Evgeny Epelbaum, Ruhr University Bochum

Physics opportunities with proton beams at SIS100
Wuppertal, February 6-9, 2024

Chiral EFT for non-strange and strange nuclear systems

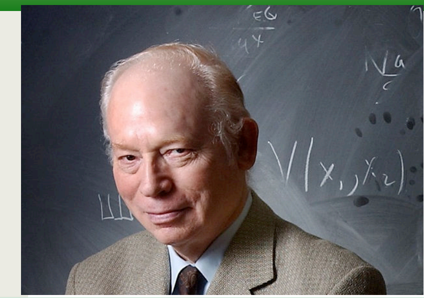
European Research Council
Bundesministerium für Bildung und Forschung
DFG Deutsche Forschungsgemeinschaft
Ministerium für Kultur und Wissenschaft des Landes Nordrhein-Westfalen
ERATO Exploratory Research for Advanced Technology

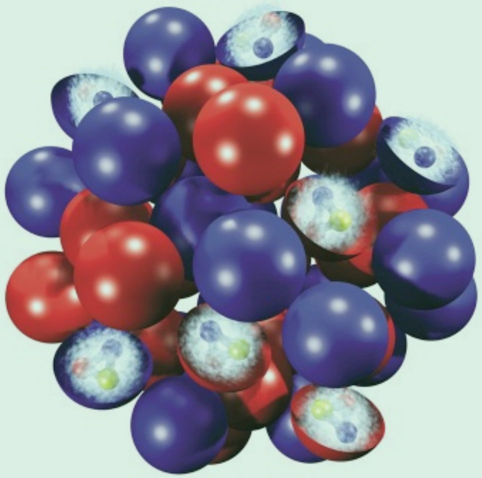
Degrees of freedom

Weinberg's 3rd law of progress in theoretical physics:

You may use any degrees of freedom you like to describe a physical system, but if you use the wrong ones, you will be sorry…

in Asymptotic Realms of Physics, MIT Press, Cambridge, 1983





Typical momenta of nucleons in nuclei:

$$\langle \Psi | \hat{p} | \Psi \rangle \sim 50 - 300 \text{ MeV}$$

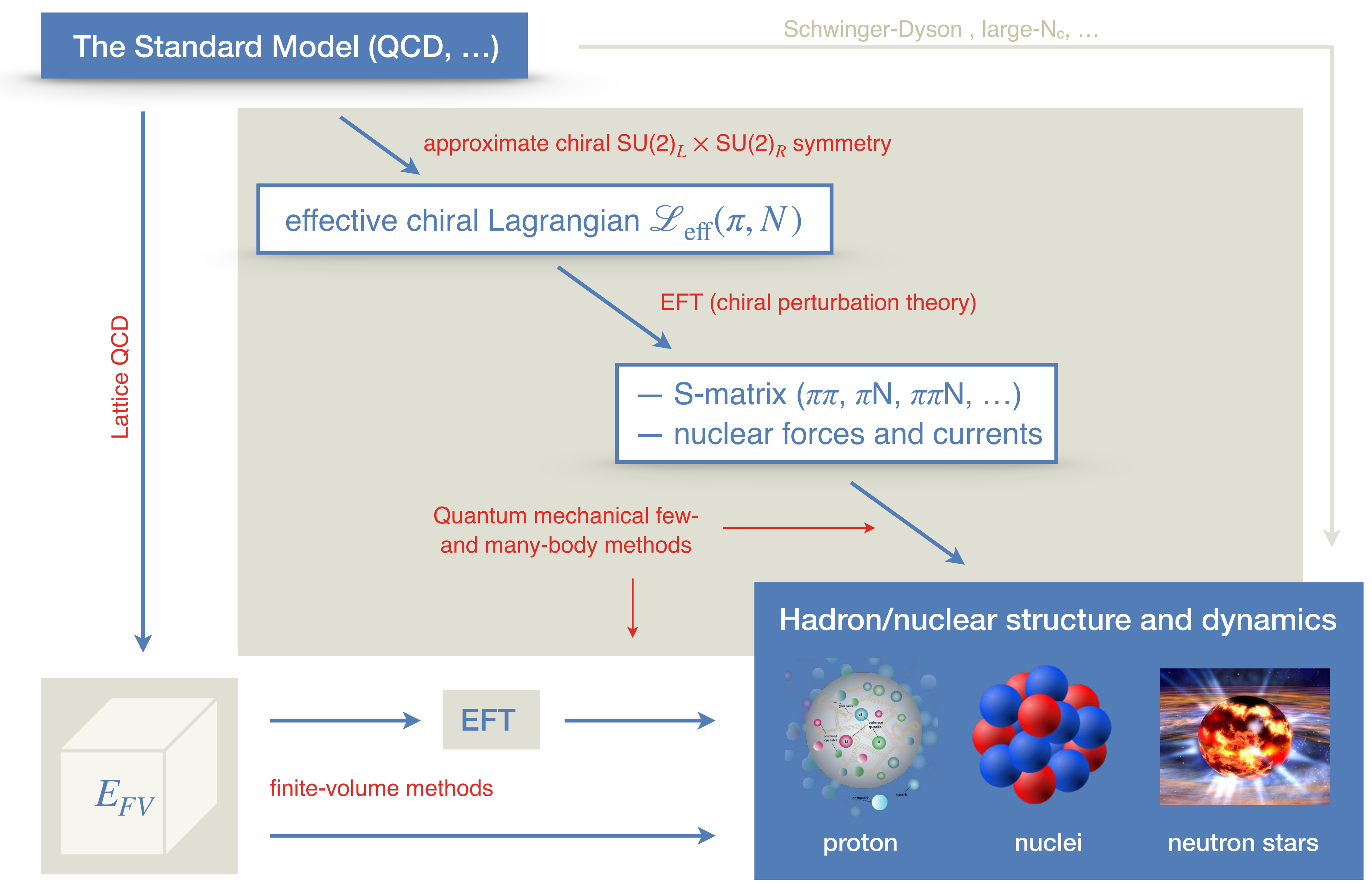
Fermi-momentum at the saturation density:

$$p_F = (3/2\pi^2 \rho)^{1/3} \sim 270 \text{ MeV}$$

$\Rightarrow$ non-relativistic description in the framework of the A-body Schrödinger equation:

$$\left[\left(\sum_{i=1}^{N} \frac{-\vec{\nabla}^2}{2m} + \mathcal{O}(m^{-3}) \right) + \underbrace{V_{2N} + V_{3N} + V_{4N} + \ldots}_{\textit{derived in ChPT}} \right] |\Psi\rangle = E |\Psi\rangle$$

The Big Picture



Chiral EFT and nuclear physics

Ordinary nuclei (non-strange) at physical quark masses

— good convergence of χEFT
— long-range interactions fixed in a parameter-free way
— huge amount of high-quality data to fix few-N LECs

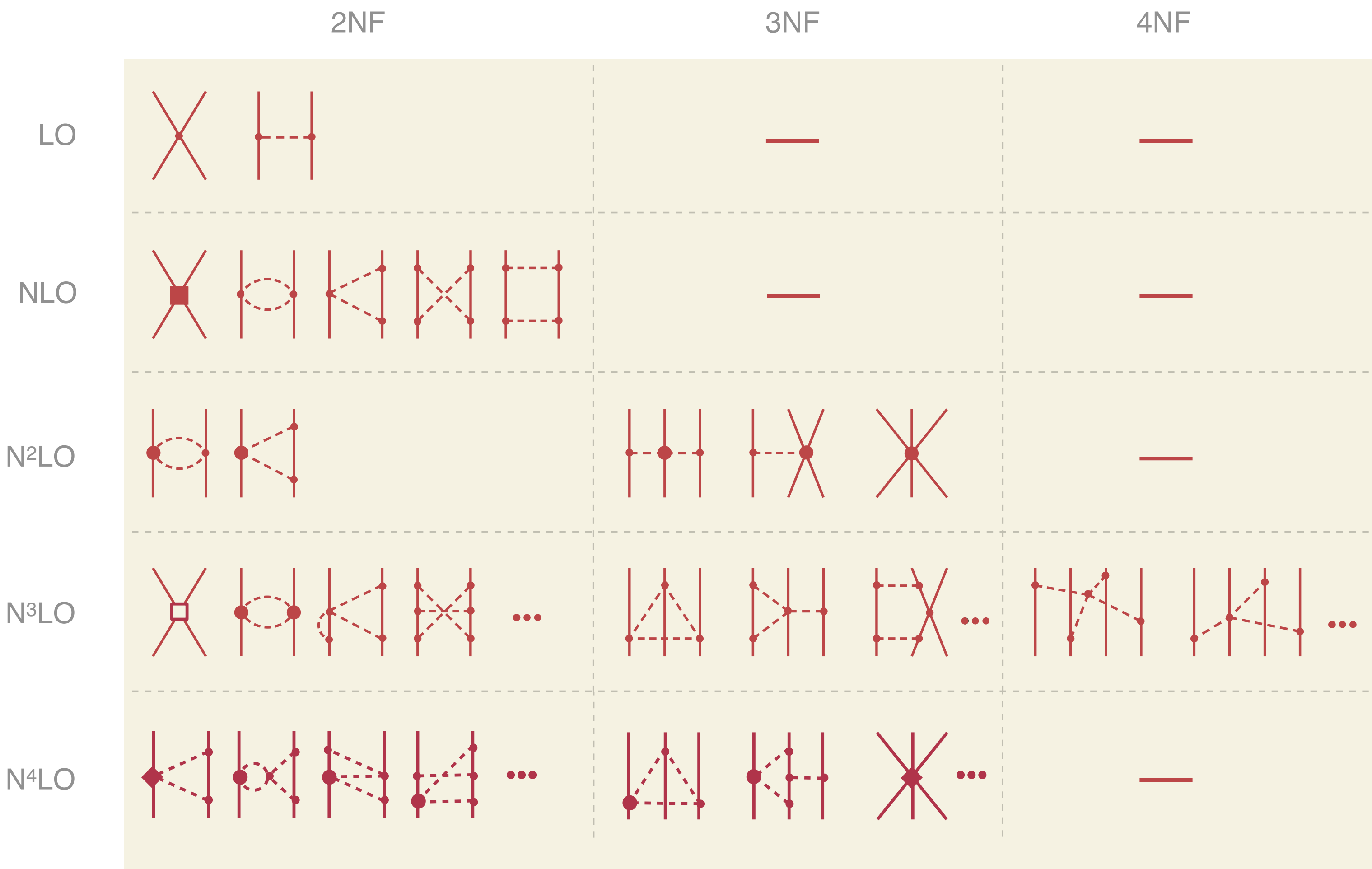
Hyper-nuclei at physical quark masses

— slow convergence of χEFT: M_K vs M_π…
— many more LECs (e.g. LO: 2 for NN vs 6 for BB)
— $SU(3)_{\text{flavor}}$ much less accurate than $SU(2)_{\text{isospin}}$
— only few experimental data available

(Hyper-) nuclei at variable quark masses

— no experimental data available
— lattice-QCD, Schwinger-Dyson?

The paradise...



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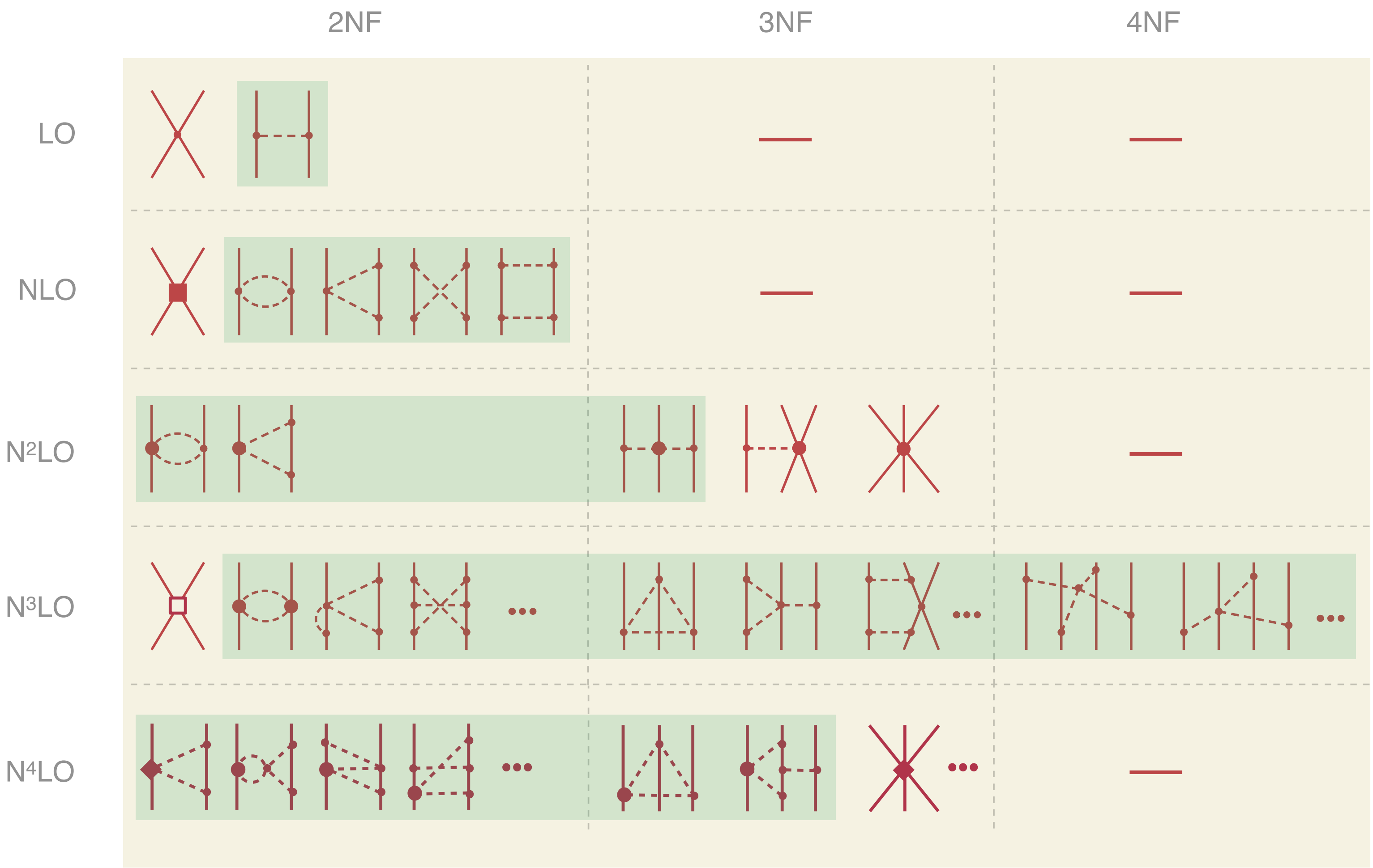
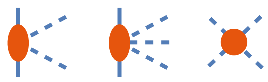


Chiral dynamics: Long-range interactions are predicted in terms of on-shell amplitudes



(πN LECs are known from the Roy-Steiner analysis Hoferichter et al.'15)

The paradise…

2NF

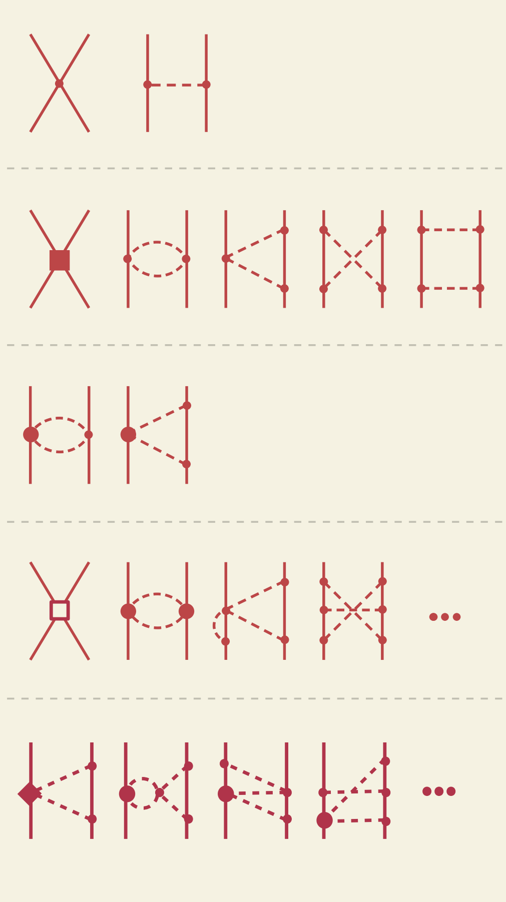
LO

NLO

N^2LO

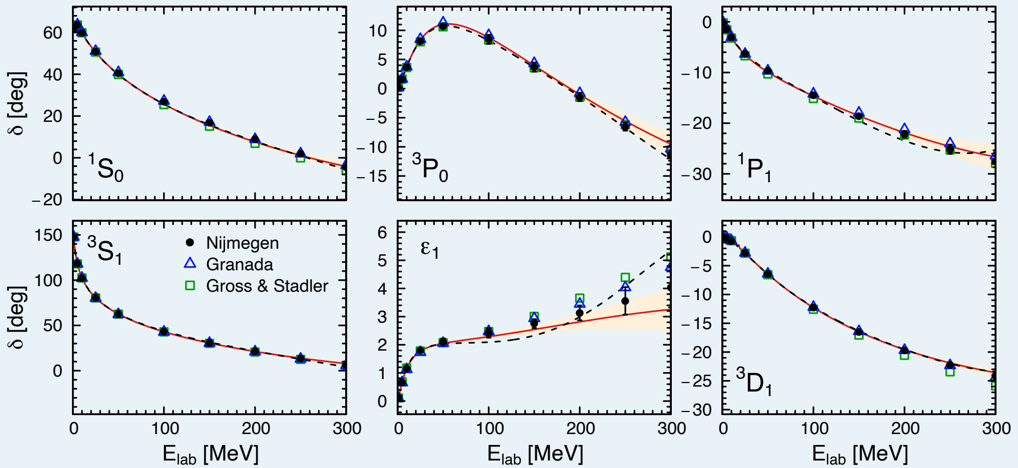
N^3LO

N^4LO

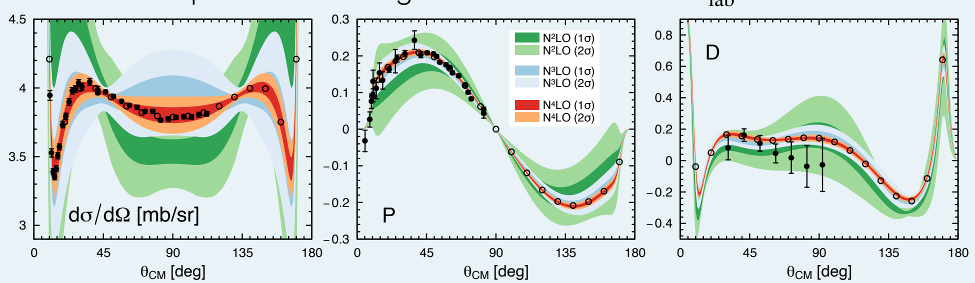


χEFT as a precision tool in the two-nucleon sector

Neutron-proton phase shifts at N^4LO^+



Proton-proton scattering observables around $E_{\rm lab} = 143$ MeV



— Determination of the neutron charge radius from the ^{2}H structure radius Filin et al., PRL 124 (20); PRC 103 (21)

$$r_{\rm str}^2 = \underbrace{r_d^2 - r_p^2}_{3.82070(31)\ \mathrm{fm}^2} - r_n^2 - 3/(4m_p^2) = 1.9729^{+0.0015}_{-0.0012}\ \mathrm{fm} \quad \Rightarrow \quad r_n^2 = -\,0.105^{+0.005}_{-0.006}\ \mathrm{fm}^2$$

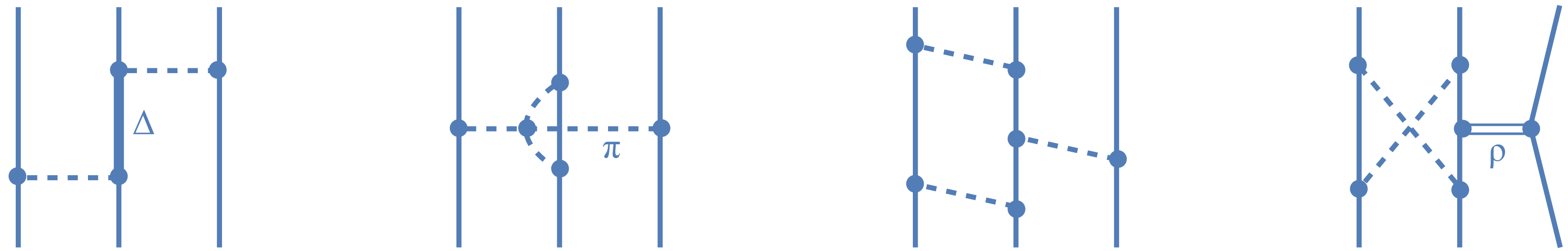
Pachucki et al. '18

— Extraction of πN coupling constant(s) from NN scattering data: $g_{\pi N} = 13.92 \pm 0.09$ Reinert, Krebs, EE, PRL 126 (21)

Three-body force: A frontier in nuclear physics

- Three-nucleon forces (3NF) are small but important corrections to the dominant NN forces

- 3NF mechanisms:



intermediate Δ-excitation
Fujita, Miyazawa '57

multi-pion interactions

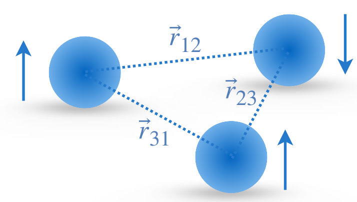
off-shell behavior of the V_{NN}
$V_{\rm ring} = \mathcal{A}_{3\pi} - V_\pi G_0 V_\pi G_0 V_\pi$

short-range

⇒ 3NF are not directly measurable and depend on the scheme (DoF, off-shell V_{NN}, …)

- 3NF have extremely rich and complex structure

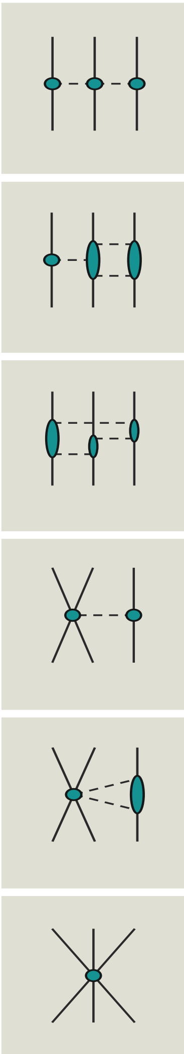
— most general ***local*** 3NF: $V_{3N} = \sum_{i=1}^{20} O_i\, f_i(r_{12}, r_{23}, r_{31}) +$ permutations
EE, Gasparyan, Krebs, Schat '15



— most general ***nonlocal*** 3NF: **320 (!)** operators Topolnicki '17

⇒ Guidance from theory indispensable — an opportunity for χEFT!

Three-body force: A frontier in nuclear physics



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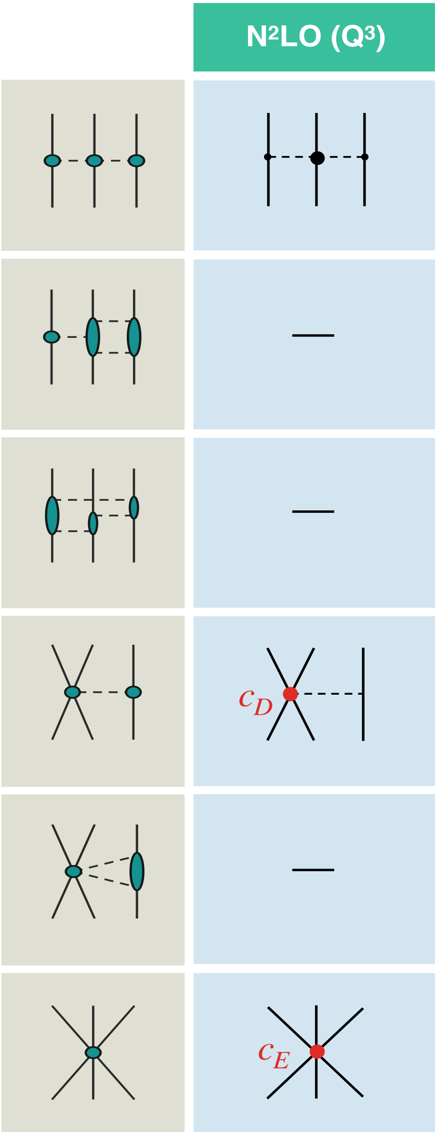

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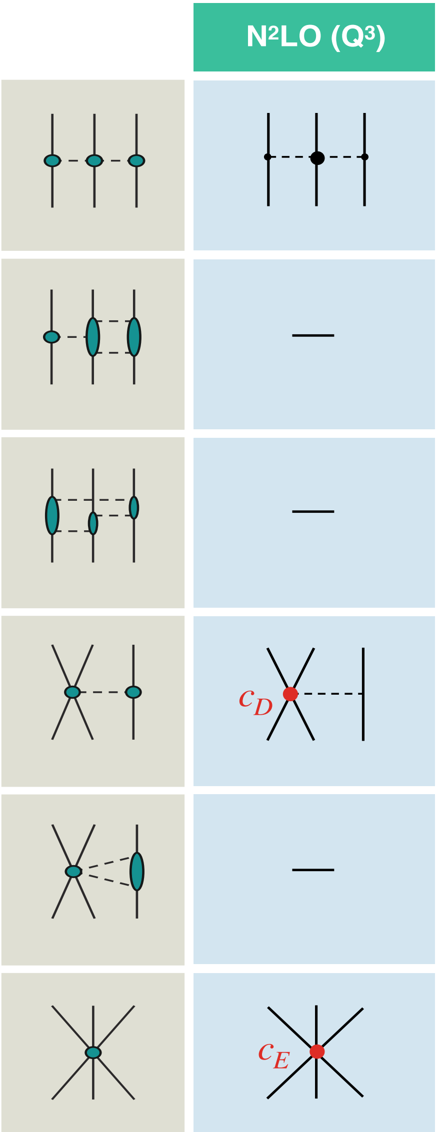
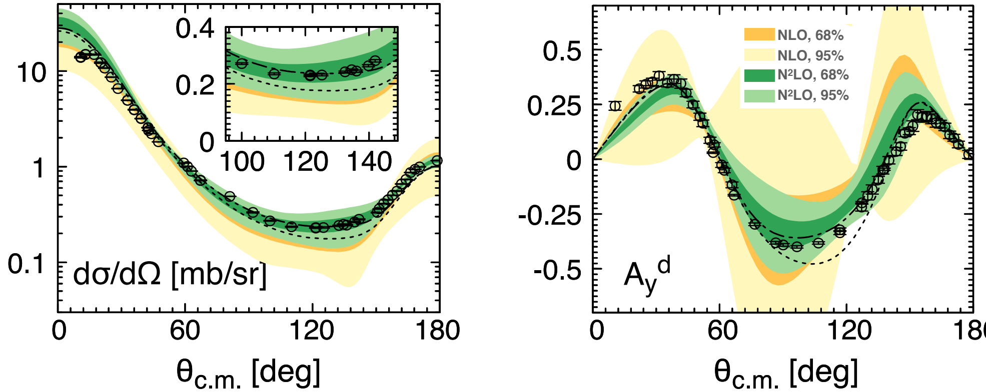
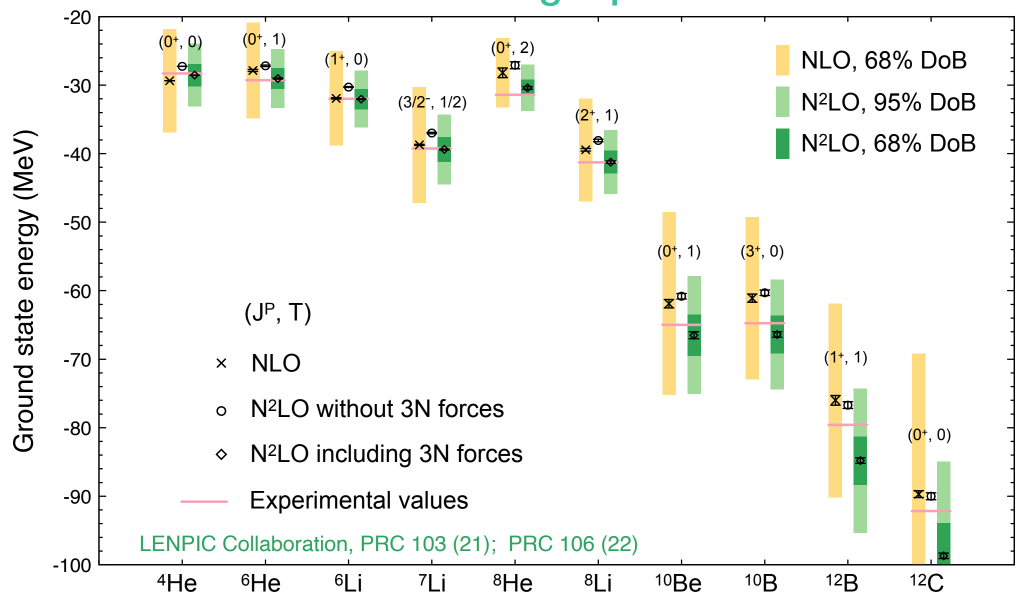


Elastic Nd scattering at 135 MeV



Predictions for light p-shell nuclei



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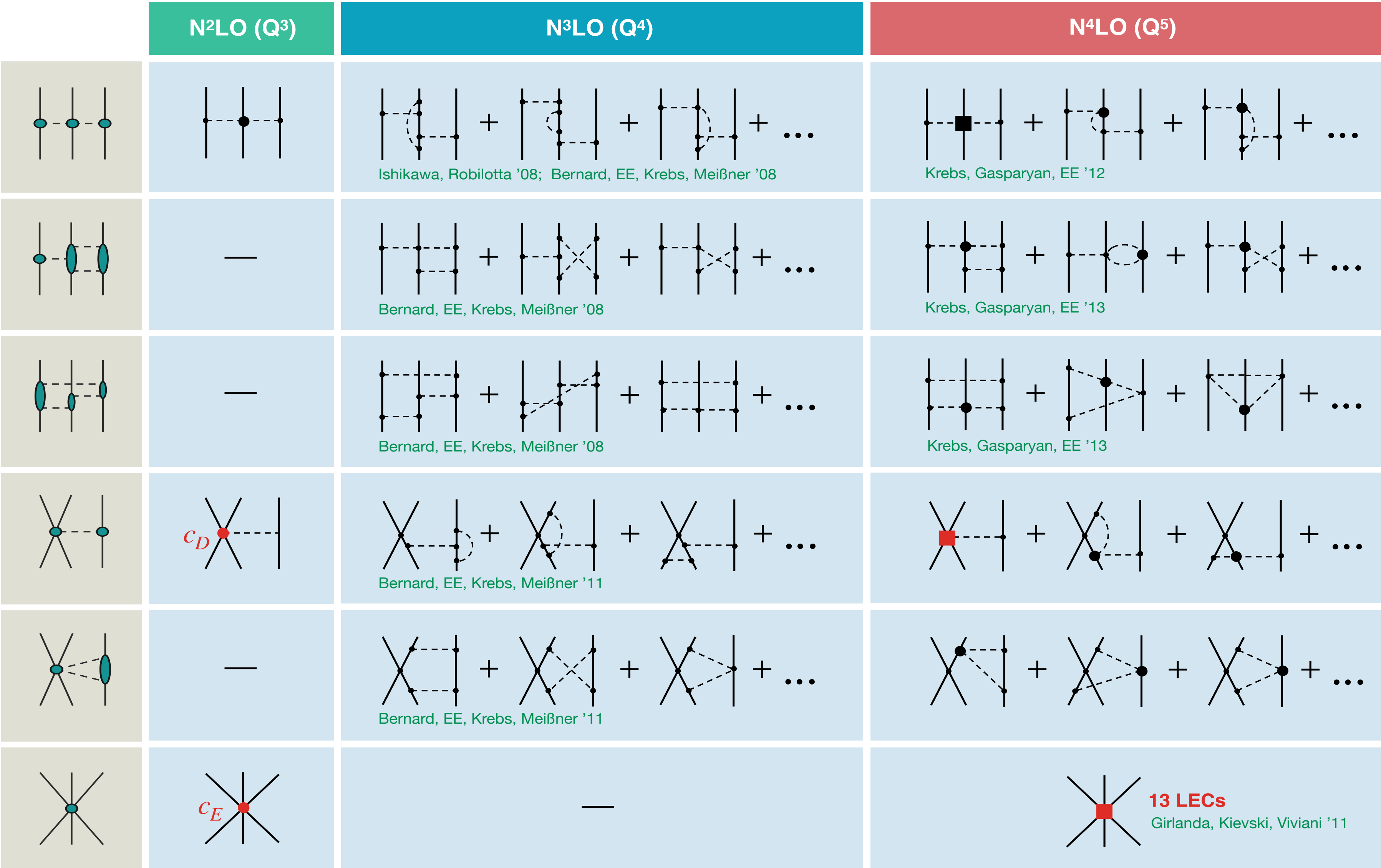
Three-body force: A frontier in nuclear physics



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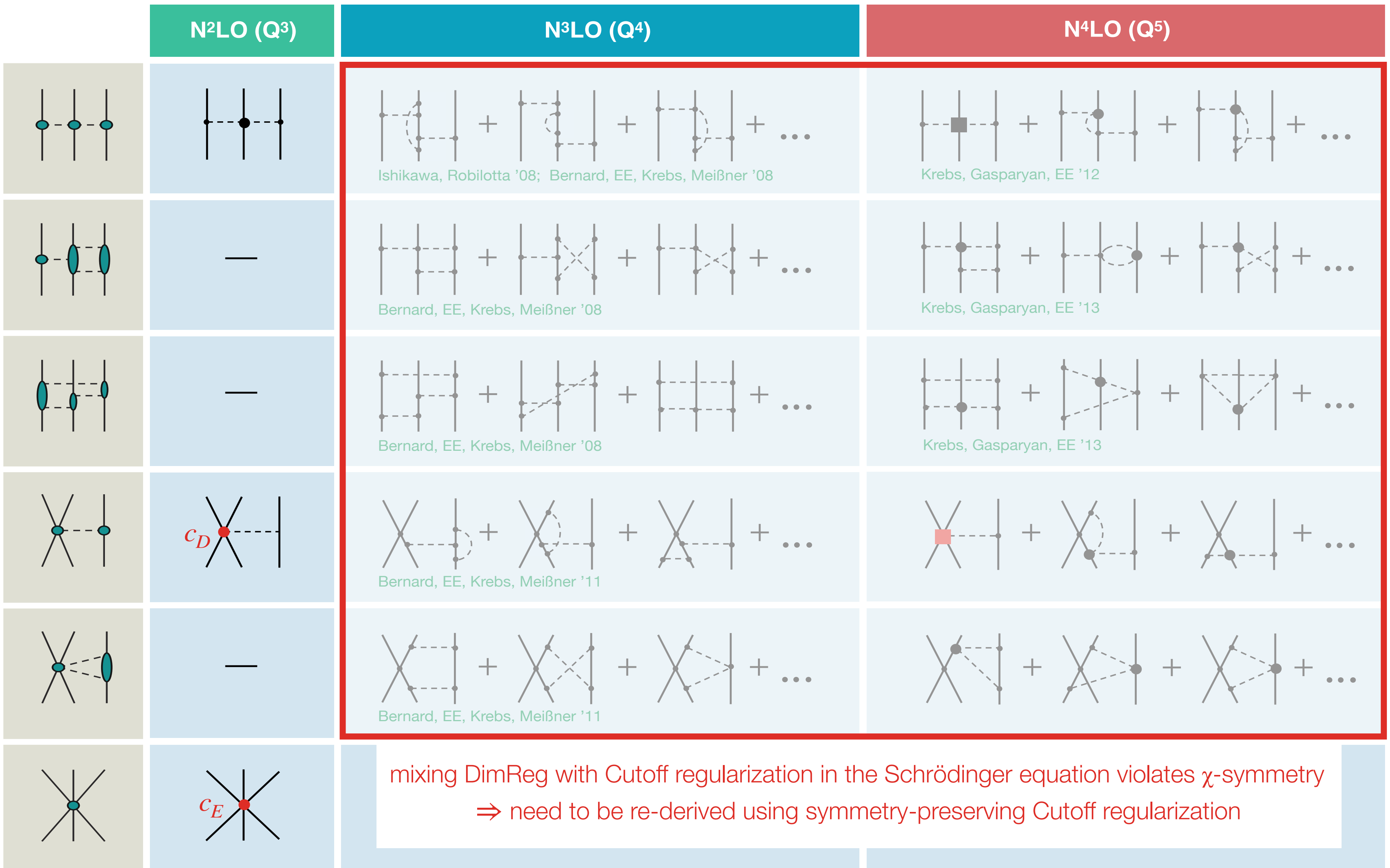

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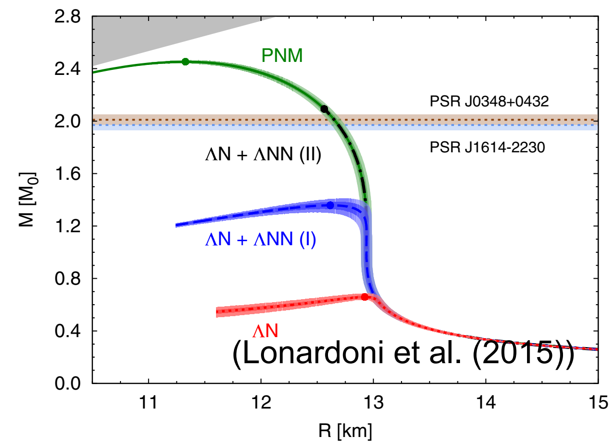
Hyper-nuclear interactions

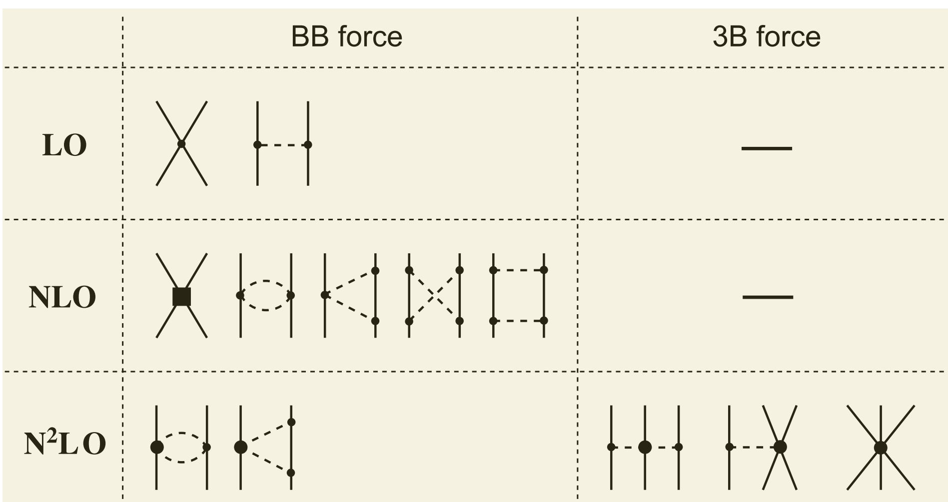
Motivation: Λ as a probe of nuclear structure, hyperon puzzle



Experimental efforts: J-PARC, Jefferson Lab Exploring the Nature of Matter, MAMI, FAIR

+ femtoscopy (ALICE, …)

Theory: much more challenging than SU(2)…



← 5 (+1) NN/YN (YY) short range parameters

← 23(+5) NN/YN (YY) short range parameters

BB and 3B interactions developed by the Jülich-Bonn-Munich group [Haidenbauer et al.]: NLO13, NLO16, NLO19, N^{2}LO22 (different regularizations & constraints)

— more in the talks by Laura Tolos and Andreas Nogga —

Long-term strategy

Lattice-QCD

input for few-B systems
(crucial for $S \neq 0$, BSM MEs and
variable m_q)

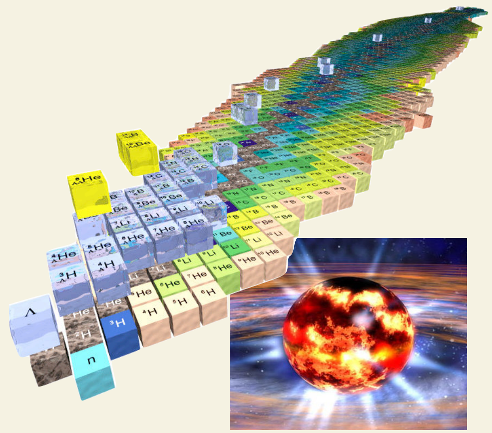
Experiment

Data for 2B & 3B
observables at
physical m_q

Chiral EFT

(Hyper-) nuclear interactions
& current operators derived from
the effective chiral Lagrangian

**(Hyper-) nuclear physics,
neutron stars, …**



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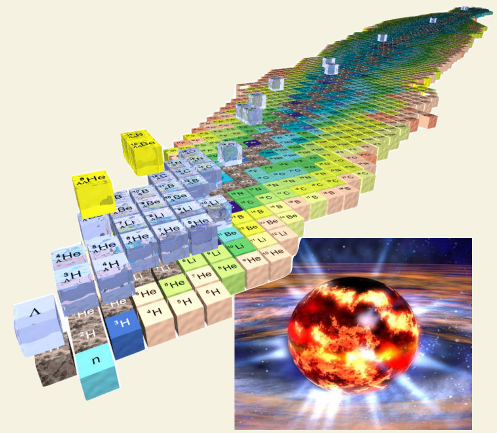
FAIR, Jefferson Lab: Exploring the Nature of Matter, MAMI, J-PARC

Chiral EFT

(Hyper-) nuclear interactions
& current operators derived from the effective chiral Lagrangian

Finite-volume energy spectra

(Hyper-) nuclear physics, neutron stars, …



Important new developments

— Finite volume energy spectra as an efficient interface between L-QCD and χEFT Lu Meng, EE, JHEP 10 (21)
(Infinite-volume extrapolations without Lüscher, no t-channel cut problem, partial-wave mixing included)

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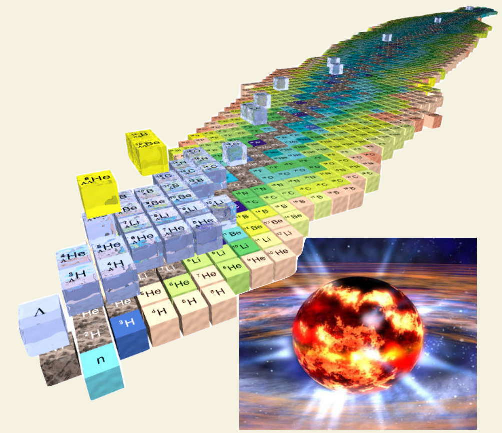

Finite-volume energy spectra

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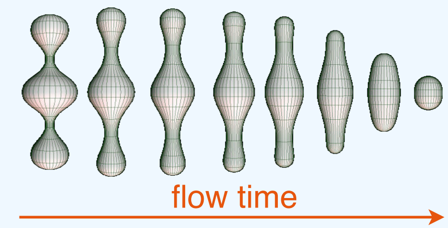
— Finite volume energy spectra as an efficient interface between L-QCD and χEFT Lu Meng, EE, JHEP 10 (21)
(Infinite-volume extrapolations without Lüscher, no t-channel cut problem, partial-wave mixing included)

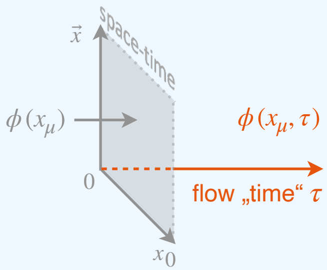
— Symmetry-preserving cutoff regularization (Gradient flow + path integral) Hermann Krebs, EE, 2311.10893, 2312.139932
(Crucial for high-precision 3NF and currents and for chiral extrapolations of few-B LQCD results)

Gradient flow

Gradient flows: methods for smoothing manifolds
(e.g., Ricci flow used in the proof of the Poincaré conjecture)



Gradient flow as a regulator in field theory



Flow equation: $$\frac{\partial}{\partial \tau}\phi(x,\tau) = -\left.\frac{\delta S[\phi]}{\delta \phi(x)}\right|_{\phi(x)\to\phi(x,\tau)}$$

subject to the boundary condition $\phi(x,0) = \phi(x)$

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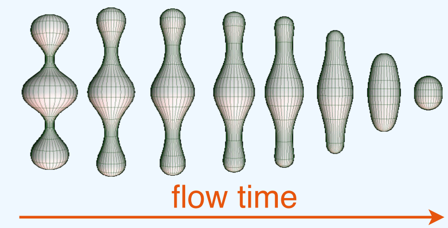

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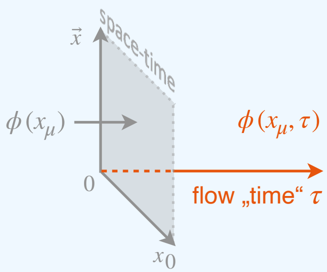

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Free scalar field:

$$\left[\partial_\tau - (\partial^x_\mu\partial^x_\mu - M^2)\right]\phi(x,\tau)=0 \quad\Rightarrow\quad \phi(x,\tau)=\underbrace{\int d^4y\, \overbrace{G(x-y,\tau)}^{G(x,\tau)=\frac{\theta(\tau)}{16\pi^2\tau^2}e^{-\frac{x^2+4M^2\tau^2}{4\tau}}}\phi(y)}_{\textit{heat kernel}} \quad\Rightarrow\quad \tilde{\phi}(q,\tau)=e^{-\tau(q^2+M^2)}\tilde{\phi}(q)$$

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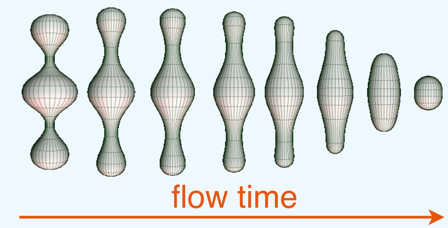

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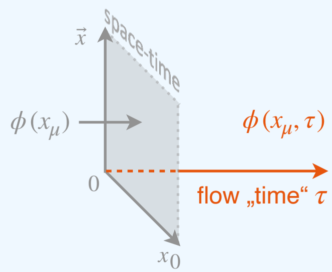

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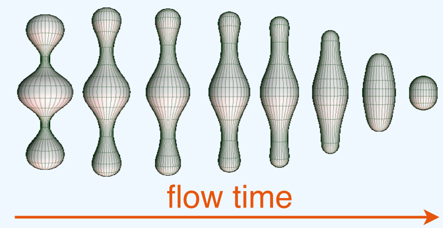
YM gradient flow Narayanan, Neuberger ’06, Lüscher, Weisz ’11: $\partial_\tau A_\mu(x,\tau) = D_\nu G_{\nu\mu}(x,\tau)$ $\leftarrow$ extensively used in LQCD

Gradient flow

Gradient flows: methods for smoothing manifolds
(e.g., Ricci flow used in the proof of the Poincaré conjecture)



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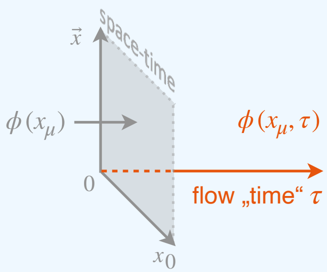

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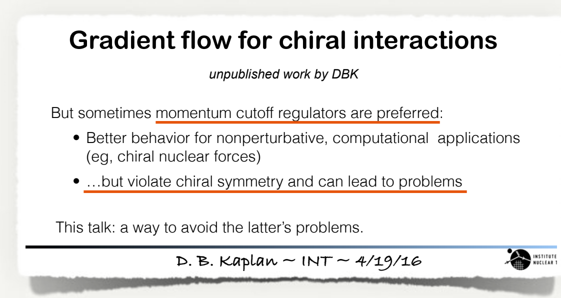
Chiral gradient flow Krebs, EE, 2312.13932

Start with $U(\boldsymbol{\pi}(x)) \in \mathrm{SU}(2) \;\to\; RU(x)L^\dagger$

Generalize $U(x)$ to $W(x,\tau)$: $\partial_\tau W = -i\underbrace{w}_{\sqrt{W}} \, \overbrace{\mathrm{EOM}(\tau)}^{[D_\mu, w_\mu] + \frac{i}{2}\chi_-(\tau) - \frac{i}{4}\mathrm{Tr}\,\chi_-(\tau)} \, w, \quad W(x,0) = U(x)$

We have proven $\forall\tau$: $W(x,\tau) \in \mathrm{SU}(2), \;\; W(x,\tau) \;\to\; RW(x,\tau)L^\dagger$



Gradient flow for chiral interactions

unpublished work by DBK

But sometimes momentum cutoff regulators are preferred:

- Better behavior for nonperturbative, computational applications (eg, chiral nuclear forces)
- …but violate chiral symmetry and can lead to problems

This talk: a way to avoid the latter’s problems.

D. B. Kaplan ~ INT ~ 4/19/16

Gradient flow regularization

Solving the chiral gradient flow equation $\partial_\tau W = -iw\,\mathrm{EOM}(\tau)\,w$

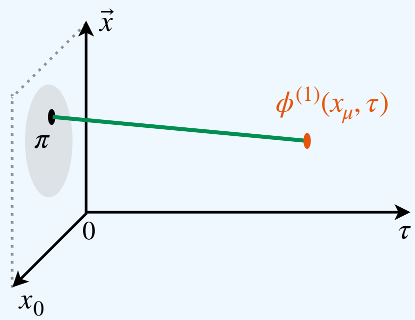
— most general parametrization of $\boldsymbol{U}$: $U = 1 + \frac{i}{F}\boldsymbol{\tau}\cdot\boldsymbol{\pi}\left(1 - \alpha\frac{\boldsymbol{\pi}^2}{F^2}\right) + \mathcal{O}(\boldsymbol{\pi}^4)$

— similarly, write $W = 1 + i\boldsymbol{\tau}\cdot\boldsymbol{\phi}\left(1 - \alpha\,\boldsymbol{\phi}^2\right) - \mathcal{O}(\boldsymbol{\phi}^4)$ and make an ansatz $\boldsymbol{\phi} = \sum_{n=0}^{\infty}\frac{\boldsymbol{\phi}^{(n)}}{F^n}$

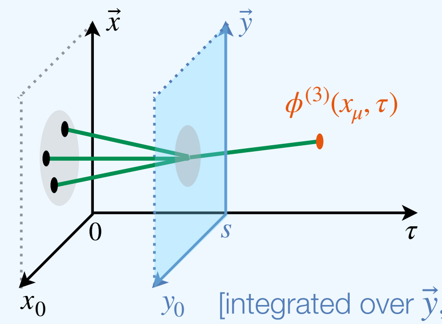
$\Rightarrow$ recursive (perturbative) solution of the GF equation in $1/F$

Gradient flow regularization

$\phi(x_\mu, \tau) \quad =$

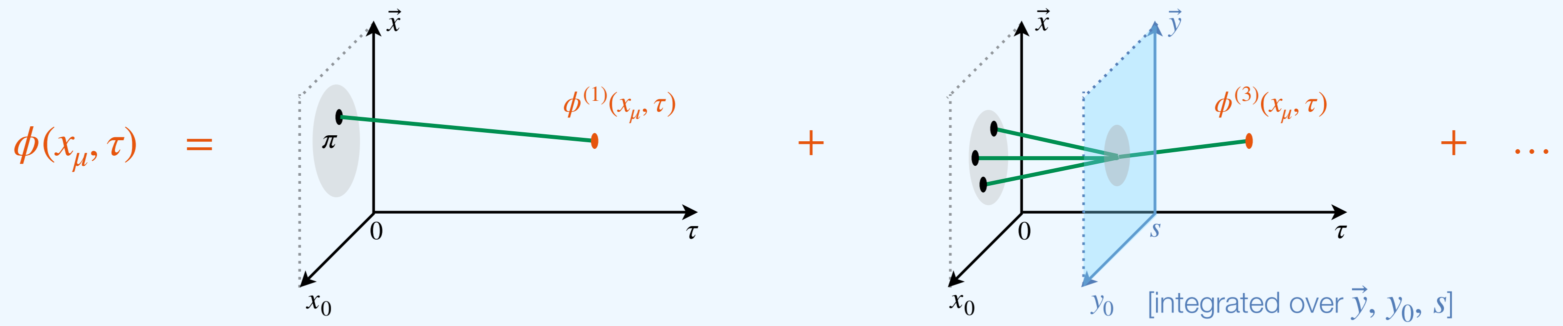


$+$

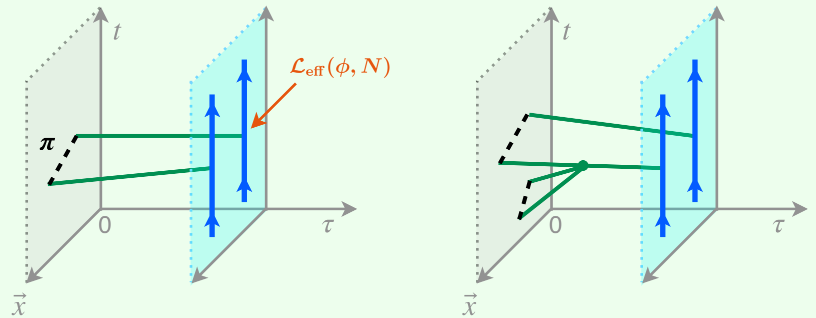


$+ \quad \dots$

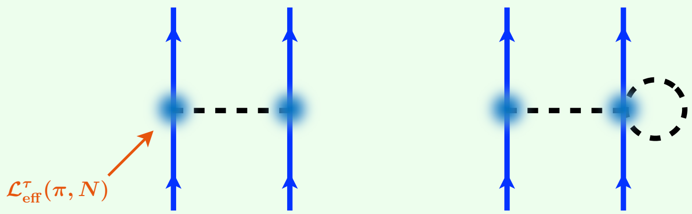
Gradient flow regularization



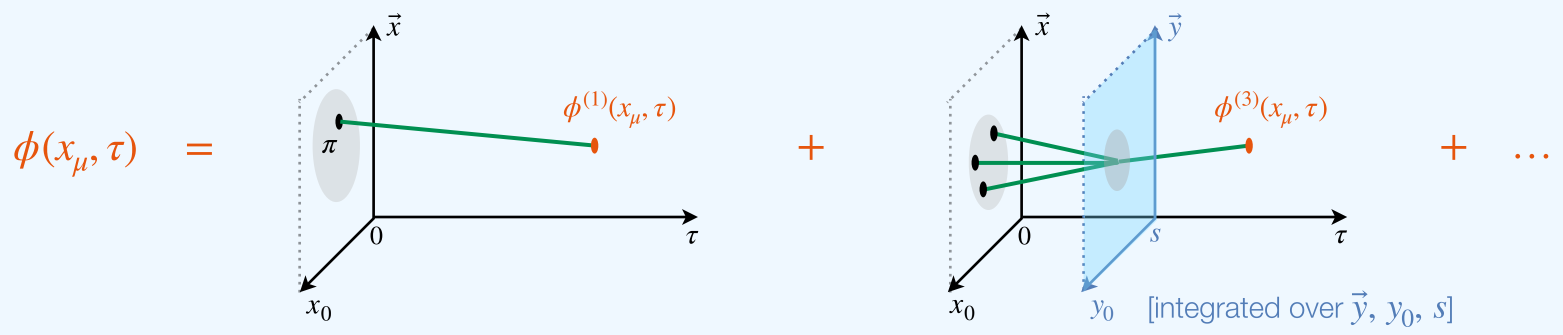
Local field theory in 5d

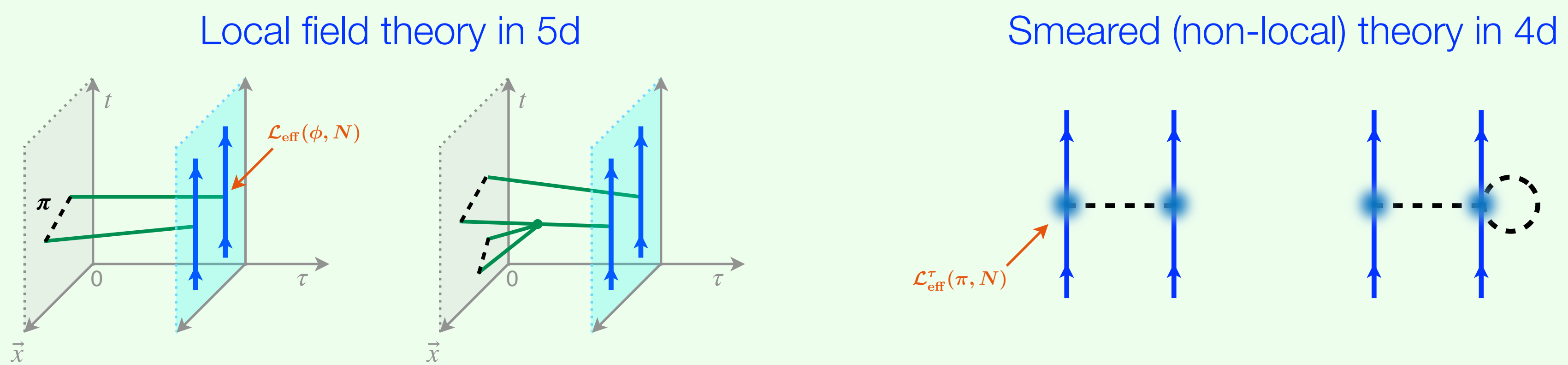


Smeared (non-local) theory in 4d



Gradient flow regularization





We now have the regularized Lagrangian, but cannot use the canonical-quantization-based UT method to derive nuclear forces ($\partial_0^n \pi$ with arbitrary n…). Path Integral approach [Krebs, EE, e-Print: 2312.13932]:

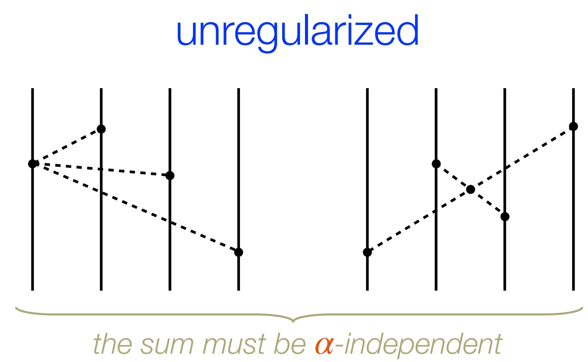
$$Z[\eta^\dagger, \eta] = A \int \mathscr{D}N^\dagger \, \mathscr{D}N \, \mathscr{D}\pi \, \exp\left(i S^\Lambda_{\rm eff} + i \int d^4x [\eta^\dagger N + N^\dagger \eta] \right)$$

instantaneous

nonlocal redefinitions of $N, N^\dagger$

$$\longrightarrow \; A \int \mathscr{D}\tilde{N}^\dagger \, \mathscr{D}\tilde{N} \exp\left(i S^\Lambda_{\rm eff,\, N} + i \int d^4x [\eta^\dagger \tilde{N} + \tilde{N}^\dagger \eta] \right)$$

Gradient flow regularization at work



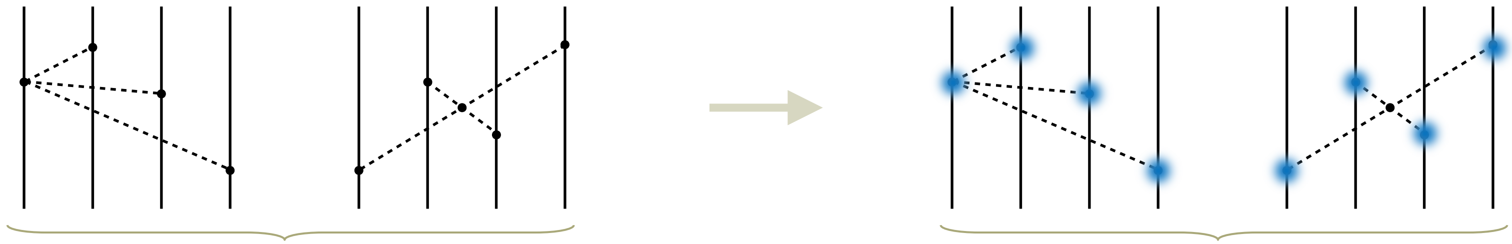
Un-regularized expression for this 4NF [EE, EPJA 34 2007]:

$\hat{O}_{[\sigma_i,\tau_i,\vec{q}_i]} = \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \boldsymbol{\tau}_3 \cdot \boldsymbol{\tau}_4 \, \vec{\sigma}_2 \cdot \vec{q}_2 \, \vec{\sigma}_3 \cdot \vec{q}_3 \, \vec{\sigma}_4 \cdot \vec{q}_4$

$$
\begin{aligned}
V^{4N} &= -\frac{g^4}{64F^6} \frac{\hat{O}_{[\sigma_i,\tau_i,\vec{q}_i]}}{(\vec{q}_2^{\,2}+M^2)(\vec{q}_3^{\,2}+M^2)(\vec{q}_4^{\,2}+M^2)} \, \vec{\sigma}_1 \cdot \vec{q}_{12} \\
&+ \frac{g^4}{128F^6} \frac{\vec{\sigma}_1 \cdot \vec{q}_1 \, \hat{O}_{[\sigma_i,\tau_i,\vec{q}_i]}}{(\vec{q}_1^{\,2}+M^2)(\vec{q}_2^{\,2}+M^2)(\vec{q}_3^{\,2}+M^2)(\vec{q}_4^{\,2}+M^2)} \left(M^2 + \vec{q}_{12}^{\,2}\right) \; + \; 23 \text{ perm.}
\end{aligned}
$$

Gradient flow regularization at work

unregularized | regularized



the sum must be α-independent | the sum must be α-independent

Gradient-flow regularization:

$$
\begin{aligned}
V_\Lambda^{4N} \; = \;& \frac{g^4}{64F^6}\frac{\hat O_{[\sigma_i,\tau_i,\vec q_i]}}{(\vec q_2^{\,2}+M^2)(\vec q_3^{\,2}+M^2)(\vec q_4^{\,2}+M^2)}\bigg[\vec\sigma_1\cdot\vec q_1\big(2g_\Lambda-4f_\Lambda^{123}+2f_\Lambda^{134}-f_\Lambda^{234}\big)-\vec\sigma_1\cdot\vec q_2 f_\Lambda^{234}\\
&+2\vec\sigma_1\cdot\vec q_1\big(5M^2+\vec q_1^{\,2}+\vec q_2^{\,2}+\vec q_3^{\,2}+\vec q_4^{\,2}+\vec q_{34}^{\,2}\big)\frac{g_\Lambda-f_\Lambda^{134}}{2M^2+\vec q_1^{\,2}+\vec q_3^{\,2}+\vec q_4^{\,2}-\vec q_2^{\,2}}\\
&-4\vec\sigma_1\cdot\vec q_1\big(3M^2+\vec q_1^{\,2}+\vec q_2^{\,2}+\vec q_3^{\,2}+\vec q_4^{\,2}-\vec q_{34}^{\,2}\big)\frac{g_\Lambda-f_\Lambda^{124}}{2M^2+\vec q_1^{\,2}+\vec q_2^{\,2}+\vec q_4^{\,2}-\vec q_3^{\,2}}\bigg]\\
+\;& \frac{g^4}{128F^6}\frac{\vec\sigma_1\cdot\vec q_1\,\hat O_{[\sigma_i,\tau_i,\vec q_i]}}{(\vec q_1^{\,2}+M^2)(\vec q_2^{\,2}+M^2)(\vec q_3^{\,2}+M^2)(\vec q_4^{\,2}+M^2)}\big(M^2+\vec q_{12}^{\,2}\big)\big(4f_\Lambda^{123}-3g_\Lambda\big)\;+\;23\text{ perm.},
\end{aligned}
$$

$$f_\Lambda^{ijk} = e^{-\frac{\vec q_i^{\,2}+M^2}{\Lambda^2}}e^{-\frac{\vec q_j^{\,2}+M^2}{\Lambda^2}}e^{-\frac{\vec q_k^{\,2}+M^2}{\Lambda^2}} \qquad g_\Lambda = e^{-\frac{\vec q_1^{\,2}+M^2}{2\Lambda^2}}e^{-\frac{\vec q_2^{\,2}+M^2}{2\Lambda^2}}e^{-\frac{\vec q_3^{\,2}+M^2}{2\Lambda^2}}e^{-\frac{\vec q_4^{\,2}+M^2}{2\Lambda^2}}$$

(coincides with the un-regularized result in the $\Lambda\to\infty$ limit)

Summary and outlook

Long term goal: Predictive and reliable theory of nuclear structure and reactions using χEFT

- Nuclear forces at physical m_q
  - — NN interactions in good shape
  - — 3NF: promising results at N^{2}LO, higher orders in progress (Gradient flow)
- Hyper-nuclear forces at physical m_q
  - — BB available up to N^{2}LO (exploratory…)
  - — Leading 3B-forces worked out
  - — Need more input to fix LECs: FAIR, lattice-QCD, femtoscopy? large-Nc?
- Hyper-nuclear physics at variable m_q
  - — Will have to rely on lattice-QCD: Match LQCD and χEFT in finite volume!
  - — Schwinger-Dyson?

Thank you for your attention

ChPT, QFT methods

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}G_a^{\mu\nu}G_{a,\mu\nu} + \bar{q}(i\gamma^\mu D_\mu - \mathcal{M})q = -\frac{1}{4}G_a^{\mu\nu}G_{a,\mu\nu} + \underbrace{\bar{q}_L i D q_L + \bar{q}_R i D q_R}_{G\,\equiv\,\mathrm{SU(N_f)_L \times SU(N_f)_R}\ \text{invariant}} - \underbrace{q_L \mathcal{M} q_R - q_R \mathcal{M} q_L}_{\text{small for } \mathrm{N_f}=2}$$

$$\underbrace{G\,\equiv\,\mathrm{SU(N_f)_L \times SU(N_f)_R}\ \text{invariant}}_{\text{SSB to } H\,\equiv\,\mathrm{SU(N_f)_V} \leq \mathrm{SU(N_f)_L \times SU(N_f)_R} \ \Rightarrow\ \mathrm{N_f}^2 - 1 \text{ GBs}}$$

Low-energy QCD dynamics can be described in terms of $\mathscr{L}_{\mathrm{eff}}\big[\mathrm{GBs} + \text{matter fields}\,(N, \Delta\ldots)\big]$

ChPT, QFT methods

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4}G_a^{\mu\nu}G_{a,\mu\nu} + \bar q (i\gamma^\mu D_\mu - \mathcal{M})q = -\frac{1}{4}G_a^{\mu\nu}G_{a,\mu\nu} + \underbrace{\underbrace{\bar q_L i D q_L + \bar q_R i D q_R}_{G\,\equiv\,\mathrm{SU(N_f)_L}\times\mathrm{SU(N_f)_R}\text{ invariant}}}_{\text{SSB to } H\,\equiv\,\mathrm{SU(N_f)_V}\,\le\,\mathrm{SU(N_f)_L}\times\mathrm{SU(N_f)_R}\;\Rightarrow\;\mathrm{N_f}^2-1\text{ GBs}} - \underbrace{q_L\mathcal{M}q_R - q_R\mathcal{M}q_L}_{\text{small for } \mathrm{N_f}=2}$$

Low-energy QCD dynamics can be described in terms of $\mathscr{L}_{\rm eff}\left[\mathrm{GBs} + \text{matter fields } (N, \Delta\ldots)\right]$

GBs are associated with coordinates ξ of the coset space G/H: $g = e^{\xi\cdot A}e^{s\cdot V} \in G, \;\; h = e^{s\cdot V} \in H.$

CCWZ nonlinear realization of G:
(becomes representation for $g_0 \in H$)

$$\begin{cases} \xi \xrightarrow{g_0} \xi' = \xi'(\xi, g_0) \\ \phi \xrightarrow{g_0} \phi' = \mathcal{D}^{(n)}\left[e^{s'\cdot V}\right]\phi \end{cases}$$

where $g_0\, e^{\xi\cdot A} =: e^{\xi'\cdot A} e^{s'\cdot V}$, with $A = -i\tau/2$, $V = -i\gamma_5\tau/2$

In practice, it is more convenient to work with

$$U \equiv e^{2\xi\cdot V} \in \mathrm{SU(2)}, \quad U = \underbrace{1 + \frac{i}{F}\boldsymbol{\tau}\cdot\boldsymbol{\pi}\left(1 - \alpha\frac{\boldsymbol{\pi}^2}{F^2}\right) + \mathcal{O}(\boldsymbol{\pi}^4)}_{\text{ambiguous, but S-matrix is parametrization-independent}}, \quad U \xrightarrow{g_0} e^{\theta_R\cdot V}\, U\, e^{-\theta_L\cdot V} \equiv RUL^\dagger.$$

$$\Rightarrow \mathcal{L}_{\rm eff} = \mathcal{L}_\pi + \mathcal{L}_{\pi N} + \ldots = \frac{F}{4}\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger) + \underbrace{l_1}_{\text{LECs}}\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger)^2 + \ldots + \bar N\left[i\gamma_\mu D^\mu - m + \frac{g_A}{2}\gamma_\mu\gamma_5 u^\mu + \ldots\right]N + \ldots$$

SU(2) matrices, complicated functions of GBs

ChPT, QFT methods

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4}G_a^{\mu\nu}G_{a,\mu\nu} + \bar q(i\gamma^\mu D_\mu - \mathcal{M})q = -\frac{1}{4}G_a^{\mu\nu}G_{a,\mu\nu} + \underbrace{\underbrace{\bar q_L i D q_L + \bar q_R i D q_R}_{G\,\equiv\,\mathrm{SU(N_f)_L}\times\mathrm{SU(N_f)_R}\text{ invariant}}}_{\text{SSB to } H\,\equiv\,\mathrm{SU(N_f)_V}\leq\mathrm{SU(N_f)_L}\times\mathrm{SU(N_f)_R}\;\Rightarrow\;\mathrm{N_f}^2-1\text{ GBs}} - \underbrace{q_L\mathcal{M}q_R - q_R\mathcal{M}q_L}_{\text{small for }\mathrm{N_f}=2}$$

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$-i\tau/2 \quad - i\gamma_5\tau/2$

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SU(2) matrices, complicated functions of GBs

$$Z[\eta^\dagger, \eta, \boldsymbol{\nu}] = A\int\mathcal{D}N^\dagger\mathcal{D}N\,\mathcal{D}\pi\,\exp\left(iS_{\rm eff} + i\int d^4x[\eta^\dagger N + N^\dagger\eta + \boldsymbol{\nu}\cdot\boldsymbol{\pi}]\right) \xrightarrow{\text{loop expansion}} \text{S-matrix}$$

GBs, 1N: perturbative expansion in $\vec p$, M_π: Feynman calculus

Regularization (typically DimReg), renormalization, Passarino-Veltman reduction of tensor integrals, ...

Mathematica, FORM, FeynCalc, TARCER, ...

BChPT using gradient flow regularization

LO pion Lagrangian:

$$\mathcal{L}_\pi^{\mathrm{E}} = \frac{F^2}{4}\mathrm{Tr}\Big[\big(\overbrace{\nabla_\mu U}^{\nabla_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu}\big)^\dagger \nabla_\mu U - U^\dagger \underbrace{\chi - \chi^\dagger}_{\chi = 2B(s+ip)} U\Big] \quad \Rightarrow \quad \mathrm{EOM} = \overbrace{\underbrace{[D_\mu, u_\mu]}_{u_\mu = i u^\dagger \nabla_\mu U u^\dagger, \; u = \sqrt{U}}}^{D_\mu = \partial_\mu + \Gamma_\mu, \; \text{where } \Gamma_\mu = \frac{1}{2}[u^\dagger, \partial_\mu u] - \frac{i}{2} u^\dagger r_\mu u - \frac{i}{2} u \, l_\mu u^\dagger} + \frac{i}{2}\chi_- - \frac{i}{4}\mathrm{Tr}\,\underbrace{\chi_-}_{\chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u}$$

Construction of the Lagrangian in BChPT

Pion Lagrangian: $U(\boldsymbol{\pi}) \to R U L^\dagger, \quad \mathcal{L}_\pi^{\mathrm{E}} = \frac{F^2}{4}\mathrm{Tr}\big[(\nabla_\mu U)^\dagger \nabla_\mu U\big] + \ldots$

BChPT using gradient flow regularization

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Generalization to N: $u(\boldsymbol{\pi}) := \sqrt{U}, \quad u \to R u K^\dagger = K u L^\dagger \quad \text{with} \quad K(L, R, U) = \sqrt{L U^\dagger R^\dagger} R \sqrt{U}$

Define [CCWZ '69] $N \to K N$ and introduce $u_\mu = i u^\dagger \nabla_\mu U u^\dagger, \; u_\mu \to K u_\mu K^\dagger, \; \ldots$

$$\Rightarrow \quad \mathcal{L}_{\pi N}^{\mathrm{E}} = N^\dagger (D_0 + g u_\mu S_\mu) N + \ldots$$

BChPT using gradient flow regularization

LO pion Lagrangian:

$$\mathcal{L}_\pi^{\rm E} = \frac{F^2}{4}\mathrm{Tr}\Big[\big(\overbrace{\nabla_\mu U}^{\nabla_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu}\big)^\dagger \nabla_\mu U - U^\dagger \underbrace{\chi}_{\chi = 2B(s+ip)} - \chi^\dagger U\Big] \quad \Rightarrow \quad \mathrm{EOM} = \big[\overbrace{D_\mu}^{D_\mu = \partial_\mu + \Gamma_\mu, \;\; \text{where} \;\; \Gamma_\mu = \frac{1}{2}[u^\dagger, \partial_\mu u] - \frac{i}{2} u^\dagger r_\mu u - \frac{i}{2} u\, l_\mu u^\dagger}, \underbrace{u_\mu}_{u_\mu = i u^\dagger \nabla_\mu U u^\dagger, \;\; u = \sqrt{U}}\big] + \frac{i}{2}\chi_- - \frac{i}{4}\mathrm{Tr}\,\underbrace{\chi_-}_{\chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u}$$

Construction of the Lagrangian in BChPT

Pion Lagrangian: $U(\boldsymbol{\pi}) \to R U L^\dagger, \quad \mathcal{L}_\pi^{\rm E} = \frac{F^2}{4}\mathrm{Tr}\big[(\nabla_\mu U)^\dagger \nabla_\mu U\big] + \ldots$

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$$\Rightarrow \quad \mathcal{L}_{\pi N}^{\rm E} = N^\dagger (D_0 + g u_\mu S_\mu) N + \ldots$$

BChPT using gradient flow regularization: $\mathcal{L}^{\rm E} = \mathcal{L}_\pi^{\rm E} + \mathcal{L}_{\pi N}^{\rm E}(\tau), \quad \underbrace{\mathcal{L}_{\pi N}^{\rm E}(\tau) = \mathcal{L}_{\pi N}^{\rm E}\Big|_{U \to W(\tau)}}_{\text{non-local (smeared) Lagrangian upon expressing in } \boldsymbol{\pi}\text{'s}}$

Solving the chiral gradient flow equation

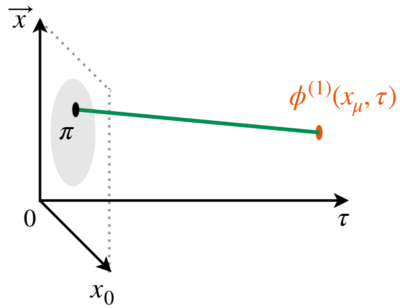
Generalized pion field $\underbrace{\boldsymbol{\phi}(x,\tau)}$: $\quad W = 1 + i\boldsymbol{\tau}\cdot\boldsymbol{\phi}(1-\alpha\,\boldsymbol{\phi}^2) - \dfrac{\boldsymbol{\phi}^2}{2}\left[1+\left(\dfrac{1}{4}-2\alpha\right)\boldsymbol{\phi}^2\right] + \ldots$

calculated by recursively solving the GF equation using $\phi = \sum_{n=0}^{\infty} \frac{\phi^{(n)}}{F^n}$

In the absence of external sources, one finds:

Solving the chiral gradient flow equation

$\phi(x_\mu, \tau) \quad =$



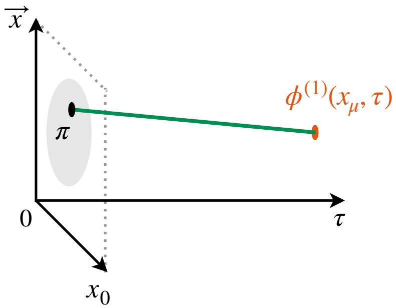
$+$

$$\left.\begin{aligned}\left[\partial_\tau - (\partial^x_\mu \partial^x_\mu - M^2)\right] \boldsymbol{\phi}^{(1)}(x,\tau) = 0 \\ \boldsymbol{\phi}^{(1)}(x,0) = \boldsymbol{\pi}(x)\end{aligned}\right\} \Rightarrow \quad \boldsymbol{\phi}^{(1)}(x,\tau) = \int d^4y \, \overbrace{G(x-y,\tau)}^{G(x,\tau) = \frac{\theta(\tau)}{16\pi^2\tau^2} e^{-\frac{x^2+4M^2\tau^2}{4\tau}}} \boldsymbol{\pi}(y) \quad \Rightarrow \quad \tilde{\boldsymbol{\phi}}^{(1)}(q,\tau) = e^{-\tau(q^2+M^2)} \tilde{\boldsymbol{\pi}}(q)$$

SMS regulator for $\tau = 1/(2\Lambda^2)$

Solving the chiral gradient flow equation

$\phi(x_\mu, \tau) \quad =$



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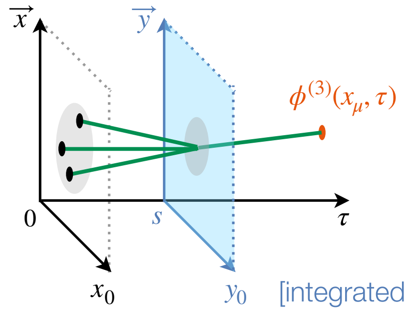


$+ \quad \dots$

[integrated over $\vec{y}$, y_0, s]

$$\left.\begin{array}{r} \left[\partial_\tau - (\partial^x_\mu \partial^x_\mu - M^2)\right] \boldsymbol{\phi}^{(1)}(x,\tau) = 0 \\ \boldsymbol{\phi}^{(1)}(x,0) = \boldsymbol{\pi}(x) \end{array}\right\} \Rightarrow \quad \boldsymbol{\phi}^{(1)}(x,\tau) = \int d^4y\, \overbrace{G(x-y,\tau)}^{G(x,\tau) = \frac{\theta(\tau)}{16\pi^2\tau^2} e^{-\frac{x^2+4M^2\tau^2}{4\tau}}} \boldsymbol{\pi}(y) \quad \Rightarrow \quad \tilde{\boldsymbol{\phi}}^{(1)}(q,\tau) = e^{-\tau(q^2+M^2)} \tilde{\boldsymbol{\pi}}(q)$$

SMS regulator for $\tau = 1/(2\Lambda^2)$

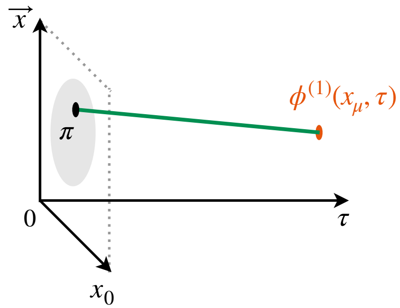
$$\left[\partial_\tau - (\partial^x_\mu \partial^x_\mu - M^2)\right] \phi_b^{(3)}(x,\tau) = \overbrace{(1-2\alpha)\partial_\mu \boldsymbol{\phi}^{(1)} \cdot \partial_\mu \boldsymbol{\phi}^{(1)} \phi_b^{(1)} - 4\alpha\, \partial_\mu \boldsymbol{\phi}^{(1)} \cdot \boldsymbol{\phi}^{(1)} \partial_\mu \phi_b^{(1)} + \frac{M^2}{2}(1-4\alpha) \boldsymbol{\phi}^{(1)} \cdot \boldsymbol{\phi}^{(1)} \phi_b^{(1)}}^{\equiv \mathrm{RHS}_b(x,\tau)}$$

$$\phi_b^{(3)}(x,0) = 0$$

$$\Rightarrow \quad \phi_b^{(3)}(x,\tau) = \int_0^\tau ds \int d^4y\, G(x-y,\tau-s)\, \mathrm{RHS}_b(y,s)$$

Solving the chiral gradient flow equation

$\phi(x_\mu, \tau) \;=$



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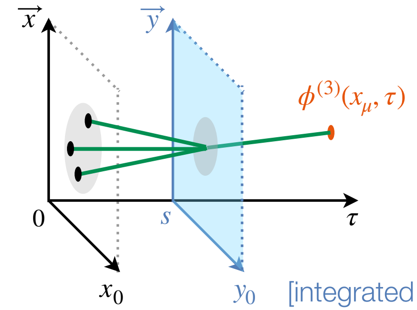


[integrated over $\vec{y}$, y_0, s]

$+ \;\;\ldots$

$$\left.\begin{array}{r} \left[\partial_\tau - (\partial^x_\mu \partial^x_\mu - M^2)\right] \boldsymbol{\phi}^{(1)}(x,\tau) = 0 \\ \boldsymbol{\phi}^{(1)}(x,0) = \boldsymbol{\pi}(x) \end{array}\right\} \Rightarrow \quad \boldsymbol{\phi}^{(1)}(x,\tau) = \int d^4y \, \overbrace{G(x-y,\tau)}^{G(x,\tau) = \frac{\theta(\tau)}{16\pi^2\tau^2} e^{-\frac{x^2 + 4M^2\tau^2}{4\tau}}} \boldsymbol{\pi}(y) \quad \Rightarrow \quad \tilde{\boldsymbol{\phi}}^{(1)}(q,\tau) = e^{-\tau(q^2+M^2)} \tilde{\boldsymbol{\pi}}(q)$$

SMS regulator for $\tau = 1/(2\Lambda^2)$

$$\left[\partial_\tau - (\partial^x_\mu \partial^x_\mu - M^2)\right] \phi^{(3)}_b(x,\tau) = \overbrace{(1-2\alpha)\partial_\mu \boldsymbol{\phi}^{(1)} \cdot \partial_\mu \boldsymbol{\phi}^{(1)} \phi^{(1)}_b - 4\alpha \, \partial_\mu \boldsymbol{\phi}^{(1)} \cdot \boldsymbol{\phi}^{(1)} \partial_\mu \phi^{(1)}_b + \frac{M^2}{2}(1-4\alpha) \boldsymbol{\phi}^{(1)} \cdot \boldsymbol{\phi}^{(1)} \phi^{(1)}_b}^{\equiv \, \mathrm{RHS}_b(x,\tau)}$$

$$\phi^{(3)}_b(x,0) = 0$$

$$\Rightarrow \quad \phi^{(3)}_b(x,\tau) = \int_0^\tau ds \int d^4y \, G(x-y,\tau-s) \, \mathrm{RHS}_b(y,s)$$

In momentum space, this solution takes the form:

$$\tilde{\phi}^{(3)}_b(q,\tau) = \int \prod_{i=1}^{3} \frac{d^4 q_i}{(2\pi)^4} (2\pi)^4 \delta^4(q - q_1 - q_2 - q_3) \, \underbrace{f_\Lambda(\{q_i\})}_{\frac{e^{-\tau(q^2+M^2)} - e^{-\tau \sum_{j=1}^{3}(q_j^2+M^2)}}{q_1^2 + q_2^2 + q_3^2 - q^2 + 2M^2}} \left[4\alpha \, q_1 \cdot q_3 - (1-2\alpha) q_1 \cdot q_2 + \frac{M^2}{2}(1-4\alpha)\right] \tilde{\boldsymbol{\pi}}(q_1) \cdot \tilde{\boldsymbol{\pi}}(q_2) \, \tilde{\pi}_b(q_3)$$

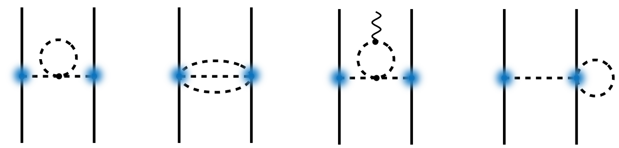
Gradient flow regularization at work

Gradient flow regularization at work

— per construction, no exponentially growing factors

— nuclear forces and currents are sufficiently regularized

(some purely pionic loops are divergent and require e.g. DR) →

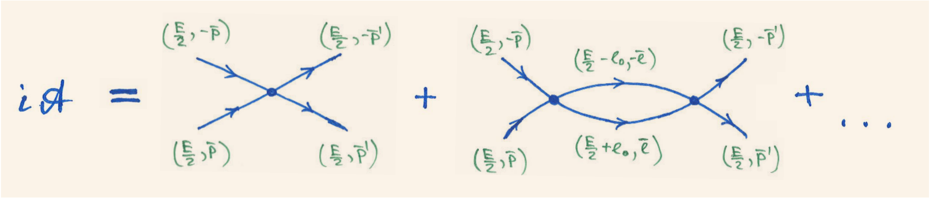


Warm-up exercise

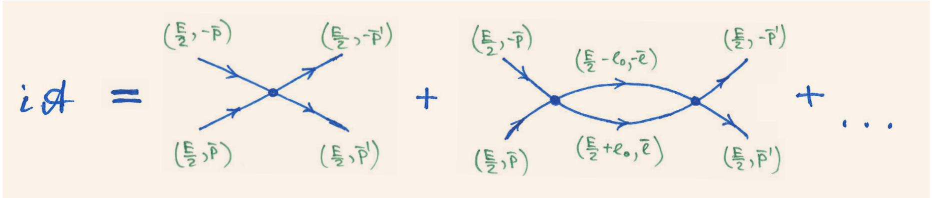
Pion-less EFT:

$$\mathcal{L} = N^\dagger \left[i\partial_0 + \frac{\vec{\nabla}^2}{2m_N} \right] N - \frac{C_S}{2} (N^\dagger N)^2 \; + \ldots$$

$$\Rightarrow \quad \mathcal{A}_{\rm tree} = \left[C_0 + C_2 (\vec{p}^{\,2} + \vec{p}^{\,\prime 2}) + \ldots \right]$$



Warm-up exercise



Pion-less EFT:

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$$\Rightarrow \quad \mathcal{A}_{\rm tree} = [C_0 + C_2(\vec{p}^{\,2} + \vec{p}^{\,\prime 2}) + \ldots]$$

Scattering amplitude to 1 loop:

$$\begin{aligned} -i\mathcal{A}_{1-{\rm loop}} &= \int \frac{d^4 l}{(2\pi)^4} \left[C_0 + C_2(\vec{p}^{\,2} + \vec{l}^{\,2}) + \ldots\right] \frac{1}{\left(\frac{E}{2} + l_0 - \frac{\vec{l}^2}{2m_N} + i\epsilon\right)\left(\frac{E}{2} - l_0 - \frac{\vec{l}^2}{2m_N} + i\epsilon\right)} \left[C_0 + \ldots\right] \\ &= -i \int \frac{d^3 l}{(2\pi)^3} \left[C_0 + C_2(\vec{p}^{\,2} + \vec{l}^{\,2}) + \ldots\right] \frac{1}{E - \frac{\vec{l}^2}{m_N} + i\epsilon} \left[C_0 + (\vec{l}^{\,2} + \vec{p}^{\,\prime 2}) \ldots\right] \end{aligned}$$

All l_0-integrals factorize $\Rightarrow$ Lippmann-Schwinger eq. $\mathcal{A} = \mathcal{V} + \mathcal{V}\, G_0\, \mathcal{A}$ with $\mathcal{V} = -\mathcal{L}_{\rm int}$

Warm-up exercise

Pion-less EFT:

$$\mathcal{L} = N^\dagger \left[i\partial_0 + \frac{\vec{\nabla}^2}{2m_N} \right] N - \frac{C_S}{2}(N^\dagger N)^2 \; + \ldots$$

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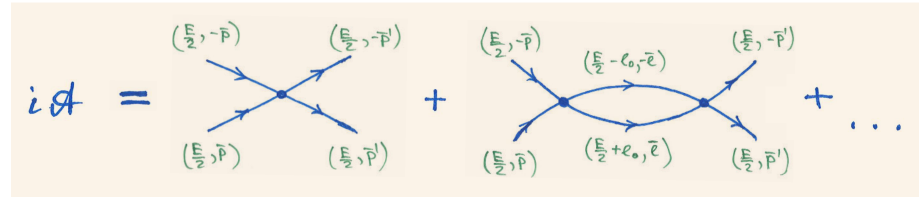

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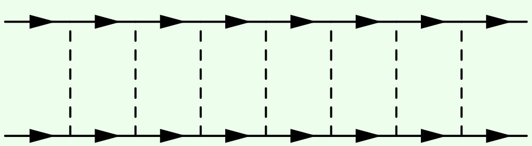

But l_0-integrals do not factorize for pions due to l_0-dependence of π-propagators…

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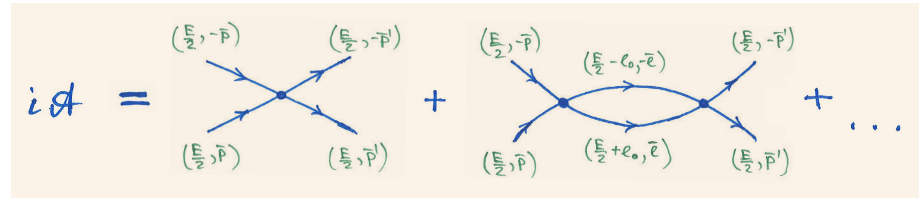

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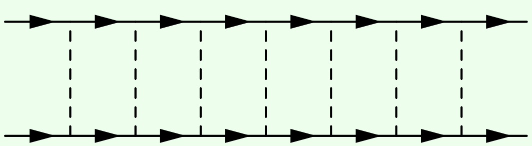

But l_0-integrals do not factorize for pions due to l_0-dependence of π-propagators…

Idea: $Z[\eta^\dagger, \eta] = A \int \mathcal{D}N^\dagger \, \mathcal{D}N \, \mathcal{D}\pi \, \exp\left(i S_{\rm eff}^\Lambda + i \int d^4x [\eta^\dagger N + N^\dagger \eta] \right)$

$$\xrightarrow{\text{nonlocal redefinitions of } N,\, N^\dagger} \; A \int \mathcal{D}\tilde{N}^\dagger \, \mathcal{D}\tilde{N} \exp\left(i S_{\rm eff,\, N}^\Lambda + i \int d^4x [\eta^\dagger \tilde{N} + \tilde{N}^\dagger \eta] \right)$$

instantaneous

Nuclear interactions from path integral

Regularized toy model: $\mathcal{L}^{\mathrm{E}}_{\pi N} = N^\dagger \left[\partial_0 - \frac{\vec{\nabla}^{\,2}}{2m} - \frac{g}{2F} \vec{\sigma} \cdot \vec{\nabla} \boldsymbol{\pi} \cdot \boldsymbol{\tau} \right] N + \frac{1}{2} \boldsymbol{\pi} \cdot \left(-\partial^2 + M^2 \right) e^{\frac{-\partial^2 + M^2}{\Lambda^2}} \boldsymbol{\pi}$

Nuclear interactions from path integral

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Nonlocal action $S^{\rm E}_N$ after integrating out pion fields (Gaussian):

$$Z[\eta^\dagger,\eta] = \int DN^\dagger DN D\boldsymbol{\pi}\, e^{-S^{\rm E}_{\pi N}+\int d^4x\,[\eta^\dagger N + N^\dagger\eta]} = A\int DN^\dagger DN\, e^{-S^{\rm E}_N+\int d^4x\,[\eta^\dagger N + N^\dagger\eta]}$$

where $$S^{\rm E}_N = \underbrace{N^\dagger_x\left[\partial_0 - \frac{\vec\nabla^2_x}{2m}\right]N_x}_{\text{integration over } d^4x \text{ not shown}} + \underbrace{\frac{g^2}{8F^2}\left[N^\dagger\vec\sigma\boldsymbol{\tau}N\right]_{x_1}\cdot\left[\vec\nabla_{x_1}\otimes\vec\nabla_{x_1}\overbrace{\Delta^{\rm E}_\Lambda(x_1-x_2)}^{\text{non-static regularized pion propagator}}\right]\cdot\left[N^\dagger\vec\sigma\boldsymbol{\tau}N\right]_{x_2}}_{\text{integration over } d^4x_1\, d^4x_2 \text{ not shown}}$$

Nuclear interactions from path integral

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Rewrite the pion propagator to the static one plus rest:

$$\Delta^{\mathrm{E}}_\Lambda(x) = \int\frac{d^4q}{(2\pi)^4}e^{iq\cdot x}\frac{e^{-\frac{q_0^2+\vec{q}^{\,2}+M^2}{\Lambda^2}}}{q_0^2+\vec{q}^{\,2}+M^2} = \underbrace{\Delta^{\mathrm{S}}_\Lambda(x)}_{\delta(x_0)\bar{\Delta}^{\mathrm{S}}_\Lambda(\vec{x})} + \Delta^{\mathrm{E}}_\Lambda(x) - \Delta^{\mathrm{S}}_\Lambda(x) = \Delta^{\mathrm{S}}_\Lambda(x) + \underbrace{\partial_0^2\Delta^{\mathrm{ES}}_\Lambda(x)}_{-\int\frac{d^4q}{(2\pi)^4}\frac{e^{iq\cdot x}}{q_0^2}\left[\tilde{\Delta}^{\mathrm{E}}_\Lambda(q)-\tilde{\Delta}^{\mathrm{S}}_\Lambda(\vec{q})\right]}$$

Nuclear interactions from path integral

Regularized toy model: $\mathcal{L}^{\mathrm{E}}_{\pi N} = N^\dagger \left[\partial_0 - \frac{\vec{\nabla}^2}{2m} - \frac{g}{2F} \vec{\sigma} \cdot \vec{\nabla} \boldsymbol{\pi} \cdot \boldsymbol{\tau} \right] N + \frac{1}{2} \boldsymbol{\pi} \cdot \left(-\partial^2 + M^2 \right) e^{\frac{-\partial^2 + M^2}{\Lambda^2}} \boldsymbol{\pi}$

Nonlocal action S^{E}_N after integrating out pion fields (Gaussian):

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Nucleon field redefinition: $N_x = \tilde{N}_x - \frac{g^2}{8F^2} \boldsymbol{\tau} \vec{\sigma} \tilde{N}_x \cdot \underbrace{\left[\vec{\nabla}_x \otimes \vec{\nabla}_x \, \partial_0 \Delta^{\mathrm{ES}}_\Lambda (x - x_2) \right] \cdot \left[\tilde{N}^\dagger \vec{\sigma} \boldsymbol{\tau} \tilde{N} \right]_{x_2}}_{\text{integration over } d^4x_2 \text{ not shown}}, \quad N^\dagger_x = \ldots$

$$\Rightarrow S^{\mathrm{E}}_{\tilde{N}} = \tilde{N}^\dagger_x \left[\partial_0 - \frac{\vec{\nabla}^2_x}{2m} \right] \tilde{N}_x + \frac{g^2}{8F^2} \left[\tilde{N}^\dagger \vec{\sigma} \boldsymbol{\tau} \tilde{N} \right]_{x_1} \cdot \left[\vec{\nabla}_{x_1} \otimes \vec{\nabla}_{x_1} \Delta^{\mathrm{S}}_\Lambda (x_1 - x_2) \right] \cdot \left[\tilde{N}^\dagger \vec{\sigma} \boldsymbol{\tau} \tilde{N} \right]_{x_2} + S^{1/m}_{2\mathrm{N}} + S_{3\mathrm{N}}$$

Nuclear interactions from path integral

To summarize:

$$Z[\eta^\dagger, \eta] \;=\; \int DN^\dagger DN D\boldsymbol{\pi}\, e^{-S^{\rm E}_{\pi N} + \int d^4x \left[\eta^\dagger N + N^\dagger \eta\right]} \;=\; \ldots \;=\; A \int D\tilde N^\dagger D\tilde N\, e^{-S^{\rm E}_{\tilde N} + \int d^4x \left[\eta^\dagger \tilde N + \tilde N^\dagger \eta\right]}$$

where the many-body action is now instantaneous (up to higher-order corrections):

$$S^{\rm E}_{\tilde N} \;=\; \tilde N^\dagger_x \left[\partial_0 - \frac{\vec\nabla_x^2}{2m}\right] \tilde N_x \;+\; \frac{g^2}{8F^2} \left[\tilde N^\dagger \vec\sigma\, \boldsymbol{\tau} \tilde N\right]_{x_1} \cdot \left[\vec\nabla_{x_1} \otimes \vec\nabla_{x_1} \Delta^{\rm S}_\Lambda(x_1 - x_2)\right] \cdot \left[\tilde N^\dagger \vec\sigma\, \boldsymbol{\tau} \tilde N\right]_{x_2} \;+\; S^{1/m}_{\rm 2N} \;+\; S_{\rm 3N}$$

$\Rightarrow$ read out $V_{\rm NN}$ directly from the action: $V^\Lambda_{\rm 2N}(\vec x_{12}) = \dfrac{g^2}{4F^2}\, \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2\, (\vec\sigma_1 \cdot \vec\nabla)\, (\vec\sigma_2 \cdot \vec\nabla)\, \Delta^{\rm S}_\Lambda(\vec x_{12})$

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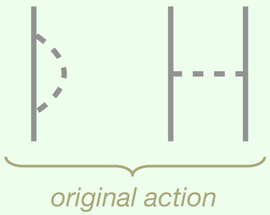
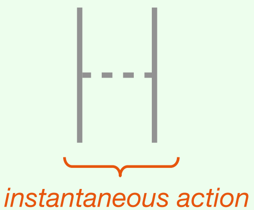
On the other hand, to order g^2:

⇒ something is missing…





Nuclear interactions from path integral

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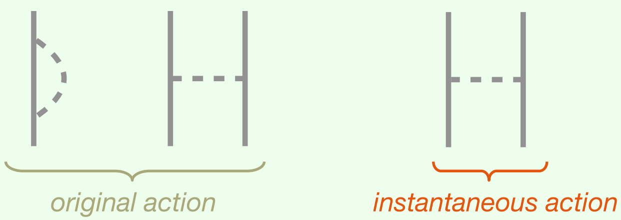

$$\int DN^\dagger DN\, e^{(\ldots)} = \int D\tilde N^\dagger D\tilde N \ \det\left[\boldsymbol{J}_{N,N^\dagger}(\tilde N, \tilde N^\dagger)\right] e^{(\ldots)} = \int D\tilde N^\dagger D\tilde N\, e^{(\ldots) + \int d^4x \tilde N^\dagger_x \Sigma_\Lambda \tilde N_x + \ldots}$$

with $$\Sigma_\Lambda = -\frac{3g^2}{8F^2} \int \frac{d^3p}{(2\pi)^3} \vec p^{\,2} \frac{e^{-\frac{\vec p^{\,2} + M^2}{\Lambda^2}}}{\vec p^{\,2} + M^2} = -\frac{3g^2}{64\pi^{3/2} F^2} \Lambda^3 + \frac{9g^2 M^2}{64\pi^{3/2} F^2} \Lambda \underbrace{- \frac{3g^2 M^3}{32\pi F^2}}_{\text{the leading non-analytic contribution to } m_N} + \mathcal{O}(\Lambda^{-1})$$

Higher-derivative regularization [Slavnov '71]