Analysis of the analytical structure of amplitudes - an area of potential discoveries and pitfalls

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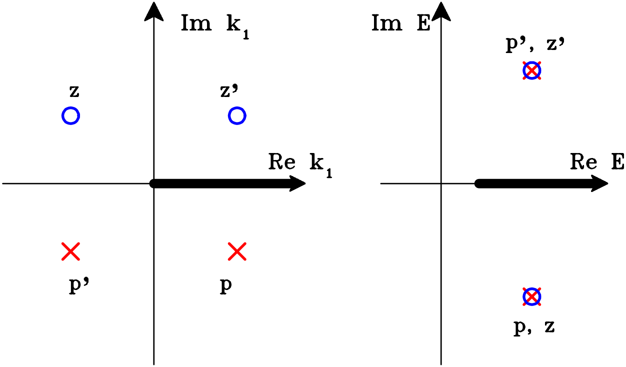
Wuppertal I 2024

Complex momentum and energy space frame

$$E = 2\sqrt{(\pm k)^2 + m^2}$$



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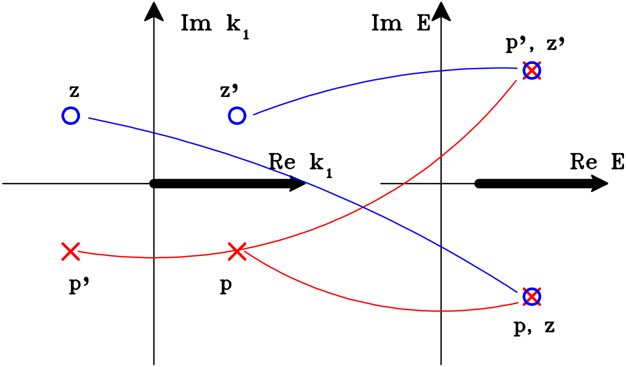
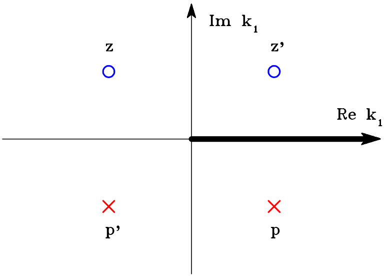
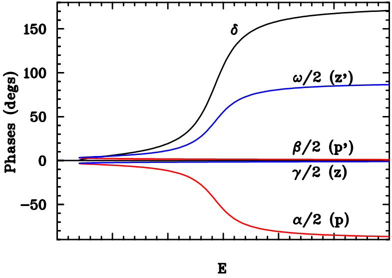


One channel scattering

- $S(k) = \frac{D(-k)}{D(k)} = e^{2i\delta}$, $|S(k)| = 1$
- $D(-k) = (-k - k_j)$
- $D(k) = (k - k_j)$
- But $|S(k)| \neq 1$ so
- $D(-k) = (-k - k_j)(-k + k_j^*)$
- $D(k) = (k - k_j)(k + k_j^*)$
- then $|S(k)| = 1$
- and $\delta = (-\alpha - \beta + \gamma + \omega)/2$

$$angle = ArcTan(\frac{-Imk_j}{k - Rek_j})$$





Pion electromagnetic form factor in the P wave

Parameter	PDG MeV	G.S. MeV	U&A MeV
m_ρ	775.26 ± 0.25	774.81 ± 0.01	763.88 ± 0.04
$m_{\rho'}$	1465.00 ± 25.00	1497.70 ± 1.07	1326.35 ± 3.46
$m_{\rho''}$	1720.00 ± 20.00	1848.40 ± 0.09	1770.54 ± 5.49
Γ_ρ	149.10 ± 0.80	149.22 ± 0.01	144.28 ± 0.01
$\Gamma_{\rho'}$	400.00 ± 60.00	442.15 ± 0.54	324.13 ± 12.01
$\Gamma_{\rho''}$	250.00 ± 100.00	322.48 ± 0.69	268.98 ± 11.40
χ^2 pdf		0.98 14 param.	1.84 11 param.

(G.S. - Gounaris-Sakurai form factor)
(U&A - unitary and analytic model)

WHAT ARE THE CORRECT $\rho^0(770)$ MESON MASS …

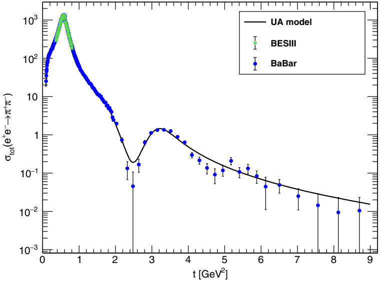


FIG. 7. Optimal description of the unified BESIII-*BABAR*

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One channel with more than one resonances

Adding resonances (for simplicity two resonances, both with $S = e^{2i\delta}$):

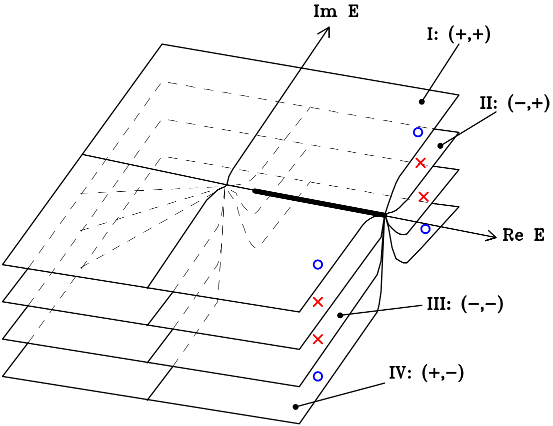
- Isobar model: adding amplitudes (even unitary ones) violates unitarity:
  $T_{1,2} = T_1 + T_2 = \frac{S_1 - 1}{2ik} + \frac{S_2 - 1}{2ik} \to S_1 + S_2 = e^{2i\delta_1} + e^{2i\delta_2}$
  of course $|S_1 + S_2| \neq 1$,

- Product of S matrices: $|S_1 S_2| = 1$ in elastic case and $|S_1 S_2| < 1$ in inelastic case ($S = \eta e^{2i\delta}$)
  For example $S_{1,2} = \frac{(-k-k_1)(-k+k_1^*)(-k-k_2)(-k+k_2^*)}{(k-k_1)(k+k_1^*)(k-k_2)(k+k_2^*)}$
  Of course $T_{1,2} = \frac{S_{1,2} - 1}{2ik}$

- Sum of K matrices: $S = 1 + 2iT = (1 + iK)/(1 - iK)$ does noit violate unitarity,
  for example $T_{1,2} = \frac{1}{k} \frac{K_1 + K_2}{1 - iK_1 - iK_2}$

More channels: $k_2 = \pm\sqrt{k_1^2 + m_1^2 - m_2^2}$

(Im(k_1), Im(k_2)): (+,+), (-,+) .... 1 pole $\longrightarrow$ $2^{(n-1)}$ poles (n-number of channels)



Multiplication and displacement of S matrix poles

- Multiplication:

  1 pole $\longrightarrow 2^{n-1}$ poles due to $(\pm k)^2$ ambiguity and 2^n Riemann sheets

- Displacement:

  $S_{11} = \frac{D_1(-k_1)D_2(k_2)}{D_1(k_1)D_2(k_2)}$ $\longleftarrow$ for decoupled channels

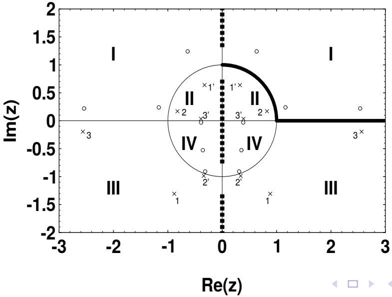
  $S = \frac{D_1(-k_1)D_2(k_2)+C(-k_1,k_2)}{D_1(k_1)D_2(k_2)+C(k_1,k_2)}$ $\longleftarrow$ for coupled channels

Example for two channels: $JI = S0$ wave

Pole	ReE_{pole} MeV	ImE_{pole} MeV	R. sheet
1	639.6	-323.9	$(-,-)$: *III*
1'	511.4	-230.6	$(-,+)$: *II*
2	982.0	-36.9	$(-,+)$: *II*
2'	432.4	-8.4	$(-,-)$: *III*
3	1431.7	-79.3	$(-,-)$: *III*
3'	1394.9	-120.6	$(-,+)$: *II*

$$z = \frac{k_1 + k_2}{\sqrt{m_K^2 - m_\pi^2}}$$

Rysunek 16: Położenie biegunów (krzyże) i zer (kółka) elementu macierzowego S_{11} macierzy rozpraszania dla dopasowania do zestawu $\mathrm{D_{CKM}}$ A. Gruba linia ciągła oznacza obszar fizyczny rozpraszania w kanałach sprzężonych $\pi\pi$ i $K\overline{K}$. Grubą linią przerywaną przedstawione jest położenie cięć funkcji Josta. Cienką linią zaznaczony jest okrąg $|z| = 1$. Numeracja poszczególnych płatów i biegunów została wyjaśniona w tekście.

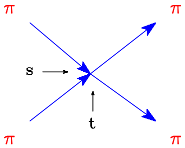


2^n Riemann sheets for n channels

channel	$C = 0$		$C = 1$		sign	sheet	
	ReE	*ImE*	*ReE*	*ImE*	Imk_π, Imk_K, Imk_3		
$\pi\pi$	658	-607	564	-279	$-,-,-$	VI	
			518	-261	$-,+,+$	II	$\leftarrow f_0(500)$
			211	0	$-,+,-$	VII	
			532	-315	$-,-,+$	III	
			235	0	$+,+,-$	VIII	
$\pi\pi$	1346	-275	1405	-74	$-,-,-$	VI	$\leftarrow f_0(1370)$
			1445	-116	$-,+,+$	II	
			1424	-94	$-,+,-$	VII	
			1456	-47	$-,-,+$	III	$\leftarrow f_0(1500)$
$K\bar{K}$	881	-498	170	0	$+,-,-$	V	
			159	0	$-,-,-$	VI	
			418	-10	$-,-,+$	III	
			1038	-204	$-,+,-$	VII	
			988	-31	$-,+,+$	II	$\leftarrow f_0(980)$
$\sigma\sigma$	118	-2227	4741	-4688	$-,-,-$	VI	
			3687	-2875	$-,+,-$	VII	
			3626	-3456	$+,-,-$	V	
			3533	-579	$+,+,-$	VIII	

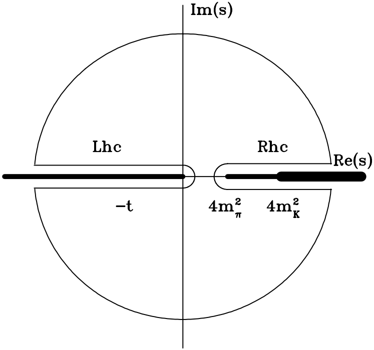
Dispersion relations with imposed crossing symmetry condition for $\pi\pi$ interactions theory $\longleftrightarrow$ experiment

crossing symmetry:



$$\rightarrow \vec{\boldsymbol{T}}_{\boldsymbol{s}}(s,t) = \hat{\boldsymbol{C}}_{st}\,\vec{\boldsymbol{T}}_{\boldsymbol{t}}(t,s)$$

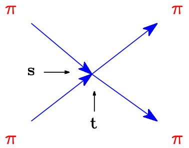
$\vec{\boldsymbol{T}}(\boldsymbol{s},\boldsymbol{t})$ + crossing symmetry $\rightarrow$ dispersion relations for $\underline{4m_\pi^2 < s < \sim (1150\ \text{MeV})^2}$



Once subtracted DR:

$$\begin{aligned}\text{Re}\,\vec{\boldsymbol{F}}(s,t) &= \text{Re}\,\vec{\boldsymbol{F}}(s_0,t) + \frac{s-s_0}{\pi}\\ &\times \left[\int_{4m_\pi^2}^{\infty} ds' \frac{\text{Im}\,\vec{\boldsymbol{F}}(s',t)}{(s'-s_0)(s'-s)}\right.\\ &\left. + \int_{-t}^{-\infty} ds' \frac{\text{Im}\,\vec{\boldsymbol{F}}(s',t)}{(s'-s_0)(s'-s)}\right]\end{aligned}$$

Dispersion relations with imposed crossing symmetry condition for $\pi\pi$ interactions theory $\longleftrightarrow$ experiment



crossing symmetry: $\rightarrow \vec{\boldsymbol{T}}_{\boldsymbol{s}}(s,t) = \hat{\boldsymbol{C}}_{st}\vec{\boldsymbol{T}}_{\boldsymbol{t}}(t,s)$

$\vec{\boldsymbol{T}}(\boldsymbol{s},\boldsymbol{t})$ + crossing symmetry → dispersion relations for $\underline{4m_\pi^2 < s < \sim (1150\ \text{MeV})^2}$

Once subtracted dispersion relations ("GKPY" for the *S* and *P* waves):

$$\boxed{\text{Re}\ t_\ell^{I(OUT)}(s) = \sum_{I'=0}^{2} C_{st}^{II'} a_0^{I'} + \sum_{I'=0}^{2}\sum_{\ell'=0}^{4} \mathop{-\!\!\!\!\!\!\int}_{4m_\pi^2}^{\infty} ds' K_{\ell\ell'}^{II'}(s,s')\ \text{Im}\ t_{\ell'}^{I'(IN)}(s')}$$

$a_0^{I'}$ - subtraction constant $= \vec{\boldsymbol{T}}_{\boldsymbol{s}}(s = 4m_\pi^2, t = 0)$ - scattering lengths from only *S* wave

due to $\text{Re}\ t_\ell^I(k) = k^{2\ell}(a_\ell^I + b_\ell^I k^2 + O(k^4))$ $\qquad \underline{\text{Re}\ t_\ell^{I(OUT)}(s) - \text{Re}\ t_\ell^{I(IN)}(s) \rightarrow 0}$

GKPY equations and $\pi\pi$ amplitudes

partial waves: *JI*

experiment

F1 **D2**

S0 **D0**

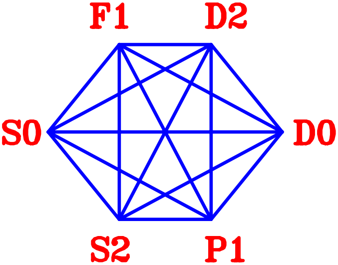
S2 **P1**

GKPY equations and $\pi\pi$ amplitudes

partial waves: *Jl*

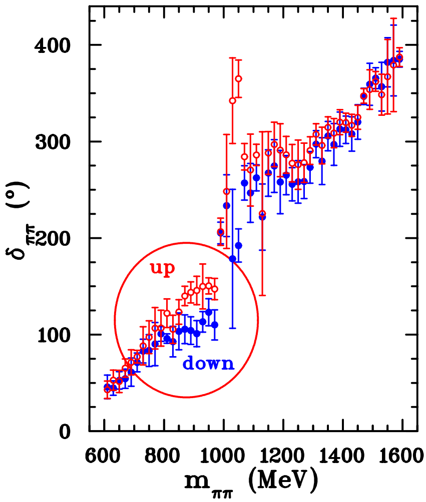
experiment + theory (GKPY)

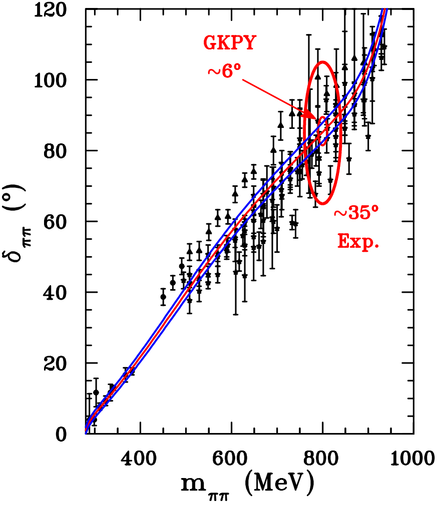
$$\int_{4m_\pi^2}^{\infty}$$



Experimental data for the $\pi\pi$ in the S0 wave (JI)

In PWA (CERN-Munich group'74) $A(s,t) \sim Cos(\theta_S - \theta_P)$

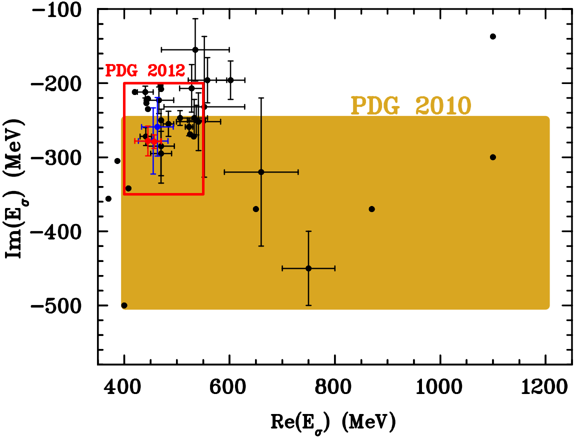




precise determination of $f_0(500)$ (σ) meson and threshold parameters

$f_0(500)$ (σ)

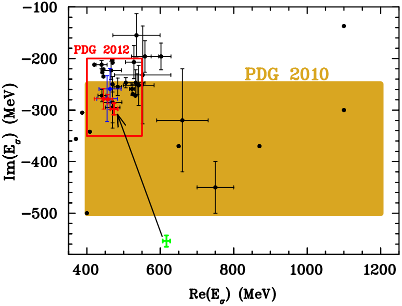
- PDG 2010:
  M = 400 – 1200 MeV
  $\Gamma = 2 \times (250 - 500)$ MeV
- PDG 2012:
  M = 400 – 550 MeV
  $\Gamma = 2 \times (200 - 350)$ MeV
- GKPY:
  $E_\sigma = 457 \pm 14 - i279^{+11}_{-7}$ MeV



threshold parameters, e.g. a_0^0:

- ChPT + Roy eqs (Bern group):
  $0.220 \pm 0.005\, m_\pi^{-1}$
- GKPY:
  $0.220 \pm 0.008\, m_\pi^{-1}$

what forces GKPY eqs to pull up-left the sigma pole?



Two things: trigonometry and crossing symmetry algebra lead to narrower and lighter σ.

Modified $\pi\pi$ amplitude with σ pole PRD 90, 116005 (2014) P. Bydzovský, 1, R. Kamiński, V. Nazari

Nothing more and nothing instead of it is needed.

Resonance near the threshold

Flatté (1976):

$$A_i \sim \frac{M_R\sqrt{\Gamma_0\Gamma_i}}{M_R^2 - E^2 - iM_R(\Gamma_1+\Gamma_2)}, \quad i = 1, 2.$$

THREE free parameters: M_R, Γ_1, Γ_2.

……………………………………………………………………………………

Leśniak (2008):

$$T_{22} = \frac{1}{\frac{1}{A} - i\,k_2 + \frac{1}{2}\,R\,k_2^2}$$

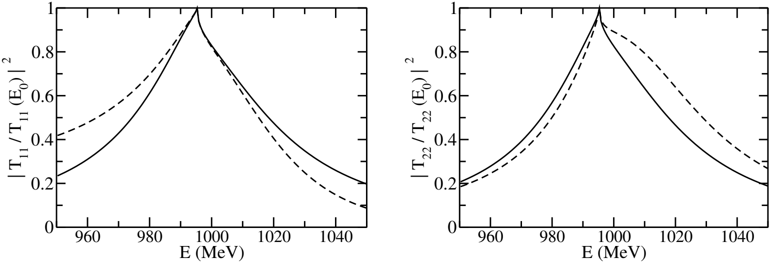
where
$$A = -i\left(\frac{1}{z_1} + \frac{1}{z_2}\right), \qquad R = \frac{2i}{z_1 + z_2}\,.$$

FOUR free parameters: z_1 and z_2 - zeroes of the S_{22} matrix element

Flatte approach: $ImR = 0 \quad (\equiv Rez_1 = -Rez_2)$

For $a_0(980)$

L. Leśniak, AIP Conf.Proc. 1030 (2008) 238

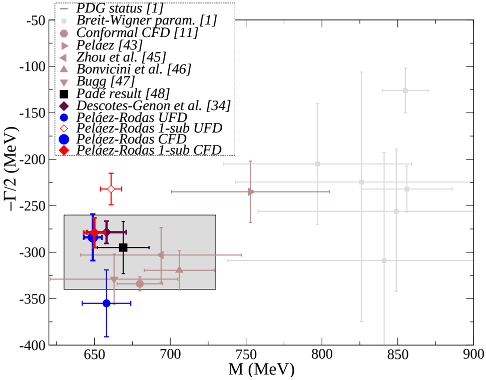


step two in testing the amplitude using dispersion relations:

The pion-kaon scattering amplitude constrained with forward dispersion relations up to 1.6 GeV

J.R. Pelaez and A.Rodas (2016)

$K\pi,\ K\bar{K}$ channels (problem of $K^{*}(800)$ (or κ))



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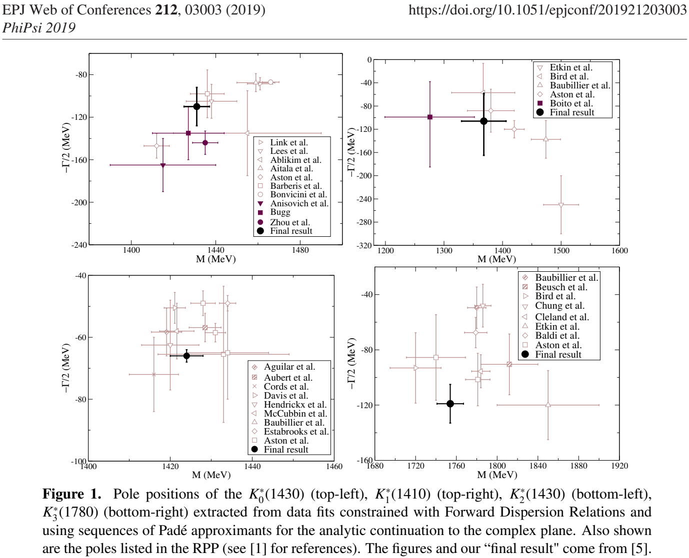


EPJ Web of Conferences **212**, 03003 (2019) https://doi.org/10.1051/epjconf/201921203003
PhiPsi 2019

Figure 1. Pole positions of the $K_0^*(1430)$ (top-left), $K_1^*(1410)$ (top-right), $K_2^*(1430)$ (bottom-left), $K_3^*(1780)$ (bottom-right) extracted from data fits constrained with Forward Dispersion Relations and using sequences of Padé approximants for the analytic continuation to the complex plane. Also shown are the poles listed in the RPP (see [1] for references). The figures and our "final result" come from [5].

within the uncertainties of the "Dispersive" one. Next we have obtained constrained fits to data (CFD) consistent with FDRs up to 1.6 GeV, see the bottom panels in Fig.1.

In Fig.2 we show the comparison of the UDF with the CDF for the S-wave, which is the most interesting one. The change is not very large, except at high energies, where it seems to prefer one data set over the other (see [17] for details and references).

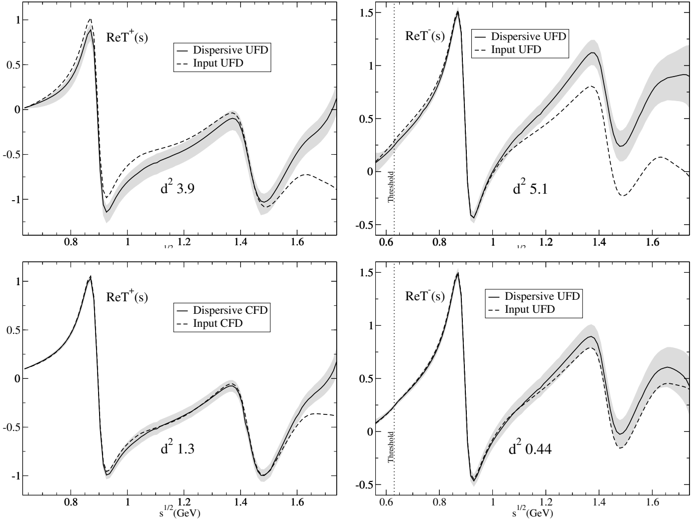


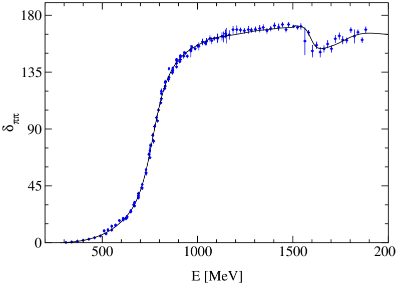
Figure 1: Comparison between the input and the dispersive result for the T^+ (left) and T^-(right) FDRs when using the UFD (top) or the CFD (bottom). The CFD set is consistent within errors up to 1.6 GeV.

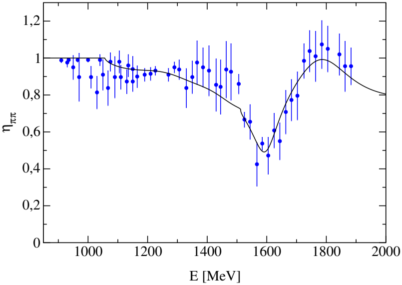
Latest analysis (Cracow 2023)

Fit to data and to dispersion relations (GKPY equations):

$\pi\pi, \omega\pi, \rho\rho$ channels in the *P*-wave

(problem $\rho(1250) - \rho(1450)$)





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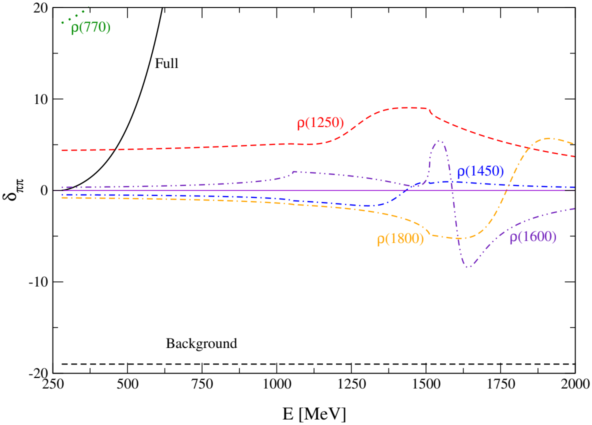

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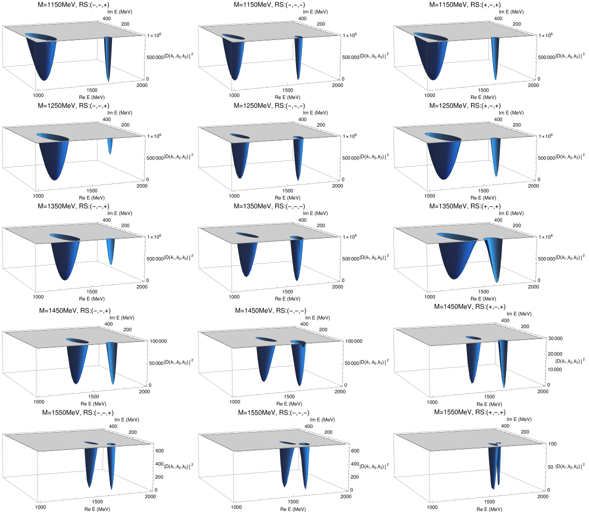

M=1150MeV, RS:(−,−,+)

M=1150MeV, RS:(−,−,−)

M=1150MeV, RS:(+,−,+)

M=1250MeV, RS:(−,−,+)

M=1250MeV, RS:(−,−,−)

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M=1450MeV, RS:(−,−,−)

M=1450MeV, RS:(+,−,+)

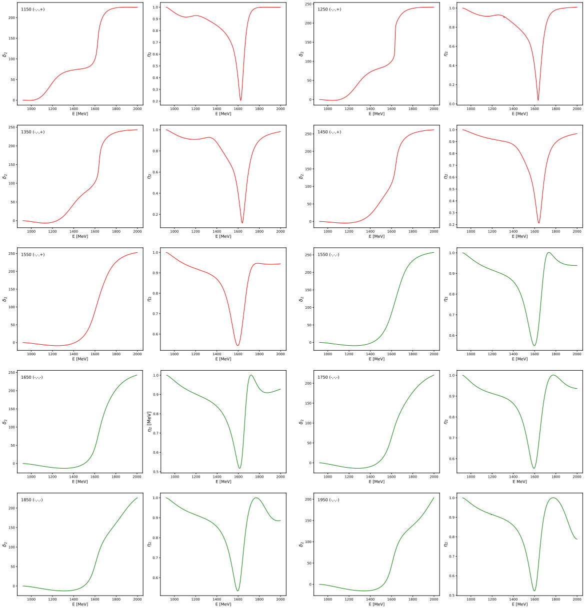
M=1550MeV, RS:(−,−,+)

M=1550MeV, RS:(−,−,−)

M=1550MeV, RS:(+,−,+)

Figure 6.1: A series of three-dimensional distributions of the Jost function for various manually inserted pole masses on the $(-,-,+)$ Riemann sheet, up to 1550 MeV. Each mass is accompanied by distributions on three Riemann sheets: $(-,-,+)$, $(-,-,-)$, and $(+,-,+)$.

Latest analysis (Cracow 2023)



Purpose

...activities of last year (Talk yesterday by Johan Messchendorp (JM)):
...activities of next years (RK):

- Formation of hadronic matter (JM),
- Hadron spectroscopy (JM),
- Underlying symmetries (JM),
- Degrees of freedom: from quarks/gluons to baryons/mesons (JM)
  - meson degrees of freedom
- Exotic states (JM)
  There is no chance for exotic states without very good knowledge of "standard meons" - spectroscopy and dynamics,
- $p - p$ interactions like e.g. $B, D \to M_1 M_2 M_3$ decays where two body interactions are parameterized via unitary amplitudes (factorisation!),
- Bring together experts from both theory and experiment (JM)
  - look at e.g. JLab before GlueX and CLAS12
  a) first the theorists,
  b) then the theorists together with the experimentalists and finally
  c) the experimentalists and go to a)
- We are ready for common works .... Thanks!

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  a) first the theorists,
  b) then the theorists together with the experimentalists and finally
  c) the experimentalists and go to a)
- We are ready for common works .... Thanks!

Purpose

...activities of last year (Talk yesterday by Johan Messchendorp (JM)):
...activities of next years (RK):

- Formation of hadronic matter (JM),
- Hadron spectroscopy (JM),
- Underlying symmetries (JM),
- Degrees of freedom: from quarks/gluons to baryons/mesons (JM)
  - meson degrees of freedom
- Exotic states (JM)
  There is no chance for exotic states without very good knowledge of "standard meons" - spectroscopy and dynamics,
- $p-p$ interactions like e.g. $B, D \to M_1 M_2 M_3$ decays where two body interactions are parameterized via unitary amplitudes (factorisation!),
- Bring together experts from both theory and experiment (JM)
  - look at e.g. JLab before GlueX and CLAS12
  a) first the theorists,
  b) then the theorists together with the experimentalists and finally
  c) the experimentalists and go to a)
- We are ready for common works .... Thanks!