Transit Time Simulation Studies

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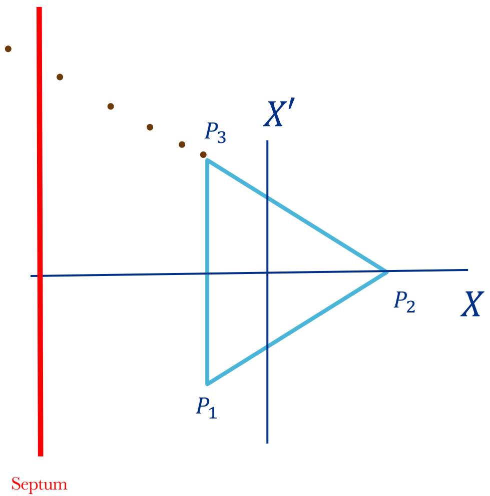
Transit Time Studies

How long does a particle take to reach the septa once it is outside the stable region?

How does the transit time change if you continue squeezing the separatrix as the particle still transits?

How does the transit time depend on the area of the stable region and other operational parameters?



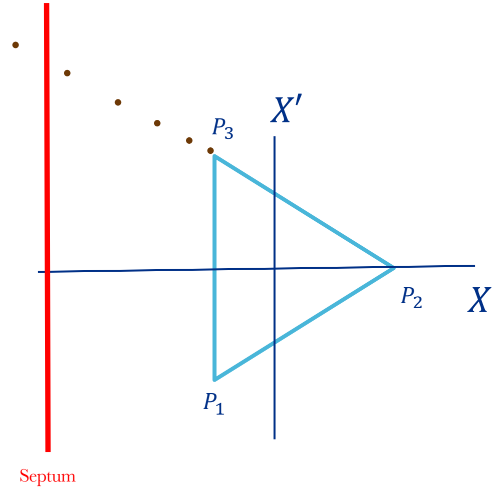
Transit time study is crucial because it determines the beam response time for the extraction.

Kobayashi Hamiltonian

The dynamics of third-integer resonance can be extracted from the Kobayashi Hamiltonian[1]:

$$H = 3\pi\delta Q\,(X^2 + X'^2) + \frac{S}{4}(3XX'^2 - X^3)$$

Linear term — Non-linear term



This simplified Hamiltonian contains only first power in δQ.

For a more detailed review, refer to Marco Pullia's PhD thesis titled "Dynamics of Slow Extraction and and its influence on transfer line designs".

Strategy to get transit time

- Get the equation of motion for X and X' through solving:

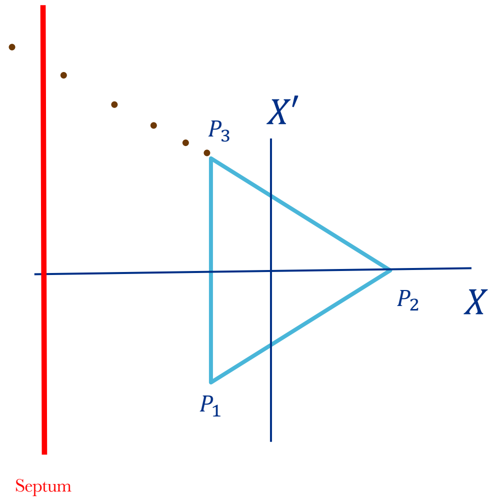
$$\frac{\Delta X}{\Delta n} = \frac{\partial H}{\partial X'} \quad \text{and} \quad \frac{\Delta X'}{\Delta n} = -\frac{\partial H}{\partial X}$$

- Since the Hamiltonian is a constant of motion,

$$H(X_0, X_0'; n) = H(X, X'; n + \Delta n)$$

- Eliminate X' in terms of X using the above equality.

- Now plug in X' gotten from the above step into $\frac{\Delta X}{\Delta n} = \frac{\partial H}{\partial X'}$ to get a RHS purely in terms of X.

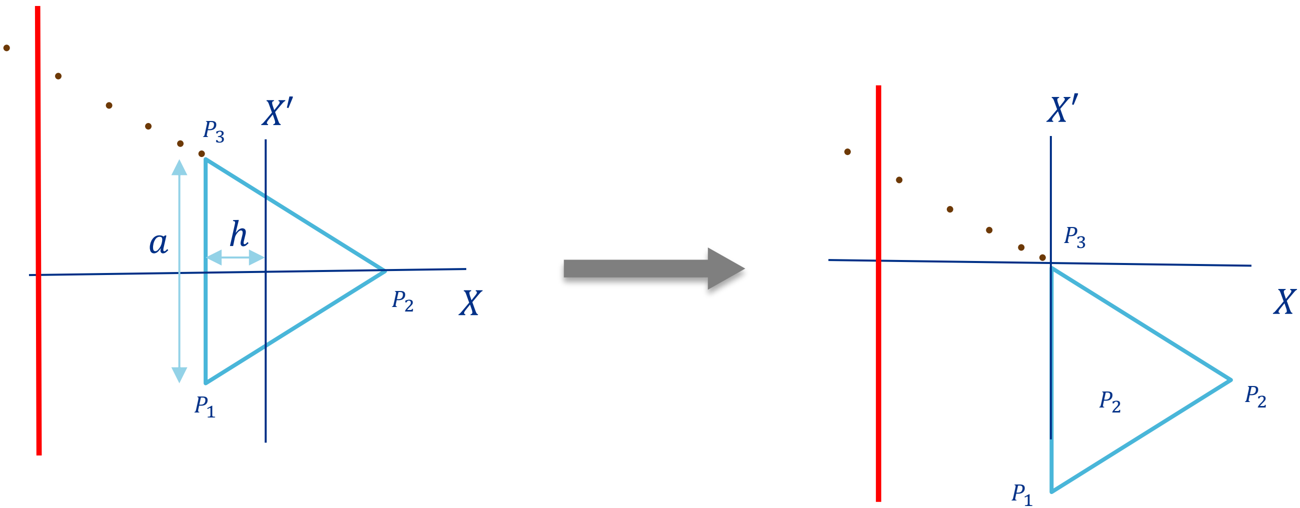


Kobayashi Hamiltonian Translated

$$H = 3\pi\delta Q\,(X^2 + X'^2) + \frac{S}{4}(3XX'^2 - X^3)$$

It is convenient to analyze the transit time when we move on of the vertices to origin.

$$X \to X - h \quad \text{and} \quad X' \to X' + \frac{a}{2}$$



Kobayashi Hamiltonian Translated

$$H = 3\pi\delta Q\,(X^2 + X'^2) + \frac{S}{4}(3XX'^2 - X^3)$$

$$X \to X - h \quad \text{and} \quad X' \to X' + \frac{a}{2}$$

$$H_{trans} = 3\pi\delta Q\left((X-h)^2 + \left(X' + \frac{a}{2}\right)^2\right) + \frac{S}{4}\left(3((X-h)^2)\left(X' + \frac{a}{2}\right)^2 - (X-h)^3\right)$$

$$= \frac{S}{4}[\,3hX^2 + 3h^3 - 6Xh^2 + 3hX'^2 + 9h^3 + 2\sqrt{3}hX'(3h)$$

$$+ 3XX' + 9Xh^2 + 6\sqrt{3}XX'h - 3hX'^2 - 9h^3 - 6\sqrt{3}h^2X'$$

$$- X^3 + 3X^2h - 3h^2X + h^3]$$

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Translated Kobayashi Hamiltonian

$$H = 3\pi\delta Q\,(X^2 + X'^2) + \frac{S}{4}(3XX'^2 - X^3)$$

$$X \to X - h \quad \text{and} \quad X' \to X' + \frac{a}{2}$$

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$$H_{trans} = \frac{S}{4}[\,6hX^2 + 4h^3 + 3XX'^2 + 6\sqrt{3}XX'h - X^3]$$

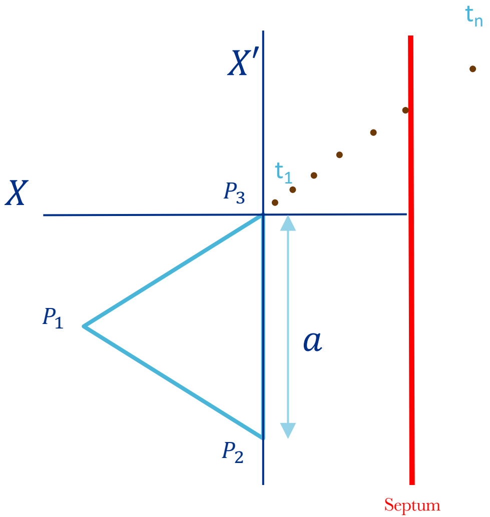
Equation of motion

$$H_{trans} = \frac{S}{4}[\, 3hX^2 + 3h^3 + \; 3XX'^2 + 6\sqrt{3}XX'h \; - \; X^3 + 3X^2h \; + h^3]$$

From the above Hamiltonian,
we get the X evolution equation as:

$$\frac{\mathrm{dX}}{\mathrm{dn}} = \frac{\partial H_{trans}}{\partial X'}$$

$$\frac{\mathrm{dX}}{\mathrm{dn}} = \frac{6S}{4} \left(XX' + \sqrt{3}Xh \right)$$



Next is to eliminate X' and get the X evolution purely in terms of X.

Equation of motion

$$H_{trans} = \frac{S}{4}[\, 3hX^2 + 3h^3 + \, 3XX'^2 + 6\sqrt{3}XX'h \, - \, X^3 + 3X^2h \, + h^3]$$

We can prove that the Hamiltonian is a constant of motion (one way is to verify using Poisson brackets).

$$H(X_0, X_0'; n) = H(X, X'; n + \Delta n)$$

Thus,

$$\frac{S}{4}[\, 3hX^2 + 3h^3 + \, 3XX'^2 + 6\sqrt{3}XX'h \, - \, X^3 + 3X^2h \, + h^3] = \frac{S}{4}[\, 3hX_0^2 + 3h^3 + \, 3X_0X_0'^2 + 6\sqrt{3}X_0X_0'h \, - \, X_0^3 + 3X_0^2h \, + h^3]$$

$$X' = \frac{X_0^2 + \sqrt{3}X_0X_0' - X^2}{\sqrt{3}X}$$

Equation of motion

$$X' = \frac{X_0^2 + \sqrt{3}X_0X_0' - X^2}{\sqrt{3}X}$$

Plugging this into the X-evolution equation, we get:

$$\frac{\mathrm{dX}}{\mathrm{dn}} = \frac{6S}{4}\left(XX' + \sqrt{3}Xh \right)$$

$$\frac{\mathrm{dX}}{\mathrm{dn}} = \frac{6S}{4}\left(X\frac{X_0^2 + \sqrt{3}X_0X_0' - X^2}{\sqrt{3}X} + \sqrt{3}Xh \right)$$

$$\frac{\mathrm{dX}}{\mathrm{dn}} = \frac{6S}{4}\left(\frac{X_0^2 + \sqrt{3}X_0X_0' - X^2}{\sqrt{3}} + \sqrt{3}Xh \right)$$

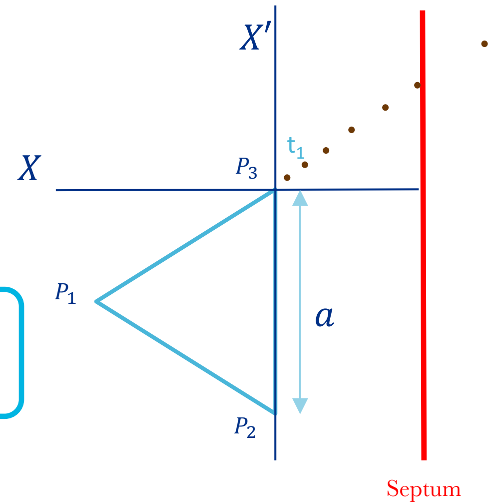
$$\frac{\mathrm{dX}}{\mathrm{dn}} = f(\mathrm{X}) \quad \text{(say)}$$

Equation of motion

$$\frac{\mathrm{dX}}{\mathrm{dn}} = \mathrm{f(X)}$$

$$\mathrm{dn} = [f(X)]^{-1} dX$$

$$T_{tt} = \int_{X_0}^{X_{sept}} \left[\frac{S}{4}\left(6\sqrt{3}hX + \frac{6}{\sqrt{3}} X_0^2 + 6\, X_0 X_0' - \frac{6}{\sqrt{3}} X^2\right)\right]^{-1} dX$$



Can be integrated by completing the squares:

$$\int \frac{1}{ax^2 + bx + c}\, dx = \frac{1}{b^2 - 4ac} \log\left|\frac{\left(2ax + b - \sqrt{b^2 - 4ac}\right)}{\left(2ax + b + \sqrt{b^2 - 4ac}\right)}\right|$$

Kobayashi Hamiltonian

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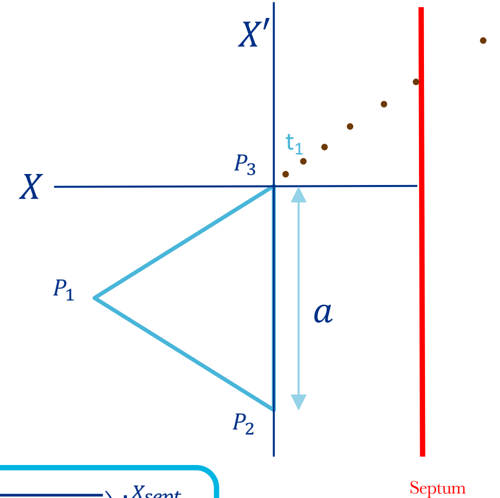


$$T_{tt} = \frac{2}{\sqrt{3}S}\frac{1}{\sqrt{9h^2 + 4(X_0^2 + \sqrt{3}X_0X_0')}}\log\left|\frac{\left(-2X + 3h - \sqrt{9h^2 + 4\left(X_0^2 + \sqrt{3}X_0X_0'\right)}\right)}{\left(-2X + 3h + \sqrt{9h^2 + 4\left(X_0^2 + \sqrt{3}X_0X_0'\right)}\right)}\right|_{X_0}^{X_{sept}}$$

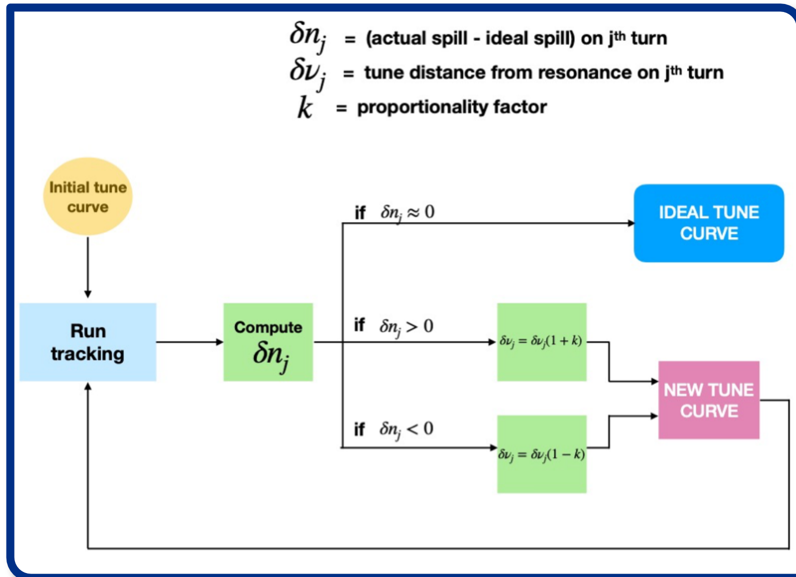
This is the analytical expression for transit time of particles when the resonance condition remains constant throughout the extraction, i.e., the separatrix size does not change.

Ideal Quad Ramp

The quad ramp for "ideal spill rate" (with error tolerance of 5%) was obtained using an adaptive learning algorithm and particle tracking.

$\nu_{new} = \nu_{old}\,(1 \pm k\%)$

for $k = 0.05\% + G_p * err_{bin}$
(with $G_p = 0.1 \rightarrow 0.01$)



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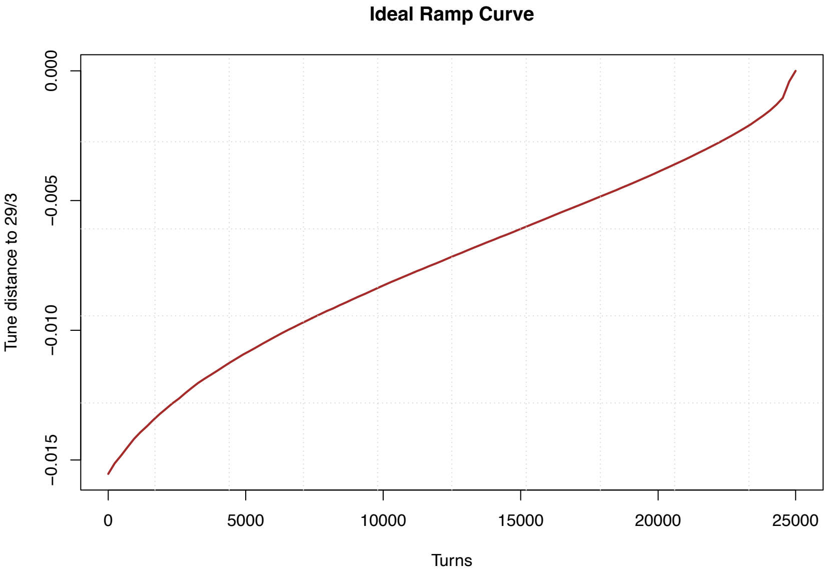

Computing the analytical transit time

Transit Time:

$$T_{tt} = \frac{2}{\sqrt{3}S}\frac{1}{\sqrt{9h^2+4(X_0^2+\sqrt{3}X_0X_0')}}\log\left|\frac{\left(-2X+3h-\sqrt{9h^2+4\left(X_0^2+\sqrt{3}X_0X_0'\right)}\right)}{\left(-2X+3h+\sqrt{9h^2+4\left(X_0^2+\sqrt{3}X_0X_0'\right)}\right)}\right|_{X_0}^{X_{sept}}$$

Plugging in sample Mu2e extraction numbers:

- $X_{sept} = (12 - h)$ mm
- $h = \frac{2}{3}\frac{6\pi\delta Q}{S}$
- $\delta Q = 9.650 \rightarrow 9.666$ (acquired from slow regulation quad ramp)
- S = 500 T/m^2
- $h_{ini} = \frac{a_{ini}}{2\sqrt{3}} \approx 2.6\, mm$
- $a_{ini} \approx 9.2$ mm (approximation)
- X_0 and X'_0 chosen from distribution at vertex

Computing the analytical transit time

Transit Time:

$$T_{tt} = \frac{2}{\sqrt{3}S} \frac{1}{\sqrt{9h^2 + 4(X_0^2 + \sqrt{3}X_0 X_0')}} \log \left| \frac{\left(-2X + 3h - \sqrt{9h^2 + 4\left(X_0^2 + \sqrt{3}X_0 X_0'\right)}\right)}{\left(-2X + 3h + \sqrt{9h^2 + 4\left(X_0^2 + \sqrt{3}X_0 X_0'\right)}\right)} \right|_{X_0}^{X_{sept}}$$

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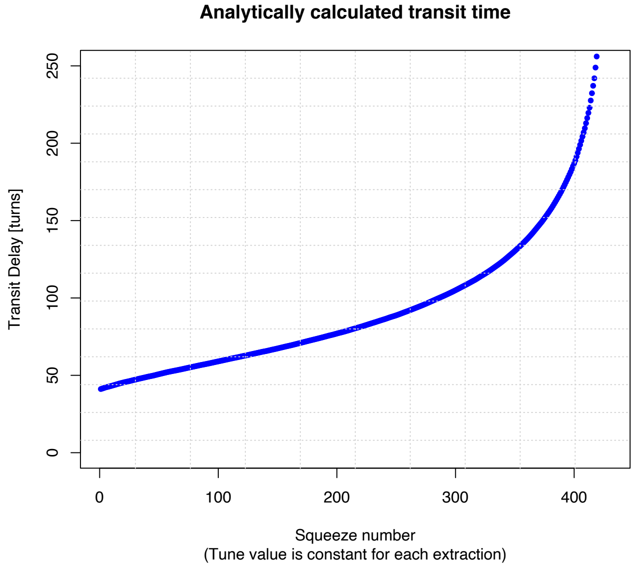
We get the analytical transit time curve to be:



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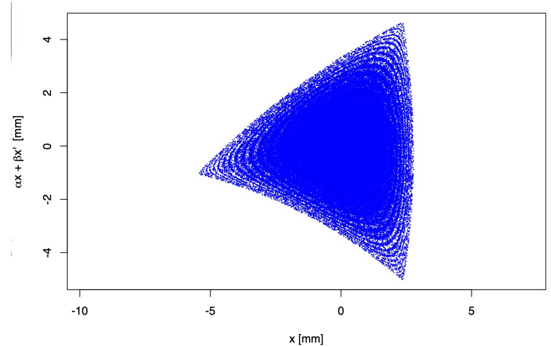
Particle Tracking to check Transit Time

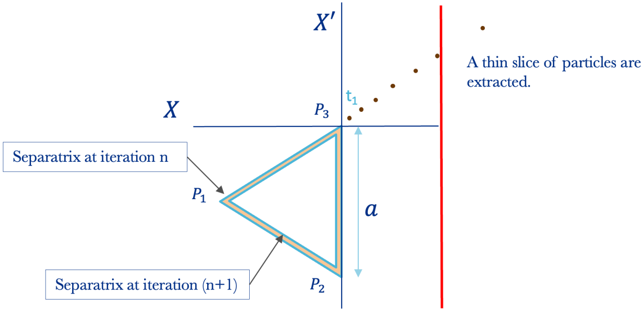
To verify the transit time expression through simulation, we prepared our initial distribution to avoid statistical noises from the beam halo. Particle tracking was done using 4 millions particles.



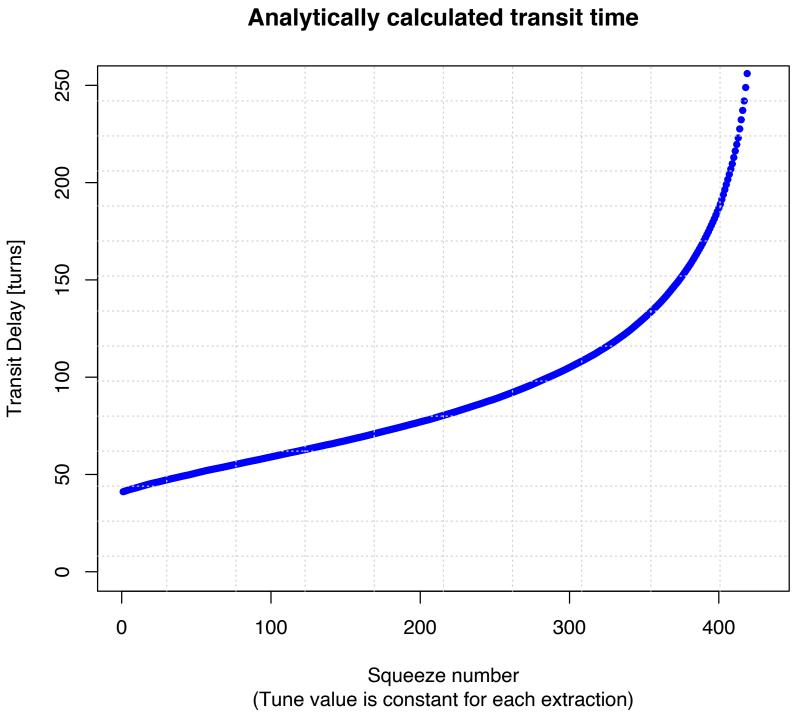
The initial distribution was prepared by running a normal distribution of particles at a constant tune of $\nu_x = 9.650$ for 2000 turns until all the halo is extracted.

Simulation strategy:

- Get the ideal tune ramp curve from Slow Regulation simulations.
- Squeeze the tune from $\Delta\nu = \nu_1 \rightarrow \nu_2$ using the tune ramp curve.
- Store at the number of particles extracted at each turn, including the transit time.
- Iterate this for all the 430 time steps until $\Delta\nu$ goes to zero.



Simulation result



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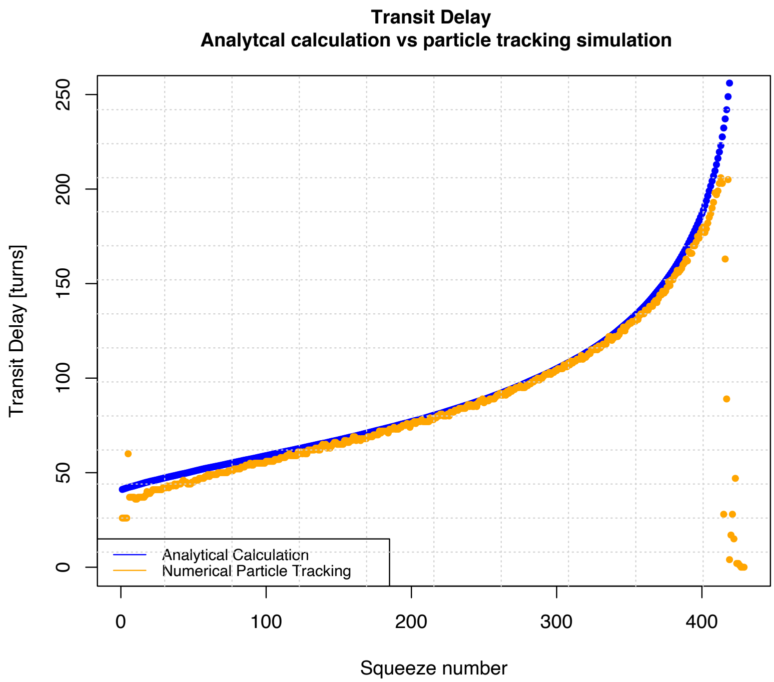
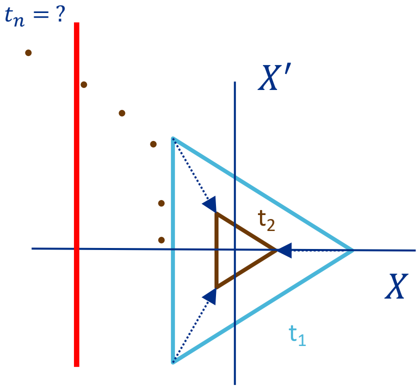


Dynamic Transit Time

The transit time function derived earlier was for when the resonance condition remains static throughout the extraction process, i.e., the stable region's size does not change while the particles are still in transition.

However, often in reality, the resonant extraction process is a continuous one where the stable region is not static but changes dynamically with time.

This begs the question: how does the transit time change with the separatrices are shrinking?



We can start from the evolution equation of X:

$$\frac{\mathrm{dX}}{\mathrm{dn}} = \frac{S}{4}\left(6\sqrt{3}hX + \frac{6}{\sqrt{3}}X_0^2 + 6X_0X_0' - \frac{6}{\sqrt{3}}X^2\right)$$

Since the particles that will get extracted first are the ones near the vertex of the triangle close to the septum, let us assume $X_0 = 0$ and $X_0' = \sqrt{3}X_0$.

Plugging in:

$$\frac{\mathrm{dX}}{\mathrm{dn}} = \frac{S}{4}\left(6\sqrt{3}h - \frac{6}{\sqrt{3}}X^2\right) = \frac{3S}{2\sqrt{3}}\left(3hX - X^2\right)$$

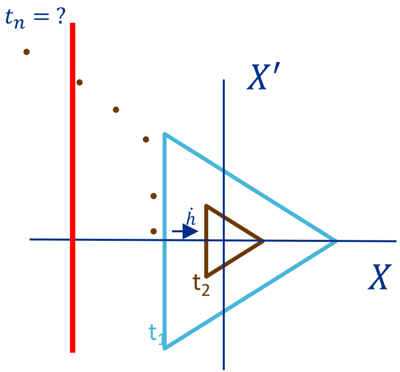
Dynamic Transit Time

Since the tune will be ramped towards resonant tune throughout the spill, the separatrix will be shrinking with the same velocity, given by:

$$\dot{h} = -\frac{4\pi}{S}\frac{dQ}{dn}$$

Since this velocity is in the opposite direction of the particle's direction (because the particle is moving away from the separatrix), we add this to the dX/dn :

$$\frac{\mathrm{dX}}{\mathrm{dn}} = \frac{3S}{2\sqrt{3}}\left(3hX - X^2\right) + \frac{4\pi}{S}\frac{dQ}{dn}$$



Now we invert the above equation and integrate to find the transit time $\mathrm{T_{ttd}} = \int dn$

$$T_{TT\,dyn} = \int dn = \int_{-\frac{X_0}{h}}^{-\frac{X_{sept}}{h}} \frac{1}{\frac{\sqrt{3}S}{2}\left(3hX - X^2\right) + \frac{4\pi}{S}\frac{dQ}{dn}}\, dX$$

Analytical Expression for Transit Time (Dynamic case)

$$T_{TT\ dyn} = \int dn = \int_{-\frac{X_0}{h}}^{-\frac{X_{sept}}{h}} \frac{1}{\frac{\sqrt{3}S}{2}(3hX - X^2) + \frac{4\pi}{S}\frac{dQ}{dn}} dX$$

We can solve this again by completing the squares.

$$T_{tt\ dyn} = \frac{2}{6\sqrt{3}\pi\delta Q} \frac{1}{\sqrt{9h^2 + 4(X_0^2 + \sqrt{3}X_0 X_0')}} \log \left| \frac{\left(\frac{2}{\sqrt{3}}\frac{X_{sept}}{h} - \frac{2}{3\delta Q}\frac{dQ}{dn}\right)\left(\frac{2}{\sqrt{3}}\frac{X_0}{h} + 2\sqrt{3} - \frac{1}{9\pi\delta Q^2}\frac{dQ}{dn}\right)}{\left(\frac{2}{\sqrt{3}}\frac{X_0}{h} - \frac{2}{3\delta Q}\frac{dQ}{dn}\right)\left(\frac{2}{\sqrt{3}}\frac{X_{sept}}{h} + 2\sqrt{3} - \frac{1}{9\pi\delta Q^2}\frac{dQ}{dn}\right)} \right|_{X_0}^{X_{sept}}$$

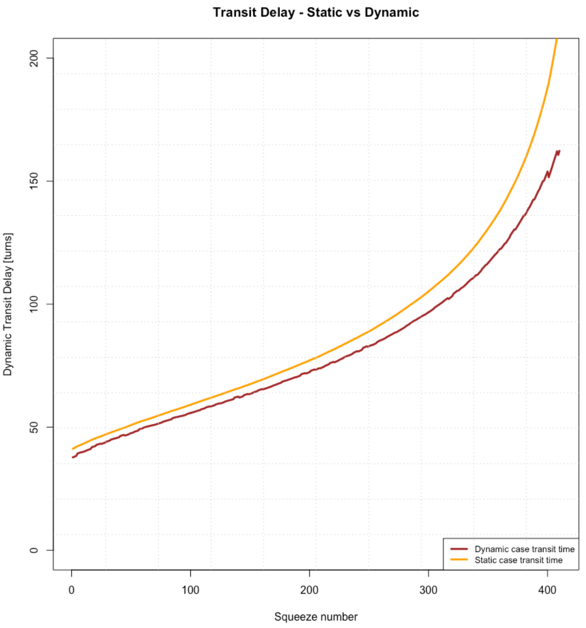
Comparison of the analytical transit time with simulation

$$T_{tt\ dyn} = \frac{2}{6\sqrt{3}\pi\delta Q}\frac{1}{\sqrt{9h^2+4(X_0^2+\sqrt{3}X_0X_0')}}\log\left|\frac{\left(\frac{2}{\sqrt{3}}\frac{X_{sept}}{h}-\frac{2}{3\delta Q}\frac{dQ}{dn}\right)\left(\frac{2}{\sqrt{3}}\frac{X_0}{h}+2\sqrt{3}-\frac{1}{9\pi\delta Q^2}\frac{dQ}{dn}\right)}{\left(\frac{2}{\sqrt{3}}\frac{X_0}{h}-\frac{2}{3\delta Q}\frac{dQ}{dn}\right)\left(\frac{2}{\sqrt{3}}\frac{X_{sept}}{h}+2\sqrt{3}-\frac{1}{9\pi\delta Q^2}\frac{dQ}{dn}\right)}\right|_{X_0}^{X_{sept}}$$

$$T_{tt} = \frac{2}{\sqrt{3}S}\frac{1}{\sqrt{9h^2+4(X_0^2+\sqrt{3}X_0X_0')}}\log\left|\frac{\left(-2X+3h-\sqrt{9h^2+4\left(X_0^2+\sqrt{3}X_0X_0'\right)}\right)}{\left(-2X+3h+\sqrt{9h^2+4\left(X_0^2+\sqrt{3}X_0X_0'\right)}\right)}\right|_{X_0}^{X_{sept}}$$

Plugging in Mu2e extraction numbers:

- $X_{sept} = (12-h)$ mm
- $h = \frac{2}{3}\frac{6\pi\delta Q}{S}$
- $N_{turns} = 500$
- $\delta Q = 6\pi \times \nu[i:i+7]$ values repeated 60 times (because $500/60 \approx 8$)
- $\delta\dot{Q} = 6\pi \times (\nu[i] - \nu[i+1])$
- $h_{ini} = a_{ini}/2\sqrt{3}$
- $a_{ini} \approx 9.2$ mm (approximation)



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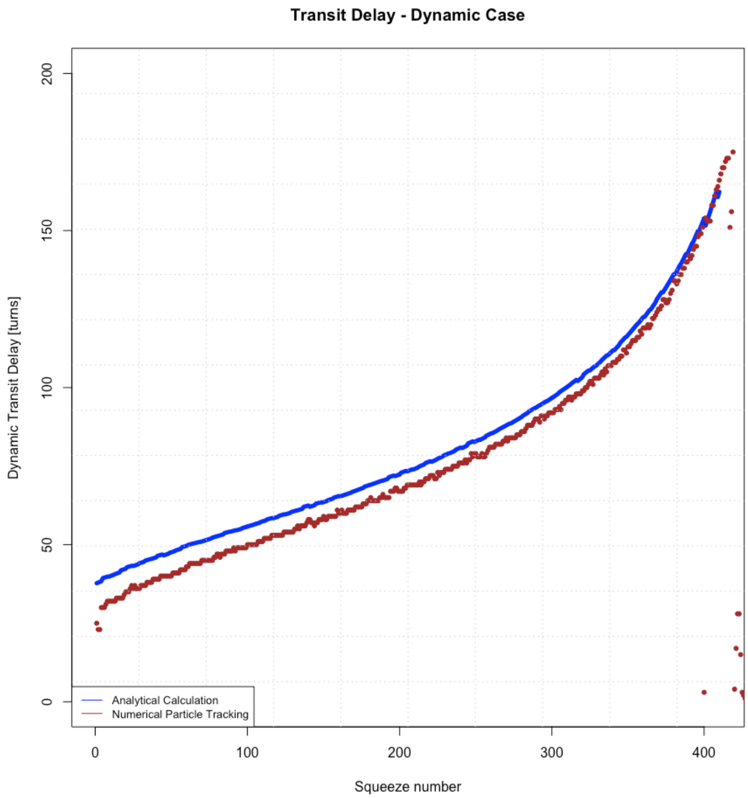
Comparison of the analytical transit time with simulation



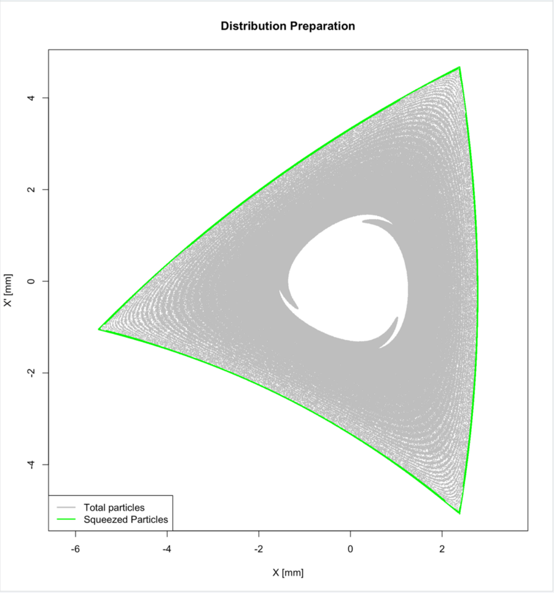
Distribution Preparation

To compare the transit time of particles in the upper and lower band just outside the separatrix, an initial distribution was prepared.

Distribution preparation could be challenging and time consuming since we require an infinitesimally thin slice of particles.

To achieve this, the distribution was prepared by squeezing the tune by 0.000128 (equivalent of about 200 turns worth of tune change).

This was achieved by assigning particle ID to each particle and backtracking the extracted slice.



# of extracted particles ~ 600,000

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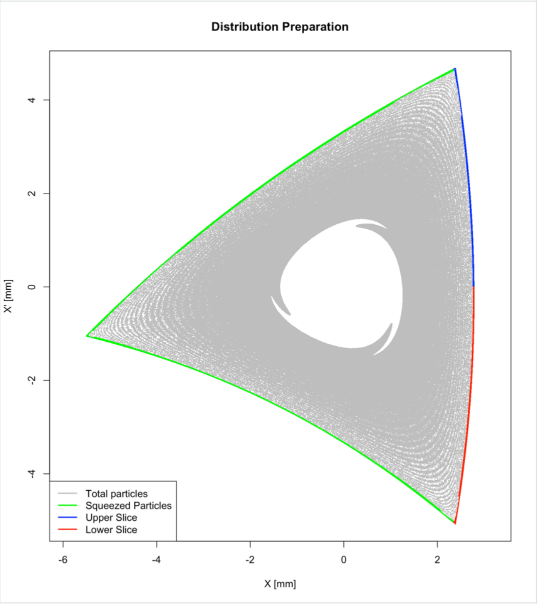

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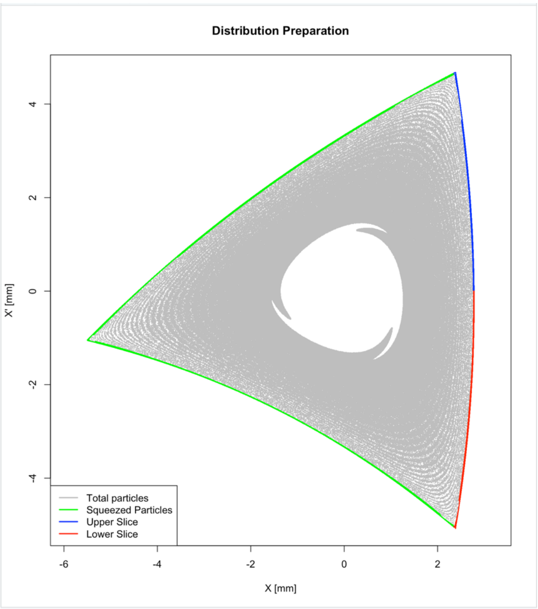
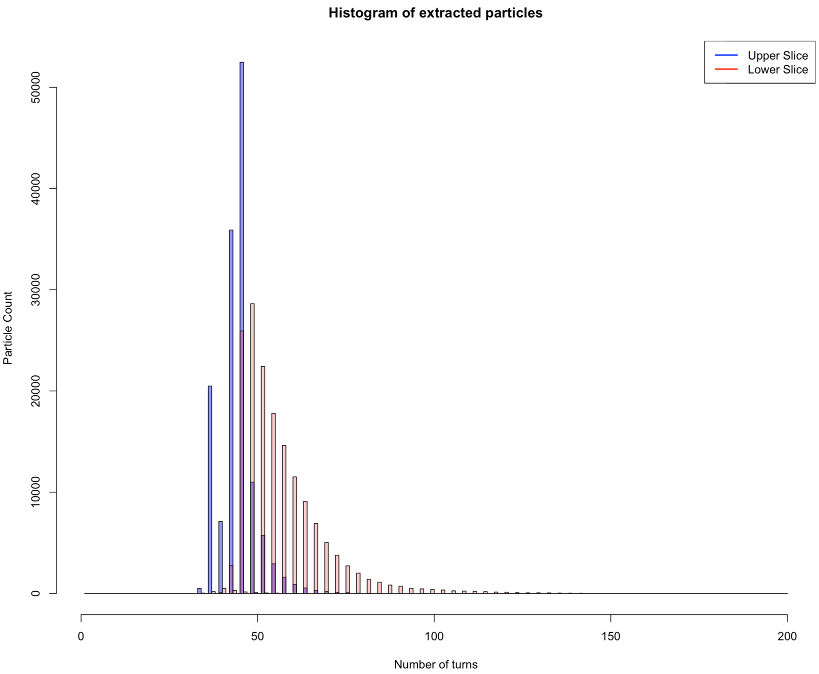
Particle TT in upper and lower slice





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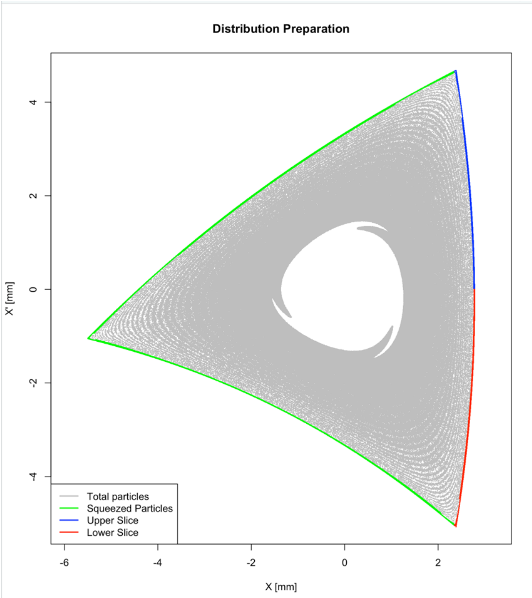

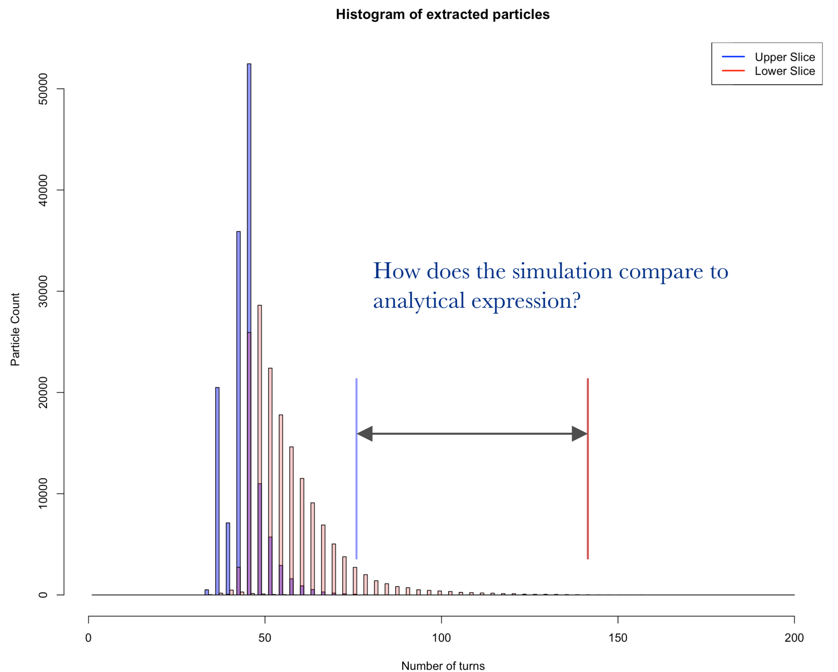

How does the simulation compare to analytical expression?

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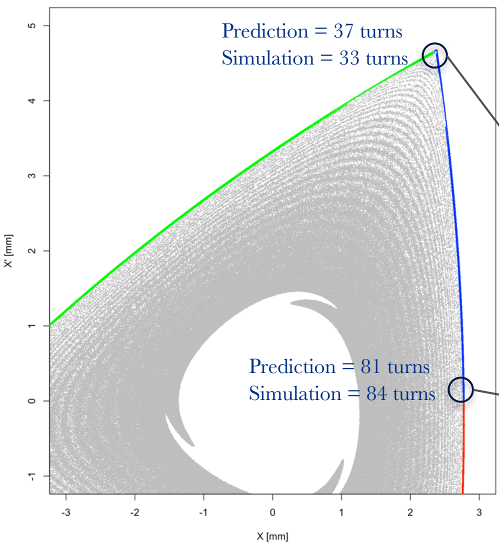
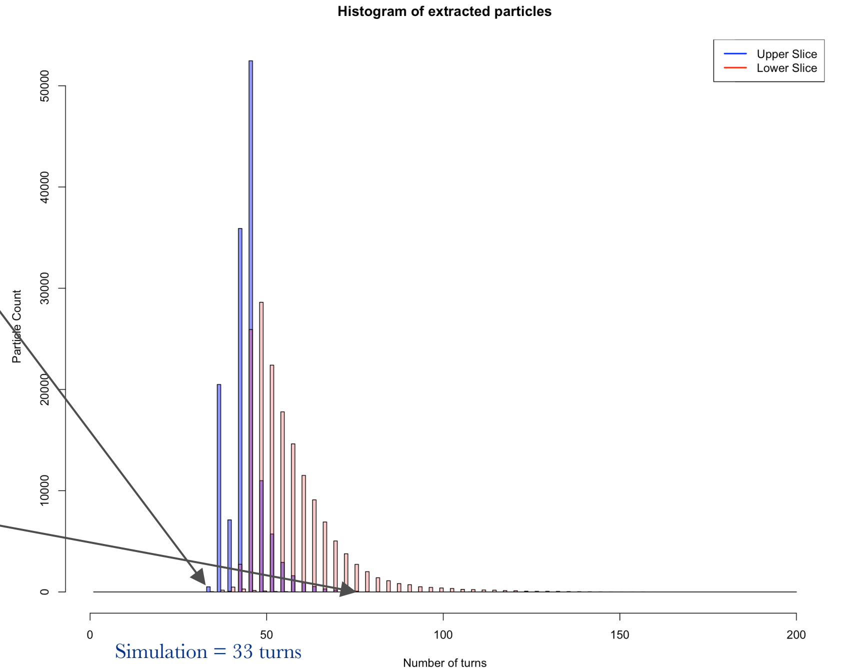
Analytical Calculation vs Simulation





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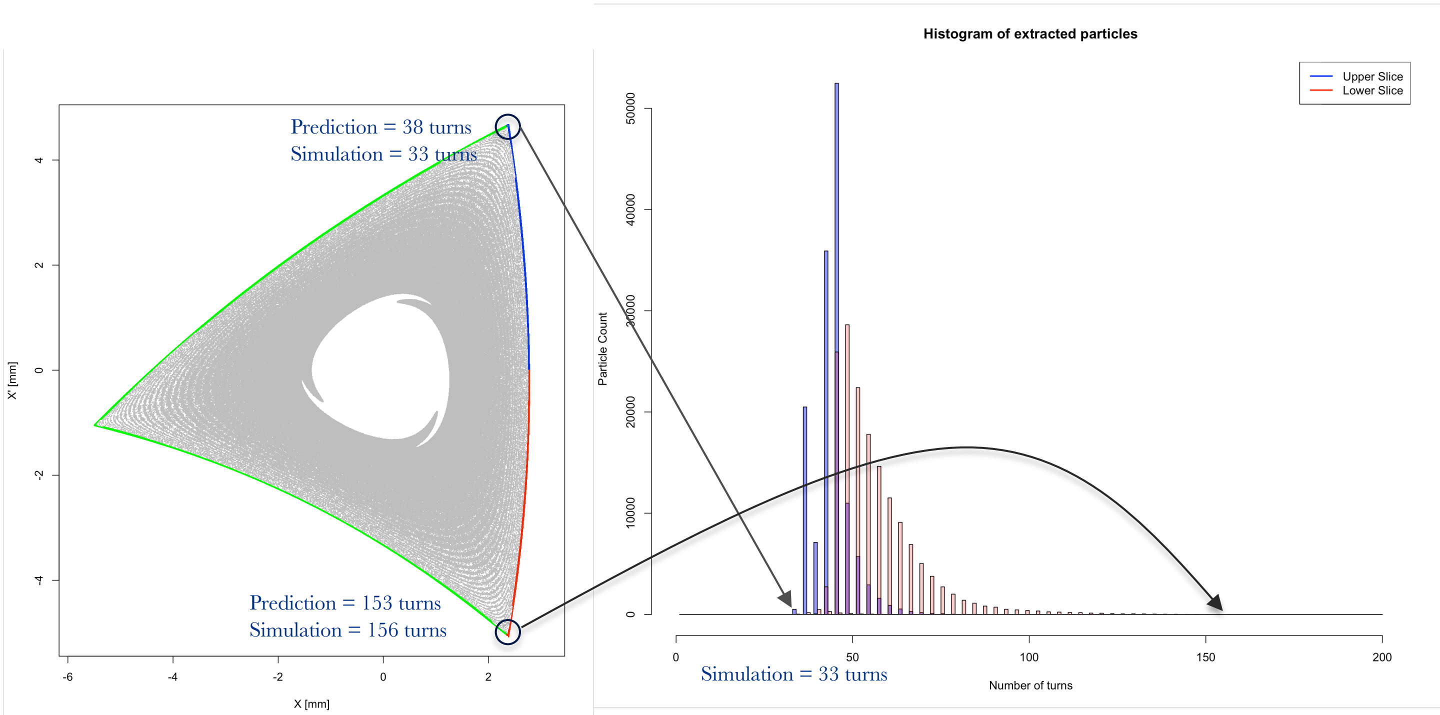

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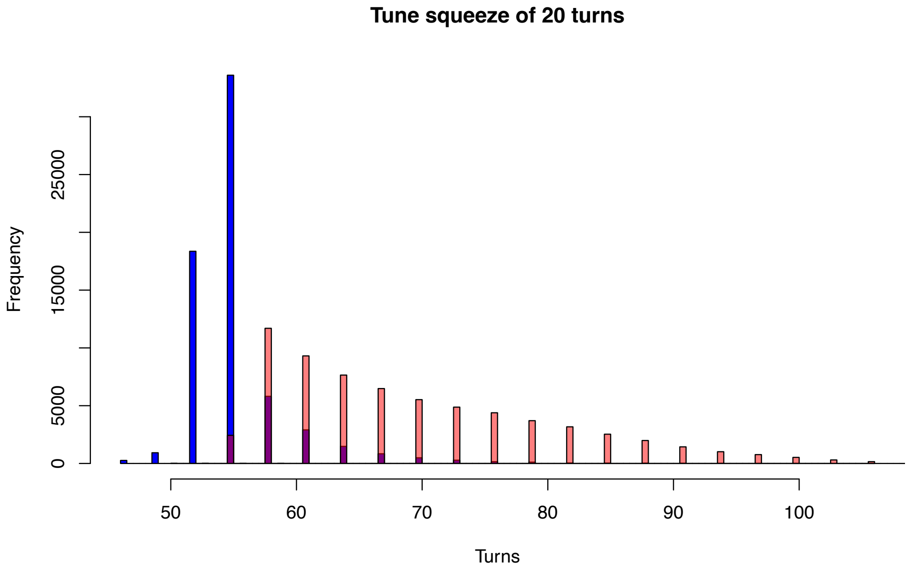
Analytical Calculation vs Simulation

Even thinner squeeze → 20 turns!!!

A tune change of 0.0000128 was done and 300,000 particles were extracted.



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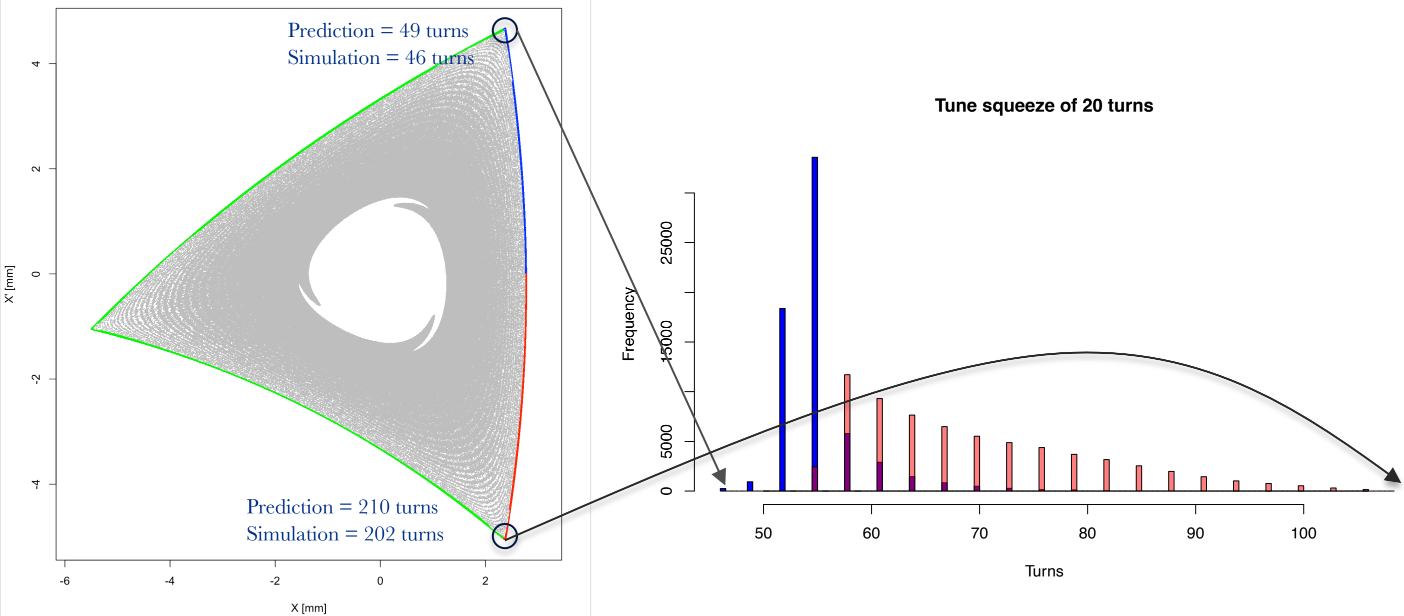

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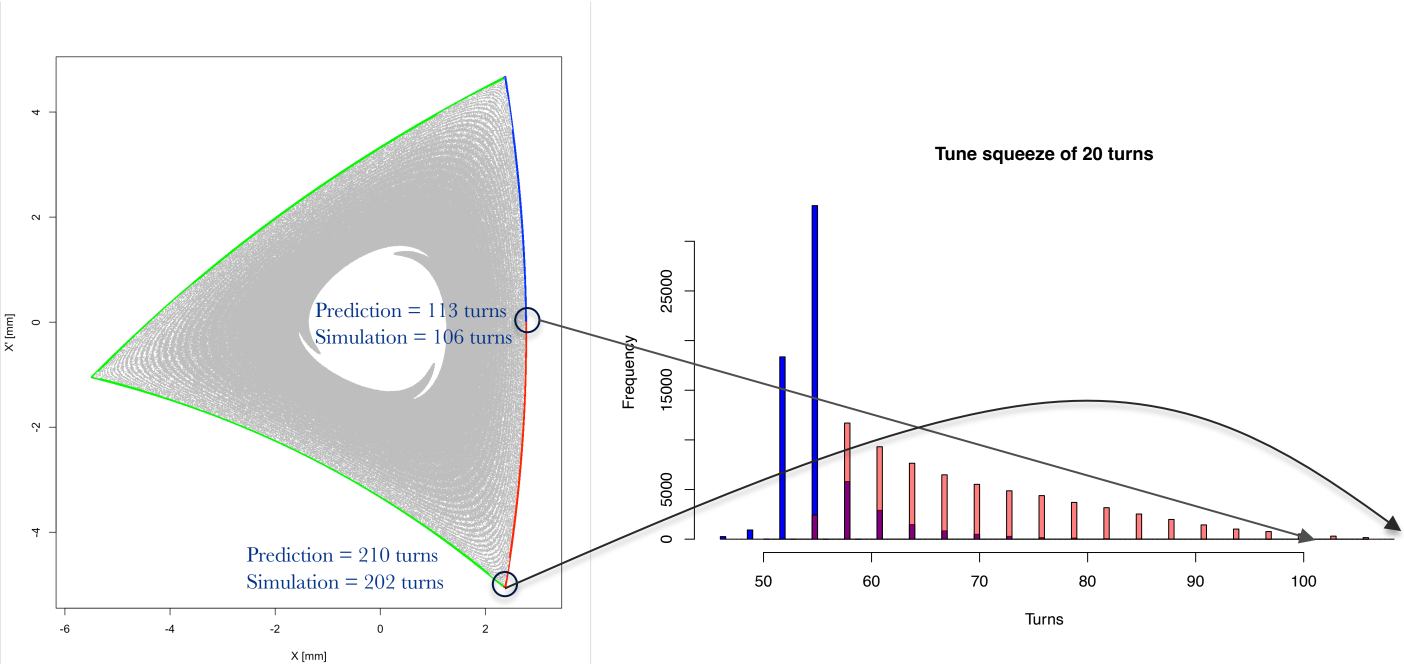
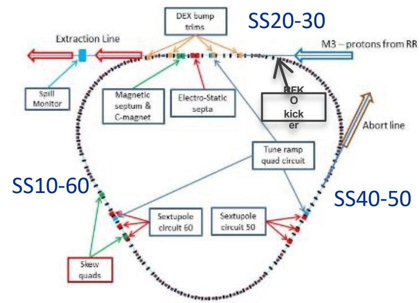
Analytical Calculation vs Simulation

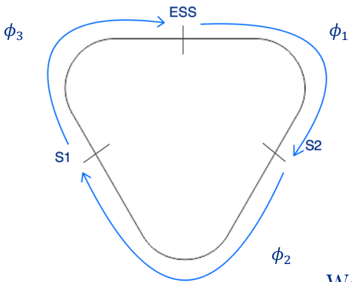


Distribution Preparation for Delivery Ring Scenario

For the resonant extraction for Mu2e, we have three dedicated fast ramping quadrupoles and six harmonic sextupoles (set of three sextupoles) in the straight sections.

To simulate the transit time, the ideal lattice file was used to get phase advances between the observation point and the sextupoles.





$$\phi_1 + \phi_2 + \phi_3 = 9.666 - \delta\nu$$

$$\phi_2 \approx \frac{\pi}{6}$$

We use the relative strength of S_1 and S_2 to rotate and orient the separatrix efficiently.

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Distribution Preparation for Delivery Ring Scenario

IN PROGRESS

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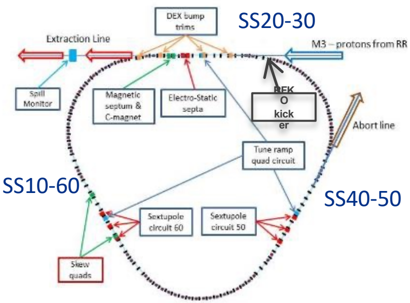

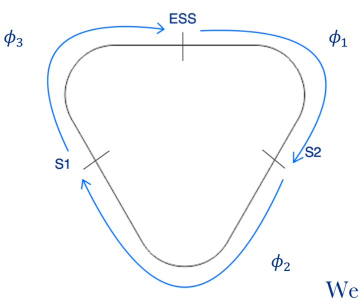

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Future Directions

- Compute the analytical histogram for all the extracted particles and compare against the histogram gotten from tracking simulation.
- Derive an expression for a truer Hamiltonian that contains higher orders in δQ, derive the equations of motion, derive the transit time and compare it against the Kobayashi Hamiltonian transit time.
- Investigate the effects of intensity dependent effects on transit time (and how one could incorporate space charge in the SX Hamiltonian) and compare with space charge tracking numerical simulations.
- Investigate ways of validating transit time not just through tracking but with the real beam.

THANK YOU