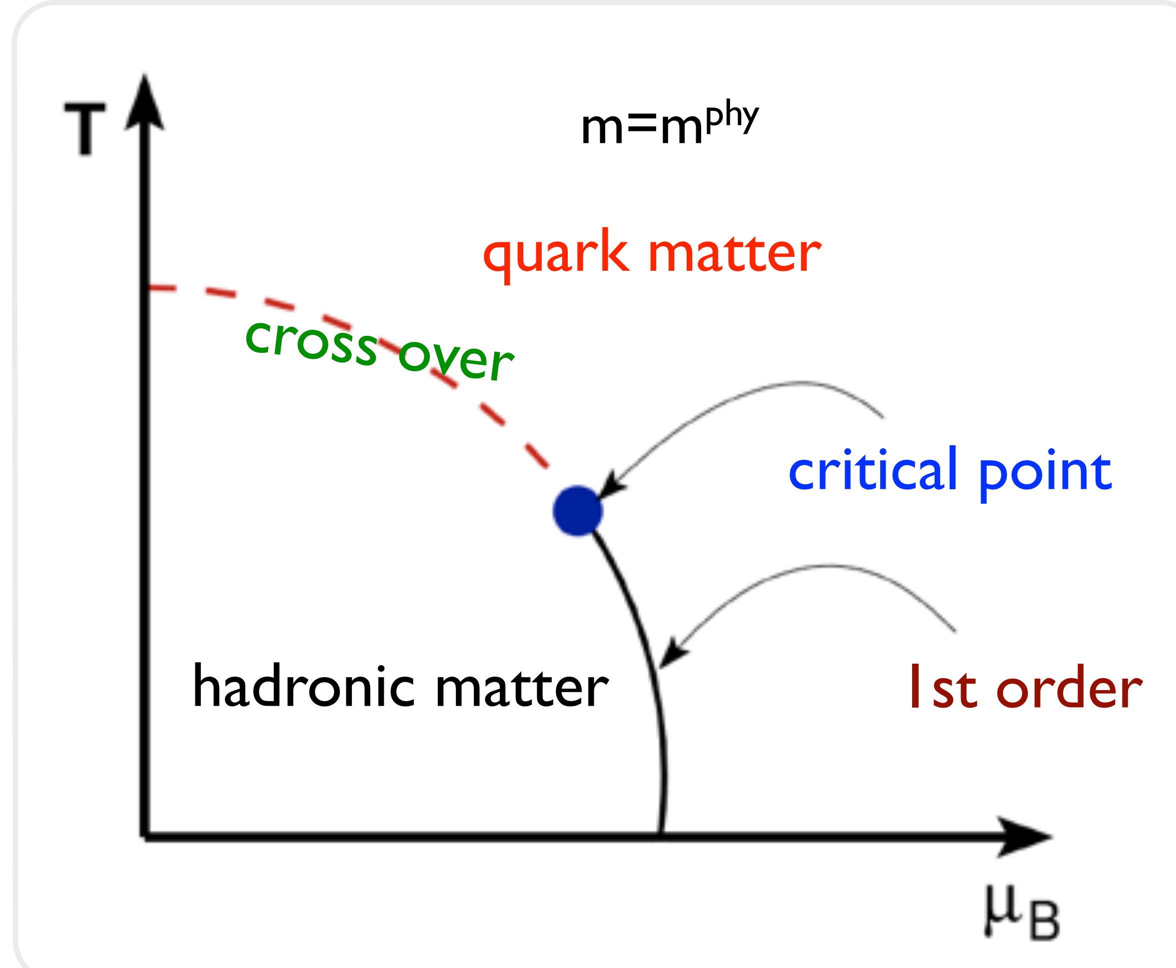
F. Wilczek, IJMPA 7(1992) 3911,6951
K. Rajagopal & F. Wilczek, NPB 399 (1993) 395
Gavin, Gocksch & Pisarski, PRD 49 (1994) 3079
Butti, Pelissetto and Vicar, JHEP 08 (2003)029

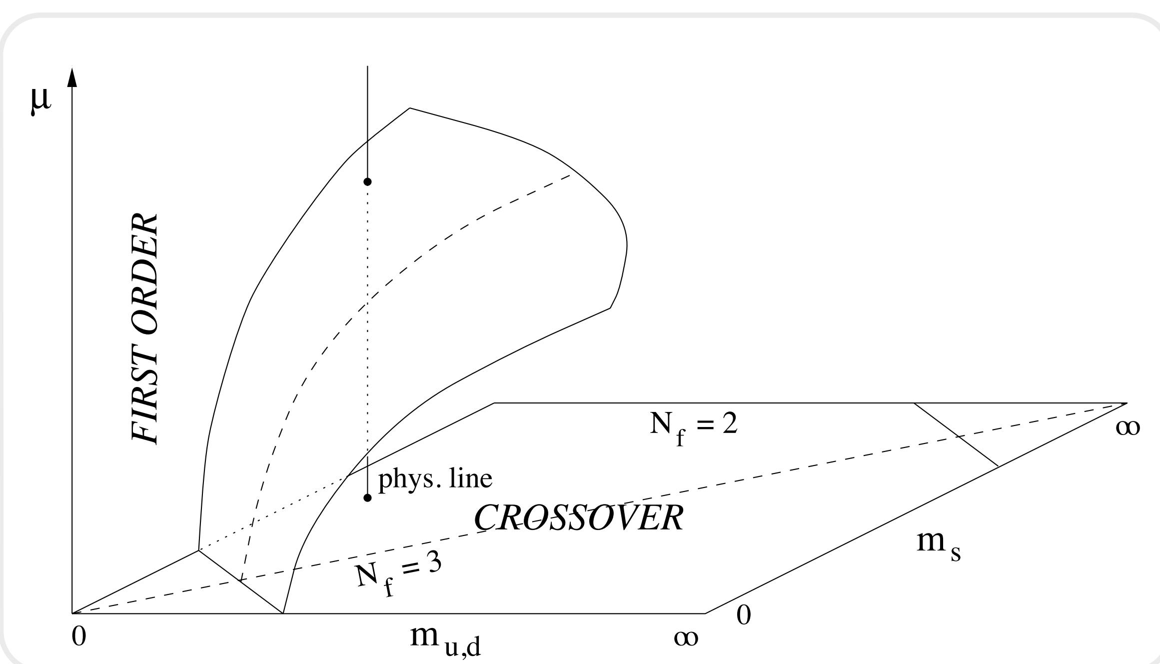
- fate of the axial U(1) symmetry at finite T ?
- The value of tri-critical point (m_s^{tri}) ?
- The location of 2^{nd} order Z(2) lines ?
- The influence of criticalities to the physical point ?

QCD transitions at the physical point

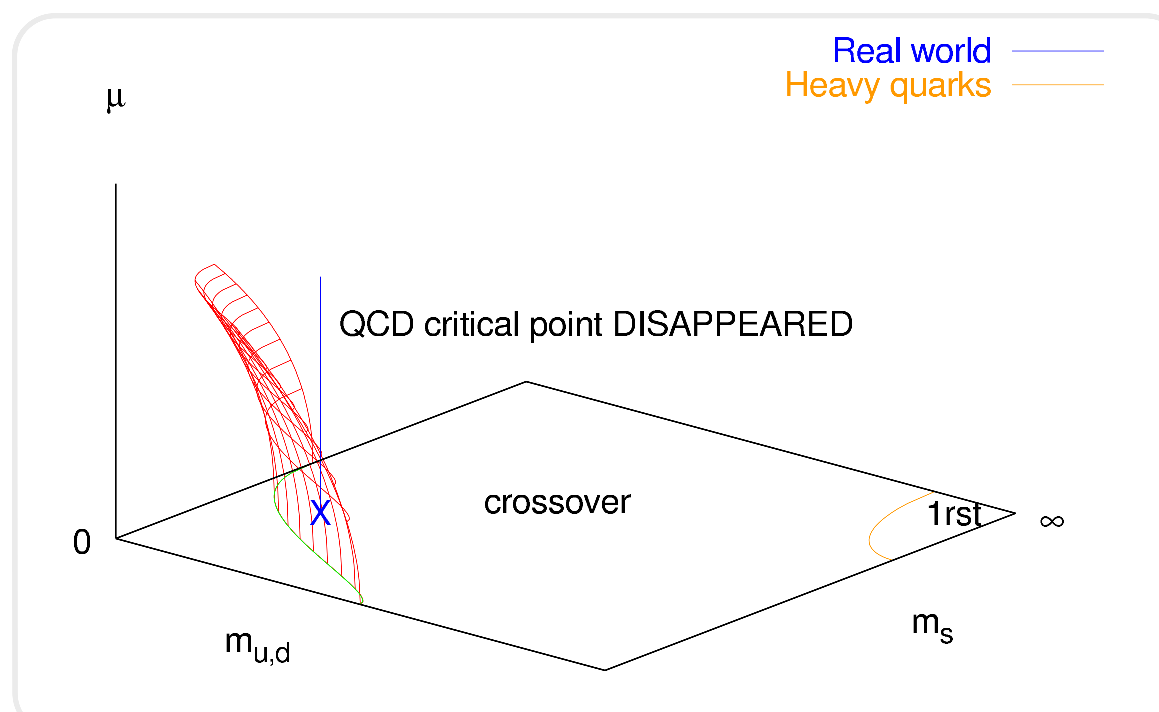


HTD, F. Karsch, S. Mukherjee, arXiv:1504.05274



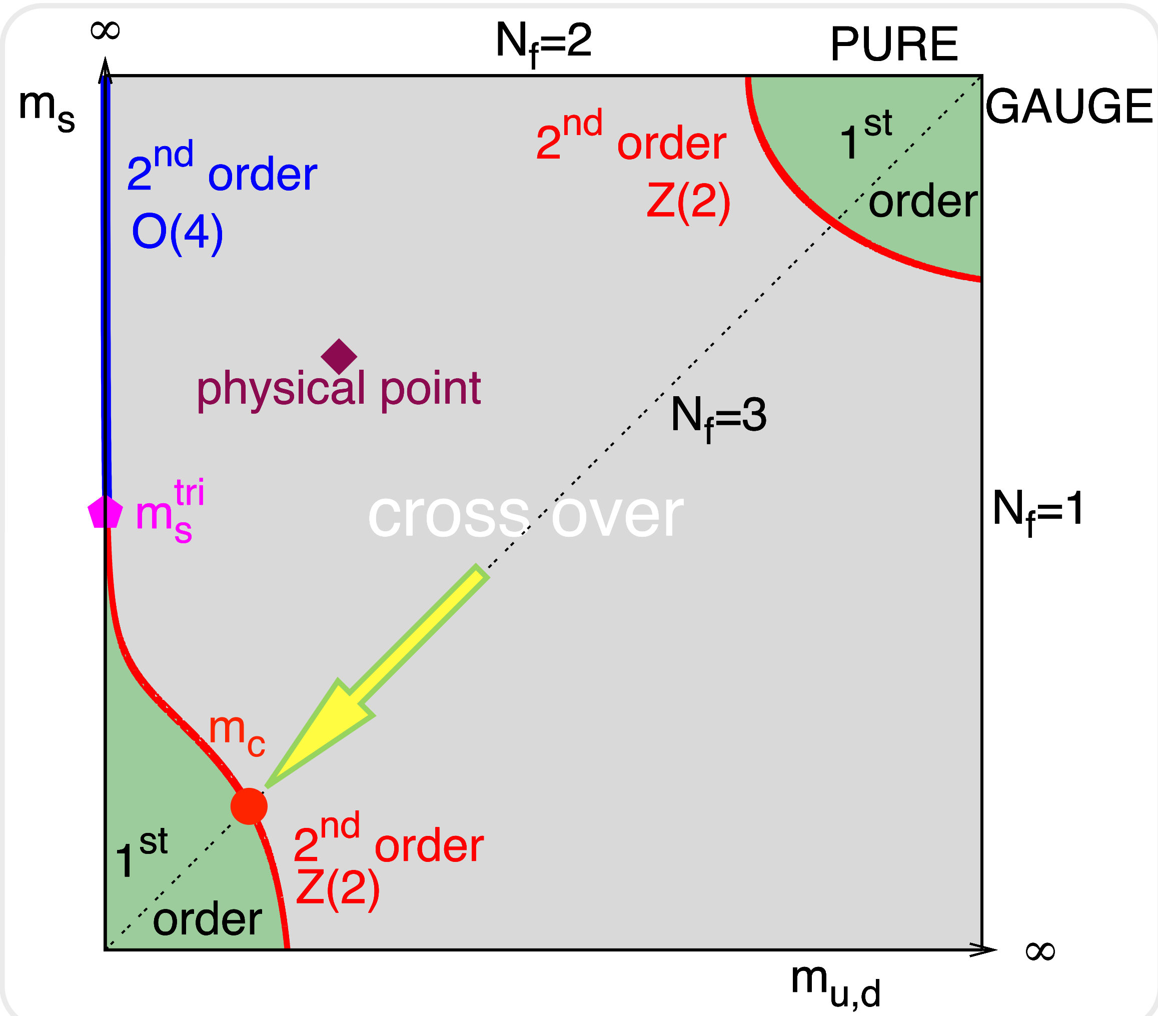


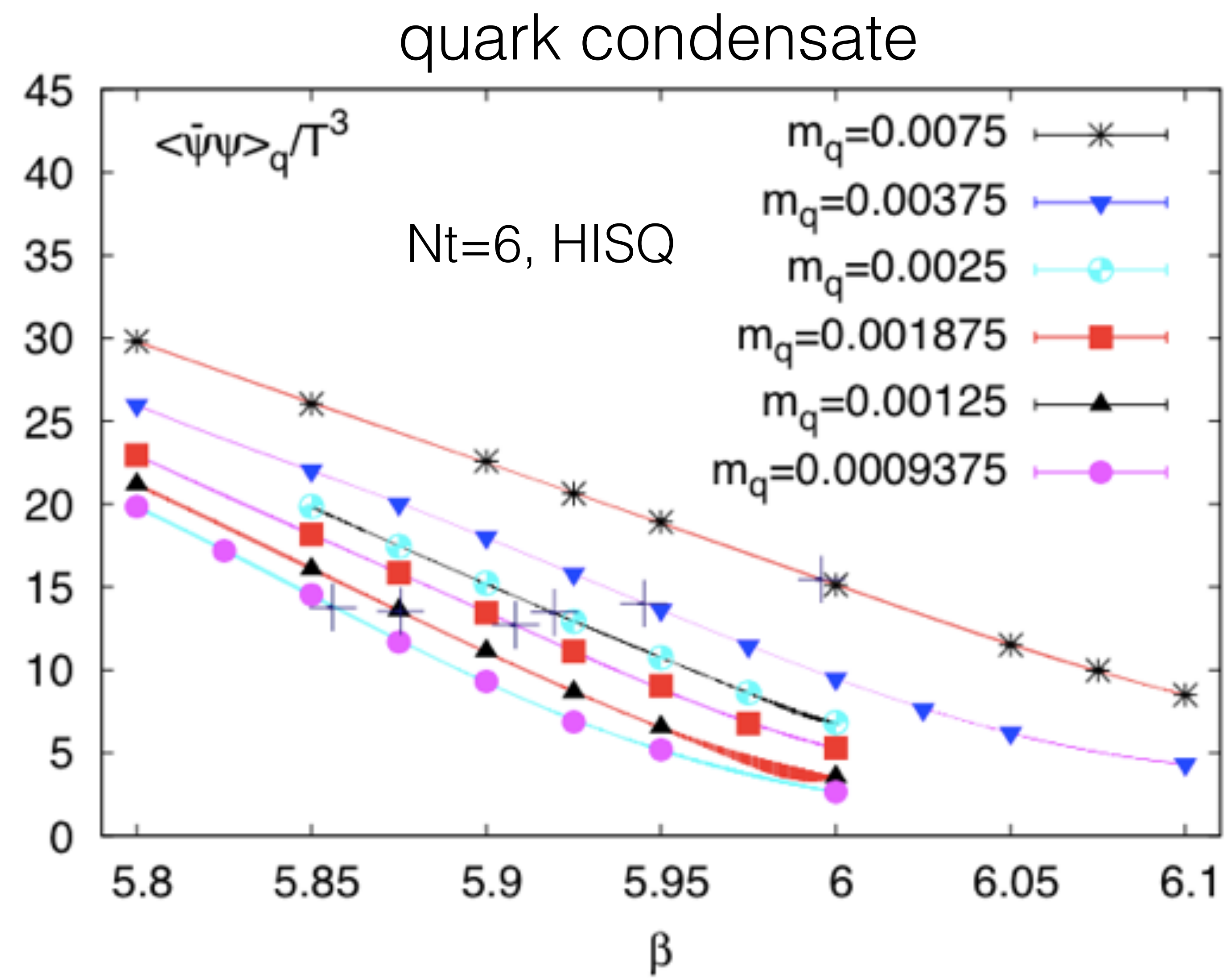
Karsch et al., '03, X.-Y. Jin et al., '15



de Forcrand & Philipsen, '07

chiral phase transition in Nf=3 QCD at μB=0



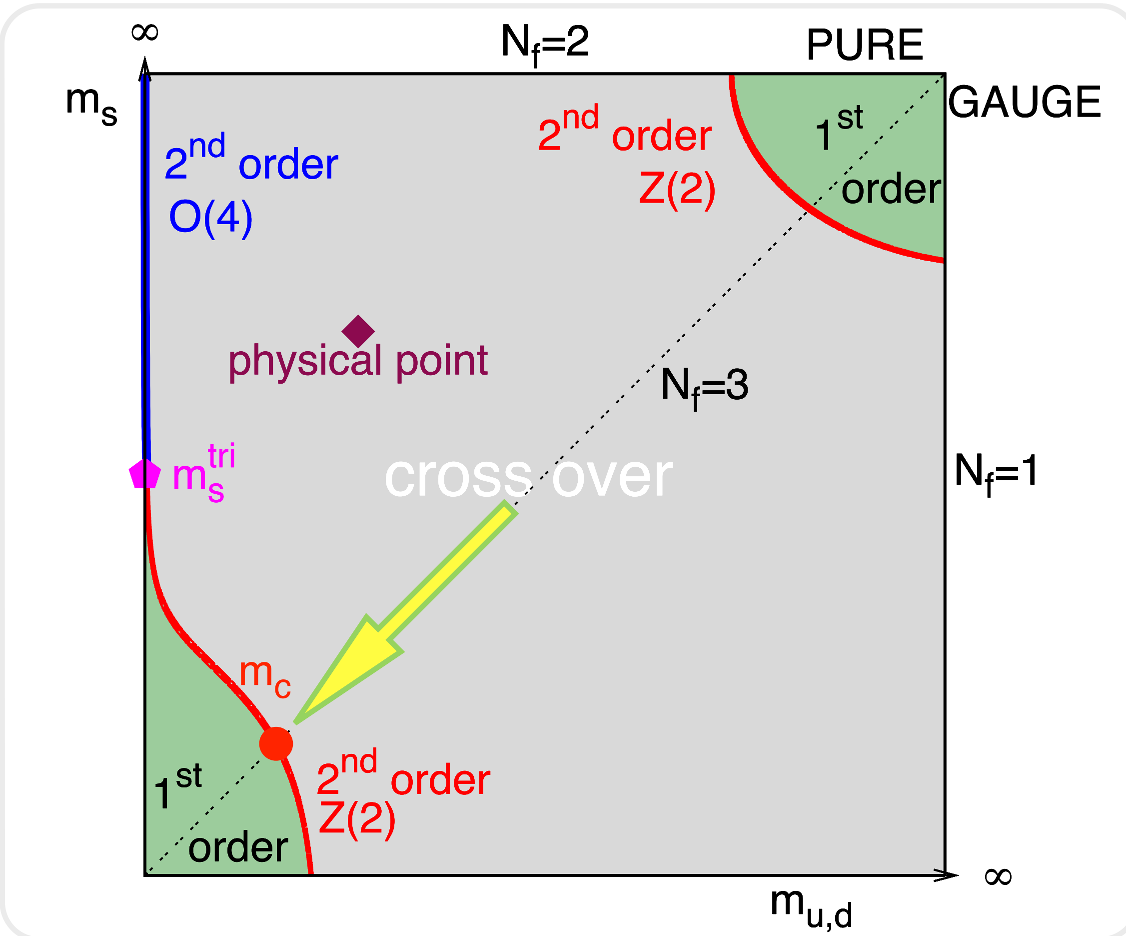


mass region: $200\ \text{MeV} \gtrsim m_\pi \gtrsim 80\ \text{MeV}$

No evidence of a first order phase transition

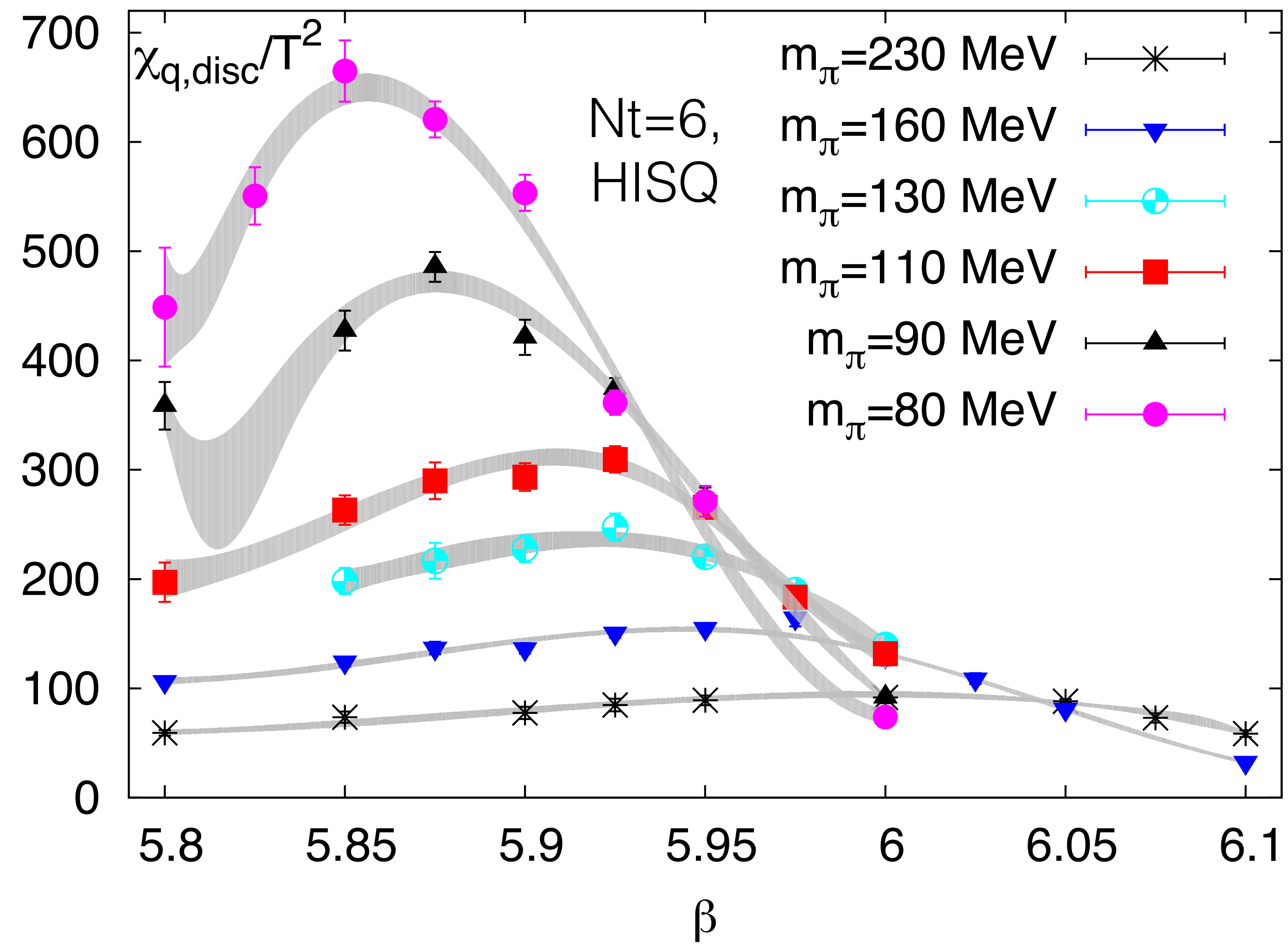
Bielefeld-BNL-CCNU,
Phys.Rev. D 95 (2017) no.7, 074505

Chiral phase transition in Nf=3 QCD at μ_B=0



physical point:
(ml, ms):(0.00375,0.10125)

disconnected chiral susceptibility

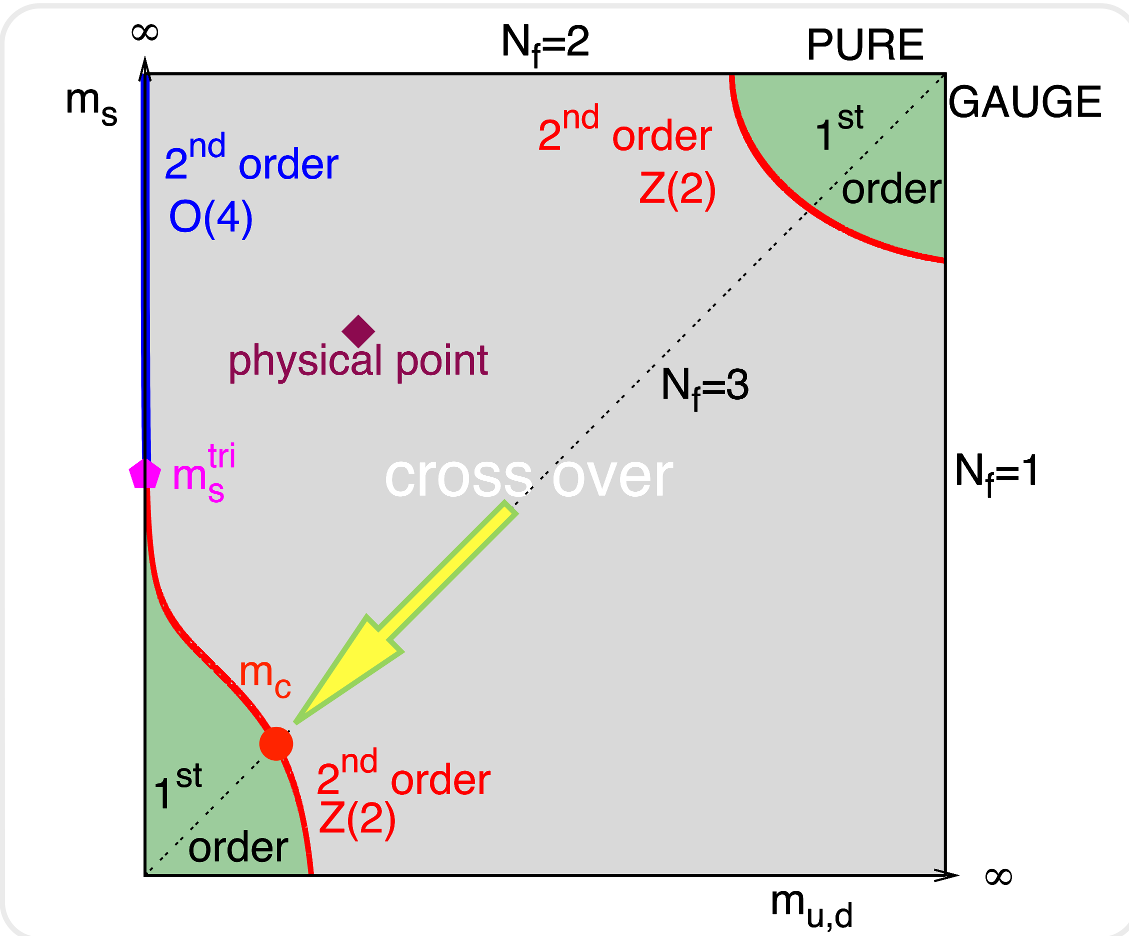


Bielefeld-BNL-CCNU
Phys.Rev. D 95 (2017) no.7, 074505

Close to Z(2) phase transition line:

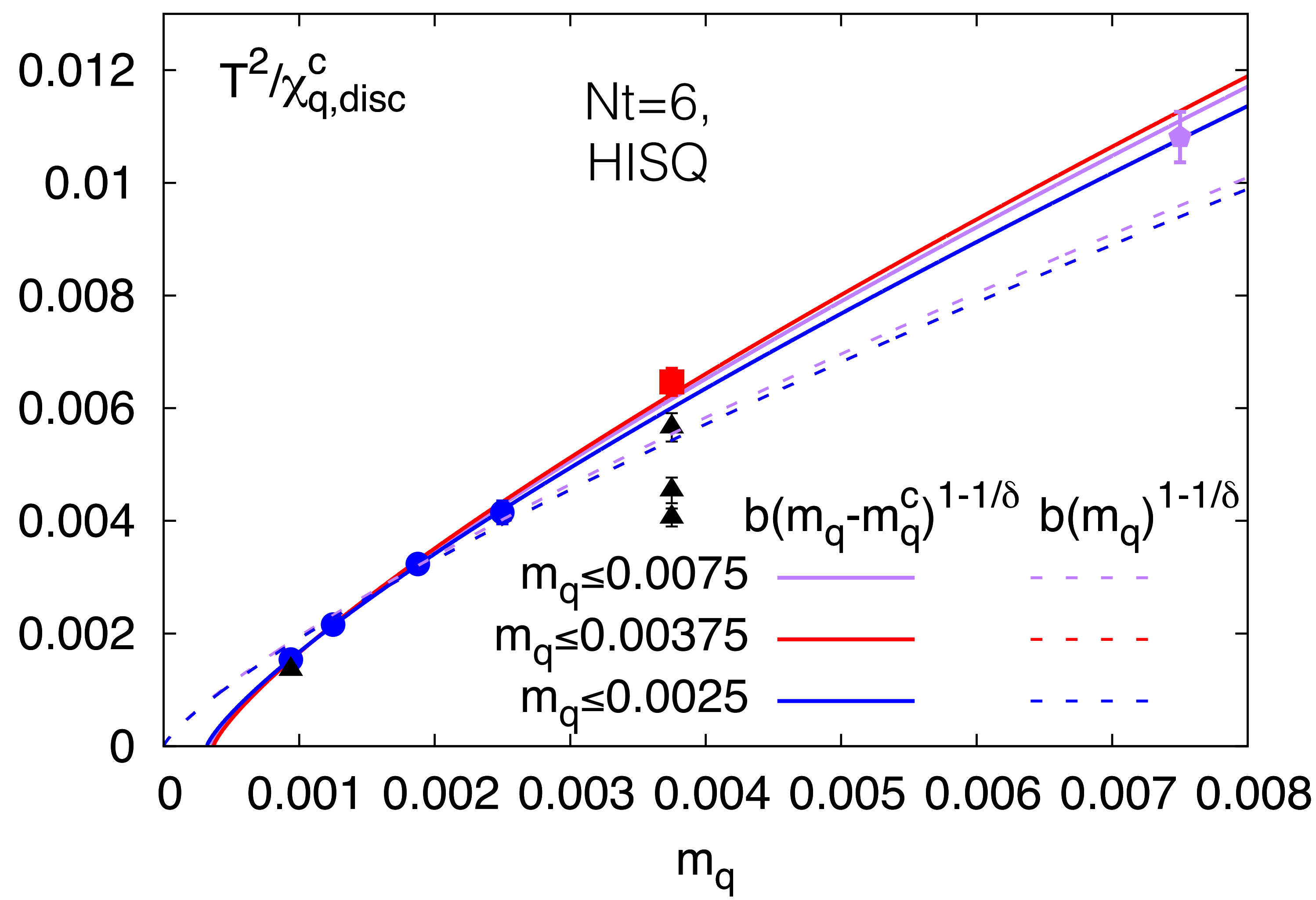
$$\chi_{q,disc}^{max} \sim (m - m_c)^{1/\delta - 1}$$

Chiral phase transition in Nf=3 QCD at μ_B=0



physical point:
(ml, ms):(0.00375,0.10125)

inverse peak height of
disconnected chiral susceptibility



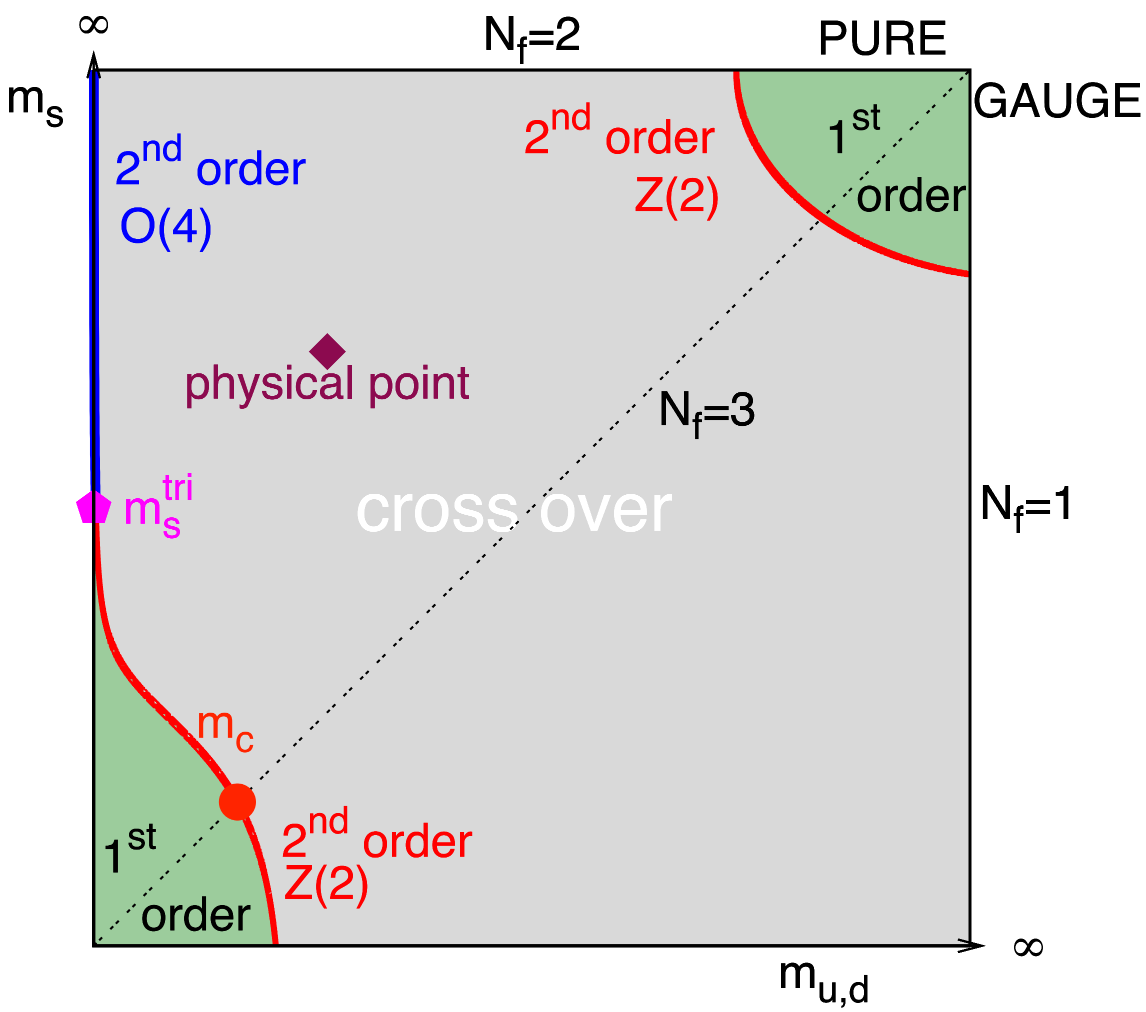
Bielefeld-BNL-CCNU
Phys.Rev. D 95 (2017) no.7, 074505

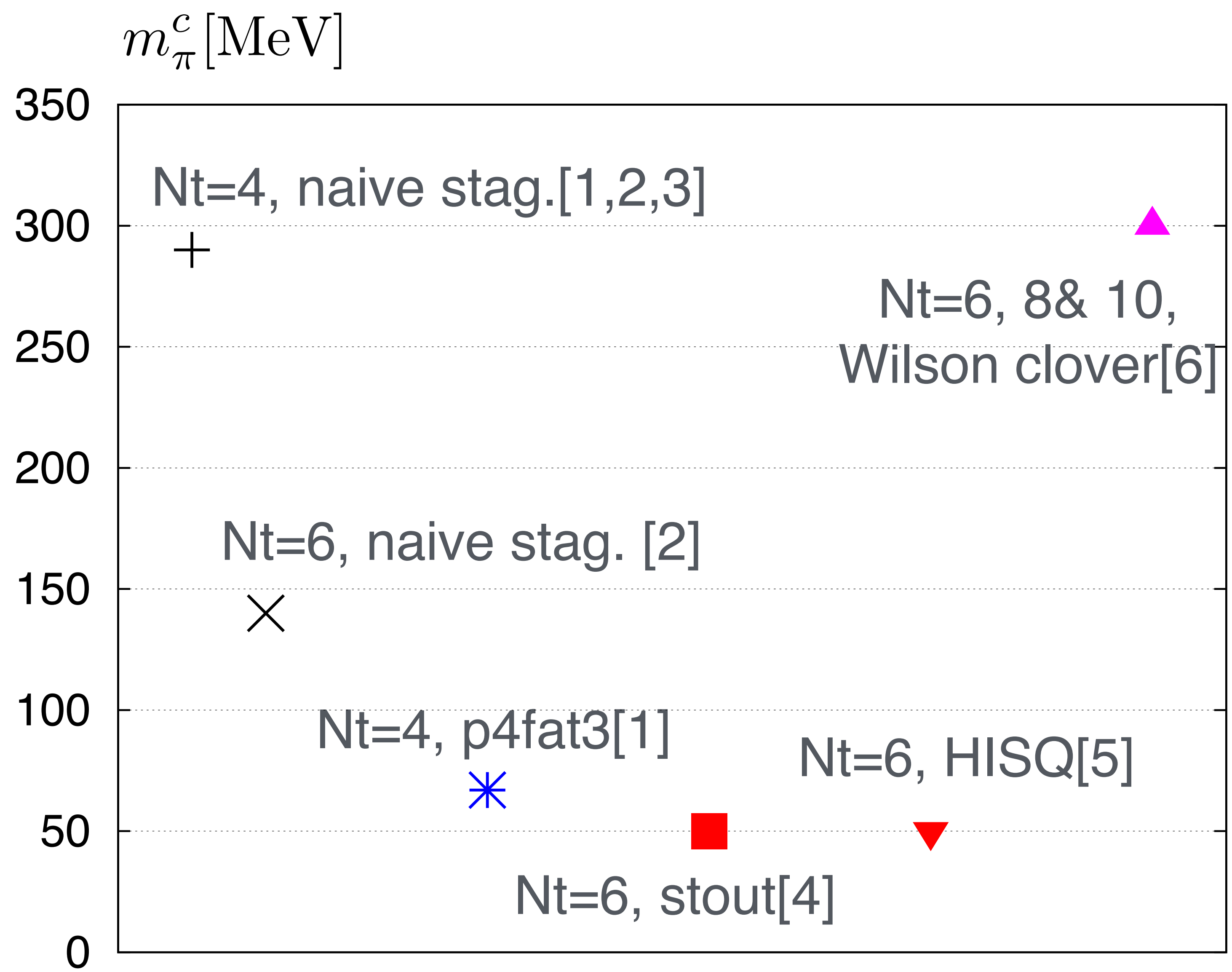
Close to Z(2) phase transition line:

$$\chi_{q,disc}^{max} \sim (m - m_c)^{1/\delta - 1}$$

critical quark mass m_c ~ 0.0004 $\Rightarrow$ $m_\pi^c \lesssim 50\mathrm{MeV}$

1st order chiral phase transition region

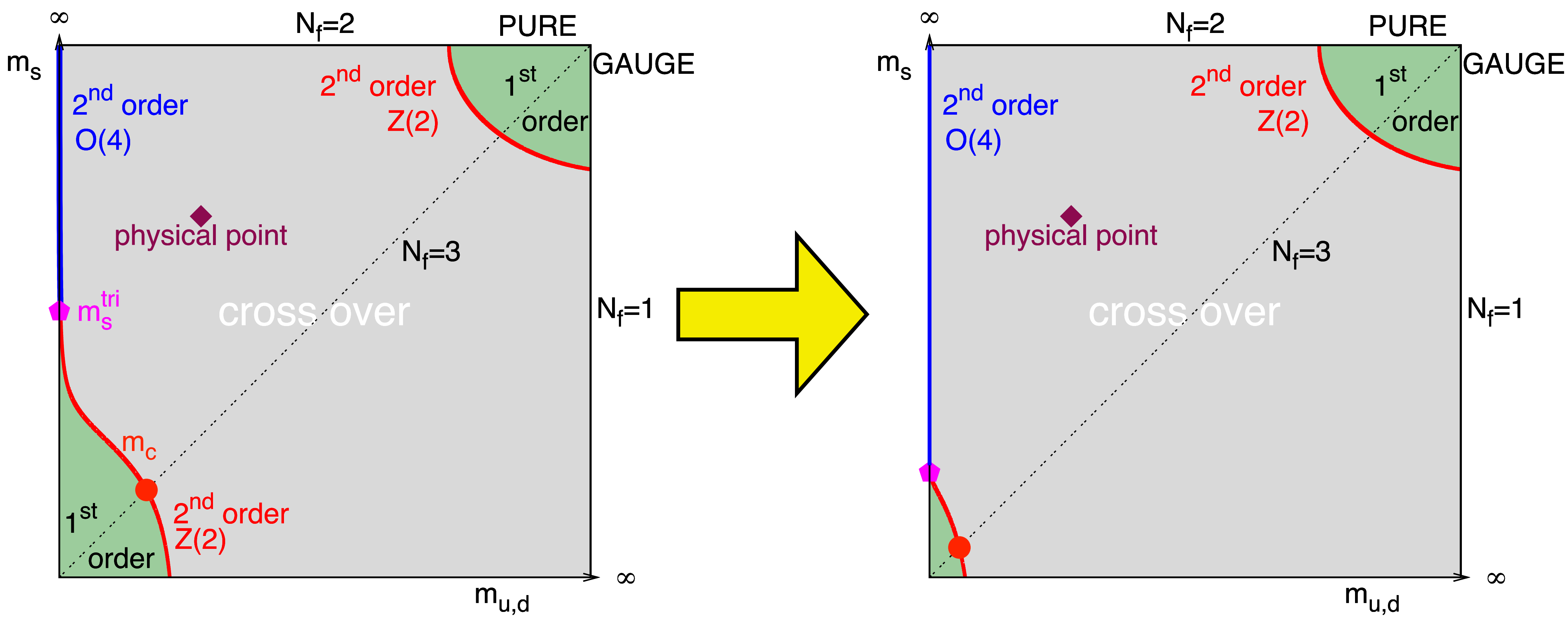




1st order chiral phase transition region shrinks towards the continuum limit

[1]F. Karsch et al., Nucl.Phys.Proc.Suppl. 129 (2004) 614 [2] P. de Forcrand et al, PoS LATTICE2007 (2007) 178
[3]D. Smith & C. Schmidt, Lattice 2011 [4]G. Endrodi et al., PoS LAT2007 (2007) 228
[5] Bielefeld-BNL-CCNU, Phys.Rev. D 95 (2017) no.7, 074505 [6]Y. Nakamura, Lattice 15',PRD92 (2015) no.11, 114511

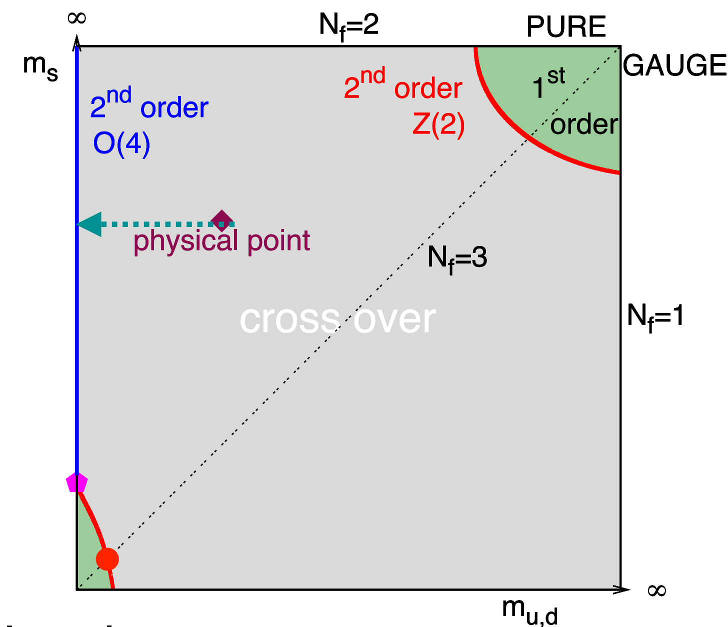
Chiral phase transition region in Nf=3 QCD at μ_B=0



1st order chiral phase transition seem to be not much relevant for thermodynamics at the physical point

How about the 2nd order O(4) transition line?

Universal behavior of chiral phase transition in N_f=2+1 QCD at μ_B=0



Behavior of the free energy close to critical lines

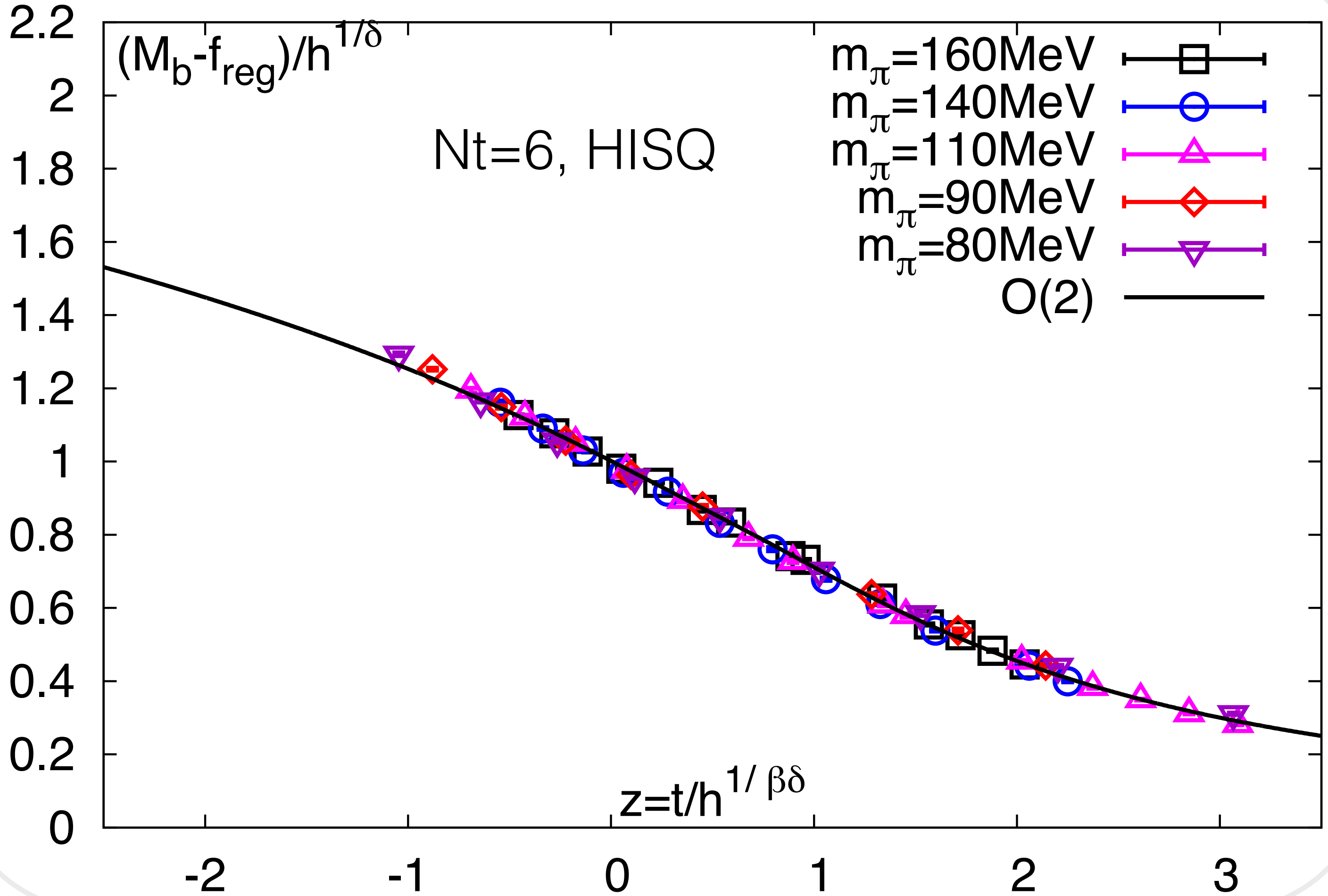
$$f(m,T)=h^{1+1/\delta} f_s(z) + f_{reg}(m,T), \qquad z=t/h^{1/\beta\delta}$$

h: external field, t: reduced temperature, β,δ: universal critical exponents

$$M = -\partial f(t,h)/\partial h = h^{1/\delta} f_G(z) + f_{reg}(t,h)$$

$h \sim m$; $t \sim T-T_C$

$f_G(z)$: O(2) scaling functions



QCD: SU(2)xSU(2)≃O(4)

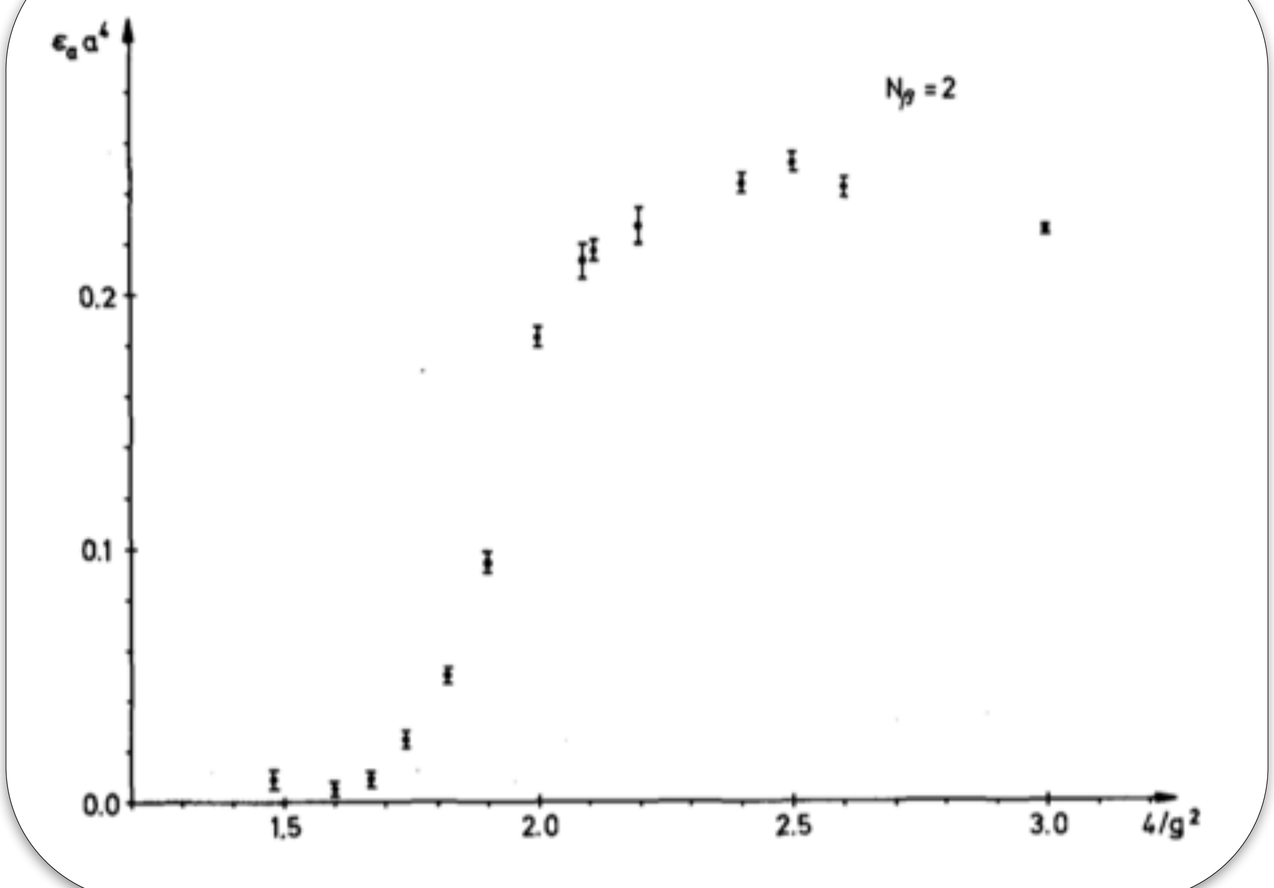
Some evidence of O(N) scaling for chiral phase transition

S.-T. Li, Lattice 2016, Bielefeld-BNL-CCNU, PoS LATTICE2016 (2017) 372

See also T. Umeda, [WHOT], arXiv:1612.09449

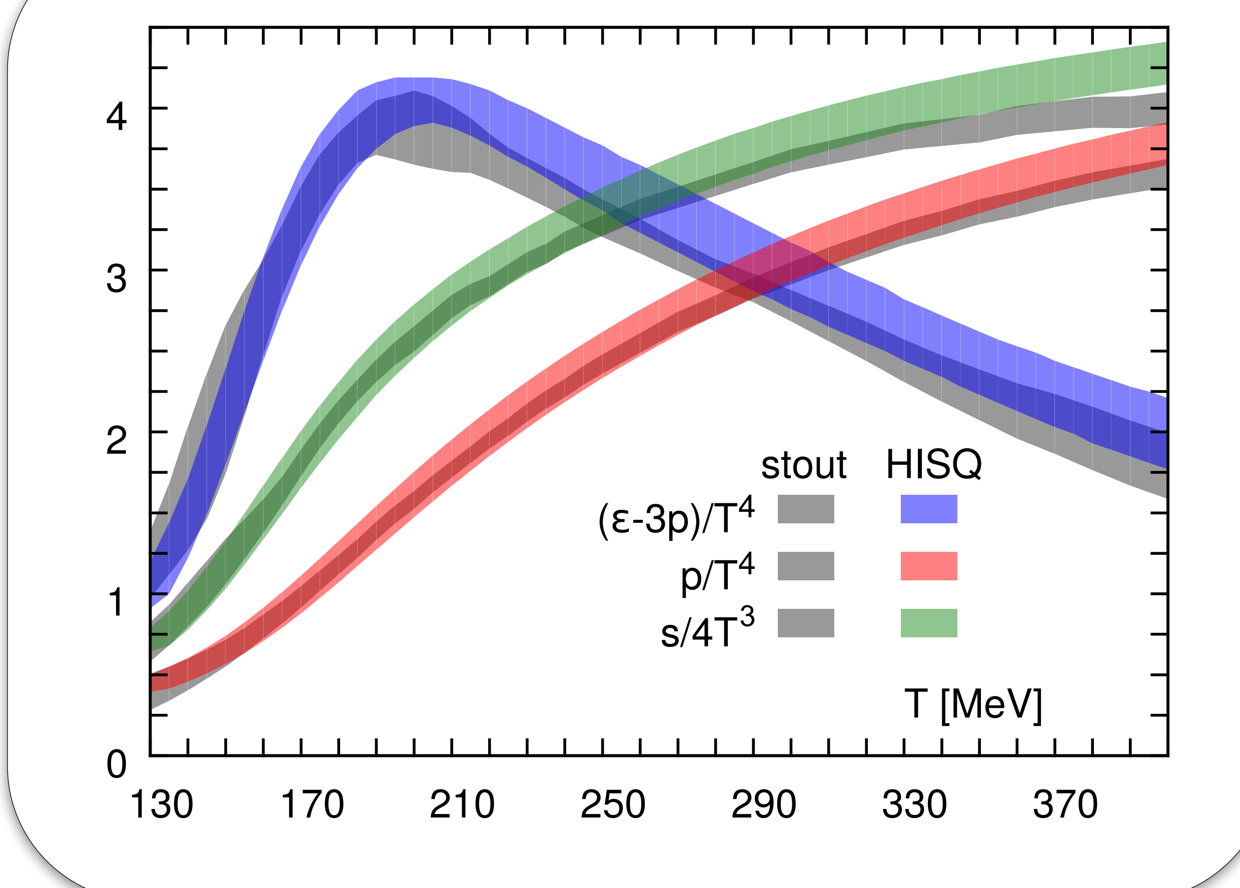
Lattice QCD calculation of EoS at μ_B =0

SU(2) pure gauge
at a finite lattice cutoff of Nt=2



J. Engels, F. Karsch, H. Satz, I. Montvay
Phys. Lett. B 101 (1981) 89-94

Nf=2+1, physical pion mass
continuum extrapolated



HotQCD, PRD 90 (2014) 094503
Wuppertal-Budapest, Phys. Lett. B730 (2014) 99

- First lattice QCD calculation of EoS was done in 1981
- Only recently a conclusive QCD EoS at μ_B=0 is obtained

Lattice simulations at nonzero μ_B

Taylor expansion of the QCD pressure:

Allton et al., Phys.Rev. D66 (2002) 074507
Gavai & Gupta et al., Phys.Rev. D68 (2003) 034506

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln \mathcal{Z}(T, V, \hat{\mu}_u, \hat{\mu}_d, \hat{\mu}_s) = \sum_{i,j,k=0}^{\infty} \frac{\chi_{ijk}^{BQS}}{i!j!k!} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k$$

- Taylor expansion coefficients at μ=0 are computable in LQCD

$$\chi_{ijk}^{BQS} \equiv \chi_{ijk}^{BQS}(T) = \frac{1}{VT^3} \frac{\partial P(T,\hat{\mu})/T^4}{\partial \hat{\mu}_B^i \partial \hat{\mu}_Q^j \partial \hat{\mu}_S^k}\Big|_{\hat{\mu}=0}$$

- Thermodynamic quantities can be obtained using relations, e.g.

$$\frac{\epsilon - 3p}{T^4} = T \frac{\partial P/T^4}{\partial T} = \sum_{i,j,k=0}^{\infty} \frac{T\, \mathrm{d}\chi_{ijk}^{BQS}/\mathrm{d}T}{i!j!k!} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k$$

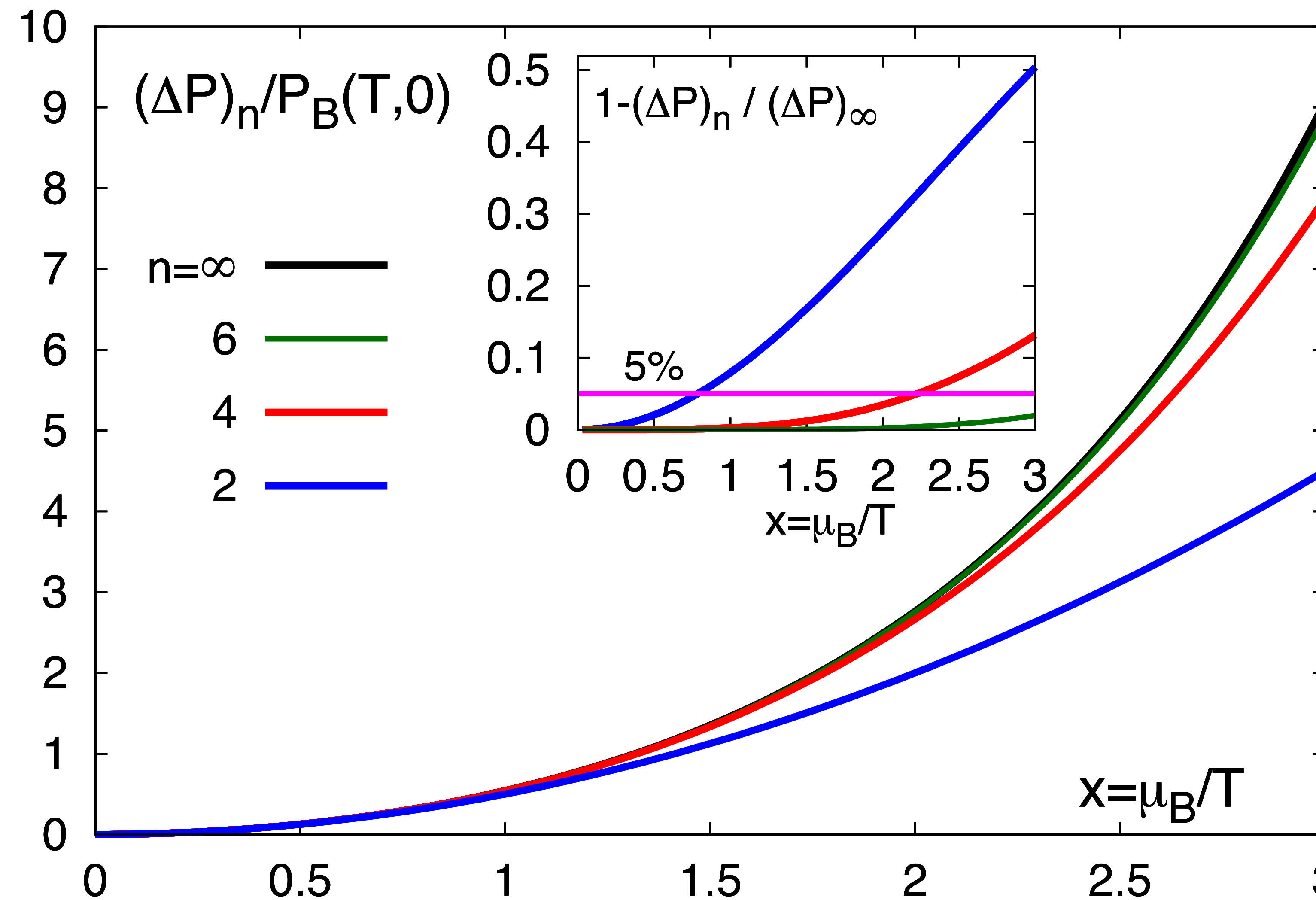
Truncation effects of pressure in HRG

Pressure of hadron resonance gas (HRG)

$$P(T,\mu_B) = P_M(T) + P_B(T,\hat{\mu}_B)$$
$$= P_M(T) + P_B(T,0) + P_B(T,0)(\cosh(\hat{\mu}_B) - 1)$$

Truncate the Taylor expansion at (2n)-th order:

$$(\Delta P)_n = \Big(P_B(T,\mu_B) - P_B(T,0)\Big)_n$$
$$= \sum_{k=1}^{n} \frac{\chi_{2k}^{B,HRG}(T)}{(2k)!}\hat{\mu}_B^{2k}$$
$$\simeq P_B(T,0)\sum_{i=1}^{n}\frac{1}{(2k)!}\hat{\mu}_B^{2k}$$



Radius of convergence from HRG is infinity

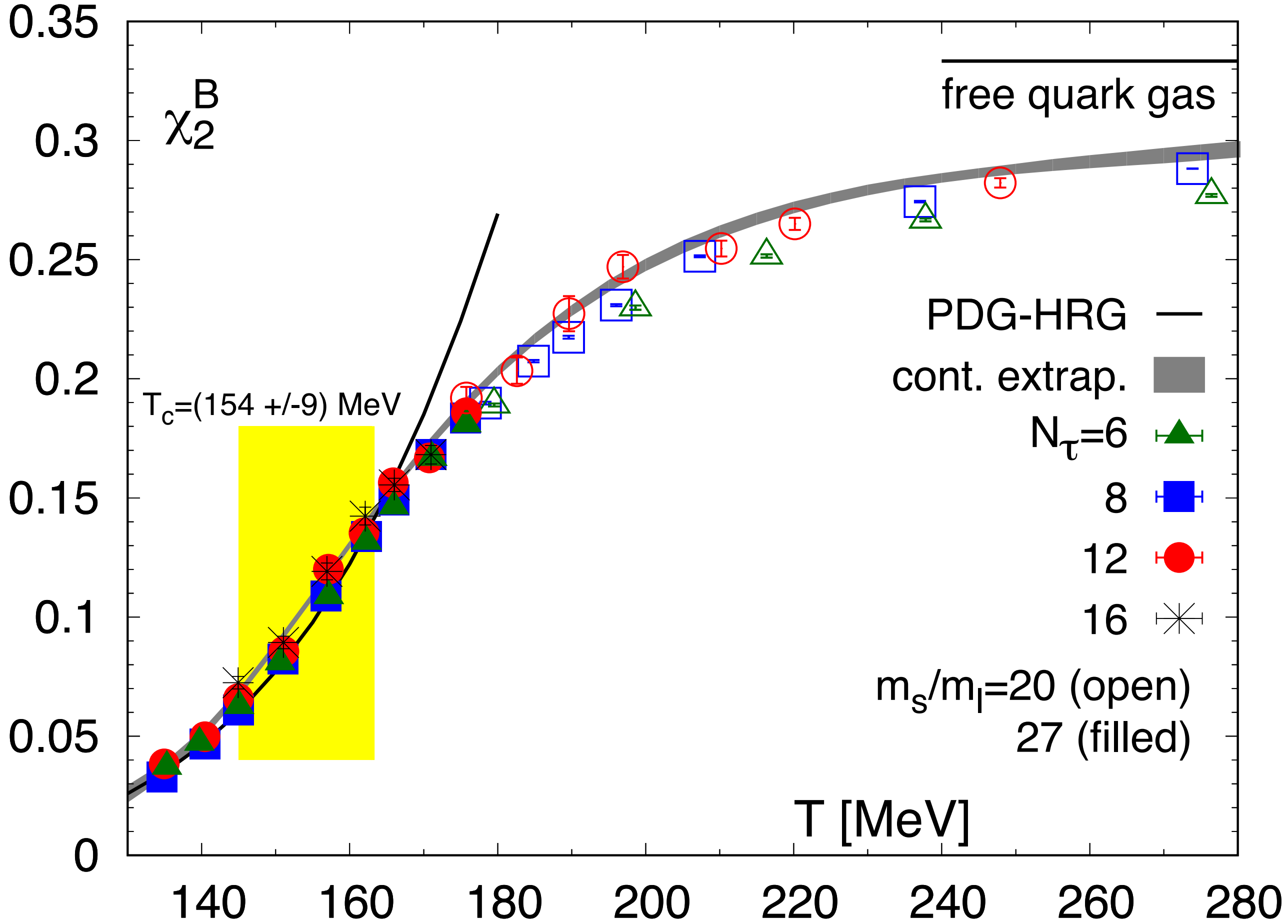
Pressure of QCD at μ_B =/=0

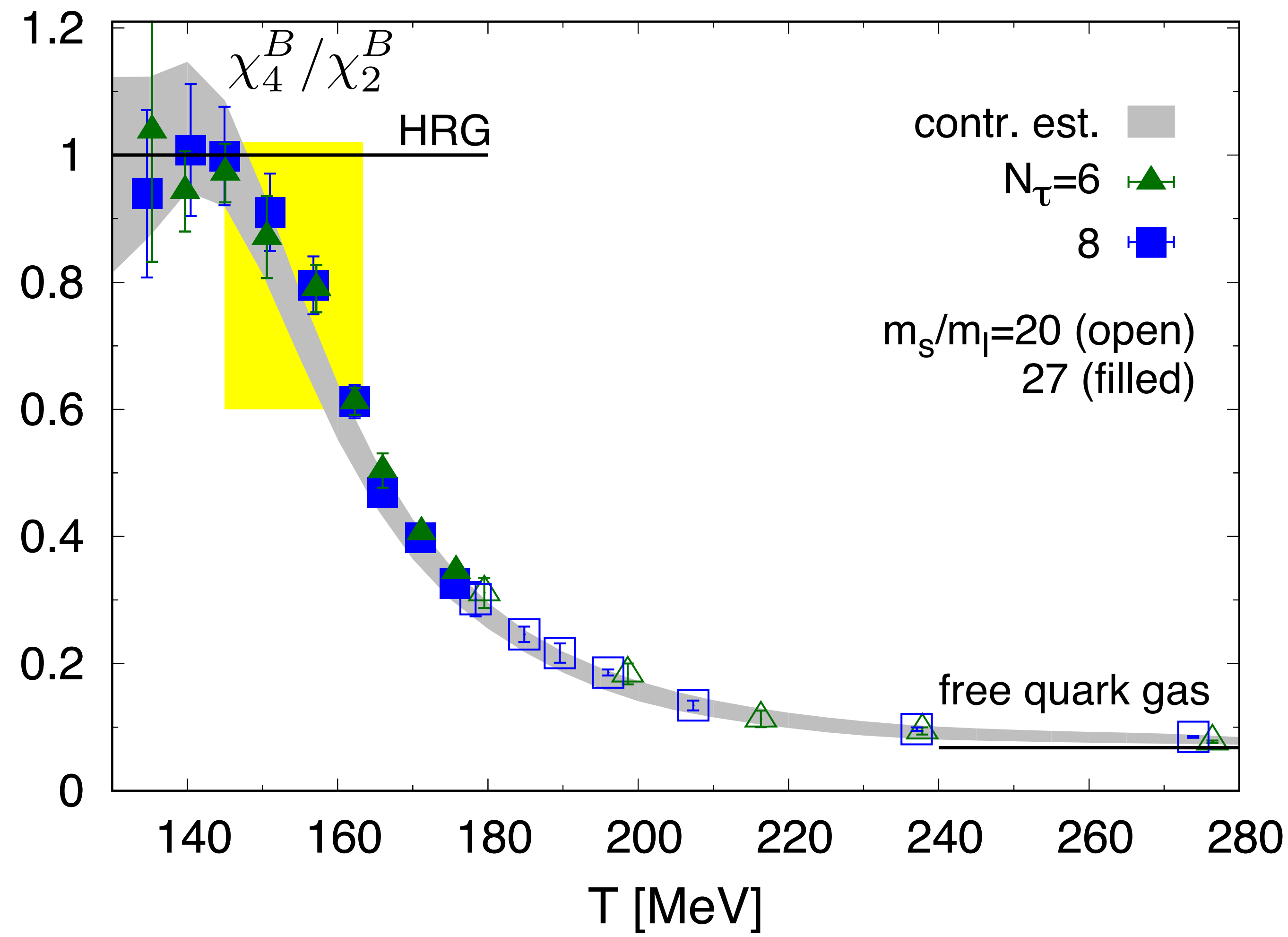
$\mu_Q=\mu_S=0$:

$$\Delta(P/T^4) = \frac{P(T,\mu_B)-P(T,0)}{T^4} = \sum_{n=1}^{\infty} \frac{\chi_{2n}^B(T)}{(2n)!}\left(\frac{\mu_B}{T}\right)^{2n}$$
$$= \frac{1}{2}\chi_2^B(T)\hat{\mu}_B^2\left(1+\frac{1}{12}\frac{\chi_4^B(T)}{\chi_2^B(T)}\hat{\mu}_B^2+\frac{1}{360}\frac{\chi_6^B(T)}{\chi_2^B(T)}\hat{\mu}_B^4+\cdots\right)$$

LO expansion coefficient
variance of net-baryon number distri.

NLO expansion coefficient
kurtosis * variance





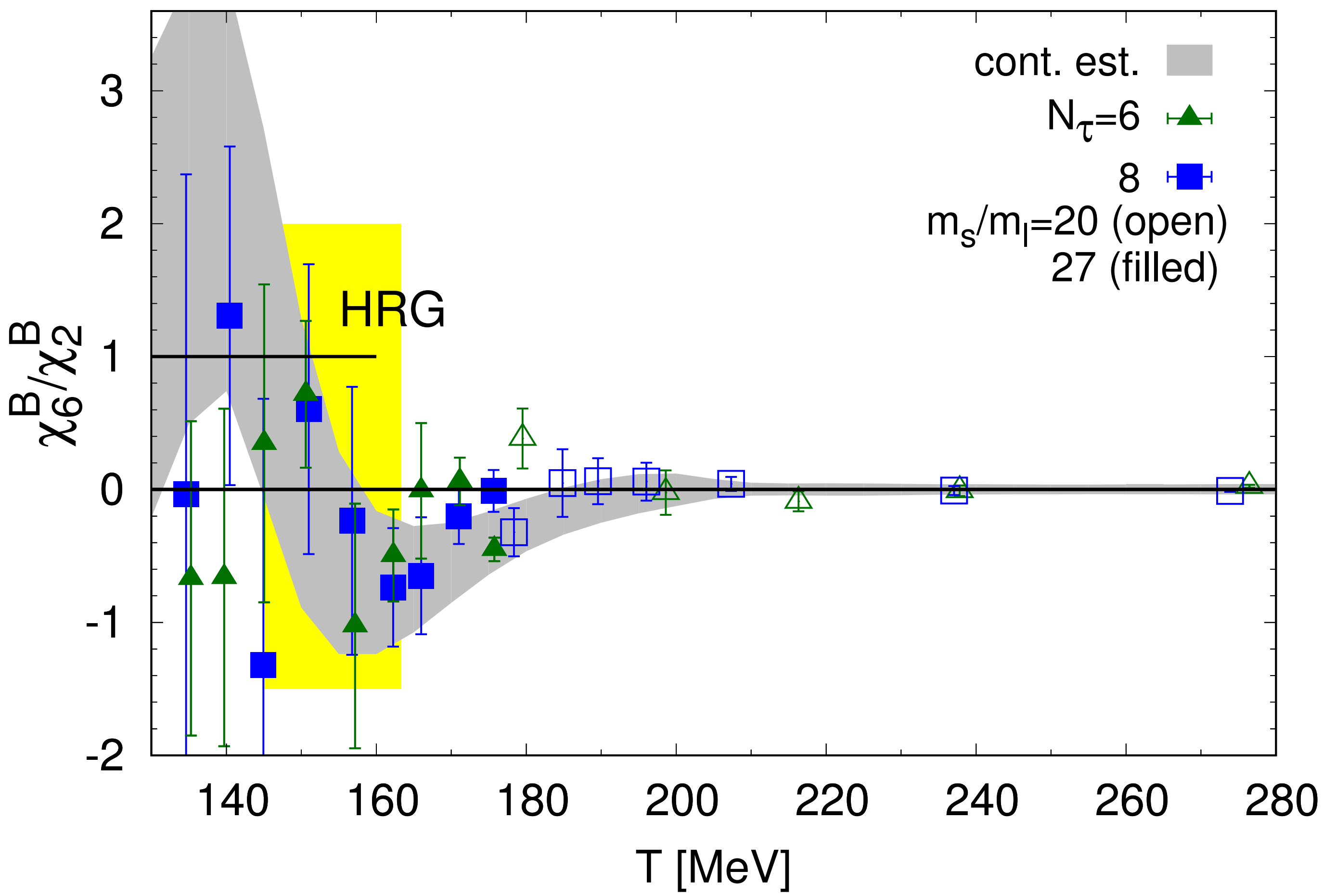
Bielefeld-BNL-CCNU, Phys.Rev. D95 (2017) no.5, 054504

Pressure of QCD at μ_B =/=0

$\mu_Q = \mu_S = 0$:

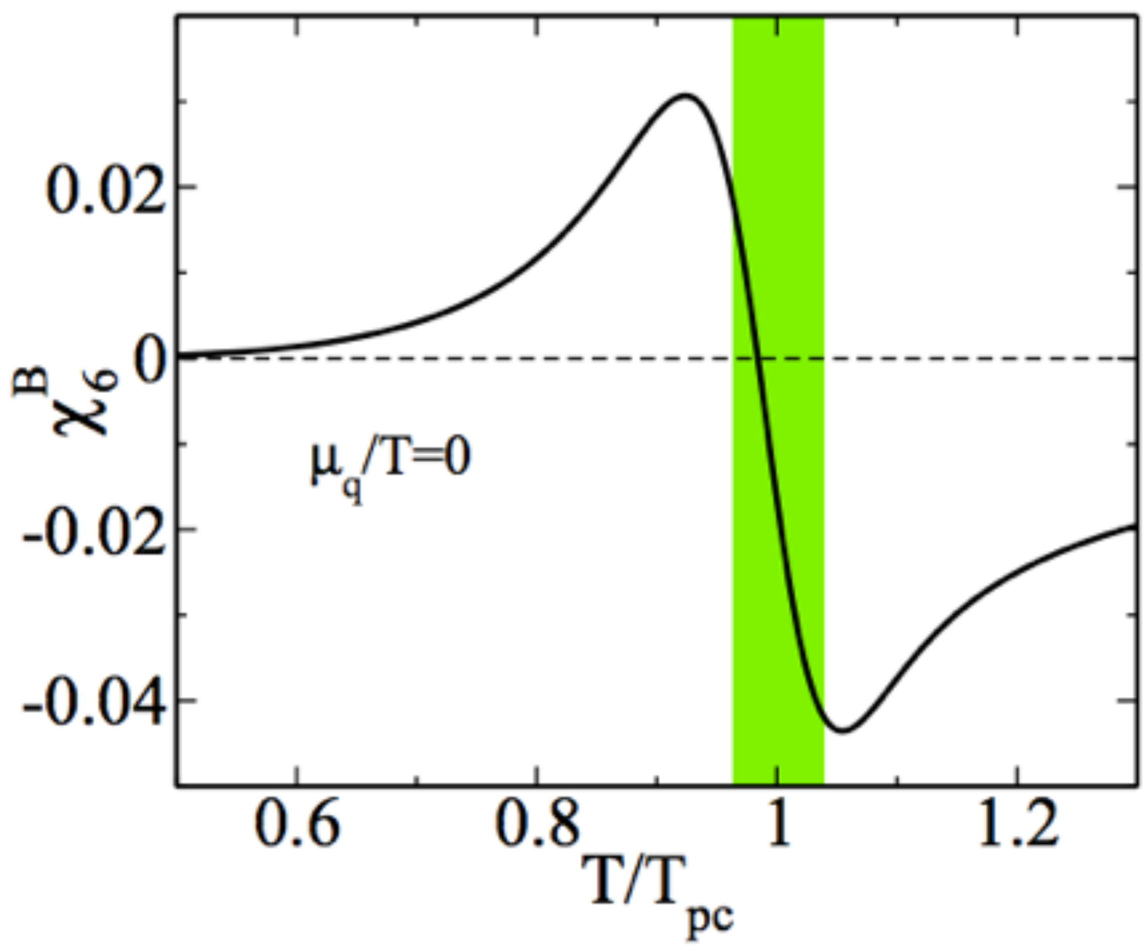
$$\Delta(P/T^4) = \frac{P(T,\mu_B) - P(T,0)}{T^4} = \sum_{n=1}^{\infty} \frac{\chi_{2n}^B(T)}{(2n)!}\left(\frac{\mu_B}{T}\right)^{2n}$$
$$= \frac{1}{2}\chi_2^B(T)\hat{\mu}_B^2\left(1 + \frac{1}{12}\frac{\chi_4^B(T)}{\chi_2^B(T)}\hat{\mu}_B^2 + \frac{1}{360}\frac{\chi_6^B(T)}{\chi_2^B(T)}\hat{\mu}_B^4 + \cdots\right)$$

NNLO expansion coefficient



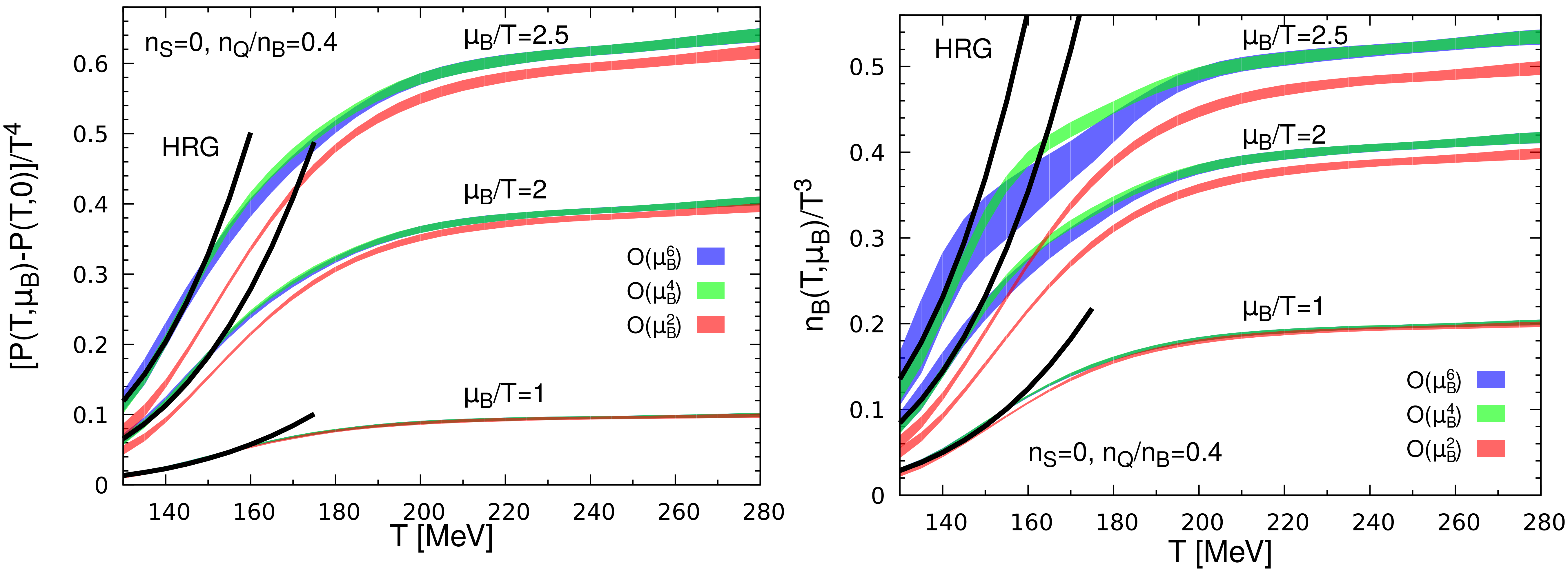
Bielefeld-BNL-CCNU, Phys.Rev. D95 (2017) no.5, 054504

PQM with O(4) symmetry



B. Friman et al.,EPJC71 (2011) 1694

Pressure and baryon number density in the strangeness neutral case



Bielefeld-BNL-CCNU, Phys.Rev. D95 (2017) no.5, 054504

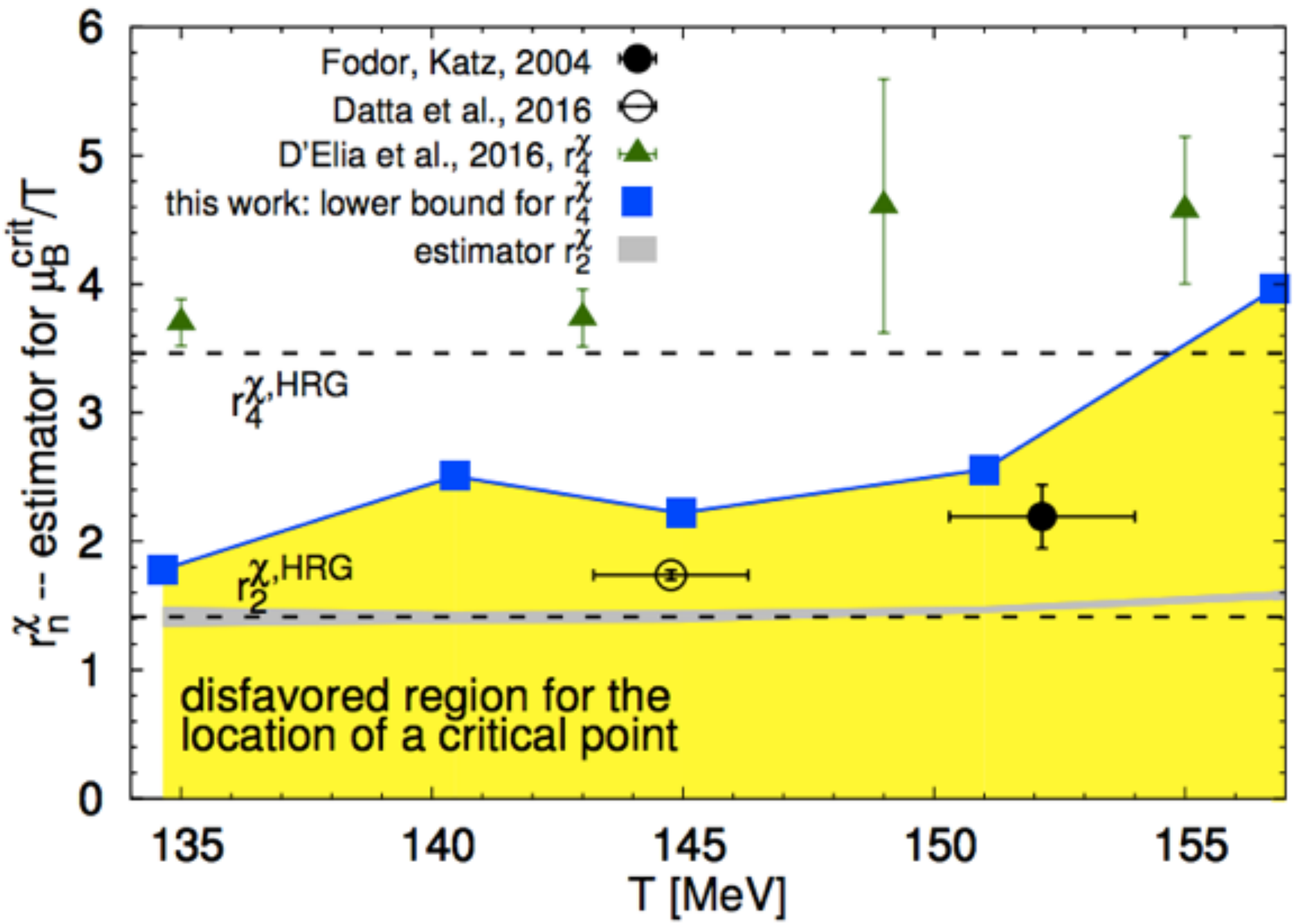
The EoS is well under control at $\mu_B/T \lesssim 2$ or $\sqrt{s_{NN}} \gtrsim 12$ GeV

Consistent results obtained using analytic continuations from the imaginary mu

Wuppertal-Budapest-Houston: EPJ Web Conf. 137(2017) 07008

Estimates of the radius of convergence

radius of convergence = $\lim_{n\to\infty} r_{2n}^{\chi} = \lim_{n\to\infty} \left| \frac{2n(2n-1)\chi_{2n}^B}{\chi_{2n+2}^B} \right|^{1/2}$



HISQ + Taylor Exp. (this work):
Nf=2+1, Nt=8
Bielefeld-BNL-CCNU,
PRD 95 (2017) no.5, 054504

stout + Img. mu:
Nf=2+1,Nt=8
D'Elia et al., PRD 95 (2017) 094503

unimproved staggered + Taylor Exp.:
Nf=2, Nt=4,6,8
Datta et al., PRD 95 (2017) 054512

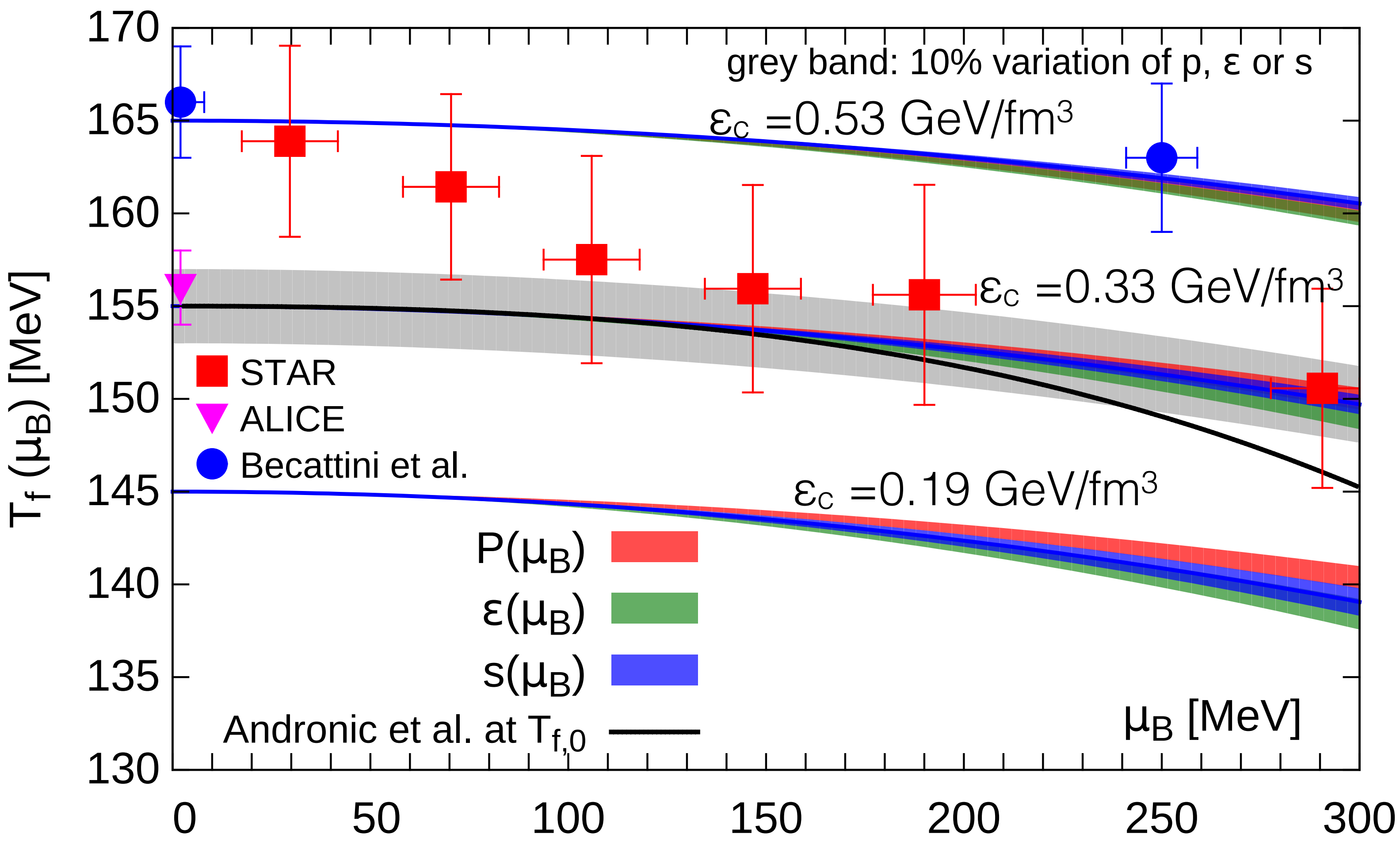
unimproved staggered + Reweighting:
Nf=2+1, Nt=4
Fodor and Katz, JHEP 0404 (2004) 050

A QCD critical point is disfavored at $\mu_B/T \lesssim 2$ at $T \gtrsim 135$ MeV

A. Bazavov, HTD et al., [Bielefeld-BNL-CCNU], Phys.Rev. D95 (2017) no.5, 054504

Line of constant physics to $O(\hat{\mu}_B^4)$ and freeze-out

Parameterization: $T(\mu_B) = T(0)(1 - \kappa_2 \hat{\mu}_B^2 + \mathcal{O}(\hat{\mu}_B^4))$



Bielefeld-BNL-CCNU, Phys.Rev. D95 (2017) no.5, 054504

curvature at constant b:

$$0.006 \leq \kappa_2^b \leq 0.012, \quad b = P, \epsilon, s$$

Bielefeld-BNL-CCNU, PRD95 (2017) no.5, 054504

curvature of transition line:

$$\kappa_2^t \approx 0.006 - 0.013$$

Cea et al., PRD 93 (2016) no. 1, 014507
Bellwied et al., PLB 751 (2015) 559
Bonati, PRD 92 (2015) no. 5, 054503
Kaczmarek et al., PRD 83 (2011) 014504,
Endrodi et al,, JHEP 1104 (2011) 001

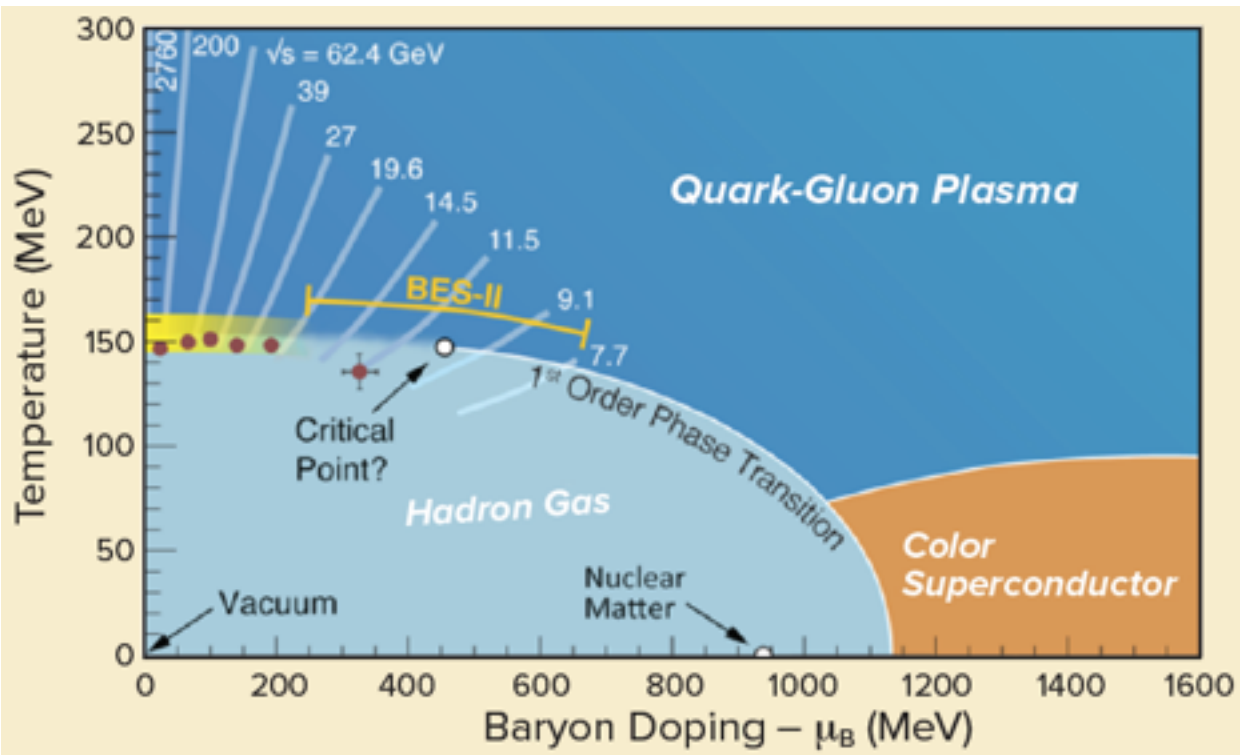
curvature of freeze-out line:

$$\kappa_2^f \lesssim 0.011$$

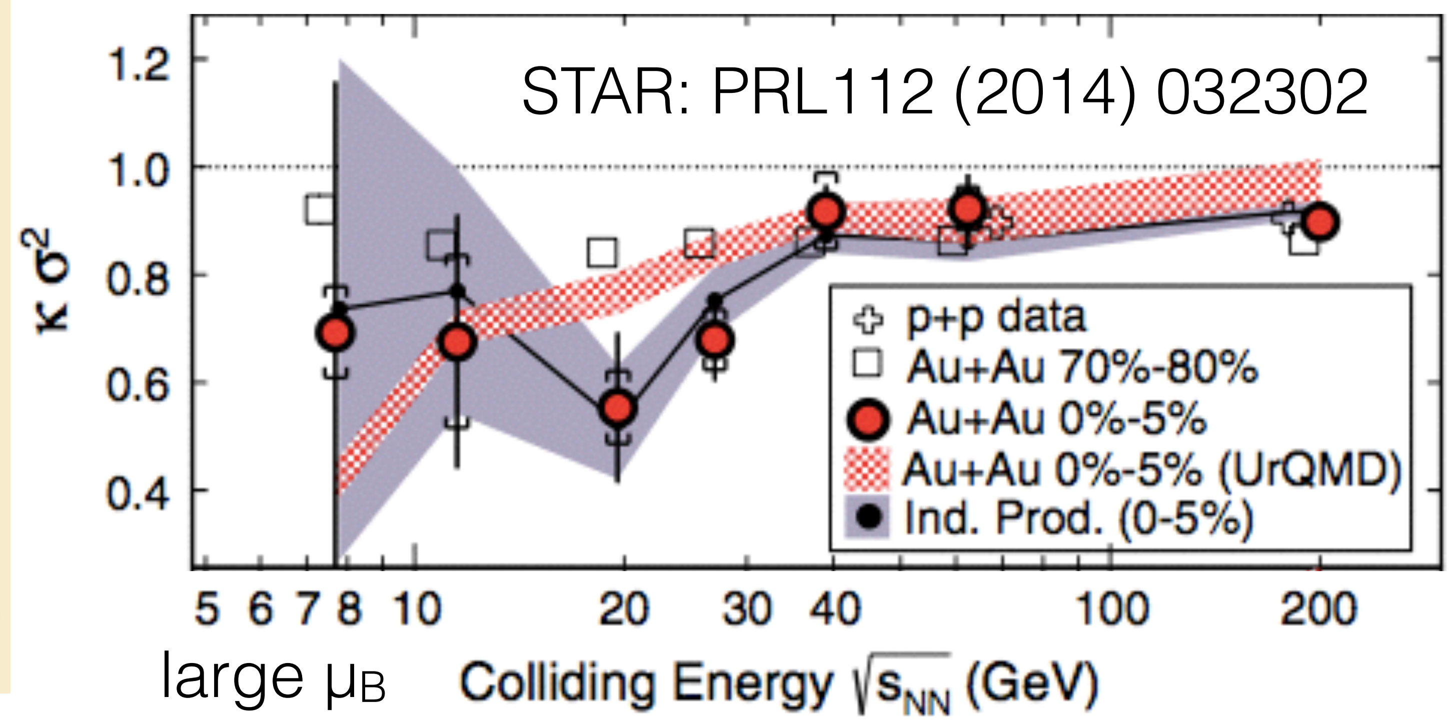
Bielefeld-BNL-CCNU, PRD93 (2016) no.1, 014512

Search for critical point in HIC

Beam Energy Scan(BES) @RHIC



Ratio of the 4th to 2nd order proton number fluctuations



large μB

Can this non-monotonic behavior be understood in terms of the QCD thermodynamics in equilibrium?

What is the relation of this intriguing phenomenon to the critical behavior of QCD phase transition?

Explore the QCD phase diagram through fluctuations of conserved charges

Comparison of experimentally measured higher order cumulants of conserved charges to those from LQCD, e.g.:

$$\frac{M_Q(\sqrt{s})}{\sigma_Q^2(\sqrt{s})} = \frac{\langle N_Q \rangle}{\langle (\delta N_Q)^2 \rangle} = \frac{\chi_1^Q(T,\mu_B)}{\chi_2^Q(T,\mu_B)} = R_{12}^Q(T,\mu_B)$$

$$\frac{S_Q(\sqrt{s})\,\sigma_Q^3(\sqrt{s})}{M_Q(\sqrt{s})} = \frac{\langle (\delta N_Q)^3 \rangle}{\langle N_Q \rangle} = \frac{\chi_3^Q(T,\mu_B)}{\chi_1^Q(T,\mu_B)} = R_{31}^Q(T,\mu_B)$$

HIC

mean: M_Q

variance: σ_Q^2

skewness: S_Q

kurtosis: κ_Q

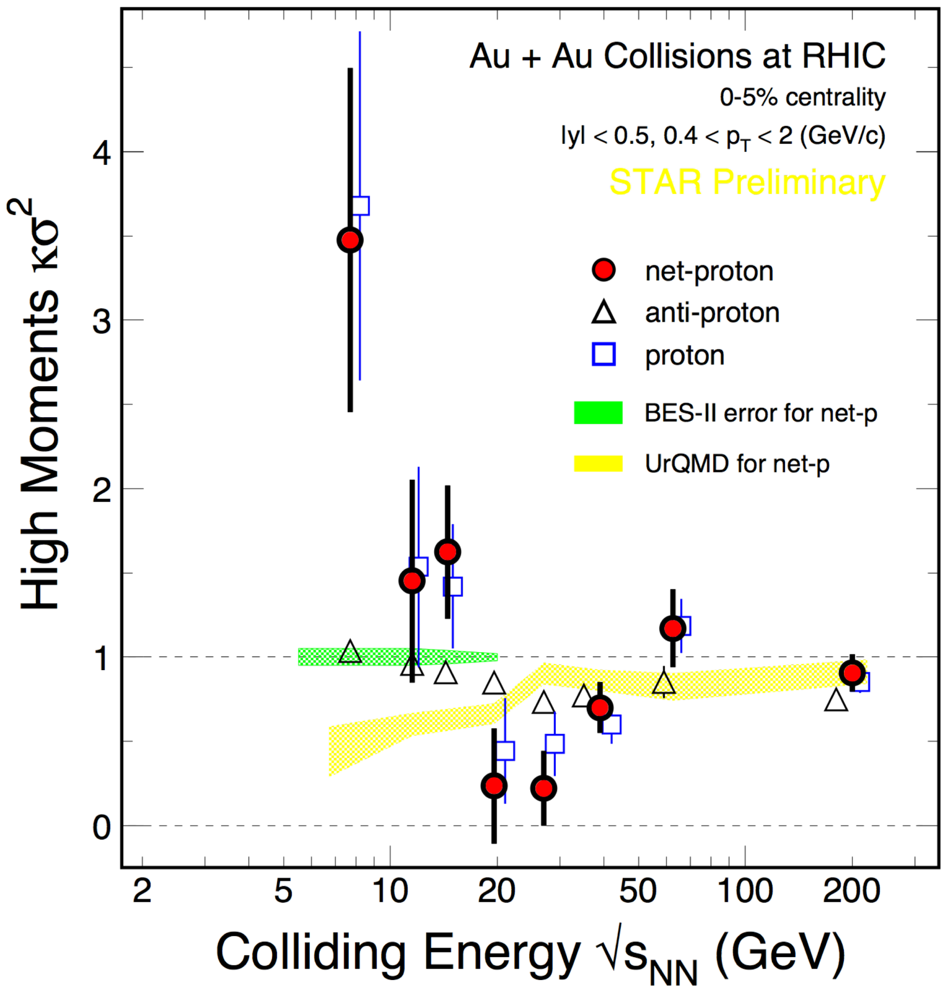
LQCD

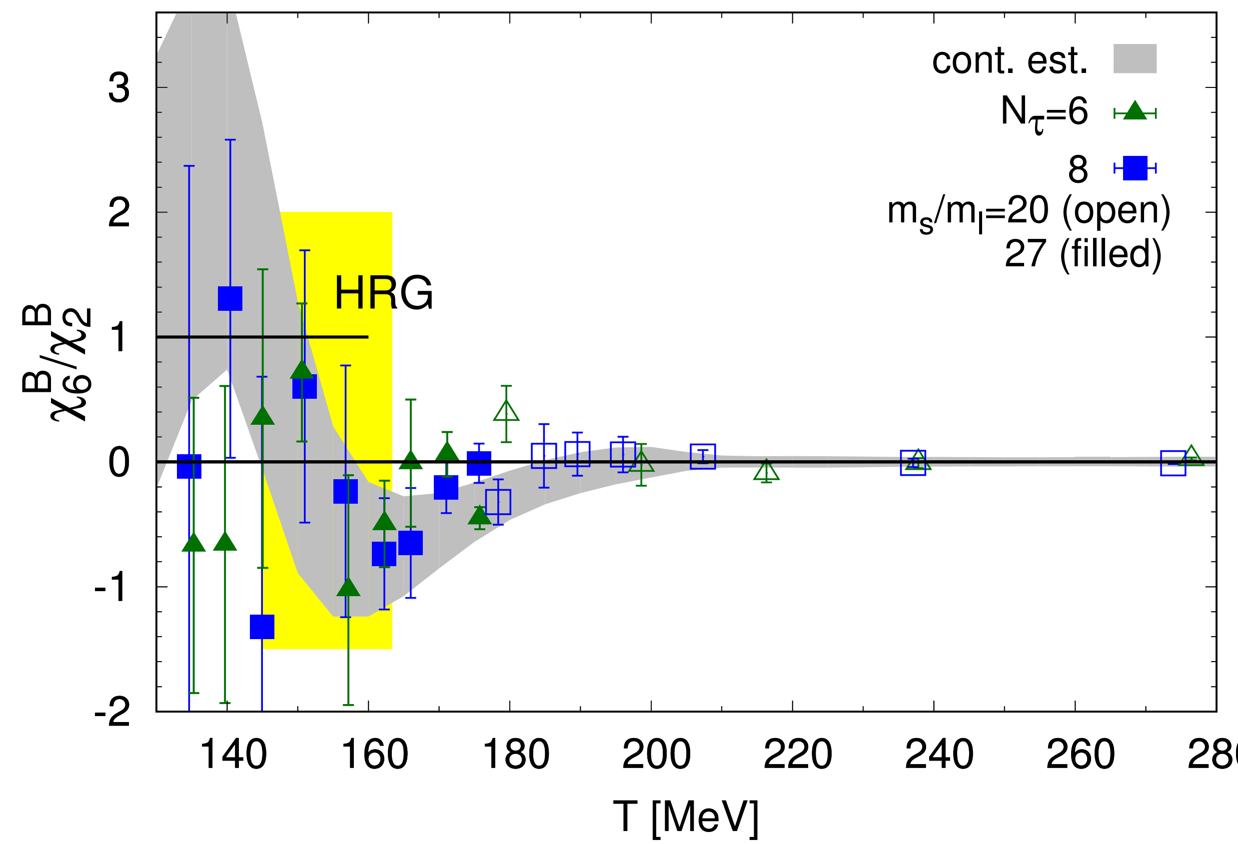
generalized susceptibilities

$$\chi_n^Q(T,\vec{\mu}) = \frac{1}{VT^3}\frac{\partial^n \ln Z(T,\vec{\mu})}{\partial(\mu_Q/T)^n}$$

BNL-Bielefeld, Phys. Rev. Lett. 109 (2012) 192302

Cumulant ratios of proton (baryon) number fluctuations: HIC data v.s. Lattice results



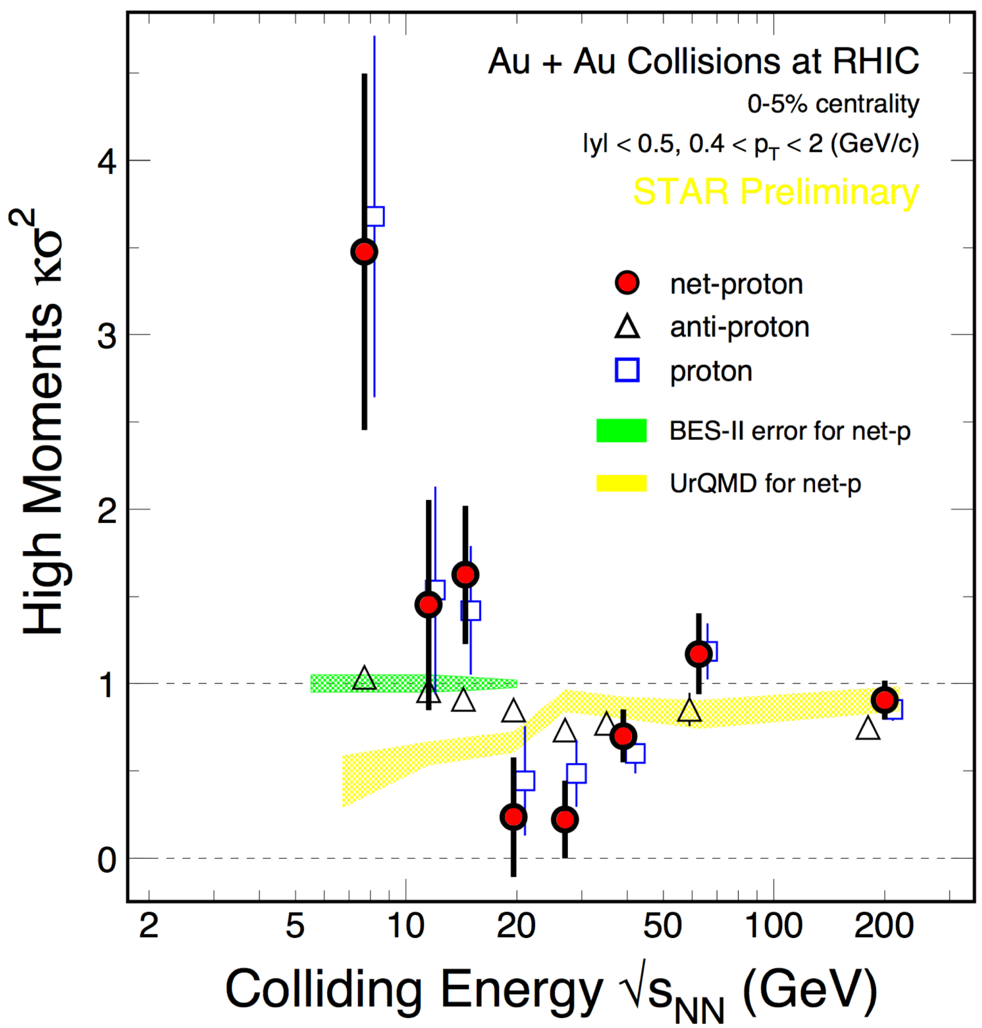


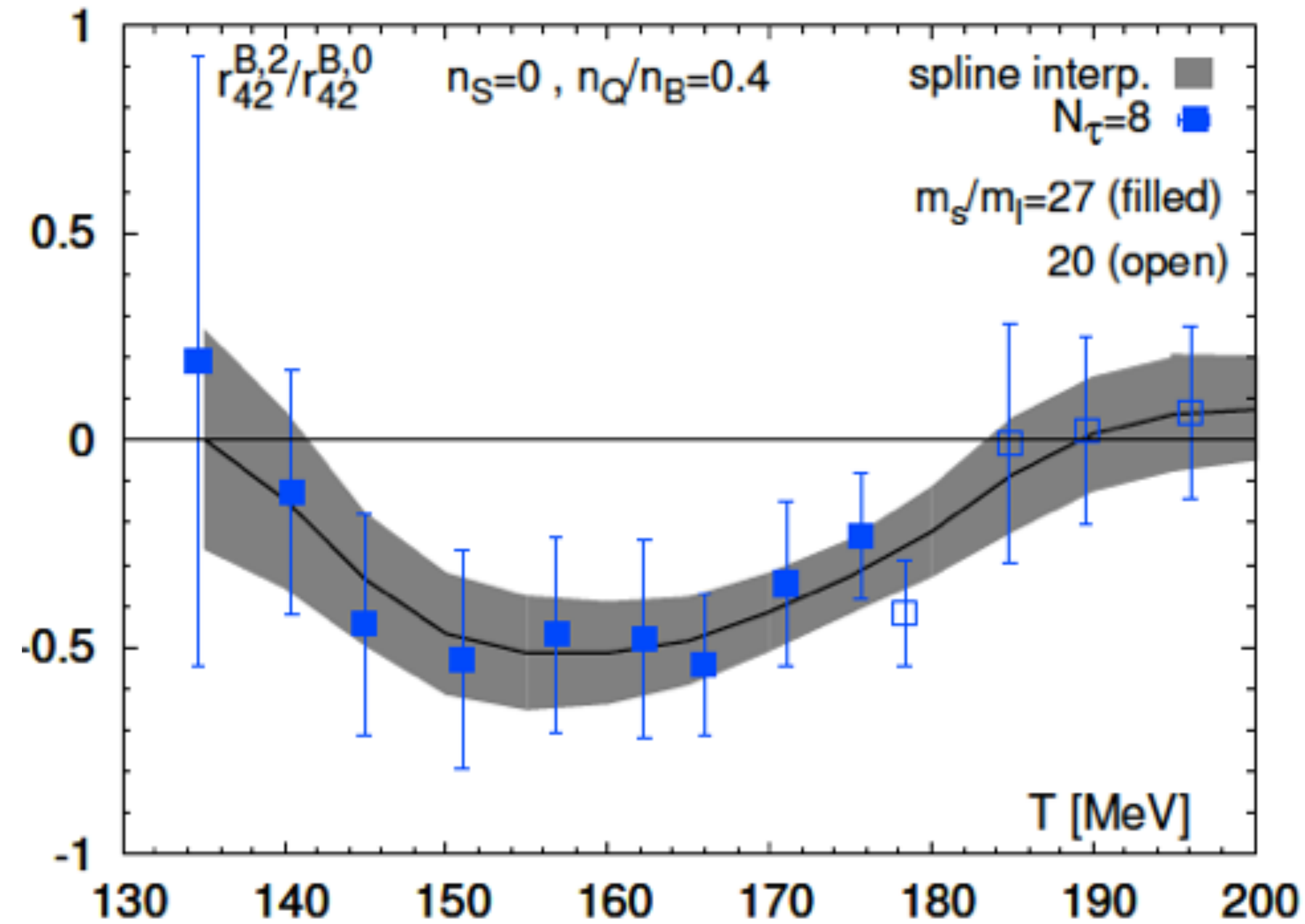
Bielefeld-BNL-CCNU, PRD95 (2017) no.5, 054504

$\mu_Q=\mu_S=0$:
$$(\kappa\sigma^2)_B = \frac{\chi^B_{4,\mu}}{\chi^B_{2,\mu}} = \frac{\chi^B_4}{\chi^B_2}\left[1+\left(\frac{\chi^B_6}{\chi^B_4}-\frac{\chi^B_4}{\chi^B_2}\right)\left(\frac{\mu_B}{T}\right)^2+\cdots\right]$$

HRG: $\chi^B_6/\chi^B_4 = \chi^B_4/\chi^B_2 = 1$, O(4) & LQCD: $\chi^B_6/\chi^B_2 < 0$ at $T \sim T_c$

Cumulant ratios of proton (baryon) number fluctuations: HIC data v.s. Lattice results





Bazavov et al., [HotQCD] arXiv:1708.04897

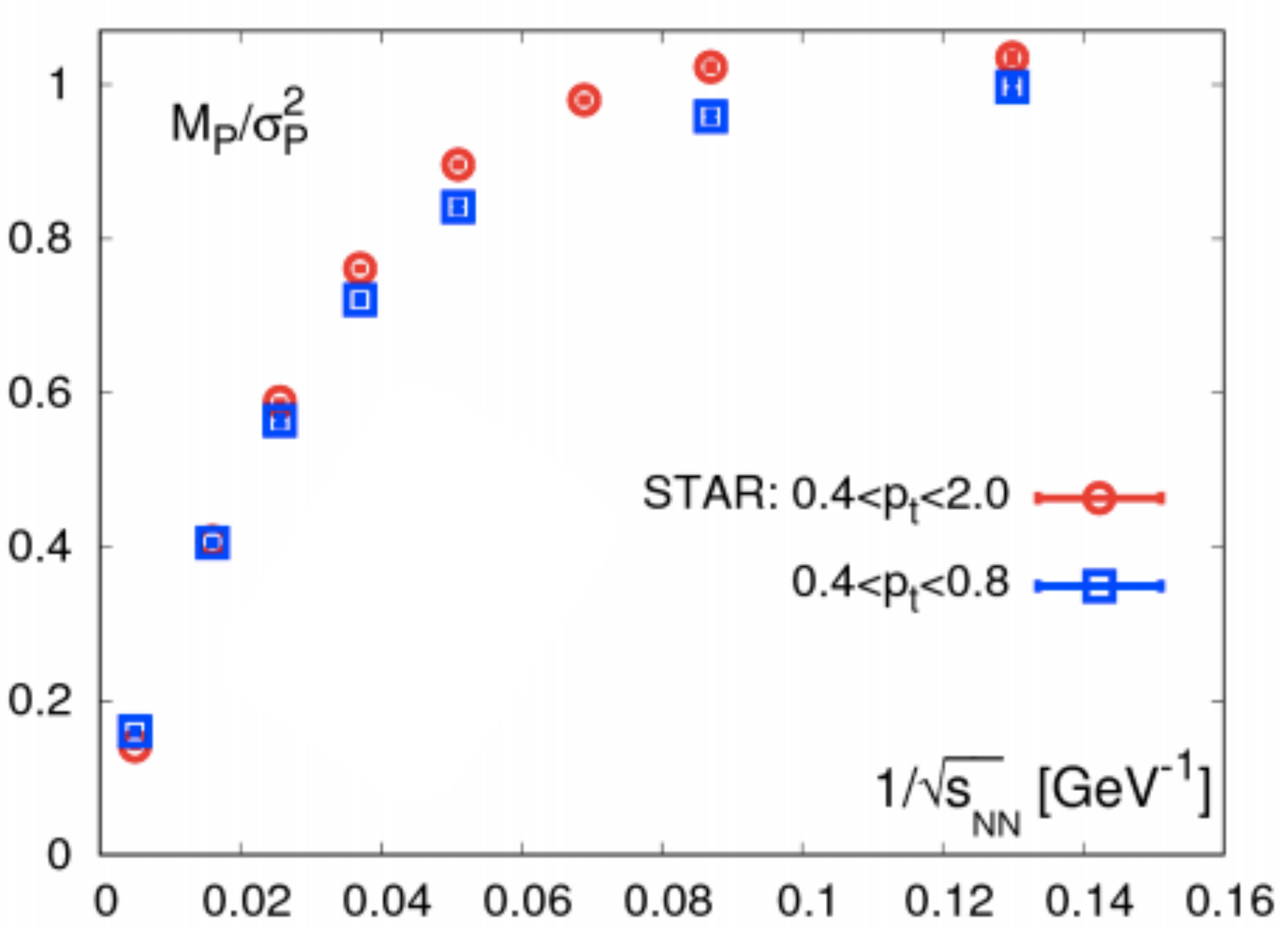
strangeness neutral case:

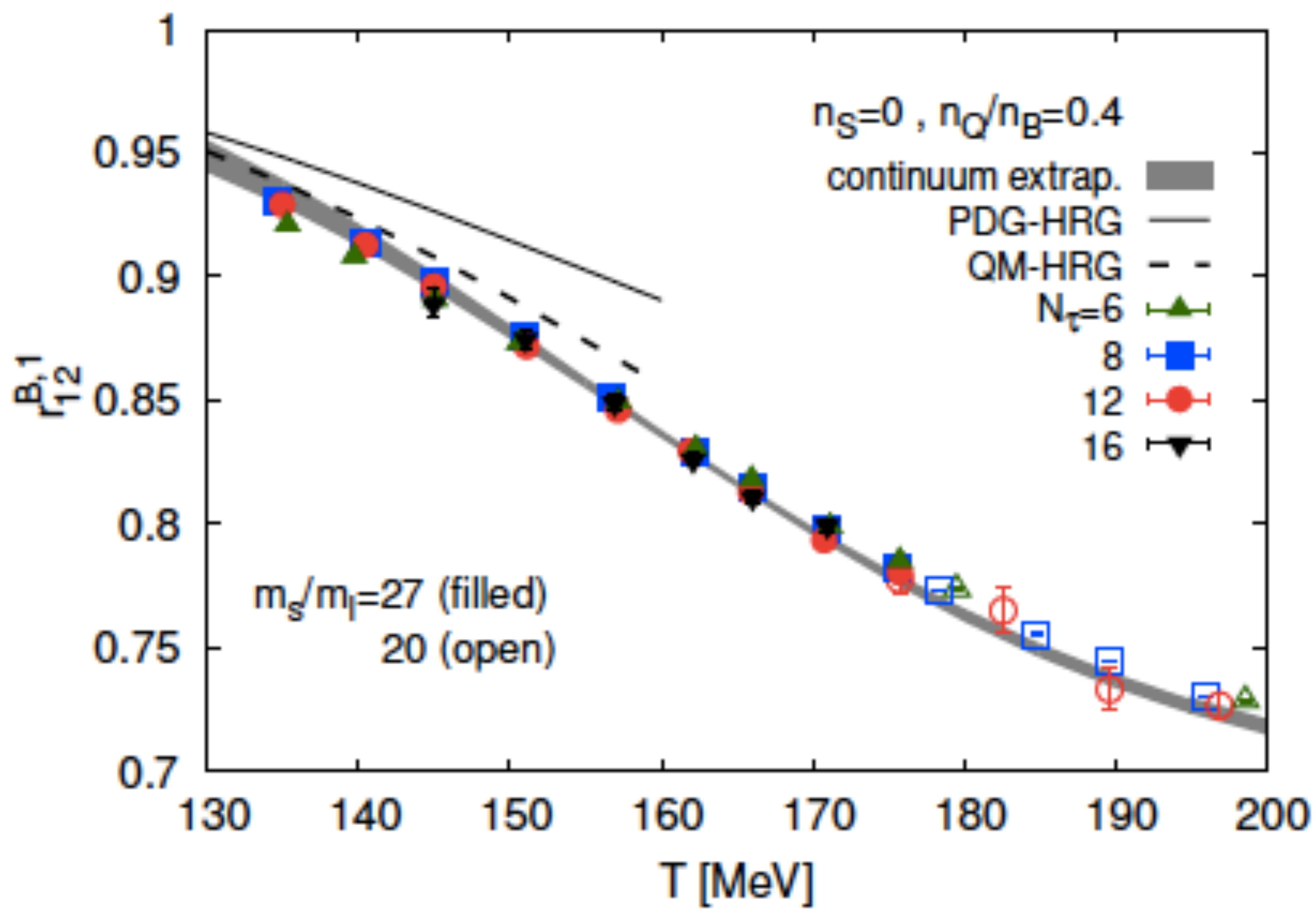
$$(\kappa\sigma^2)_B = \frac{\chi_{4,\mu}^{B,\mathrm{SN}}}{\chi_{2,\mu}^{B,\mathrm{SN}}} = r_{42}^{B,0}\left[1 + \frac{r_{42}^{B,2}}{r_{42}^{B,0}}\left(\frac{\mu_B}{T}\right)^2 + \cdots\right]$$

$\sqrt{s_{NN}} \gtrsim 20$ GeV:

$\kappa\sigma^2$ is consistent with QCD in equilibrium

Cumulant ratios of proton (baryon) number fluctuations: HIC data v.s. Lattice results



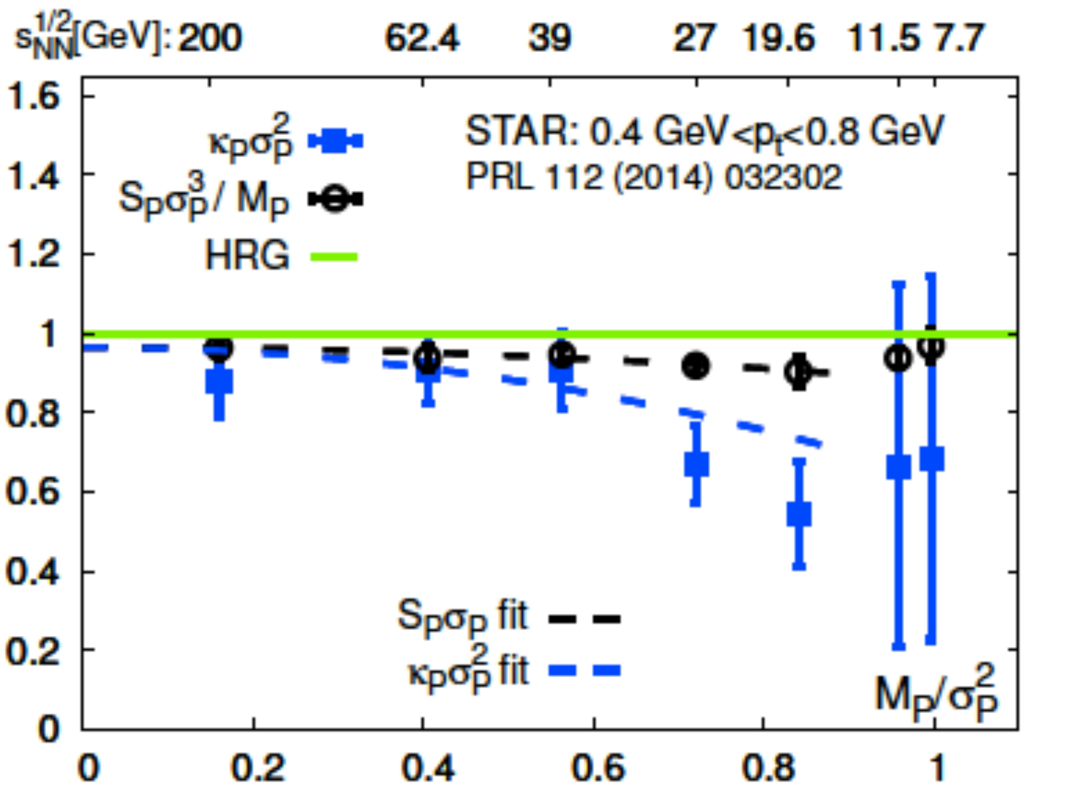


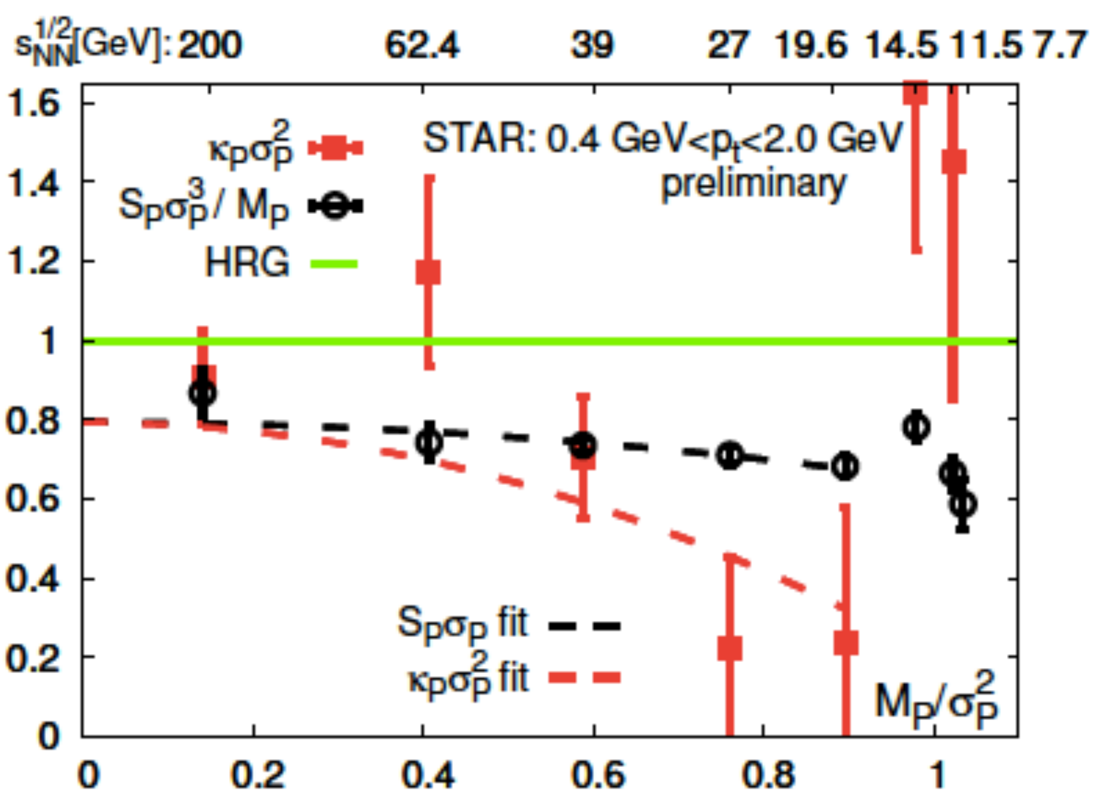
BNL-Bielefeld-CCNU, PRD 93 (2016)014512

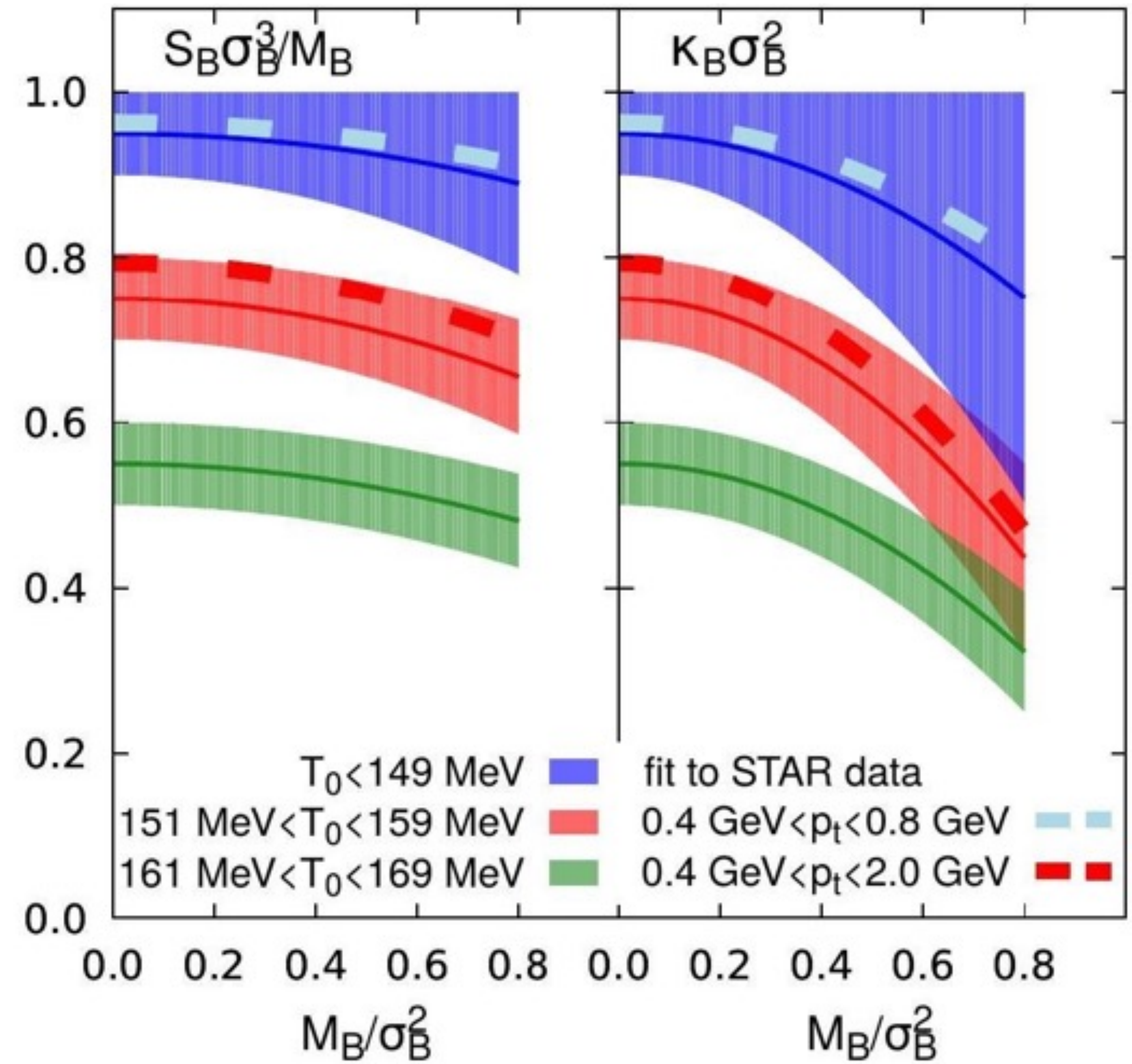
HIC: a function of M_p/σ_p^2 rather than $\sqrt{s_{NN}}$

Lattice: $\mu_B \Leftrightarrow M_B/\sigma_B^2 / R_{12}^{B,1}$, $R_{12}^B(T_f,\mu_B) \equiv \frac{M_B}{\sigma_B^2}(T_f,\mu_B) = \underbrace{\frac{\partial R_{12}^B}{\partial \hat{\mu}_B}\Big|_{\hat{\mu}_B=0}}_{\parallel\; r_{12}^{B,1}} \hat{\mu}_B + \mathcal{O}(\hat{\mu}_B^3)$

Cumulant ratios of proton (baryon) fluctuations: HIC data v.s. Lattice results







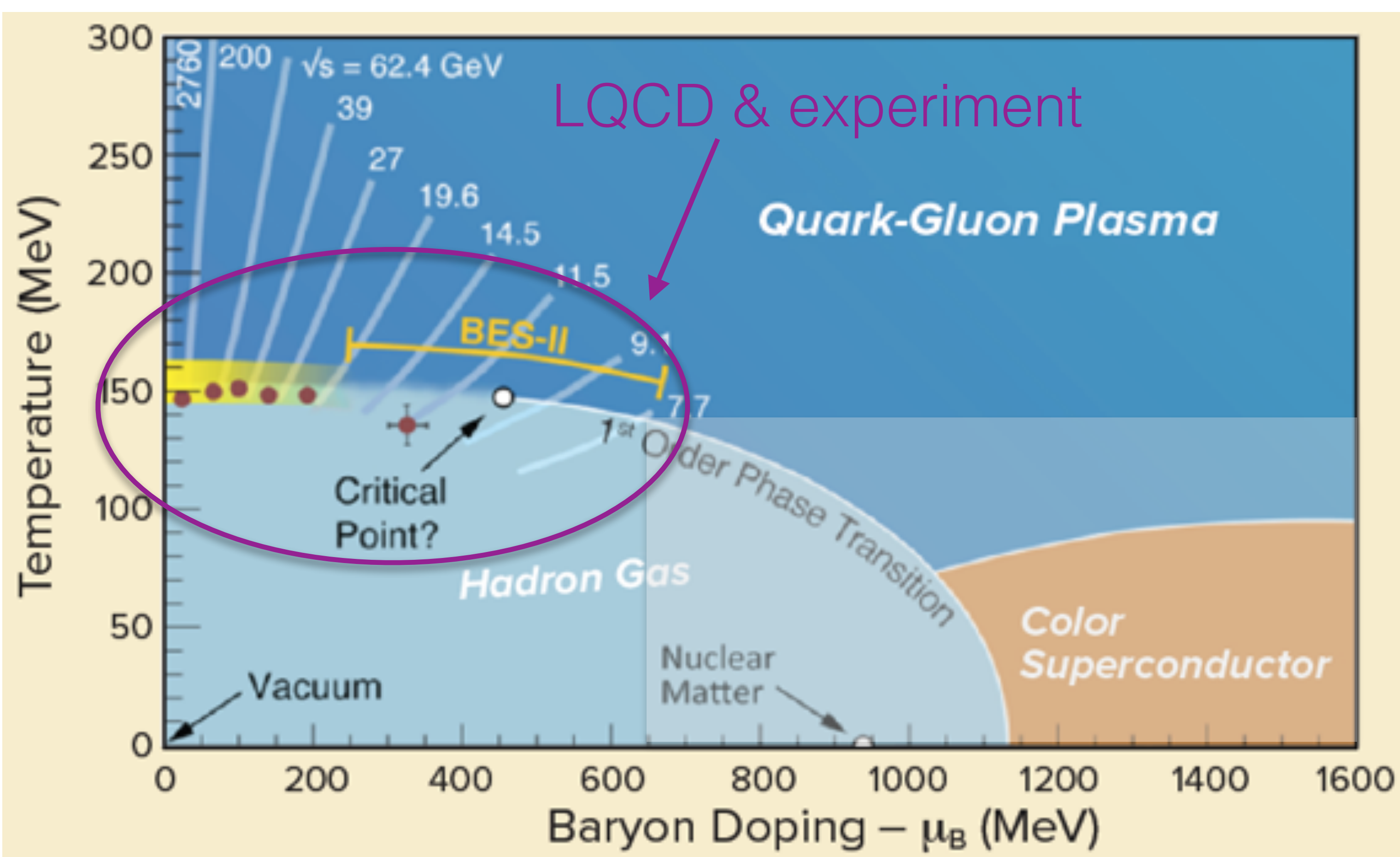
Bazavov et al., [HotQCD] arXiv:1708.04897

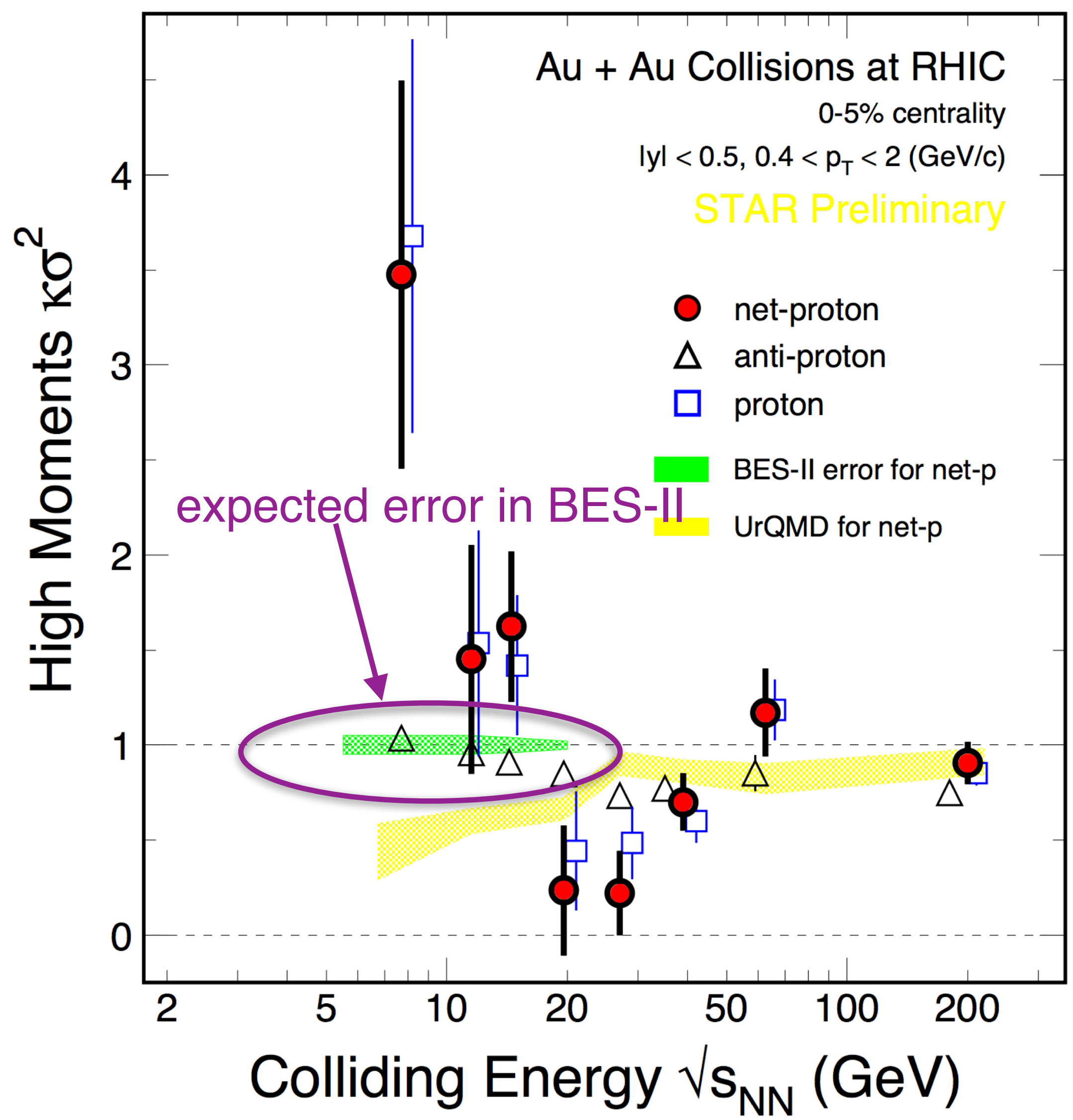
Outlook: Mapping out the QCD phase diagram

RHIC Beam Energy Scan, Phase II (BES-II)

2019-2020: at least 10 times more statistics for each $\sqrt{s_{NN}}$

LQCD: higher accuracy for the 6th & 8th or even higher order Taylor expansion coefficients





hot & dense lattice QCD

Other topics not covered but very important

electrical conductivity & baryon diffusion

energy loss of heavy quark in hot & dense medium

thermal dilepton & photon emission from QGP

shear & bulk viscosities

fate of heavy quarkonia

QCD in the external magnetic field

…

See recent reviews:
HTD, F. Karsch, S. Mukherjee, Int. J. Mod. Phys. E 24 (2015) no.10, 1530007
plenary talks@lattice conference: HTD, arXiv:1702.00151, S. Kim, arXiv:1702.02297
C. Schmidt & S. Sharma, arXiv:1701.04707
G. Endrodi, PoS CPOD2014 (2015) 038

Summary

In our quest for understanding the properties & phases of strong-interaction matter in extreme conditions

hot & dense lattice QCD is an essential component

Interpreting the phenomena observed in HIC experiments needs theory inputs based on lattice QCD

A lot of progress in hot & dense lattice QCD has been made to have close connection with experiments